1 Column Space and Nullspace (L6)

1.1 Column Space

Given a 4×3 matrix A whose columns are in R^4 space:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix}$$

Column space $C(A)$ is the linear combination of A's columns, which forms a subspace of R^4.

$$C(A) = c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

Does $Ax = b$ have a solution for every b?

$$Ax = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = b$$

No, because there are 4 equations and 3 unknowns.

Which b allows the system to be solved?

$$\text{e.g.} \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

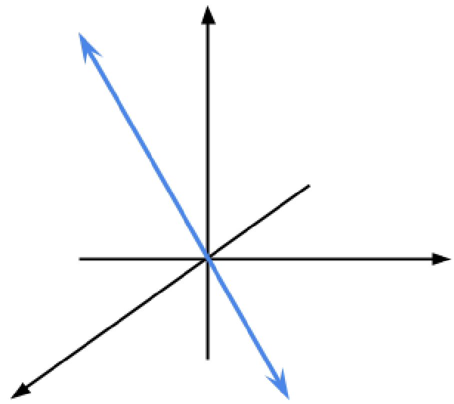
$Ax = b$ is solvable if b is the linear combination of A's columns i.e. $C(A)$

1.2 Null Space

$$Ax = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Null space $N(A)$ is all the solutions x to $Ax = 0$, which forms a subspace of R^3.

$$x = c\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$



If $Av = 0$ and $Aw = 0$, then $A(c_1v + c_2w) = c_1Av + c_2Aw = 0$, therefore **solutions to** $Ax = 0$ **always give a subspace.**

2 $Ax = 0$, Special Solutions, Reduced Row Form (L7)

2.1 Example: Solving $Ax = 0$

Given a matrix A for 3 equations and 4 unknowns:

$$A = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{pmatrix}$$

Third row of A is dependent on the first and second i.e. $r_3 = r_1 + r_2$. Second column of A is dependent on the first i.e. $c_2 = 2c_1$.

Run eliminations to derive U in an echelon (staircase) form:

$$\begin{pmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 2 & 2 & 2 \\ 2-2*1 & 4-2*2 & 6-2*2 & 8-2*2 \\ 3-3*1 & 6-3*2 & 8-3*2 & 10-3*2 \end{pmatrix} = \begin{pmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0-0 & 0-0 & 2-2 & 4-4 \end{pmatrix} = \begin{pmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

- Pivot Column: c_1 and c_3
- Pivot Rows: r_1 and r_2
- Free Columns: c_2 and c_4 $\Rightarrow$ Any values can be assigned to x_2 and x_4
- Rank r: Number of pivots = 2 (1 at U_{11} and 2 at U_{23})
- Number of free variables: $n - r = 4 - 2 = 2$ (x_2 and x_4)

Equations we have are:

$$\begin{cases} x_1 & +2x_2 & +2x_3 & +2x_4 & = 0 \\ & & 2x_3 & +4x_4 & = 0 \end{cases}$$

We can assign specific values to x_2 and x_4 and solve equations to derive x_1 and x_3. We'll call these special solutions:

$$\begin{cases} x_2 &= 1 \\ x_4 &= 0 \end{cases} \Rightarrow \begin{cases} x_1 &= -2 \\ x_3 &= 0 \end{cases} \qquad \begin{cases} x_2 &= 0 \\ x_4 &= 1 \end{cases} \Rightarrow \begin{cases} x_1 &= 2 \\ x_3 &= -2 \end{cases}$$

Null space (all solutions to $Ax = 0$) is the linear combination of the special solutions:

$$N(A) = c \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

2.2 Reduced Row Echelon Form (RREF)

Matrix is in a RREF when all pivots are 1s and there are all 0s below and above the pivots. For example, starting with U from the previous section:

$$U = \begin{pmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1-0=1 & 2-0=2 & 2-2=0 & 2-4=-2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0/2=0 & 0=0 & 2/2=1 & 4/2=2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R = RREF(A) = \begin{pmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice how the pivot columns and rows of R forms an identity matrix I:

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{|c|c|} \hline 1 & 0 \\ 0 & 1 \\ \hline \end{array} = I$$

Similarly the free columns and rows of R forms a matrix F:

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{|c|c|} \hline 2 & -2 \\ 0 & 2 \\ \hline \end{array} = F$$

Those matrices I and F form the special solutions from the previous section:

$$\begin{pmatrix} -2 \\ \boxed{1} \\ 0 \\ \boxed{0} \end{pmatrix} \begin{pmatrix} 2 \\ \boxed{0} \\ -2 \\ \boxed{1} \end{pmatrix} \Rightarrow \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} = I \qquad \begin{pmatrix} \boxed{-2} \\ 1 \\ \boxed{0} \\ 0 \end{pmatrix} \begin{pmatrix} \boxed{2} \\ 0 \\ \boxed{-2} \\ 1 \end{pmatrix} \Rightarrow \begin{array}{|c|c|} \hline -2 & 2 \\ \hline 0 & -2 \\ \hline \end{array} = -F$$

In general, R (A in RREF) can be broken down into I, F, and zeros. I and F form the null space matrix N i.e. the columns of I and F form special solutions.

$$Rx = \begin{pmatrix} I & F \end{pmatrix} \begin{pmatrix} x_{pivot} \\ x_{free} \end{pmatrix} = 0$$

$$x_{pivot} = -Fx_{free} \Rightarrow \begin{cases} x_{pivot} & = & -F \\ x_{free} & = & I \end{cases}$$

$$RN = 0 \Rightarrow N = x = \begin{pmatrix} -F \\ I \end{pmatrix}$$

2.3 Additional Example

Eliminations from A to U:

$$A = \begin{pmatrix} \boxed{1} & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{0} & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = U$$

- Pivot columns: c_1 and $c_2 \Rightarrow$ Rank $= 2$
- Free columns: c_3

Assign $x_3 = 1$:

$$\begin{cases} x_1 & +2x_2 & +3x_3 & = 0 \\ & 2x_2 & +2x_3 & = 0 \end{cases} \Rightarrow \begin{cases} x_1 & = -1 \\ x_2 & = -1 \end{cases} \Rightarrow x = c \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = N(A)$$

Eliminations from U to R:

$$U = \begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R$$

3 Solving $Ax = b$ (L8)

Is there a solution? Is the solution unique?

$$\begin{cases} x_1 & +2x_2 & +2x_3 & +2x_4 & = b_1 \\ 2x_1 & +4x_2 & +6x_3 & +8x_4 & = b_2 \\ 3x_1 & +6x_2 & +8x_3 & +10x_4 & = b_3 \end{cases} \Rightarrow Ax = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Run eliminations on an augmented matrix $\left(\begin{array}{c|c} A & b \end{array} \right)$ until the left side is in an echelon form:

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array} \right) \Rightarrow \left(\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array} \right) \Rightarrow \left(\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array} \right)$$

The third row shows the equations are solvable if $b_3 - b_2 - b_1 = 0$ e.g. $b = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$

Solvability

$Ax = b$ is solvable when b is in $C(A)$. If a combination of A's rows give a zero row, some combination of b's entries must also give zero.

3.1 Complete Solution for $Ax = b$

Find the particular solution x_p by setting all free variables to 0s and solving $Ax = b$ for pivot variables:

$$\begin{cases} x_1 & +2x_2 & +2x_3 & +2x_4 & = & b_1 \\ & & +2x_3 & +4x_4 & = & b_2 - 2b_1 \end{cases} \qquad b = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$$

$$\begin{cases} x_2 = 0 \\ x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2 \\ x_3 = \frac{3}{2} \end{cases}$$

$$x_p = \begin{pmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{pmatrix}$$

Complete solution for $Ax = b$ is a combination of x_p and the null space x_n:

$$Ax_p = b \quad Ax_n = 0 \quad \Rightarrow \quad A(x_p + x_n) = b$$

$$x_{complete} = x_p + x_n = \begin{pmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

3.2 Full Ranks

A is a $m \times n$ matrix with rank $r \Rightarrow r \leq m, r \leq n$

Full Column Rank: $r = n < m$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{pmatrix} \Rightarrow R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- Every column is a pivot column. No free columns.
- Null space $N(A)$ contains only the zero vector.
- $Ax = b$ 0 or 1 solutions, depending on b.
- If $Ax = b$ has a solution, $x = x_p$ i.e. unique solution.

Full Row Rank: $r = m < n$

$$A = \begin{pmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{pmatrix} \Rightarrow R = \begin{pmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{pmatrix}$$

- m pivot columns, $n - m$ free columns.
- $Ax = b$ has ∞ solutions for any b.

Square Full Rank: $r = m = n$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \Rightarrow R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- R is an identity matrix.

- $Ax = b$ has 1 solution for any given b.

r, m, n	Number of Solutions	R
$r = n = m$	1	$\begin{pmatrix} I \end{pmatrix}$
$r = n < m$	0 or 1	$\begin{pmatrix} I \\ 0 \end{pmatrix}$
$r = m < n$	∞	$\begin{pmatrix} I & F \end{pmatrix}$
$r < m, n$	0 or ∞	$\begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix}$

4 Independence, Basis, Dimension (L9)

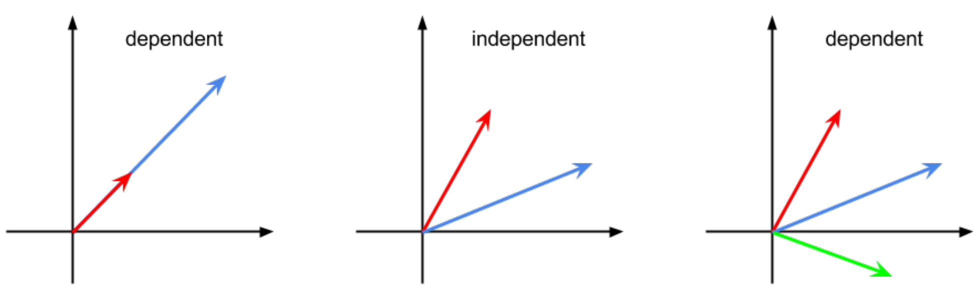
Suppose $m < n$ for a $m \times n$ matrix A, there are non-zero solutions to $Ax = 0$. Why?

- More unknowns than equations.
- There will be free variables.

4.1 Independence

Vectors $x_1, x_2, ..., x_n$ are independent if no combination gives a zero vector except when all coefficients are zeros:

$$c_1 x_1 + c_2 x_2 + ... + c_n x_n \neq 0 \quad \text{if } c_i \neq 0 \text{ for some } i$$



Given A's columns consist of $x_1, x_2, ..., x_n$:

$$A = \begin{pmatrix} x_1 & x_2 & ... & x_n \\ x_1 & x_2 & ... & x_n \end{pmatrix}$$

If $x_1, x_2, ..., x_n$ are independent:

$$\begin{pmatrix} x_1 & x_2 & ... & x_n \\ x_1 & x_2 & ... & x_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{only if} \quad \begin{cases} c_1 & = 0 \\ c_2 & = 0 \\ c_3 & = 0 \end{cases}$$

In other words, they are independent if $N(A)$ contains only the zero vector i.e. $rank(A) < n$ and there are no free variables. They're dependent if $Ac = 0$ for some non-zero c.

4.2 Span

Vectors $x_1, x_2, ..., x_n$ span a space when the space consists of all combinations of those vectors.

4.3 Basis

Vectors $x_1, x_2, ..., x_n$ are <u>basis</u> for a space when they are independent *and* span the space.
Standard basis for space R^3 is:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Another example of basis for space R^3 is:

$$\begin{pmatrix}1\\1\\2\end{pmatrix} \quad \begin{pmatrix}2\\2\\5\end{pmatrix} \quad \begin{pmatrix}3\\8\\1\end{pmatrix}$$

- Basis of R^n: n vectors give basis of R^n if $n \times n$ matrix with those columns is invertable.
- Given a space, every basis for the space has the same number of vectors. This number is the <u>dimension</u> of the space.

4.4 Example

$$A = \begin{pmatrix}1 & 2 & 3 & 1\\1 & 1 & 2 & 1\\1 & 2 & 3 & 1\end{pmatrix} \quad N(A) = \begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}$$

Given the space $C(A)$: Column vectors of A span the space.
There are 2 pivot columns because only the first and second columns are independent. Therefore rank of A i.e. dimension of $C(A)$ is 2.

One example of basis for $C(A)$:

$$2\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}1\\1\\1\end{pmatrix} + 2\begin{pmatrix}3\\2\\3\end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix}2\\2\\2\end{pmatrix} \quad \begin{pmatrix}7\\5\\7\end{pmatrix}$$

They are independent, both in $C(A)$. We need two vectors since the dimension is 2.

Now given the space $N(A)$ instead, the dimension of $N(A)$ i.e. number of free variables is 2.

One example of basis for $N(A)$: $\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix} \quad \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$

5 The Four Fundamental Subspaces (L10)

Given a $m \times n$ matrix A:

- Column space $C(A)$ in R^m
- Null space $N(A)$ in R^n
- Row space = All combinations of A's rows i.e. A^T's columns = $C(A^T)$ in R^n
- Null space of $A^T = N(A^T)$ in R^m

	Space	Basis	Dimension
$C(A)$	R^m	pivot columns	$rank(A) = r$
$N(A^T)$	R^m		$m - r$
Row space	R^n		r
$N(A)$	R^n	special solutions	$n - r$

5.1 Row Space Example

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 0 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R$$

- Different column spaces: $C(A) \neq C(R)$
- Same row spaces: $R(A) = R(R)$
- Basis for row space is the first r rows of R

5.2 Null Space of A^T

$$A^T y = 0 \quad \Rightarrow \quad y^T A^{TT} = y^T A = 0^T$$

$$rref(\left(A_{m\times n} \mid I_{m\times m} \right)) = E \left(A_{m\times n} \mid I_{m\times m} \right) = \left(R_{m\times n} \mid E_{m\times m} \right)$$

$$\text{e.g.} \quad EA = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ \hline -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{pmatrix} = R$$

Third row of E is the combination of rows that gives zeros i.e. y^T