Convex sets

Presented by: Roya Soltani

- Generally, consider the following mixed integer linear optimization (MILP) problem

$$\begin{aligned} \max \quad & \sum_m c_m x_m + \sum_k d_k y_k \\ \text{s.t.} \quad & \sum_m \tilde{a}_{im} x_m + \sum_k \tilde{b}_{ik} y_k \le \tilde{p}_i \quad \forall\, i \end{aligned}$$

$$\tilde{a}_{im} = a_{im} + \xi_{im}\hat{a}_{im}, \quad \forall\ m \in M_i$$

$$\tilde{b}_{ik} = b_{ik} + \xi_{ik}\hat{b}_{ik}, \quad \forall\ k \in K_i$$

$$\tilde{p}_i = p_i + \xi_{i0}\hat{p}_i$$

With a predefined uncertainty set U for the random variables $\xi = \{\xi_{i0}, \xi_{im}, \xi_{ik}\}$, the objective is to find solutions that remain feasible for any ξ in the set so as to immunize against infeasibility; that is,

$$\sum_m a_{im}x_m + \sum_k b_{ik}y_k + \max_{\xi \in U}\{-\xi_{i0}\hat{p}_i + \sum_{m \in M_i} \xi_{im}\hat{a}_{im}x_m + \sum_{k \in K_i} \xi_{ik}\hat{b}_{ik}y_k\} \leq p_i$$

Box-Uncertainty set

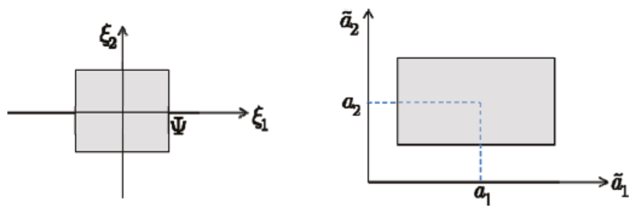


Figure 1. Illustration of box uncertainty set.

$$U_\infty = \{\xi \mid \|\xi\|_\infty \le \Psi\} = \{\xi \mid |\xi_j| \le \Psi,\ \forall\ j \in J_i\}$$

Ellipsoidal Uncertainty set

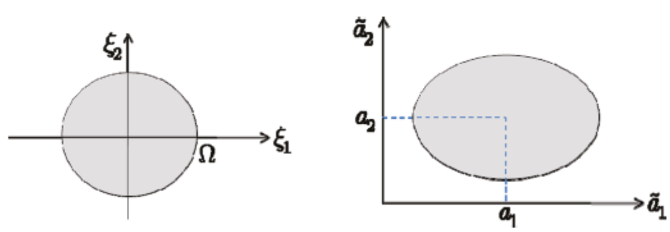


Figure 2. Illustration of ellipsoidal uncertainty set.

$$U_2 = \{\xi \mid \|\xi\|_2 \le \Omega\} = \left\{\xi \mid \sqrt{\sum_{j \in J_i} \xi_j^{\,2}} \le \Omega\right\}$$

Polyhedral Uncertainty set

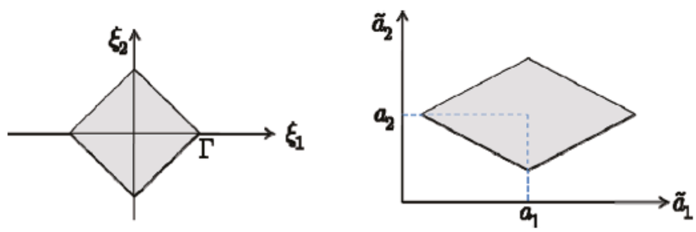


Figure 3. Illustration of polyhedral uncertainty set.

$$U_1 = \{\xi | \|\xi\|_1 \leq \Gamma\} = \left\{\xi | \sum_{j \in J_i} |\xi_j| \leq \Gamma\right\}$$

Interval-Ellipsoidal

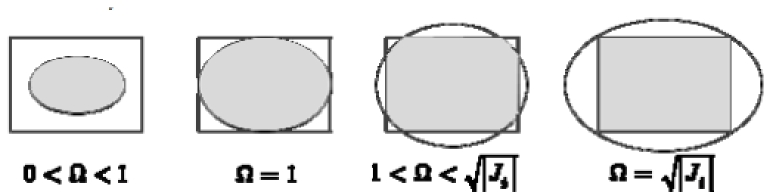
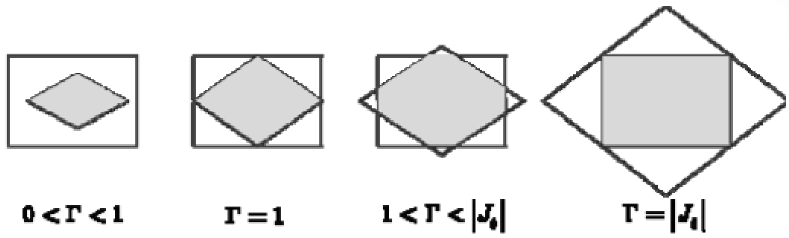


Figure 1. The "interval+ellipsoidal" uncertainty set

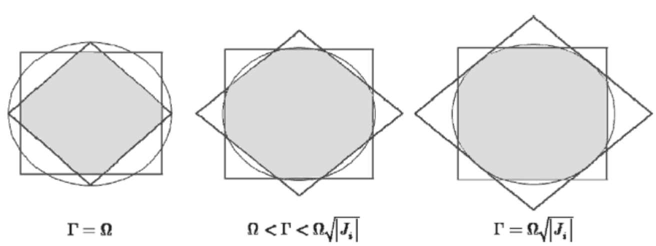
$$U_{2 \cap \infty} = \{\xi | \sum_{j \in J_i} \xi_j^2 \leq \Omega^2, \quad |\xi_j| \leq \Psi, \quad \forall j \in J_i\}$$

Interval-Polyhedral



$$U_{1 \cap \infty} = \{\xi | \sum_{j \in J_i} |\xi_j| \leq \Gamma, \quad |\xi_j| \leq \Psi, \quad \forall j \in J_i\}$$

Box+Ellipsoidal+Polyhedral" Uncertainty Set



$$U_{1 \cap 2 \cap \infty} = \{\xi | \sum_{j \in J_i} |\xi_j| \leq \Gamma, \quad \sum_{j \in J_i} \xi_j^2 \leq \Omega^2,$$
$$|\xi_j| \leq \Psi, \quad \forall j \in J_i\}$$

Robust counterpart for LHS and RHS uncertainty

Uncertainty set	Robust counterpart formulation
Box	$\sum_j a_{ij}x_j + \Psi\left[\sum_{j\in J_i}\hat{a}_{ij}\lvert x_j\rvert + \hat{b}_i\right] \le b_i$
Ellipsoidal	$\sum_j a_{ij}x_j + \left[\Omega\sqrt{\sum_{j\in J_i}\hat{a}_{ij}^2x_j^2 + \hat{b}_i^2}\right] \le b_i$
Polyhedral	$\begin{cases}\sum_j a_{ij}x_j + z_i\Gamma \le b_i \\ z_i \ge \hat{a}_{ij}\lvert x_j\rvert \quad \forall j \in J_i, \qquad z_i \ge \hat{b}_i\end{cases}$
Interval+ Ellipsoidal	$\sum_j a_{ij}x_j + \left[\sum_{j\in J_i}\hat{a}_{ij}\lvert x_j - z_{ij}\rvert + \hat{b}_i\lvert 1 + z_{i0}\rvert + \Omega\sqrt{\sum_{j\in J_i}\hat{a}_{ij}^2z_{ij}^2 + \hat{b}_i^2z_{i0}^2}\right] \le b_i$
Interval+ Polyhedral	$\begin{cases}\sum_j a_{ij}x_j + \left[z_i\Gamma + \sum_{j\in J_i}p_{ij} + p_{i0}\right] \le b_i \\ z_i + p_{ij} \ge \hat{a}_{ij}\lvert x_j\rvert \quad \forall j \in J_i, \quad z_i + p_{i0} \ge \hat{b}_i \\ z_i \ge 0, p_{ij} \ge 0, p_{i0} \ge 0\end{cases}$
Interval+ Ellipsoidal+ Polyhedral	$\begin{cases}\sum_j a_{ij}x_j + \left[z_i\Gamma + \sum_{j\in J_i}\lvert p_{ij}\rvert + \lvert p_{i0}\rvert + \Omega\sqrt{\sum_{j\in J_i}w_{ij}^2 + w_{i0}^2}\right] \le b_i \\ z_i \ge \lvert\hat{a}_{ij}x_j - p_{ij} - w_{ij}\rvert \quad \forall j \in J_i, \qquad z_i \ge \lvert\hat{b}_i + p_{i0} + w_{i0}\rvert\end{cases}$

RHS Uncertainty

$$\sum_j a_{ij}x_j \le \tilde{b}_i \qquad\qquad \tilde{b} = b_i + \xi_i\hat{b}_i$$

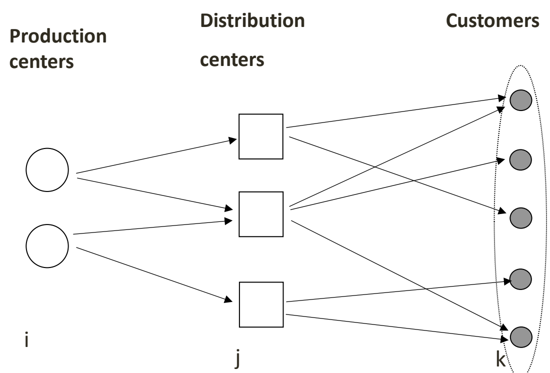
$$\sum_j a_{ij}x_j + \left[\max_{\xi \in U}\{-\xi_i\hat{b}\}\right] \le b_i$$

$$\sum_j a_{ij}x_j + \Delta\hat{b}_i \le b_i$$

where Δ is defined as Ψ, Ω, Γ, $\min(\Omega,1)$, $\min(\Gamma,1)$, $\min(\Omega, \Gamma, 1)$ for the box, ellipsoidal, polyhedral, "interval+ellipsoidal", "interval+polyhedral", and "interval+polyhedral+ellipsoidal" uncertainty sets, respectively.

زنجیره تامین الوار (MDF)

- 3 کارخانه، 10 محل بلقوه برای مراکز توزیع و 32 مرکز مشتری



پارامترهای مساله

- d_k : تقاضای مشتری k
- g_j: هزینه ثابت احداث مرکز توزیع j
- c_{ij} a_{jk} هزینه حمل و نقل بین مراکز j و k
- cw_i : ظرفیت تولید در مرکز تولید i
- cy_j : ظرفیت مرکز توزیع j
-

متغیر های مساله

- x_{ij} : مقدار محصولی که از مرکز تولید i به مرکز توزیع j حمل می شود.
- u_{jk} : مقدار محصولی که از مرکز توزیع j به مشتری kحمل می شود.
- y_j : متغیر باینری نشان دهنده احداث مرکز توزیع در مکان j

مدل مساله قطعی

$$\min \sum_j g_j\, y_j + \sum_i \sum_j c_{ij} x_{ij} + \sum_j \sum_k a_{jk} u_{jk}$$

s.t.

$$\sum_j u_{jk} \geq d_k \quad ,\forall k$$

$$\sum_i x_{ij} - \sum_k u_{jk} = 0 \quad ,\forall j$$

$$\sum_i x_{ij} \leq cw_i \quad \forall i$$

$$\sum_k u_{jk} \leq cy_j\, y_j \quad \forall j$$

مفروضات مساله

- تقاضا d_k غیر قطعی است.
- هزینه حمل و نقل بین مراکز i و j c_{ij} غیر قطعی است.

Gams code

```
Sets
      i      "factory index"             /1*3/
      j      "Distribution center index"   /1*10/
      k       "customer zone index"        /1*32/
;

Positive variables
      x(i,j)       "amount of products shipped from factory i to DC j under scenario s"
      u(j,k)       "amount of products shipped from DC j to customer zone k under scenario"
      v(i,j)
      v1(j)
     ;

Binary variable
      y(j)          "binary variable representing activation of DC" ;

Variables
      z , OBJ     "objective function value"
   ;
```

- Parameters
- d(k) "demand of customer k under scenario s"
- c(i,j) "transportation cost of unit product from factory i to DC j"
- a(j,k) "transportation cost of unit product from DC j to customer zone k“;

Demands (consider the first column from the Excel file)

- d(k)
- /1 487
- 2 490
- 3 519
- 4 478
- 5 522
- 6 491
- 7 499
- 8 493
- 9 498
- 10 513
- 11 525
- 12 542
- 13 521
- 14 505
- 15 522
- 16 514
- 17 494
- 18 477
- 19 486
- 20 506
- 21 481
- 22 514
- 23 474
- 24 504
- 25 489
- 26 501
- 27 526
- 28 492
- 29 525
- 30 506
- 31 502
- 32 498/
- ;

Read data from Excel

- $CALL GDXXRW.EXE stochastic.xlsx   par=c rng=c!B3:L7 Rdim=1 Cdim=1   par=a rng=a!B3:AH14 Rdim=1 Cdim=1

- $GDXIN stochastic.gdx
- $LOAD   c, a
- $GDXIN

- Display c, a ;

```
Equations
        demand(k)          "demand satisfy constraint"
        balance(j)         "balance constraint"
        cap1(i)            "capacity limitation at factory i"
        cap2(j)            "capacity limitation at DC j"

        objective          "objective function"
         Cons
         cons1(i,j)
         cons3(j)
         cons4(j)
         cons2(i,j)
                ;
```

Scalars

- scalar SAY/1/;
- scalar deltad/0.2/;
- scalar deltac/0.1/;

Box uncertainty

Deltac=0.1 , deltad=0.2

```
objective..     z =e=OBJ ;
Cons..sum(j, g(j) * y(j)) + sum((i,j), c(i,j) * x(i,j)) + sum((j,k),  a(j,k) *
u(j,k))+SAY*( sum((i,j),deltac * c(i,j) * Ux(i,j)))=l=OBJ;
cons1(i,j)..  Ux(i,j)=g=x(i,j) ;
cons2(i,j)..  -Ux(i,j)=l=x(i,j) ;

demand(k)..   -sum(j,u(j,k))+ d(k)+SAY*deltad*d(k)=l=0 ;

balance(j)..  sum(i, x(i,j)) =e= sum(k, u(j,k)) ;

cap1(i)..     sum(j, x(i,j)) =l= cw(i) ;

cap2(j)..     sum(i, x(i,j)) - (cy(j)*y(j))  =l= 0 ;
```

```
Model stochastic /all/ ;
stochastic.reslim=1000000;
stochastic.optCR= 0;
Solve stochastic using minlp minimizing z ;

Display z.l, y.l, x.l, u.l;
```

Gams results

SAY=1

---- 152 VARIABLE z.L = 4496251.880

---- 152 VARIABLE y.L binary variable representing activation of DC

2 1.000, 6 1.000, 7 1.000, 9 1.000, 10 1.000

---- 152 VARIABLE x.L amount of products shipped from factory i to DC j unde
r scenario s

	2	6	7	9	10
1	3001.000	3889.000	3966.000	4456.000	4000.800

Ellipsoidal Uncertainty

```
objective..     z =e=OBJ ;
Cons..sum(j, g(j) * y(j)) + sum((i,j), c(i,j) * x(i,j)) + sum((j,k),  a(j,k)
* u(j,k))+OMG*sqrt(sum((i,j), (0.1*c(i,j) * x(i,j))**2))=l=OBJ;

demand(k)..   -sum(j,u(j,k))+ d(k)+OMG1*0.2*d(k)=l=0 ;

balance(j)..  sum(i, x(i,j)) =e= sum(k, u(j,k)) ;

cap1(i)..     sum(j, x(i,j)) =l= cw(i) ;

cap2(j)..     sum(i, x(i,j)) =l= cy(j)*y(j) ;
```

Gams Results

```
OMG=OMG1=1
----    143 VARIABLE z.L                 =  4484149.497

----    143 VARIABLE y.L  binary variable representing activation of DC

2  1.000,    6  1.000,    7  1.000,    9  1.000,    10 1.000


----    143 VARIABLE x.L  amount of products shipped from factory i to DC j unde
                r scenario s

          2           6           7           9          10

1    1500.500    3858.800    3966.000    4456.000    2015.500
2    1500.500
3                                                    2015.500
```

Polyhedral uncertainty

objective.. z =e=OBJ ;

Cons..sum(j, g(j) * y(j)) + sum((i,j), c(i,j) * x(i,j)) + sum((j,k), a(j,k) * u(j,k))+Gama0*Z0=l=OBJ;

cons1(i,j).. Z0 =g= deltac * c(i,j) * x(i,j) ;

demand(k).. -sum(j,u(j,k))+ d(k)+Gama1*deltad*d(k)=l=0 ;

balance(j).. sum(i, x(i,j)) =e= sum(k, u(j,k)) ;

cap1(i).. sum(j, x(i,j)) =l= cw(i) ;

cap2(j).. sum(i, x(i,j)) - (cy(j)*y(j)) =l= 0 ;

GAMS Results

Gama0=Gama1=1

---- 150 VARIABLE z.L = 4480306.000

---- 150 VARIABLE y.L binary variable representing activation of DC

2 1.000, 6 1.000, 7 1.000, 9 1.000, 10 1.000

---- 150 VARIABLE x.L amount of products shipped from factory i to DC j unde

r scenario s

	2	6	7	9	10
1	3001.000	3858.800	3966.000	4456.000	4031.000

Interval-Ellipsoidal

```
objective..     z =e=OBJ ;
Cons..sum(j, g(j) * y(j)) + sum((i,j), c(i,j) * x(i,j)) + sum((j,k),  a(j,k)
* u(j,k))+sum((i,j),deltac*c(i,j)*v(i,j))+OMG0*sqrt(sum((i,j),
power((deltac*c(i,j)),2) * power(Z0(i,j),2)))=l=OBJ;

cons1(i,j)..  x(i,j)-Z0(i,j)=l=v(i,j);

cons2(i,j)..  x(i,j)-Z0(i,j)=g=-v(i,j);

demand(k)..   -sum(j,u(j,k))+d(k)+min(OMG1,1)*deltad*d(k)=l=0
;
```

```
balance(j)..  sum(i, x(i,j)) =e= sum(k, u(j,k)) ;

cap1(i)..     sum(j, x(i,j)) =l= cw(i) ;

cap2(j)..     sum(i, x(i,j)) - (cy(j)*y(j)) =l= 0 ;
```

GAMS Results for Interval-Ellipsoidal

OMG0=OMG1=2

---- 153 VARIABLE z.L = 4492425.548

---- 153 VARIABLE y.L binary variable representing activation of DC

2 1.000, 6 1.000, 7 1.000, 9 1.000, 10 1.000

---- 153 VARIABLE x.L amount of products shipped from factory i to DC j unde
r scenario s

	2	6	7	9	10
1	1500.500	3858.800	3965.994	4456.000	2015.500
2	1500.500		0.006		
3				2015.500	

Interval-Polyhedral

Positive variables

x(i,j) "amount of products shipped from factory i to DC j under scenario s"

u(j,k) "amount of products shipped from DC j to customer zone k under scenario"

P(i,j)

P1(j)

Z0 "auxiliary variable for objective function"

Z1(j) "auxiliary variable for constraint" ;

;

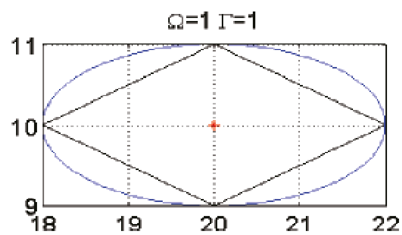
```
objective..     z =e=OBJ ;
Cons..sum(j, g(j) * y(j)) + sum((i,j), c(i,j) * x(i,j)) + sum((j,k),  a(j,k)
* u(j,k))+Gama0*Z0+sum((i,j), P(i,j))=l=OBJ;
cons1(i,j)..  Z0+P(i,j) =g= deltac * c(i,j) * x(i,j) ;

demand(k)..   -sum(j,u(j,k))+
d(k)+min(Gama1,1)*deltad*d(k)=l=0 ;

balance(j)..  sum(i, x(i,j)) =e= sum(k, u(j,k)) ;

cap1(i)..     sum(j, x(i,j)) =l= cw(i) ;

cap2(j)..     sum(i, x(i,j)) - (cy(j)*y(j)) =l= 0 ;
```

GAMS Results

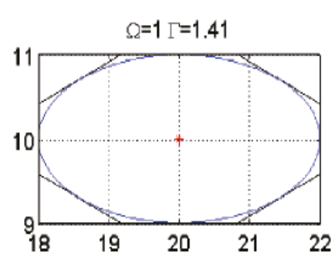
GAMA0=GAMA1=0.5

```
----    150 VARIABLE z.L                =  4110712.170

----    150 VARIABLE y.L  binary variable representing activation of DC

4 1.000,    5 1.000,    7 1.000,    9 1.000


----    150 VARIABLE x.L  amount of products shipped from factory i to DC
j unde
                  r scenario s

        4         5         7         9

1    4678.000    4603.400    3966.000    4456.000
```

- Deterministic= 3725152
- Box= 4496251
- Ellipsoidal= 4484149.497 Φ=Ω=1
- Polyhedral= 4480306 Γ= Ω =1 polyhedral is inscribed by ellipsoid

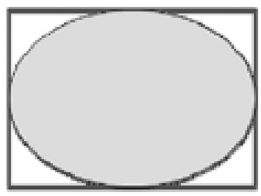


- Polyhedral= 4785281.283 Γ= 1.414 ($\Gamma = \Omega\sqrt{2}$) ellipsoid is inscribed by polyhedral
- Ellipsoidal= 4484149.497 Ω=1



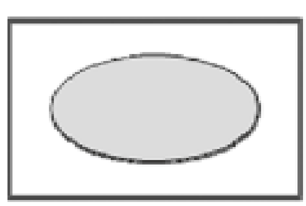
Interval-ellipsoidal= 4484149.497

- Interval-ellipsoidal= 4484149.497
- Ellipsoidal= 4484149.497 Ω=1



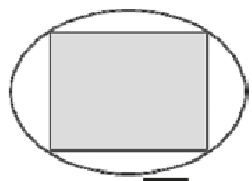
$\Omega = 1$

- Interval-ellipsoidal= 4112738.541 Ω=0.5
- Ellipsoidal= 4484149.497 Ω=1
- Box= 4496251



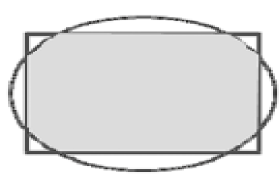
$0 < \Omega < 1$

- Interval-ellipsoidal= 4496251.878 $\Omega_1=(32)^{1/2}=5.477$ $\Omega_2=(32)^{1/2}=5.66$
- Box= 4496251



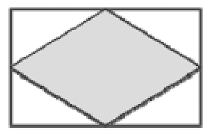
$$\Omega = \sqrt{|J_i|}$$

- Interval-ellipsoidal= 4492425.548 $\Omega_{1=}\ \Omega_2=2$
- Box= 4496251 $\Phi=1$
- Ellipsoidal= 5245066.789 $\Omega_{1=}\ \Omega_2=2$



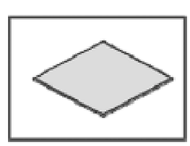
$$1 < \Omega < \sqrt{|J_i|}$$

- Interval-polyhedral=4480306 Γ=1
- Polyhedral= 4480306 Γ=1



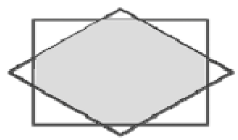
$$\Gamma = 1$$

- Interval-polyhedral= 4110712.170 Γ=0.5
- Polyhedral= 4480306 Γ =1



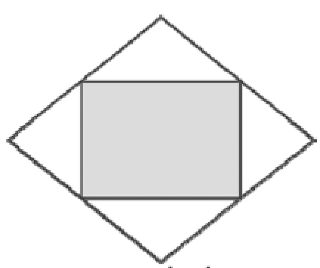
$$0 < \Gamma < 1$$

- Interval-polyhedral= 4492950.780 Γ=5,5
- Polyhedral= 7497541.667 Γ=5,5
- Box= 4496251 Φ=1



$$1 < \Gamma < |J_i|$$

- Interval-polyhedral= 4496251 Γ=30,32
- Box= 4496251
-



$$\Gamma = |J_i|$$

- Interval-ellipsoidal= 4484149.497 Ω=1
- Interval-polyhedral= 4493738.465 $\Gamma_1=1*\sqrt{30}$, $\Gamma_1=1*\sqrt{32}$

Comparing the "interval+ellipsoidal" and the "interval +polyhedral" set based model from Figures 10a, 12a, and 13a, when $\Gamma = \Omega(|J_i|)^{1/2}$, the "interval+polyhedral" set based solution is always worse than the "interval +ellipsoidal" based solution, which is verified by the

fact that the "interval+polyhedral" uncertainty set is larger and completely covers the "interval+ellipsoidal"

- Interval-ellipsoidal= 4484149.497 Ω=1
- Interval-polyhedral=4480306 Γ=1

set; when $\Gamma = \Omega$, the "interval+polyhedral" set based solution is always better than the "interval+ellipsoidal" based solution because the "interval+polyhedral" uncertainty set is smaller and completely covered by the "interval+ellipsoidal" set.