

Applications of Stable Marriage and Related Problems

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based on paper, A Survey of the Stable Marriage Problem and Its Variants,
Iwama et al. 2008

Isfahan University of Technology
Game Theory Course, Dr. Javadi
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Outline

Stable Marriage (SM)

Some Extensions of Preference Lists

- Incomplete Preference Lists (SMI)

- Preference Lists with Ties (SMT)

- Incomplete Preference Lists with Ties (SMTI)

The Number of Stable Matchings

Optimal Stable Matchings

Other Variants

- Stable Roommates Problem (SR)

- Hospitals/Residents Problem (HR)

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- ▶ Gale-Shapley algorithm finds stable marriage in $O(n^2)$

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- ▶ Gale-Shapley algorithm with a slight modification can be applied here too

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- ▶ Three type of stability similar to SMT
- ▶ similar to SMT, There is an algorithm that decides about existence of super and strong stabilities
- ▶ Weakly stable marriage always exists and can be find in polynomial time but finding the largest one (called MAX SMTI) is NP-hard

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- ▶ Nontrivial upper bounds are still open (in 2008)

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 - ▶ regret: $r(M) = \max_{(m,w) \in M} \max\{p_m(w), p_w(m)\}$
 - ▶ egalitarian: $c(M) = \sum_{(m,w) \in M} p_m(w) + \sum_{(m,w) \in M} p_w(m)$
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- ▶ Finding optimal weakly stable marriage in SMT, for all three costs is hard, even to approximate

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- ▶ finding a matching with minimum number of blocking pairs is NP-hard and also hard to approximate

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- ▶ Rural Hospitals Theorem
 - ▶ any stable matching assigns the same number of residents to all hospitals
 - ▶ if a hospital obtains residents fewer than its quota in one stable matching, then the hospital gets the same set of residents in any stable matching

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- ▶ One-Sided Preference Lists

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- ▶ Large-scale application of stable marriage is in assigning users to servers in a large distributed Internet service
 - ▶ each user prefer servers that have a faster response time
 - ▶ each server prefer user with lower cost
 - ▶ Content Delivery Networks (CDN) try to solve this large and complex stable marriage problem

Thank You