Design and Simulation of LQR Controller with the Linear Inverted Pendulum

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***Abstract*—This paper focused on modeling and performance analysis of linear inverted pendulum and design and simulation of LQR controller. Main to introduce how to built the mathematic model and the analysis of it's system performance, then design a LQR controller in order to get the much better control. Simulation is done to show the efficiency and feasibility of proposed approach.**

Keywords-inverted pendulum; system performance; LQR controller; simulation

I. INTRODUCTION

Inverted pendulum system is a typical model of multi-variable, nonlinear, essentially unsteady system, which is perfect experiment equipment not only for pedagogy but for research because many abstract concepts of control theory can be demonstrated by the system-based experiments. The research on such a complex system involves many important theory problems about system control, such as nonlinear problems, robustness, ability and tracking problems. Therefore, as an ideal example of the study, the inverted pendulum system in the control system has been universal attention. And it has been recognized as control theory, especially the typical modern control theory research and test equipment. So it is not only the best experimental tool but also an ideal experimental platform. The research of inverted pendulum has profound meaning in theory and methodology, and has valued by various countries' scientists [1].

Linear quadratic optimal control theory(LQR) is the most important and the most comprehensive of a class of optimization-based synthesis problem, obtain the performance index function for the quadratic function of the points system, not only taking into account both performance requirements, but also taking into account the control energy requirements.

II. MATHEMATICAL MODEL

First, confirm that you have the correct template for your paper size. This template has been tailored for output on the A4 As a linear inverted pendulum system, the mathematical mode will be established by the method of mechanical analysis as follows. The Schematic diagram of the inverted pendulum is shown in Figure 1.

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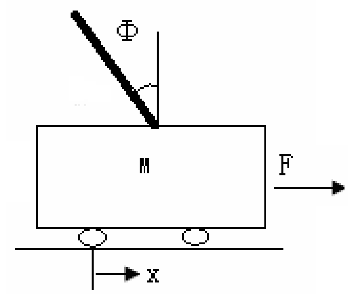

Figure 1. Schematic diagram of linear inverted pendulum

Isolation force analysis of the car as shown in Figue 2, and isolation force analysis of the rod as shown in Figure 3.

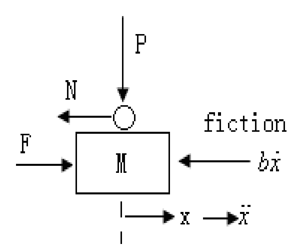

Figure 2. Isolation force analysis of the car

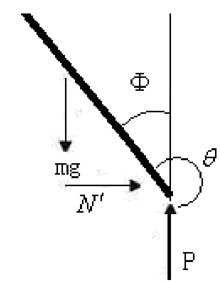

Figure 3. Isolation force analysis of the rod

Analysis the horizontal direction force of the car via the Newton's law, we can get the following equation:

$$M\ddot{x} = F - b\dot{x} - N$$

Similarly, from the analysis of the pendulum which suffered the horizontal force ,we can get the following equation:

$$N' = m\frac{d^2}{dt^2}(x + l\sin\Phi)$$

Because:

$$\Phi = \theta - \pi\,,\quad N' = -N$$

So:

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$$N' = m\frac{d^2}{dt^2}(x + l\sin(\theta - \pi)) = m\frac{d^2}{dt^2}(x - l\sin\theta)$$

That is:

$$N' = m\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta \qquad (1)$$

Substituting this equation into the equation, we can get the system equations of motion.

$$(M+m)\ddot{x} + b\dot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = F \qquad (2)$$

Similarly, from the analysis of the pendulum which suffered the vertical force ,we can get the following equation:

$$P - mg = -ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta \qquad (3)$$

Suppose $\theta = \pi + \Phi$,and make u to represent the controlled object with the input force F, the latter two linear equations of motion are as follows:

$$\begin{cases} \left(l + ml^2\right)\ddot{\Phi} - mgl\Phi = ml\ddot{x} \\ (M+m)\ddot{x} + b\dot{x} - ml\ddot{\Phi} = u \end{cases} \qquad (4)$$

Take the equations(4)with Laplace transform, to be:

$$\begin{cases} \left(l + ml^2\right)\Phi(s)s^2 - mgl\Phi(s) = mlX(s)s^2 \\ (M+m)X(s)s^2 + bX(s)s - ml\Phi(s)s^2 = U(s) \end{cases}$$

Note: the transfer function is derived assuming the initial condition is 0,as the output angle is Φ ,solving the equations ,we can get:

$$\frac{\Phi(s)}{X(s)} = \frac{mls^2}{\left(I + ml^2\right)s^2 - mgl}$$

$$(M+m)\left[\frac{\left(I+ml^2\right)}{ml} - \frac{g}{s}\right]\Phi(s)s^2 + b\left[\frac{\left(I+ml^2\right)}{ml} - \frac{g}{s^2}\right]\Phi(s)s$$

$$-ml\Phi(s)s^2 = U(s)$$

After finishing, obtain the transfer function:

$$\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}s^2}{s^4 + \frac{b\left(I+ml^2\right)}{q}s^3 - \frac{(M+m)mgl}{q}s^2 - \frac{bmgl}{q}s}$$

Meanwhile：

$$q = \left[(M+m)\left(I+ml^2\right) - \left(ml^2\right)\right]$$

By the principles of modern control theory, and then substituted the inverted pendulum system parameters which is designed by ourselves into the state space equation[2 ~ 4]:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\Phi} \\ \ddot{\Phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.098 & 0.629 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.237 & 26.853 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \Phi \\ \dot{\Phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.883 \\ 0 \\ 2.356 \end{bmatrix} u$$

$$y = \begin{bmatrix} x \\ \dot{x} \\ \Phi \\ \dot{\Phi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \Phi \\ \dot{\Phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

III. CHARACTERISTIC ANALYSIS OF SYSTEM

After obtaining the mathematical model of the system features, we need to analyze the stability, controllability and observability of system's in order to further understand the characteristics of the system [5~7].

A. Stability Analysis

If the closed-loop poles are all located in the left “s” plane, the system must be stable, otherwise the system instability. In MATLAB, to strike a linear time-invariant system, the characteristic roots can be run by eig (a,b) function. According to the sufficient and necessary conditions for stability of the system, we can see the inverted pendulum system is unstable.

B. Controllability Analysis

Linear time-invariant controllablility systems necessary and sufficient condition is:

$$rank\left[B \;\; AB \;\; \cdots \;\; A^{n-1}B\right] = n$$

The dimension of the matrix A is n.

In MATLAB, the function ctrb (a,b) is used to test the controllability of matrix ,through the calculation we can see that the system is status controllable.

C. Observability Analysis

Linear time-invariant observability systems necessary and sufficient condition is:

$$rank\left[C \quad CA \quad \cdots \quad CA^{n-1}\right]^T = n$$

In MATLAB, the function obsv(a,b) is used to test the observability of matrix ,through the calculation we can see that the system is status considerable.

IV. DESIGN AND SIMULATION OF LQR CONTROLLER

A. LQR optimal control strategy

LQR control theory, is an important tool in modern control theory. It provides an effective analysis method multi-variable for feedback system design, adapt to time-varying systems

,handle the perturbation signal and measurement noise problems and deals with finite and infinite time horizon.

The basic principles of LQR linear quadratic optimal control, by the system equation:

$$\begin{cases} \dot{X} = AX + Bu \\ y = CX + Du \end{cases}$$

And the quadratic performance index function:

$$J = \frac{1}{2}\int_0^\infty \left(X^T Q X + u^T R u \right) dt \qquad (5)$$

Q is a positive semi-definite matrix, R is positive definite matrix.

If the inverted pendulum deviates from the zero state system due to the outside interference, then u^* can make the system back to zero state's vicinity ,at the same time satisfy type (5) and obtain the minimum value, from optimal control theory, we can see that if(5) type obtain the minimum, the optimal control law is :

$$u^* = v - KX \qquad (6)$$

Type (6), K is linear optimal feedback gain matrix, v is the reference signal input.

In MATLAB, utility function lqr,can get the corresponding feedback gain matrix K = lqr (A, B, Q, R), thus completing the LQR control strategy design.

On the face of the inverted pendulum system modeling, parameter determination and the analysis of characteristics of the system, because the system is unstable, anti-jamming ability is poor, but it is status controllable and status considerable so it is necessary to design a corresponding controller in order to ensure that the system can be controlled by stable manner. As we have got the system state equation, so it is using to accept the linear quadratic optimal control and other control methods to control it [8][9].

B. Setting and simulation of LQR control parameter

Using the LQR method, the effect of optimal control depends on the selection of weighting matrices Q and R,if Q and R selected not properly, it make the solution can not meet the actual system performance requirements. In general, Q and R are taken the diagonal matrix, the current approach for selecting weighting matrices Q and R is simulation of trial,after finding a suitable Q and R, it allows the use of computers to find the optimal gain matrix K easily.

According to the state equation which we had got according to straight line of the inverted pendulum system,the four state variables $x, \dot{x}, \Phi, \dot{\Phi}$ representing the cart displacement, car speed, the pendulum angle and pendulum angular velocity, the output $y = \begin{bmatrix} x & \dot{x} & \Phi & \dot{\Phi} \end{bmatrix}^T$ including the trolley position and pendulum angle.

Assuming the whole state feedback can be achieved (four state variables can be measured), so the feedback control law can be found to determine the vector K. In MATLAB, the function Lqr allows you to select two parameters——R & Q, These two parameters are used to balance the input and the weight of state, the simplest situation is assume R = 1, $Q = C \times C'$ Of course, you can also change the Q matrix nonzero elements to adjust the controller to get the desired response. Here the pursuit of matrix K, In MATLAB, statement is K = lqr(A, B,Q, R) [10] [11].

The following is the analysis according to Q and R which are selected based on actual.

- when Q=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1], R=1, part of the MATLAB procedure is as follows

```
>> A=[];B=[];
>> Q=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];R=1;
>> K=lqr(A,B,Q,R)
K =
-1.0000   -2.3738  -37.9020   -6.6154
>> C=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];D=0;
>> sys=ss(A-B*K,B*K(1),C,D);
>> step(sys,5)
```

Get the system step response curves shown in Figure 4.

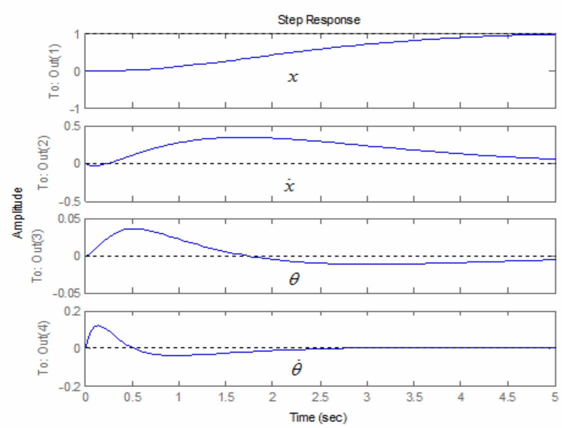

Figure 4. LQR system step response

We can see from the figure 4, the system response overshoot is small, but the settling time and rise time is too large.

That increase the displacement of the car pendulum and the swinging angle of the rod, MATLAB procedure is as follows:

- when Q=[100 0 0 0;0 1 0 0;0 0 100 0;0 0 0 1], R=1

```
>> A=[];B=[]';
>> Q=[100 0 0 0;0 1 0 0;0 0 100 0;0 0 0 1];R=1;
```

```
>> K=lqr(A,B,Q,R)
K =
-10.0000   -9.3840  -57.8514  -10.1083
>> C=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];D=0;
>> sys=ss(A-B*K,B*K(1),C,D);
>>step(sys,5)
```

Get the system step response curves shown in Figure 5:

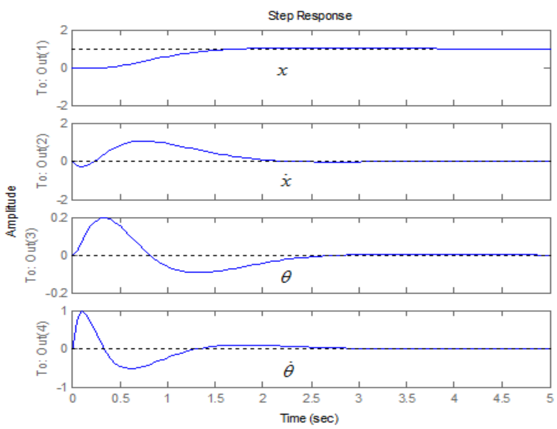

Figure 5 . Q11, Q33 increased after the LQR system step response

Comparing Figure 4 with Figure 5 we can see that with Q11, Q33 increase, R constant ,then the system feedback gain matrix K is larger, through the corresponding results of the system ,we will find the time rise and reduced overshoot. the system can stabilize in the 3s after the Q increase.

V. CONCLUSION

This paper introduces the modeling and performance analysis of a linear inverted pendulum, designs LOR controller based on its performance index .Then use MATLAB language to implement the control system simulation, get all the state variables and the control variable's response curve of straight line inverted pendulum. The simulation shows the designed controller. is effective and feasible.

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