ML-HALET1

Thursday, November 13, 2025 5:36 PM

1- $E_{out}(g^{(0)}) = E_{x,y}\left[(g^{(0)} - y(x))^2\right] = E_{x,y}\left[(g^{(0)} - (f(x)+\varepsilon))^2\right] =$

$y = f(x)+\varepsilon$

$\varepsilon \sim \mathcal{N}(0,\sigma^2)$

$E_y(\varepsilon) = 0$

$E_y(\varepsilon^2) = \sigma^2$

$$= E_{x,y}\left[(g^{(0)})^2 + f(x)^2 + \varepsilon^2 + 2\varepsilon f(x) - 2g^{(0)}f(x) - 2\varepsilon g^{(0)}\right] =$$

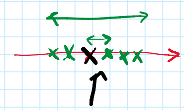
$$= \underline{E_{x,y}}\left[(g^{(0)} - f(x))^2 + 2\varepsilon(f(x) - g^{(0)}) + \varepsilon^2\right] =$$

$$= E_x\left[E_y\left[(g^{(0)} - f(x))^2 + 2\varepsilon(f(x) - g^{(0)}) + \varepsilon^2\right]\right] =$$

$$= E_x\left[(g^{(0)} - f(x))^2 + 2\underbrace{E_y(\varepsilon)}_{0}(f(x) - g^{(0)}) + \underbrace{E_y[\varepsilon^2]}_{\sigma^2}\right] =$$

$$= E_x\left[(g^{(0)} - f(x))^2\right] + \sigma^2$$

$$E_D\left[E_{out}(g^{(0)})\right] = \underbrace{E_D E_x\left[(g^{(0)} - \overline{f(x)})^2\right]}\, + \sigma^2 = \overbrace{bias^2 + var} + \sigma^2$$



2- a) $X \sim \tilde{y} \rightarrow \tilde{w} = (X^TX)^{-1}X^T\tilde{y}$

$X\tilde{w} = \tilde{y}$ $\quad \tilde{f}(z) = z^T\tilde{w} = z^T(X^TX)^{-1}X^T\tilde{y}$

Moore-Penrose inverse

b) $E\left[(\tilde{y} - \tilde{f}(z))^2\right] = \underline{var[\varepsilon_y]} + \underline{(y - E(\tilde{f}(z)))^2} + \underline{Var[\tilde{f}(z)]}$ $\quad Xw = y$

c) $\tilde{w} = (X^TX)^{-1}X^T\underbrace{\tilde{y}}_{y+\varepsilon} = \underbrace{(X^TX)^{-1}X^Ty}_{w} + (X^TX)^{-1}X^T\varepsilon$ $\quad E[\tilde{w}] = w$

bias $= 0$

d) $\underline{var(\tilde{w})} = var((X^TX)^{-1}X^T\varepsilon) = (X^TX)^{-1}X^T I_n X(X^TX)^{-1} = (X^TX)^{-1}$

$var(\tilde{f}(z)) = var(z^T\tilde{w}) = z^T(X^TX)^{-1}z$

e) ✓

↳ positive semi definite

Mathematics for Machine Learning

M.P. Deisenroth

$\underbrace{z^T(X^TX)^{-1}z} \geq 0$

3- 1) $k_1(a,b) = (a_1b_1 + a_2b_2)^2 = a_1^2b_1^2 + a_2^2b_2^2 + 2a_1a_2b_1b_2 =$

$$= \boxed{a_1^2}\boxed{b_1^2} + \boxed{a_2^2}\boxed{b_2^2} + \boxed{\sqrt{2}a_1a_2}\boxed{\sqrt{2}b_1b_2} = \phi(a)^T\phi(b) = \begin{bmatrix} a_1^2 & a_2^2 & \sqrt{2}a_1a_2 \end{bmatrix}\begin{bmatrix} b_1^2 \\ b_2^2 \\ \sqrt{2}b_1b_2 \end{bmatrix}$$

$$\rightarrow \phi(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2}x_1x_2 \end{bmatrix}$$

2) $k_2(a,b) = (1 + a^Tb)^2 = (1 + a_1b_1 + a_2b_2)^2 = 1 + a_1^2b_1^2 + a_2^2b_2^2 + 2a_1b_1 + 2a_2b_2 + 2a_1a_2b_1b_2 = \phi(a)^T\phi(b)$

$$\rightarrow \phi(x) = \begin{bmatrix} 1 \\ x_1^2 \\ x_2^2 \\ \sqrt{2}x_1 \\ \sqrt{2}x_2 \\ \sqrt{2}x_1x_2 \end{bmatrix}$$

4- $h_0 = b$ $\quad D = \{(-1,0), (0,1), (1,0)\}$ (1, 2, 3)

$h_1 = ax + b$

h_0 : $\hat{b} = \underset{b}{\operatorname{argmin}} \sum_i (y_i - b)^2 = \frac{\sum_i y_i}{n}$

LOOCV:

1 : $\hat{b} = 1/2 \longrightarrow e_1 = (\frac{1}{2})^2$

2 : $\hat{b} = 0 \longrightarrow e_2 = 1$

3 : $\hat{b} = 1/2 \longrightarrow e_3 = (\frac{1}{2})^2$

$2: \quad b = 0 \longrightarrow e_2 \ldots$

$3: \quad \hat{b} = 1/2 \longrightarrow e_3 = \left(\frac{1}{2}\right)^2$

$$\rightarrow CV_1 = \frac{1}{3}\left[1 + \frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2}$$

$h_1: \quad \hat{a}, \hat{b} = \underset{a,b}{\operatorname{argmin}} \sum_i (y_i - b - a x_i)^2$

$$\rightarrow \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} \hat{b} \\ \hat{a} \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum y_i x_i \end{bmatrix}$$

LOOCV

$2: \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{b} \\ \hat{a} \end{bmatrix} = \vec{0} \rightarrow \hat{b}, \hat{a} = 0 \rightarrow e_2 = y_2^2 = 1$

$3,1:$ challenge yourself! (use RREF)

$CV_2 = \frac{1}{3}\left[e_1 + e_2 + e_3\right]$

$CV_1 = CV_2 \longrightarrow \Theta$ ✓

---

5 - if $x^{(t)}$ misclassified. $w_{t+1} = w_t + \eta x^{(t)} y^{(t)}$

induction:

- base: $w^{(0)} = 0 = \sum_i 0 \cdot x^{(i)}$
- step: $w^{(t)} = \sum_i \alpha_i x^i \longrightarrow w^{t+1} = \sum_i \alpha_i' x^i \qquad \alpha_i' = \begin{cases} \alpha_i & i \neq t \\ \alpha_i + \eta y^{(t)} & i = t \end{cases}$ ✓

on convergence $\longrightarrow \quad w = \sum_i \underbrace{\eta y^{(i)} n_i}_{\text{why?}} x^{(i)}$
