An Introduction to Noise Reduction Using Singular Value Decomposition

AliReza Tabatabaeian [1], Farnaz Sedighin[2]

Abstract
Nonlinear methods in Noise reduction has a tremendous significance because inherent linear methods in which the noise can be reduced without applying a low-pass filter are rare. So a new nonlinear approach is needed to reduce additive white gaussian noise in signals. A novel approach for noise reduction in 1-D signals is to embed a signal by applying a rolling window which makes a Hankel matrix(or a trajectory matrix), decompose the resultant matrix using singular value decomposition, choose the best singular value for approximation, approximate it with a low-rank matrix, and reconstruct a noiseless signal using anti-diagonal averaging. Different approaches and visualizations are used here to understand what will happen if a low-rank approximation is applied to a noisy signal with additive white gaussian noise. Different criteria will be considered in the below context[1, 2, 3, 4].

Keywords
Denoising — Singular Value Decomposition — Singular Spectrum Analysis — Noise Reduction

[1] *Department of Electrical Engineering, AmirKabir University of Technology, Tehran, Iran*
[2] *Research Assistant Professor, Medical Image and Signal Processing Research Center, Isfahan University of Medical Sciences, Isfahan, Iran*

Contents

Introduction

Noise reduction or Denoising in some topics is an important and favorable subject of signal processing. There are lots of linear filters and linear methods which can be applied to increase signal to noise ratio (SNR) in data sequences.
However, in the world of nonlinear methods and approaches, which have special importance in nonlinear signal processing, noise reduction using singular value decomposition has a particular position because of outstanding results in denoising white gaussian noise.
There are some difficulties in the above mentioned method, such as choosing the best window length, determining an adaptive r for the best denoising result, or finding a good criterion for comparing the original signal with a denoised signal. Each challenge will be described respectively.

1. Mathematical Prerequisites

1.1 Embedding

Embedding is an algorithm in which a vector of dimensions $1*N$ is converted to a Hankelized trajectory matrix.
A moving window of length L is used to embed a signal.

1.1.1 Hankelization Method A

In the first method, a moving window of length L with overlaps is used to embed a vector signal called S of dimensions $1*N$ to a matrix of dimensions $L*K$ called H_{Hankel} matrix that $K = N - L + 1$.

$$S_{1*N} = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 & . & . & . & . & \mathbf{s}_{N-1} & \mathbf{s}_N \end{bmatrix}$$

$$H_{L*K} = \begin{bmatrix} s_1 & s_2 & \cdot & \cdot & \cdot & s_K \\ s_2 & s_3 & \cdot & \cdot & \cdot & s_{K+1} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ s_L & s_{L+1} & \cdot & \cdot & \cdot & s_N \end{bmatrix}$$

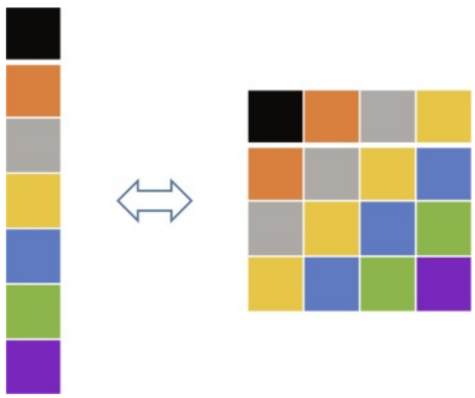

Figure 1. Hankelization Method A

1.1.2 Hankelization Method B

In the second method, a moving window without overlaps with length L_1 and a moving window with overlaps with length L_2 is used to embed a vector signal like below figure :

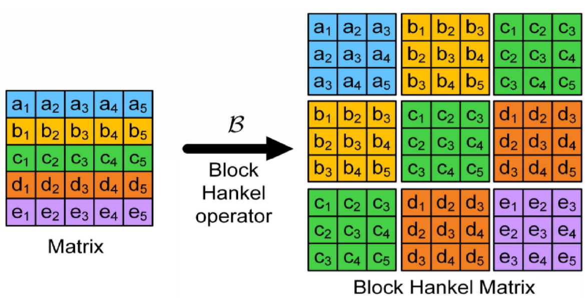

Figure 2. Hankelization Method B

1.2 Singular Value Decomposition

Singular value decomposition is a decomposition method which can decompose a matrix into product of 3 matrices called $\mathbf{U}, \mathbf{S}, \mathbf{V}$ which leads to :

$$\mathbf{M} = \mathbf{U} * \Sigma * \mathbf{V}^T$$

U is an $L * L$ unitary matrix with the columns of left-singular vectors.
Σ is an $L * K$ semi-diagonal matrix with decreasing singular values on the diagonal.
V is a $K * K$ unitary matrix with the columns of right-singular vectors.

1.3 Low Rank Approximation

If we replace some special singular values(lower ones) with 0 and compute the product of $\mathbf{U} * \Sigma_{new} * \mathbf{V}^T$, an approximation of matrix M will be resulted.

1.3.1 Best Low Rank Approximation

Rank reduction is based on lowering the Frobenius Norm of the difference between the main data and denoised data as much as possible.
All possible cases($r = 1, r = 2, ..., r = min(L,K)$) will be considered and the lowest Frobenius norm will be choosen as the best choice.
r = Number of the singular values from the top-left corner without replacing with 0 .

1.4 De-Embedding

1.4.1 In Accordance With Method A

It's a reconstruction method which turns the $L*K$ approximated matrix into a $1*N$ vector using anti-diagonal averaging.

$$H_{new\ (L*K)} = \begin{bmatrix} s_{11} & s_{22} & s_{33} & s_{44} & \cdot & \cdot \\ s_{21} & s_{32} & s_{43} & \cdot & \cdot & \cdot \\ s_{31} & s_{42} & \cdot & \cdot & \cdot & \cdot \\ s_{41} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & s_{(N-1)2} \\ \cdot & \cdot & \cdot & \cdot & s_{(N-1)1} & s_{N1} \end{bmatrix}$$

$$\begin{pmatrix} S_1 & S_2 & S_3 & S_4 \\ S_2 & S_3 & S_4 & S_5 \\ S_3 & S_4 & S_5 & S_6 \\ S_4 & S_5 & S_6 & S_7 \end{pmatrix}$$

Figure 3. De-Embedding Method A

$$(S_{new\ (1*N)})^T = \begin{bmatrix} s_{11} \\ \frac{s_{21}+s_{22}}{2} \\ \frac{s_{31}+s_{32}+s_{33}}{3} \\ \frac{s_{41}+s_{42}+s_{43}+s_{44}}{4} \\ \cdot \\ \cdot \\ \cdot \\ \frac{s_{(N-1)1}+s_{(N-1)2}}{2} \\ s_{N1} \end{bmatrix}$$

1.4.2 In Accordance With Method B

In this reconstruction method, as you see in figure 2, all of the block matrices will be de-embeded and then the semi-color vectors will be averaged.

2. Experiment A (Using Hankelization Method A)

2.1 Stage 1 - Arbitrary Signal Loading

as the first stage, an arbitrary sequence need to be saved into a vector called S.
as an example, we loaded an EEG signal with length $N = 500$ and saved it into vector $\mathbf{S}_{1*500}$.

2.2 Stage 2 - Signal Normalization

The signal should be normalized (divided) to the highest absolute value in order to avoid from large numbers in calculations.

2.3 Stage 3 - Add White Gaussian Noise to Signal

In order to test the algorithm, a noisy signal is essential. This algorithm will be tested on signals with additive white gaussian noise with $SNR = 3$.

2.4 Stage 4 - Signal Embedding

The noisy vector should be converted into a matrix using the first method of hankelization in order to enable us to perform a singular value decomposition.
The most efficient window length can vary between $N/2$ to $N/20$ but $N/3$ or $N/4$, practically can be an efficient length.
$L = N/4$
$K = N - L + 1$

2.5 Stage 5 - Find The Best Approximation

As mentioned before, the best approximation is based on lowering the Frobenius norm.
The choosen r can vary the Frobenius norm so the best r which leads to the lowest Frobenius norm should be selected.

Then, the signal could be approximated using the singular value decomposition as mentioned in mathematical prerequisites.

2.6 Stage 6 - Signal DeEmbedding

The matrix should be reconstructed using anti-diagonal averaging as mentioned before.
Averaging makes an smooth signal while without it, no data can be preserved.

2.7 Results

In figure 4 , the time result of denoising using method A can be seen. Also the main data and noisy data have been plotted.

In figure 5 , the frequency domain results can be seen.

In figure 6 , the singular values of main data and noisy data are plotted. Also the frobenius norm of difference between main data and denoised data has been plotted.

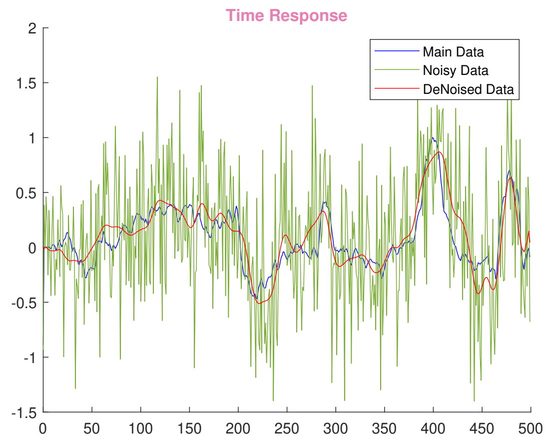

Figure 4. Method A Time Result

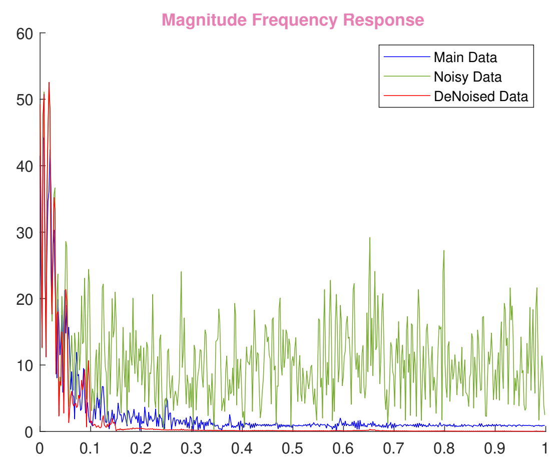

Figure 5. Method A Frequency Result

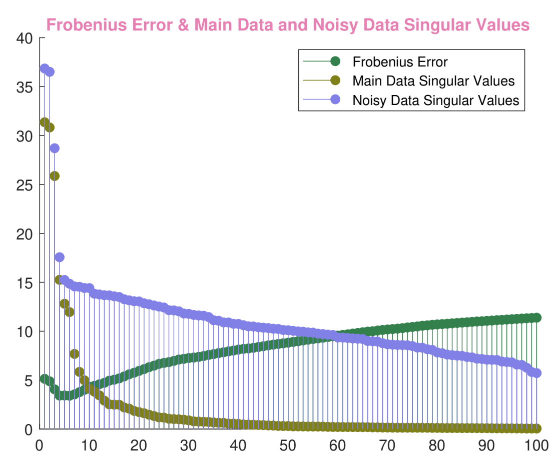

Figure 6. Singular Values

3. Experiment B (Using Hankelization Method B)

3.1 Stages and Results

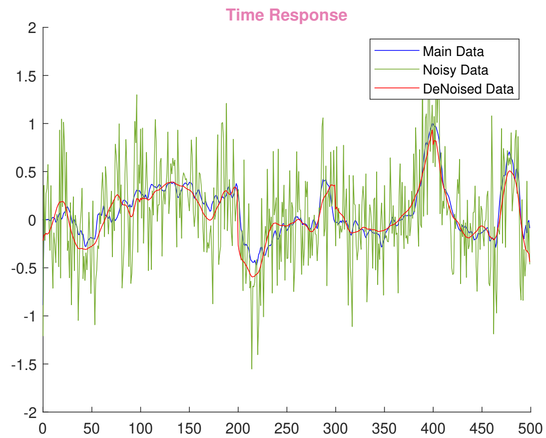

Figure 7. Method B Time Result

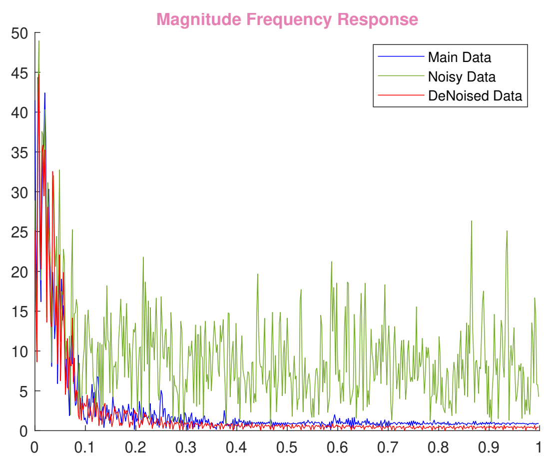

Figure 8. Method B Frequency Result

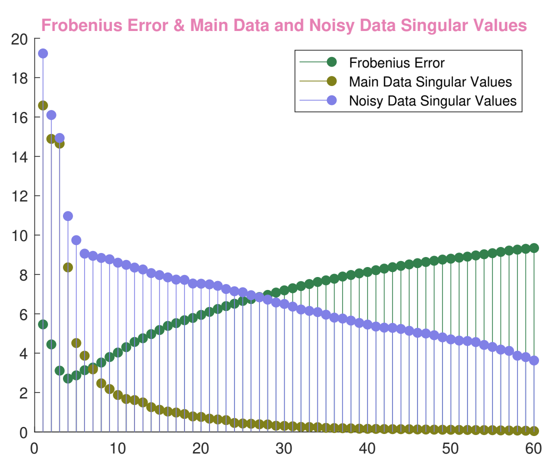

Figure 9. Singular Values

All levels are as same as experiment 1 except level 4 and 6. In those two levels you must use method B for hankelization and de-hankelization .

In figure 7, 8 and 9 everything has been plotted like above but for method B.

4. Final Results

1) As it can be seen, the second method reserves higher frequencies although it's noisier than the first denoised signal because of high frequency noise. It can be visualized in figure 10 & 11. As a trade off, the more frequency reserves, the more noise remains.

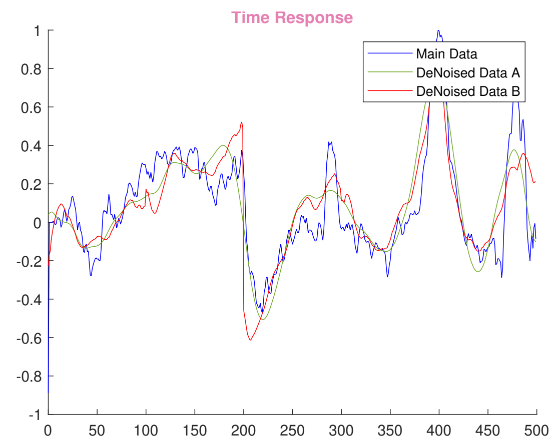

Figure 10. Both Method Time Results

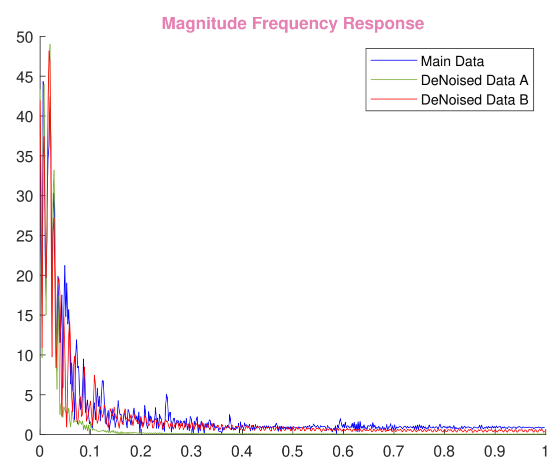

Figure 11. Both Method Frequency Results

2) In low power noise, the first method has lower frobenius error but in high power noise, both methods approximately have the same result in frobenius norm although the second method reserves high frequencies better. Figure 12 shows it.

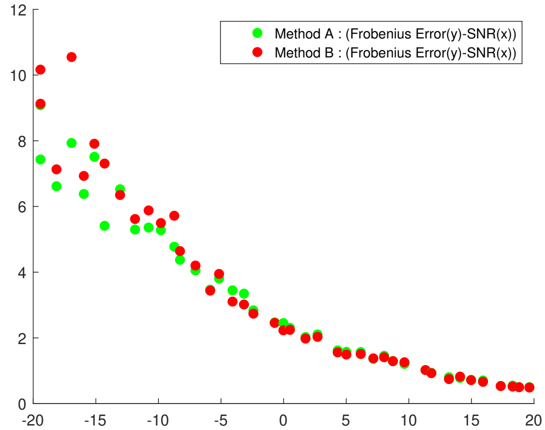

Figure 12. Frobenius Error - Standard Deviation

3) As a result, make a more complex Hankelization matrix doesn't necessarily means a better result in noise reduction.

5. Future Works

1) In this research, more parameters such as hankelization length in method A or window length in method B have been introduced which have been considered constant. These parameters synchronous in remaining singular values(r) can be optimized in next works.

2) Real sound test combined with short time algorithm can be fantastic in this area.

References

[1] Vicente E. Oropeza. The singular spectrum analysis method and its application to seismic data denoising and reconstruction. *University of Alberta*, Fall, 2010.

[2] Thomas Schanze. Removing noise in biomedical signal recordings by singular value decomposition. *Current Directions in Biomedical Engineering*, 3:253–256, 2017.

[3] Nina Golyandina • Anatoly Zhigljavsky. Singular spectrum analysis for time series. *springer*, 2013.

[4] Kai Zhou . Mingzhi Li . Yuan Li . Min Xie and Yonglu Huang. An improved denoising method for partial discharge signals contaminated by white noise based on adaptive short-time singular value decomposition. *energies*, September 2019.