

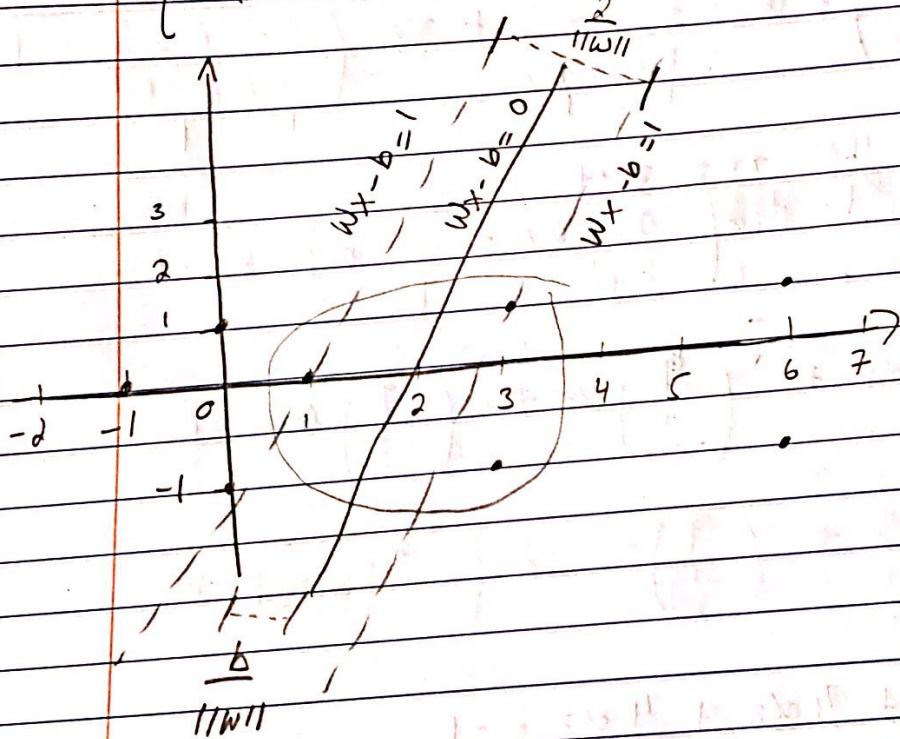
Linearly Separable case truly

(a) Given the following labeled data points: \mathbb{R}^2

$$\left\{ \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \end{pmatrix} \right\}$$

And the following truly labeled data points \mathbb{R}^2

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$$



∴ the three support vectors are :-

$$\left\{ s_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, s_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, s_3 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}$$

$$\alpha_1 \bar{g}(s_1) + \alpha_2 \bar{g}(s_2) + \alpha_3 \bar{g}(s_3) = -1$$

$$\alpha_1 \bar{g}(s_1) + \alpha_2 \bar{g}(s_2) + \alpha_3 \bar{g}(s_3) = +1$$

$$\alpha_1 \bar{g}(s_1) + \alpha_2 \bar{g}(s_2) + \alpha_3 \bar{g}(s_3) = +1$$

For now we have $\phi() = I$, this reduces to:-

$$\alpha_1 \tilde{S}_1 \cdot \tilde{S}_1 + \alpha_2 \tilde{S}_2 \cdot \tilde{S}_2 + \alpha_3 \tilde{S}_3 \cdot \tilde{S}_1 = 1$$

$$\alpha_1 \tilde{S}_1 \cdot \tilde{S}_2 + \alpha_2 \tilde{S}_2 \cdot \tilde{S}_2 + \alpha_3 \tilde{S}_3 \cdot \tilde{S}_2 = 1$$

$$\alpha_1 \tilde{S}_1 \cdot \tilde{S}_3 + \alpha_2 \tilde{S}_2 \cdot \tilde{S}_3 + \alpha_3 \tilde{S}_3 \cdot \tilde{S}_3 = 1$$

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 1$$

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ -1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 1$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \alpha_1 + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \alpha_2 + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \alpha_3 = 1$$

$$\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \alpha_1 + \begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix} \alpha_2 + \begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix} \alpha_3 = 1$$

$$\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \alpha_1 + \begin{pmatrix} 9 \\ -1 \\ 1 \end{pmatrix} \alpha_2 + \begin{pmatrix} 9 \\ 1 \\ 1 \end{pmatrix} \alpha_3 = 1$$

$$2\alpha_1 + 4\alpha_2 + 4\alpha_3 = 1$$

$$4\alpha_1 + 11\alpha_2 + 9\alpha_3 = 1$$

$$4\alpha_1 + 9\alpha_2 + 11\alpha_3 = 1$$

$$\alpha_1 = \frac{-1 - 4\alpha_3 - 4\alpha_2}{2}$$

$$\alpha_2 = \frac{2 - 5\alpha_3}{7}$$

$$4\left(\frac{-1 - 4\alpha_3 - 4\alpha_2}{2}\right) + 11\alpha_2 + 9\alpha_3 = 1$$

$$14\alpha_1 = -1 - 28\alpha_3 - 56$$

$$-1 - 4\alpha_3 - 4\alpha_2 + 11\alpha_2 + 9\alpha_3 = 1$$

$$5\alpha_3 + 7\alpha_2 = 2$$

$$\alpha_1 = -1 - 4\alpha_3 - 8 + 10\alpha_3 / 7$$

$$\alpha_1 = -3.5, \alpha_2 = 0.75, \alpha_3 = 0.75$$

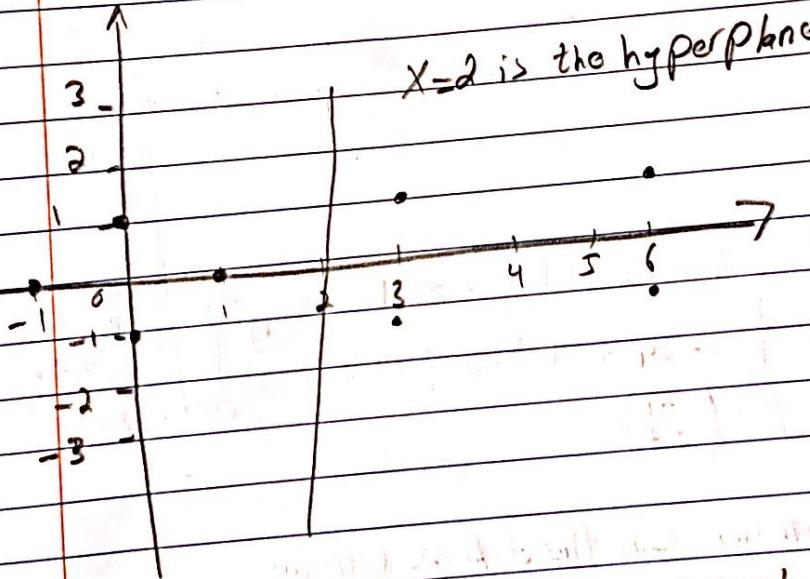
$$\tilde{w} = \sum_i \alpha_i s_i$$

$$\tilde{w} = -3.5 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 0.75 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + 0.75 \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$w \Rightarrow \boxed{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}} \quad : \quad y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x - 2$$

$y = x - 2$

So, the hyperplane is :-



See final plot at end of assignment

(b)

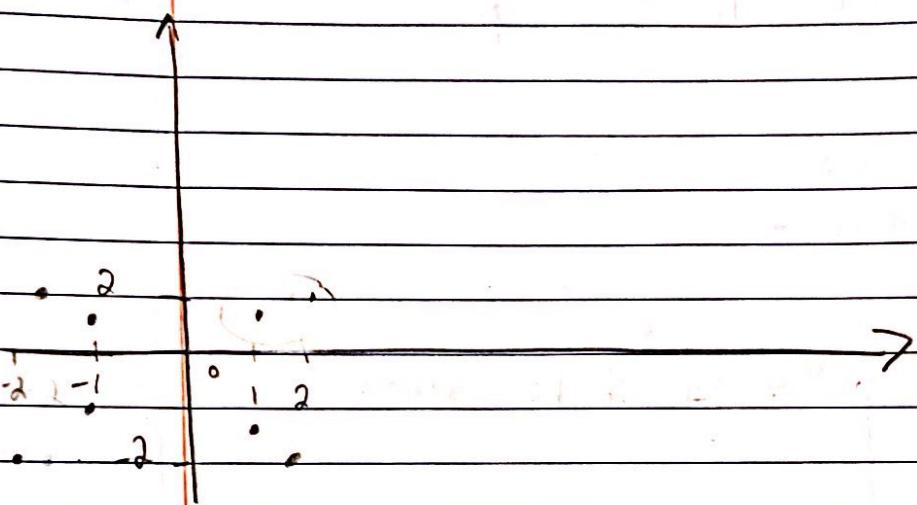
Non-linearly Separable Case

(i)

Given the following labeled data points:-

$$\left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$



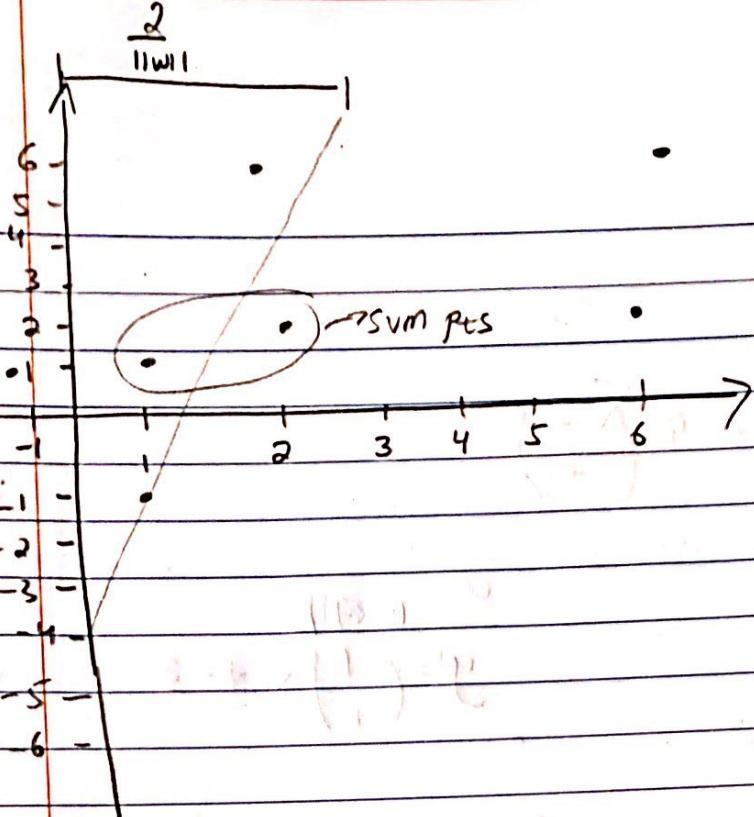
Now Define

$$\Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } \sqrt{x_1^2 + x_2^2} > 2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \text{otherwise} \end{cases}$$

So, now this will transform the data as follows

$$\left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \end{pmatrix} \right\} \text{ +ves}$$

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \text{ -ves}$$



$$\left\{ \mathbf{s}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{s}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$$

(*) $\alpha_1 \phi(s_1) \cdot \phi(s_1) + \alpha_2 \phi(s_2) \cdot \phi(s_1) = -1$
 $\alpha_1 \phi(s_1) \cdot \phi(s_2) + \alpha_2 \phi(s_2) \cdot \phi(s_2) = 1$

this reduces to

$$\alpha_1 \tilde{s}_1 \cdot \tilde{s}_1 + \alpha_2 \tilde{s}_2 \cdot \tilde{s}_1 = -1$$

$$\alpha_1 \tilde{s}_1 \cdot \tilde{s}_2 + \alpha_2 \tilde{s}_2 \cdot \tilde{s}_2 = 1$$

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1$$

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 1$$

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} = -1 \quad 3\alpha_1 + 5\alpha_2 = -1$$

$$\Rightarrow 5\alpha_1 + 9\alpha_2 = 1$$

$$\alpha_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 1 \quad \alpha_1 = \frac{1 - 9\alpha_2}{5}$$

$$3 - 27\alpha_2 + 25\alpha_2^2 = -5$$

$$(3 \left(1 - \frac{9\alpha_2}{5}\right) + 5\alpha_2^2 = -1) 5$$

$$\begin{cases} \alpha_2 = 4 \\ \alpha_1 = -2 \end{cases}$$

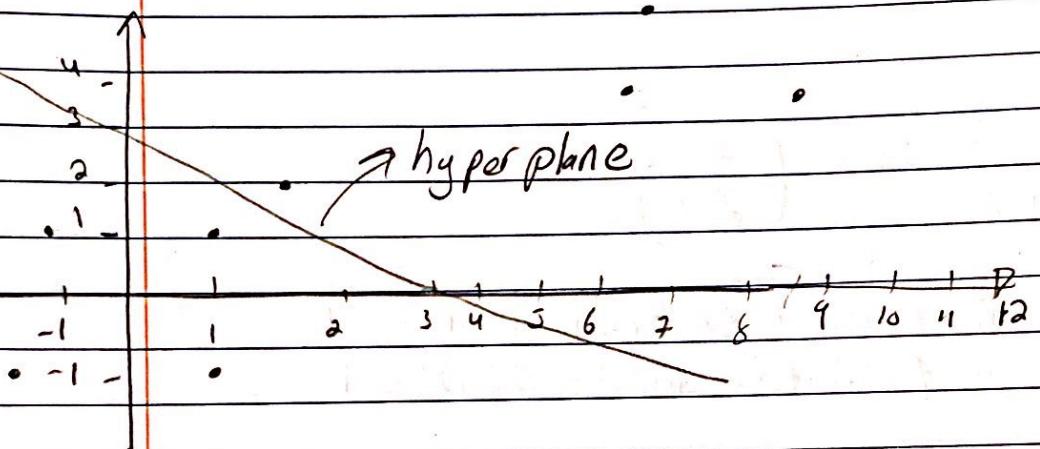
$$\vec{w} = \sum_i \alpha_i \vec{s}_i$$

$$\Rightarrow -2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 2 \\ ? \\ ? \end{pmatrix}$$

$$\vec{w} \Rightarrow \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

$$y = \vec{w} \cdot \vec{x} + b$$

$$y = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + b$$



(ii) Kernel trick

Here we will use a new mapping function

$$\phi_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & 4x_1 - 3 \\ x_2 & 2x_2 - 3 \\ (x_1^2 + x_2^2) - 5 \\ 3 \end{pmatrix}$$

This will transform it from a 2D - 3D space

$$\left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \right\} \rightarrow \text{res}$$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\} \rightarrow \text{res}$$

See plot attached

$$\left\{ \begin{array}{l} S_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, -S_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \end{array} \right.$$

$$\alpha_1 \tilde{S}_1 \cdot \tilde{S}_1 + \alpha_2 \tilde{S}_2 \cdot \tilde{S}_1 = -1$$

$$\alpha_1 \tilde{S}_1 \cdot \tilde{S}_2 + \alpha_2 \tilde{S}_2 \cdot \tilde{S}_1 = +1$$

$$\alpha_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = -1$$

$$\alpha_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = 1$$

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} \alpha_1 - \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \alpha_2 = -1$$

$$-\alpha_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \alpha_2 = 1$$

$$9\alpha_1 - 5\alpha_2 = -1$$

$$-5\alpha_1 + 3\alpha_2 = 1$$

$$\alpha_2 = \frac{1+5\alpha_1}{3}$$

$$2\alpha_1 = -2$$

$$2(9\alpha_1 - 5\left(\frac{1+5\alpha_1}{3}\right)) = -1$$

$$-\frac{7}{46} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \frac{1}{46} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$21\alpha_1 + 5 + 25\alpha_1 = -2$$

$$\boxed{\alpha_2 = \frac{1}{46}}$$

$$\boxed{\alpha_1 = \frac{-2}{46}}$$

$$\begin{pmatrix} -\frac{14}{46} \\ -\frac{14}{46} \\ -\frac{7}{46} \end{pmatrix} + \begin{pmatrix} \frac{-1}{46} \\ \frac{-1}{46} \\ \frac{-1}{46} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{-15}{46} \\ \frac{-15}{46} \\ \frac{-8}{46} \end{pmatrix}$$

Syed Ali

$$\bar{w} = \sum_i \alpha_i \bar{s}_i \rightarrow \text{passes through the origin}$$

$$\bar{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$y = w \cdot x + b = y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x + 0$$

$$b = 0$$

See final plots attached at the end of this assignment

SVM Example Assignment 6

Part a using Linearly Separable Case

```
In [1]: import numpy as np
import matplotlib.pyplot as plt

def __intersect(rect, line):
    l = []
    xmin,xmax,ymin,ymax = rect
    a,b,c = line

    assert a!=0 or b!=0

    if a == 0:
        y = -c/b
        if y<=ymax and y>=ymin:
            l.append((xmin, y))
            l.append((xmax, y))
        return l
    if b == 0:
        x = -c/a
        if x<=xmax and x>=xmin:
            l.append((x, ymin))
            l.append((x, ymax))
        return l

    k = -a/b
    m = -c/b
    for x in (xmin, xmax):
        y = k*x+m
        if y<=ymax and y>=ymin:
            l.append((x,y))

    k = -b/a
    m = -c/a
    for y in (ymin, ymax):
        x = k*y+m
        if x<=xmax and x>=xmin:
            l.append((x,y))
    return l

def plotline(coef, *args, **kwargs):
    '''plot line: y=a*x+b or a*x+b*y+c=0'''
    coef = np.float64(coef[:])
    assert len(coef)==2 or len(coef)==3
    if len(coef) == 2:
        a, b, c = coef[0], -1., coef[1]
    elif len(coef) == 3:
        a, b, c = coef
    ax = plt.gca()

    limits = ax.axis()
    points = __intersect(limits, (a,b,c))
    if len(points) == 2:
```

```

pts = np.array(points)
ax.plot(pts[:,0], pts[:,1], *args, **kwargs)
ax.axis(limits)

def circle_out(x, y, s=20, *args, **kwargs):
    '''Circle out points with size 's' and edgecolors'''
    ax = plt.gca()
    if 'edgecolors' not in kwargs:
        kwargs['edgecolors'] = 'g'
    ax.scatter(x, y, s, facecolors='none', *args, **kwargs)

def plotSVM(coef, support_vectors=None):
    coef1 = coef[:]
    coef2 = coef[:]
    coef1[2] += 1
    coef2[2] -= 1
    plotline(coef, 'b', lw=2)
    plotline(coef1, 'b', ls='dashed')
    plotline(coef2, 'b', ls='dashed')

from pylab import *

X = array([[3.0, 3.0, 6.0, 6.0],
           [1.0, -1.0, 1.0, -1.0]])

Y = array([[1.0, 0.0, 0.0, -1.0],
           [0.0, 1.0, -1.0, -1.0]])

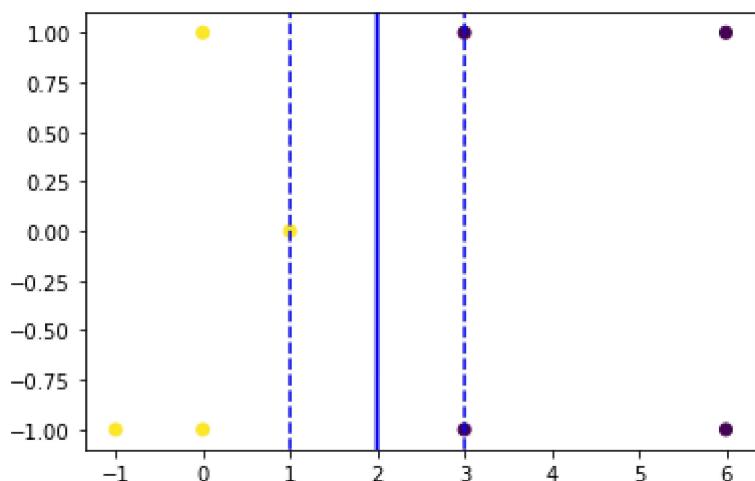
data = hstack((X,Y)).T
label = hstack((zeros(X.shape[1])), ones(Y.shape[1])))

from sklearn.svm import SVC

clf = SVC(kernel='linear')
clf.fit(data, label)

coef = [clf.coef_[0,0], clf.coef_[0,1], clf.intercept_[0]]
scatter(data[:,0], data[:,1], c=label)
plotSVM(coef, clf.support_vectors_)
show()

```



Part b (i) using Non-Linearly Separable Case

In [2]:

```

import numpy as np
import matplotlib.pyplot as plt

def __intersect(rect, line):
    l = []
    xmin,xmax,ymin,ymax = rect
    a,b,c = line

    assert a!=0 or b!=0

    if a == 0:
        y = -c/b
        if y<=ymax and y>=ymin:
            l.append((xmin, y))
            l.append((xmax, y))
    return l

    if b == 0:
        x = -c/a
        if x<=xmax and x>=xmin:
            l.append((x, ymin))
            l.append((x, ymax))
    return l

    k = -a/b
    m = -c/b
    for x in (xmin, xmax):
        y = k*x+m
        if y<=ymax and y>=ymin:
            l.append((x,y))

    k = -b/a
    m = -c/a
    for y in (ymin, ymax):
        x = k*y+m
        if x<=xmax and x>=xmin:
            l.append((x,y))
    return l

def plotline(coef, *args, **kwargs):
    '''plot line: y=a*x+b or a*x+b*y+c=0'''
    coef = np.float64(coef[:])
    assert len(coef)==2 or len(coef)==3
    if len(coef) == 2:
        a, b, c = coef[0], -1., coef[1]
    elif len(coef) == 3:
        a, b, c = coef
    ax = plt.gca()

    limits = ax.axis()
    points = __intersect(limits, (a,b,c))
    if len(points) == 2:
        pts = np.array(points)
        ax.plot(pts[:,0], pts[:,1], *args, **kwargs)
        ax.axis(limits)

def circle_out(x, y, s=20, *args, **kwargs):
    '''Circle out points with size 's' and edgecolors'''

```

```

ax = plt.gca()
if 'edgecolors' not in kwargs:
    kwargs['edgecolors'] = 'g'
ax.scatter(x, y, s, facecolors='none', *args, **kwargs)

def plotSVM(coef, support_vectors=None):
    coef1 = coef[:]
    coef2 = coef[:]
    coef1[2] += 1
    coef2[2] -= 1
    plotline(coef, 'b', lw=2)
    plotline(coef1, 'b', ls='dashed')
    plotline(coef2, 'b', ls='dashed')

from pylab import *

X = array([[2.0, 6.0, 6.0, 2.0],
           [2.0, 2.0, 6.0, 6.0]])

Y = array([[1.0, 1.0, -1.0, -1.0],
           [1.0, -1.0, -1.0, 1.0]])

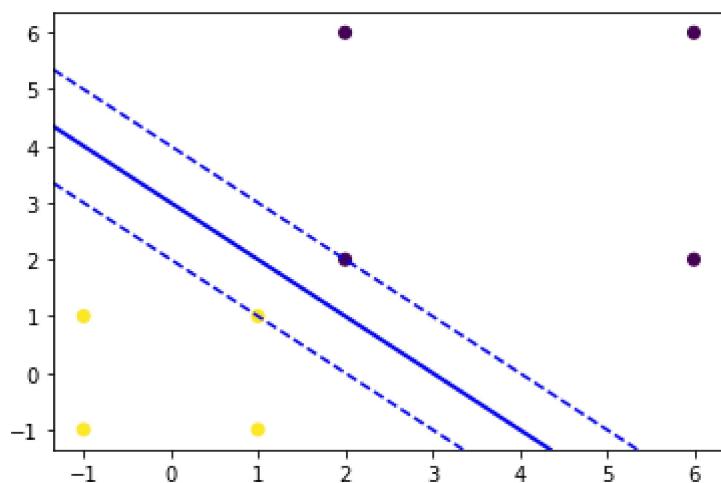
data = hstack((X,Y)).T
label = hstack((zeros(X.shape[1]), ones(Y.shape[1])))

from sklearn.svm import SVC

clf = SVC(kernel='linear')
clf.fit(data, label)

coef = [clf.coef_[0,0], clf.coef_[0,1], clf.intercept_[0]]
scatter(data[:,0], data[:,1], c=label)
plotSVM(coef, clf.support_vectors_)
show()

```



Part b (ii) using Kernel Trick

In [3]:

```

import numpy as np
import matplotlib.pyplot as plt

```

```

def __intersect(rect, line):
    l = []
    xmin, xmax, ymin, ymax = rect
    a, b, c = line

    assert a!=0 or b!=0

    if a == 0:
        y = -c/b
        if y<=ymax and y>=ymin:
            l.append((xmin, y))
            l.append((xmax, y))
        return l
    if b == 0:
        x = -c/a
        if x<=xmax and x>=xmin:
            l.append((x, ymin))
            l.append((x, ymax))
        return l

    k = -a/b
    m = -c/b
    for x in (xmin, xmax):
        y = k*x+m
        if y<=ymax and y>= ymin:
            l.append((x,y))

    k = -b/a
    m = -c/a
    for y in (ymin, ymax):
        x = k*y+m
        if x<=xmax and x>= xmin:
            l.append((x,y))
    return l

def plotline(coef, *args, **kwargs):
    '''plot line: y=a*x+b or a*x+b*y+c=0'''
    coef = np.float64(coef[:])
    assert len(coef)==2 or len(coef)==3
    if len(coef) == 2:
        a, b, c = coef[0], -1., coef[1]
    elif len(coef) == 3:
        a, b, c = coef
    ax = plt.gca()

    limits = ax.axis()
    points = __intersect(limits, (a,b,c))
    if len(points) == 2:
        pts = np.array(points)
        ax.plot(pts[:,0], pts[:,1], *args, **kwargs)
        ax.axis(limits)

def circle_out(x, y, s=20, *args, **kwargs):
    '''Circle out points with size 's' and edgecolors'''
    ax = plt.gca()
    if 'edgecolors' not in kwargs:
        kwargs['edgecolors'] = 'g'
    ax.scatter(x, y, s, facecolors='none', *args, **kwargs)

```

```
def plotSVM(coef, support_vectors=None):
    coef1 = coef[:,]
    coef2 = coef[:,]
    coef1[2] += 1
    coef2[2] -= 1
    plotline(coef, 'b', lw=2)
    plotline(coef1, 'b', ls='dashed')
    plotline(coef2, 'b', ls='dashed')

from pylab import *

X = array([[2.0, 2.0, -2.0, -2.0],
           [2.0, -2.0, -2.0, 2.0],[1.0,1.0,1.0,1.0]])

Y = array([[1.0, 1.0, -1.0, -1.0],
           [1.0, -1.0, -1.0, 1.0],[-1.0,-1.0,-1.0,-1.0]])

data = hstack((X,Y)).T
label = hstack((zeros(X.shape[1])), ones(Y.shape[1])))

from sklearn.svm import SVC

clf = SVC(kernel='linear')
clf.fit(data, label)

coef = [clf.coef_[0,0], clf.coef_[0,1], clf.intercept_[0]]
scatter(data[:,0], data[:,1], c=label)
plotSVM(coef, clf.support_vectors_)
show()
```

