MATH 8680: PROJECT 1 REPORT

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1 Preprocessing of Data

The input data is:

	$Dose_i$	N_i	A_i
$i = 1$	1.69	60	6
$i = 2$	1.72	62	13
$i = 3$	1.76	63	20
$i = 4$	1.78	60	30
$i = 5$	1.81	64	53
$i = 6$	1.84	60	55
$i = 7$	1.86	62	61
$i = 8$	1.88	64	62

For each Binomial experiment i, we created N_i Bernoulli random variable y_{ij} in a way that A_i of them have value 1 and $(N_i - A_i)$ of them 0. Following is our final y vector:

```
print(y)
[1] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
[32] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 0 0
[63] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
[94] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0
[125] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
[156] 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0
[187] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1
[218] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0
[249] 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
[280] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0
[311] 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
[342] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1
[373] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
[404] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 1
[435] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
[466] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

2 Problem Formulation

We are using same method introduced in [1] The fully conditional distribution for parameters β is:

$$\boldsymbol{\beta}|\boldsymbol{z}, \boldsymbol{X} \sim \mathcal{N}\Big((\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{z}, (\boldsymbol{X}^T\boldsymbol{X})^{-1}\sigma^2\Big)$$

And the fully conditional for latent variables z is:

$$z_i|\boldsymbol{\beta}, y_i, \boldsymbol{x}_i \sim \begin{cases} \mathcal{TN}(\boldsymbol{x}_i\boldsymbol{\beta}, 1, 0, \infty) & \text{if } y_i = 1 \\ \mathcal{TN}(\boldsymbol{x}_i\boldsymbol{\beta}, 1, -\infty, 0) & \text{if } y_i = 0 \end{cases}$$

3 Results

3.1 Non Informative Prior $\tau(\beta) = 1$

	β_1	β_2	β_3
Mean Posterior	94.67	-125.43	40.60
MLE	83.87	-113.02	37.05

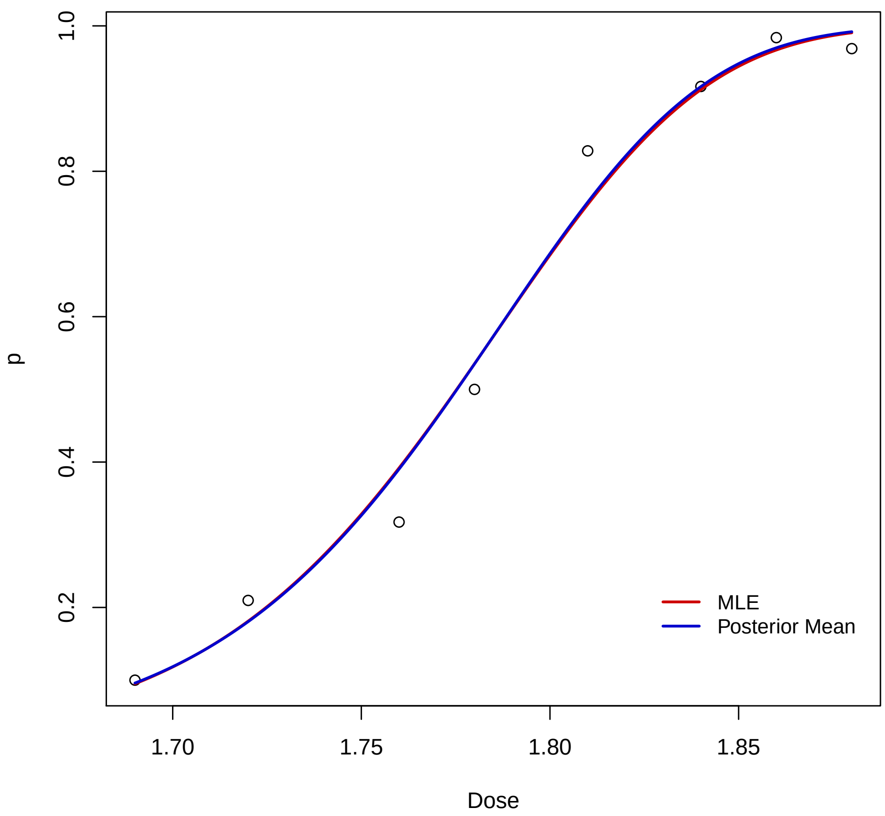

Figure 1: P(d) by estimated parameters (Non informative)

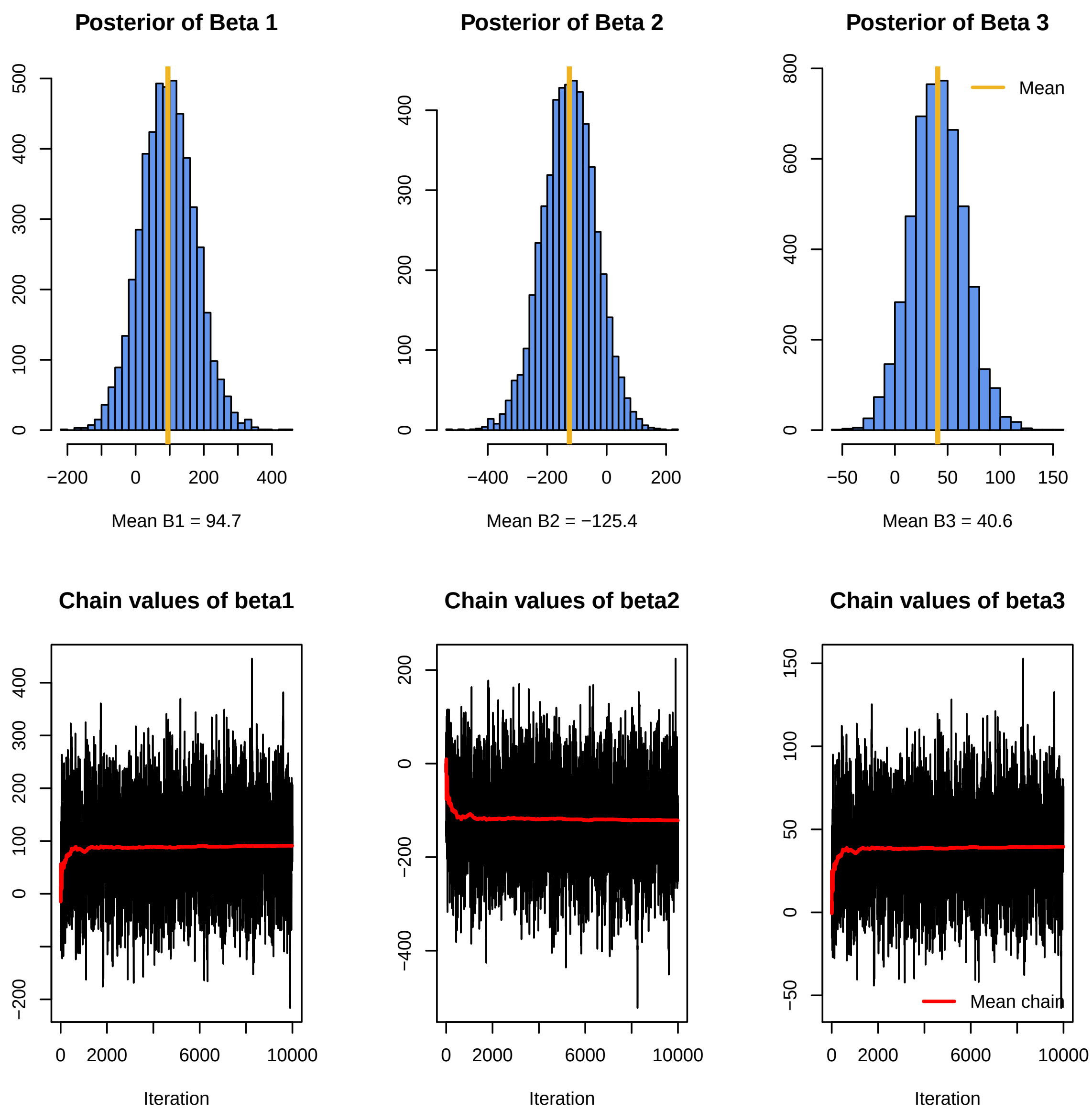

Figure 2: Histograms and Trace plots (Non informative)

3.2 Non Informative Prior $\tau(1/\sigma^2) \sim Gamma(1/2, 0.005)$

We decided to use three step Gibbs sampling to estimate β, z and σ^2 in the way presented in this paper. [2]. For this purpose, we need to have fully conditional distribution of $1/\sigma^2$, using the prior $\tau(1/\sigma^2) \sim Gamma(1/2, 0.005)$

For simplicity we put $\theta = \frac{1}{\sigma^2}$ and find the posterior of that:

$$\begin{aligned} f(y|\beta, \theta) &= \frac{1}{(2\pi)^2} \theta^{\frac{n}{2}} e^{\frac{-\theta}{2}(\underline{y} - X\underline{\beta})'(\underline{y} - X\underline{\beta})} \\ &= L(\beta, \theta|\underline{y}) \end{aligned}$$

Given prior

$$\tau(\theta) = Gamma(1/2, 0.005) = \frac{0.005^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\theta^{-\frac{1}{2}}e^{-0.005}$$

So

$$\begin{aligned}
\tau(\beta,\theta|\underline{y}) &\propto L(\beta,\theta)\tau(\theta) \\
&\propto \theta^{\frac{n}{2}}e^{-\frac{\theta}{2}(y-X\hat{\beta})'(y-X\hat{\beta})}e^{-\frac{\theta}{2}(\beta-\hat{\beta})'X'X(\beta-\hat{\beta})}\theta^{-\frac{1}{2}}e^{-0.005} \\
\tau(\theta|\beta,\underline{y},X) &\propto \theta^{\frac{n}{2}}e^{-\frac{\theta}{2}(\underline{y}-X\beta)'(\underline{y}-X\beta)}\theta^{-\frac{1}{2}}e^{-0.005} \\
&\propto \theta^{\frac{n-1}{2}}e^{-0.005\theta-\frac{\theta}{2}(\underline{y}-X\beta)'(\underline{y}-X\beta)} \\
&\propto \theta^{\frac{n-1}{2}}e^{-\theta(0.005+\frac{1}{2}(\underline{y}-X\beta)'(\underline{y}-X\beta)} \\
&\sim Gamma(\frac{n-1}{2}, 0.005+\frac{1}{2}(\underline{y}-X\beta)'(\underline{y}-X\beta))
\end{aligned}$$

Using this method the results are:

	β_1	β_2	β_3
Mean Posterior	91.34	-121.60	39.51
MLE	83.87	-113.02	37.05

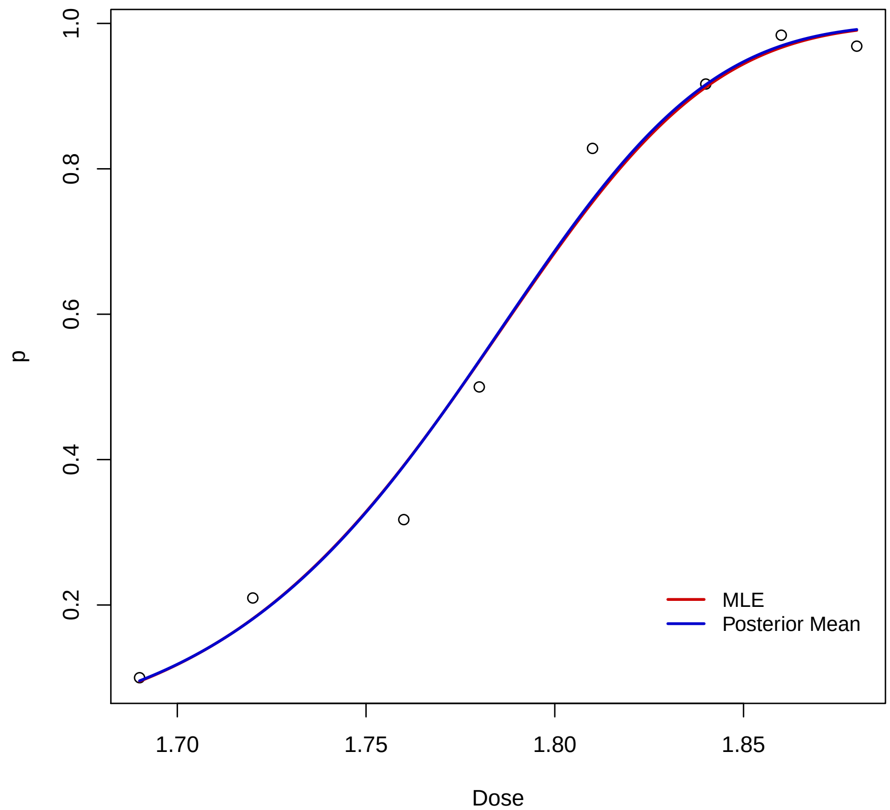

Figure 3: P(d) by estimated parameters (Non informative, $\tau(1/\sigma^2)$)

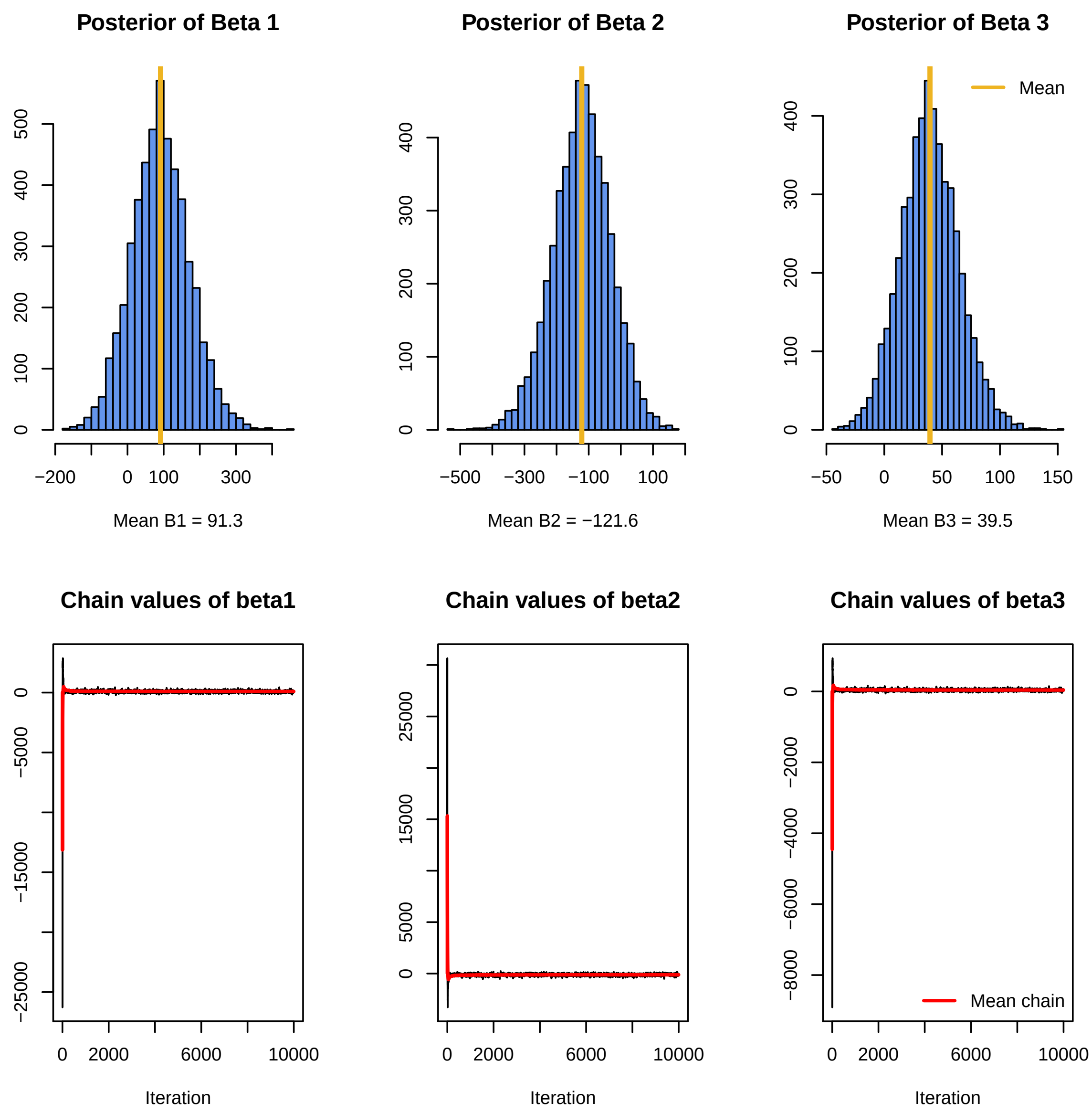

Figure 4: Histograms and Trace plots (Non informative, $\tau(1/\sigma^2)$)

3.3 Informative Prior $\tau(\boldsymbol{\beta}) = \mathcal{N}(\boldsymbol{\beta}_0, \boldsymbol{Q}_0)$

In which $\beta_0 = [000]$ and

$$\beta_0 = \left(\begin{smallmatrix} 0 & 0 & 0 \end{smallmatrix} \right)$$

and

$$Q_0 = \left(\begin{smallmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{smallmatrix} \right)$$

The fully conditional posterior will be:

$$\boldsymbol{\beta}|\boldsymbol{z}, \boldsymbol{X} \sim \mathcal{N}(\boldsymbol{M}, \boldsymbol{V})$$
$$\boldsymbol{M} = \boldsymbol{V}(\boldsymbol{Q}_0^{-1}\boldsymbol{\beta}_0 + \boldsymbol{X}^T\boldsymbol{z})$$
$$\boldsymbol{V} = (\boldsymbol{Q}_0^{-1} + \boldsymbol{X}^T\boldsymbol{X})^{-1}$$

[?] Using this prior the result are:

	β_1	β_2	β_3
Mean Posterior	-9.24	-6.45	6.59
MLE	83.87	-113.02	37.05

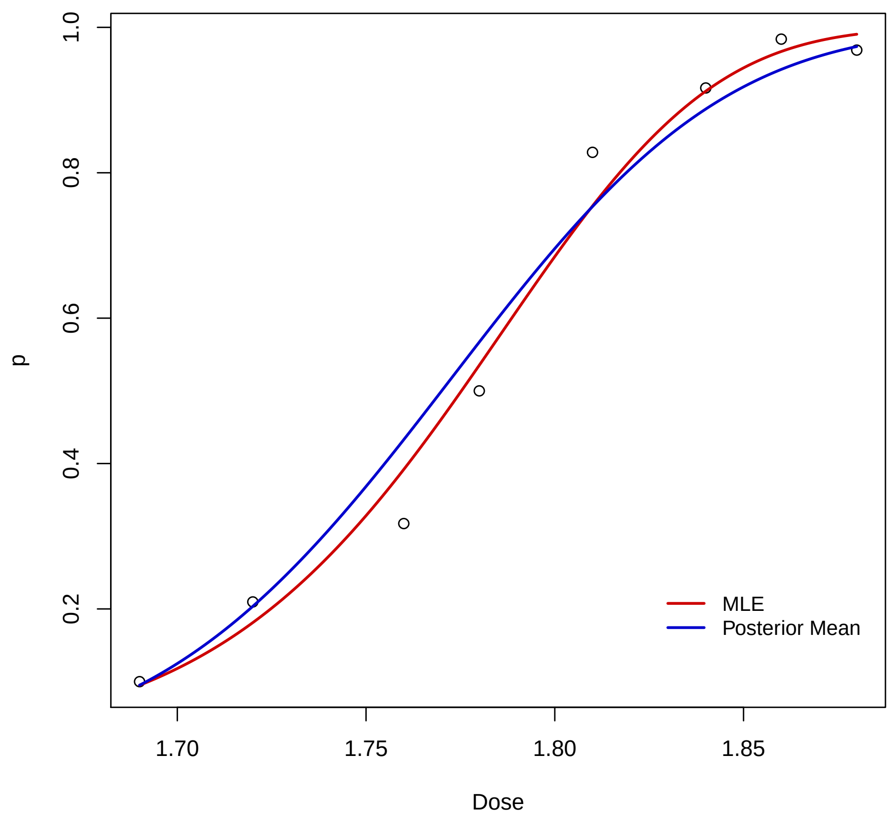

Figure 5: P(d) by estimated parameters (Informative)

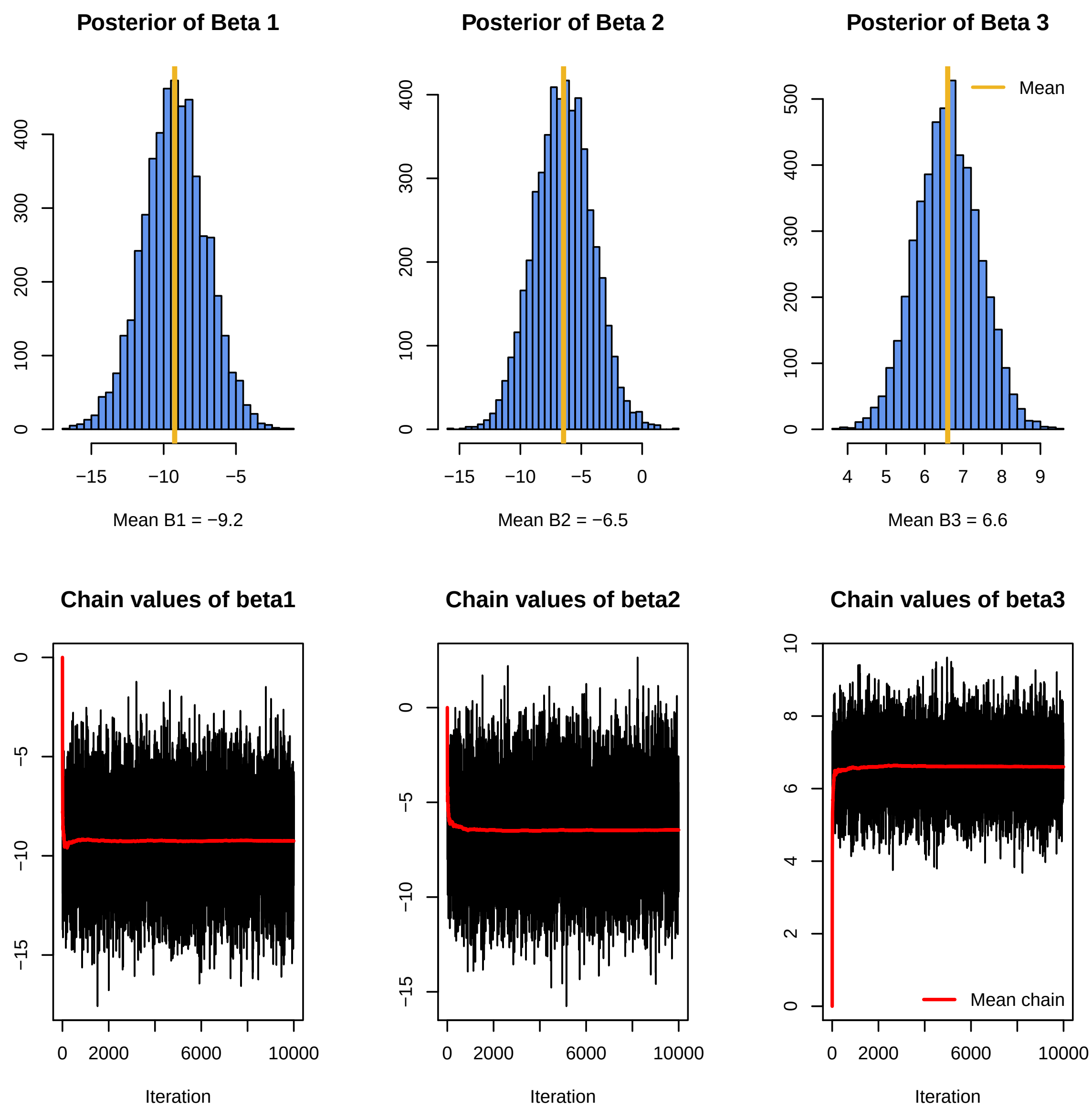

Figure 6: Histograms and Trace plots (Informative)

4 Implementation In R

References

[1] J. H. Albert and S. Chib, "Bayesian analysis of binary and polychotomous response data," *Journal of the American statistical Association*, vol. 88, no. 422, pp. 669–679, 1993.

[2] C. Czado, "Bayesian inference of binary regression models with parametric link," *Journal of Statistical Planning and Inference*, vol. 41, no. 2, pp. 121–140, 1994.