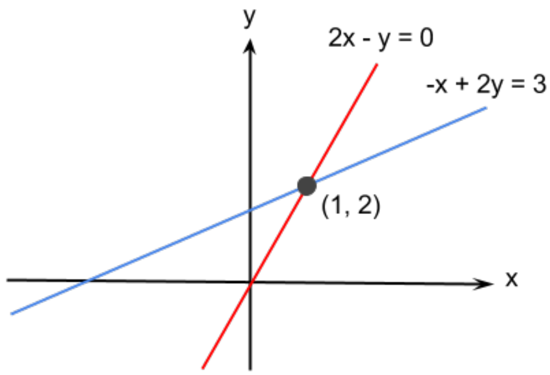
1 The Geometry of Linear Equations (L1)

1.1 Example 1: 2 Unknowns

$$\begin{cases} 2x & -y & = 0 \\ -x & +2y & = 3 \end{cases}$$

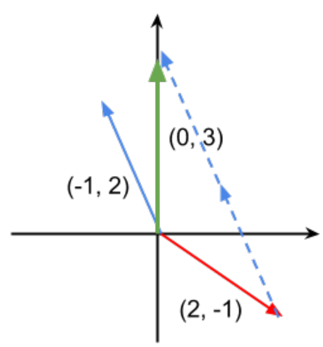


The two lines intersect at (1,2) $\Rightarrow x = 1, y = 2$ satisfy the two equations.
The equations represented in $Ax = b$ form:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Represented as a linear combination of columns:

$$x \begin{pmatrix} 2 \\ -1 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$



$x = 1, y = 2$ also satisfy the combination of columns.

1.2 Example 2: 3 Unknowns

$$\begin{cases} 2x & -y & & = 0 \\ -x & +2y & -z & = -1 \\ & -3y & +4z & = 4 \end{cases}$$

Each equation can be represented as a plane. The solution then is the point where three planes intersect.

The equations represented in $Ax = b$ form:

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$$

Represented as a linear combination of columns:

$$x \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + z \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$$

We can easily see that $x = 0, y = 0, z = 1$ satisfy the combination of columns.

Similarly, if we change the right hand side to:

$$x \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + z \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$$

We can also easily see that $x = 1, y = 1, z = 0$ satisfy the combination of columns.

1.3 Solution Spaces

Q. Given we have 3 unknowns, do the linear combinations of columns fill the 3-D space? i.e. Can we solve $Ax = b$ for any b?

A. YES for the columns from the example above because the matrix is non-singular and invertible. NO if the column vectors lie on the same plane i.e. 2-D space.

Q. How about 9 unknowns i.e. '9-D' space?

A. NO if the column vector lie on the same '8-D' space.

2 Elimination with Matrices (L2)

2.1 Gaussian Elimination

$$\begin{cases} x & +2y & +z & = 2 \\ 3x & +8y & +z & = 12 \\ & 4y & +z & = 2 \end{cases}$$

When the equations are represented in $Ax = b$ form:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

First pivot is 1 at (1, 1). We want to multiply the first row by a multiplier and subtract from the second row so that 3 at (2, 1) becomes 0. Such multiplier is 3:

$$\begin{pmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 2 & 1 \\ 1-3*1 & 8-3*2 & 1-3*1 \\ 0 & 4 & 1 \end{pmatrix} = \begin{pmatrix} \boxed{1} & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{pmatrix}$$

Second pivot is 2 at (2, 2). We now want to multiply the second row by a multiplier and subtract from the third row so that 4 at (3, 2) becomes 0. Such multiplier is 2:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0-2*0 & 4-2*2 & 1-2*(-2) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & 5 \end{pmatrix}$$

Third pivot is 5 at (3,3):

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \boxed{5} \end{pmatrix} = U$$

Pivots cannot be 0.

For example, if

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

After subtracting three times the first row from the second row, the second pivot becomes 0:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1-3*1 & 6-3*2 & 1-3*1 \\ 0 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & \boxed{0} & -2 \\ 0 & 4 & 1 \end{pmatrix}$$

We must switch the second and third rows before proceeding so 4 becomes the new pivot.

Another example, if

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{pmatrix}$$

After subtracting three times the first row from the second row, then subtracting twice the second row from the third row, the third pivot becomes 0:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 - 2 * 0 & 4 - 2 * 2 & -4 - 2 * (-2) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \boxed{0} \end{pmatrix}$$

Elimination fails because we cannot switch the rows to make the third pivot non-zero.

2.2 Back Substitution

Augment the original matrix A with b:

$$\left(\begin{array}{c|c} A & b \end{array} \right) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array} \right)$$

Perform Gaussian elimination:

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array} \right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 - 3 * 1 & 8 - 3 * 2 & 1 - 3 * 1 & 12 - 3 * 2 \\ 0 & 4 & 1 & 2 \end{array} \right) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array} \right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 - 2 * 0 & 4 - 2 * 2 & 1 - 2 * (-2) & 2 - 2 * 6 \end{array} \right) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & \boxed{5} & -10 \end{array} \right)$$

Now write down the rows as equations and solve the unknowns by substitutions:

$$\begin{cases} x \quad +2y \quad +z \quad = 2 \\ \qquad 2y \quad -2z \quad = 6 \\ \qquad\qquad 5z \quad = -10 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 1 \\ z = -2 \end{cases}$$

2.3 Elementary Matrices

We will translate the elimination process into matrix operations.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

Subtracting three times the first row from the second:

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{-3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow E_{21}A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{pmatrix}$$

- First row of E_{21} has 1 at its first column because the first row of $E_{21}A$ should be the same as the first row of A.
- Second row of E_{21} has -3 at its first and 1 at its second column because the second row of $E_{21}A$ should be three times the first row of A subtracted from the second row of A.
- First row of E_{21} has 1 at its third column because the third row of $E_{21}A$ should be the same as the third row of A.

Similarly, subtracting twice the second row from the third:

$$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{-2} & 1 \end{pmatrix} \Rightarrow E_{32}(E_{21}A) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{pmatrix} = U$$

In summary, we found elementary matrices E_{21} and E_{32} such that $E_{32}E_{21}A = U$

3 Multiplication and Inverse Matrices (L3)

3.1 Matrix Multiplications

Matrix $\times$ Column $\Rightarrow$ Column

$$\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \times \text{1st column} \\ y \times \text{2nd column} \\ z \times \text{3rd column} \end{pmatrix}$$

Row $\times$ Matrix $\Rightarrow$ Row

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix} = \begin{pmatrix} x \times \text{1st row} & y \times \text{2nd row} & z \times \text{3rd row} \end{pmatrix}$$

A (m $\times$ n) $\times$ B (n $\times$ p) $\Rightarrow$ C (m $\times$ p)

$$\begin{pmatrix} \boxed{- \quad - \quad -} \\ - \quad - \quad - \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} - \\ - \\ - \end{matrix}} & \begin{matrix} - \\ - \\ - \end{matrix} \end{pmatrix} = \begin{pmatrix} \boxed{-} & - \\ - & - \end{pmatrix}$$

$$(i\text{-th row of A}) \times (j\text{-th column of B}) = \sum_{k=1}^{n} a_{ik} b_{kj} = C_{ij}$$

3.1.1 A $\times$ B by Columns

Each column of C is a combination of A's columns.

$$(\mathrm{A}) \begin{pmatrix} \boxed{\begin{matrix} - \\ - \\ - \end{matrix}} & \begin{matrix} - \\ - \\ - \end{matrix} \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} - \\ - \end{matrix}} & \begin{matrix} - \\ - \end{matrix} \end{pmatrix}$$

$$\mathrm{A} \times (j\text{-th column of B}) = j\text{-th column of C}$$

3.1.2 A $\times$ B by Rows

Each row of C is a combination of B's rows.

$$\begin{pmatrix} \boxed{- \quad - \quad -} \\ - \quad - \quad - \end{pmatrix} (\mathrm{B}) = \begin{pmatrix} \boxed{- \quad -} \\ - \quad - \end{pmatrix}$$

$$(i\text{-th row of A}) \times \mathrm{B} = i\text{-th row of C}$$

3.1.3 A × B by Column-Row Pairs

AB is the sum of (j-th column of A) × (i-th row of B)

$$\begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix} \begin{pmatrix} b_{i1} & b_{i2} \end{pmatrix} = \begin{pmatrix} a_{1j}b_{i1} & a_{1j}b_{i2} \\ a_{2j}b_{i1} & a_{2j}b_{i2} \end{pmatrix}$$

3.1.4 A × B by Blocks

We can also divide matrices into blocks of smaller matrices to multiply them.

$$\begin{pmatrix} (A_1) & (A_2) \\ (A_3) & (A_4) \end{pmatrix} \begin{pmatrix} (B_1) & (B_2) \\ (B_3) & (B_4) \end{pmatrix} = \begin{pmatrix} (A_1B_1 + A_2B_3) & (A_1B_2 + A_2B_4) \\ (A_3B_1 + A_4B_3) & (A_3B_2 + A_4B_4) \end{pmatrix}$$

3.2 Matrix Inverses

Given a square matrix A, its inverse A^{-1} exists if A is non-singular and invertible.

$$A^{-1}A = I = AA^{-1}$$

Singular, non-invertible A:

- Determinant of A is zero.
- There exists a non-zero vector x such that $Ax = 0$

  Proof by contradiction: If A^{-1} existed and $Ax = 0$, $A^{-1}Ax = x = 0$, but $x \neq 0$

For example, $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ is singular and non-invertible. $Ax = 0$ when $x = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

3.2.1 Finding A^{-1} with Gauss-Jordan

$$\underset{A}{\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}} \quad \underset{A^{-1}}{\begin{pmatrix} a & c \\ b & d \end{pmatrix}} \quad = \quad \underset{I}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Augment the matrix A with I:

$$\left(\; A \mid I \; \right) = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array} \right)$$

We will repeat eliminations until the left side becomes I, which is when the right side is A^{-1}. Subtract twice the first row from the second:

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2-2*1 & 7-2*3 & 0-2*1 & 1-2*0 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

Subtract three times the second row from the first:

$$\Rightarrow \left(\begin{array}{cc|cc} 1-3*0 & 3-3*1 & 1-3*(-2) & 0-3*1 \\ 0 & 1 & -2 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}$$

Why is the right side A^{-1}?
As shown in L2, we can represent the eliminations with an elementary matrix E:

$$E\left(\begin{array}{c|c} A & I \end{array}\right) = \left(\begin{array}{c|c} I & ? \end{array}\right)$$

Since $EA = I$, $E = A^{-1}$ therefore $? = EI = E = A^{-1}$

4 Factorization into A = LU (L4)

4.1 Matrix Inverses

$$\text{Inverse of } A: \quad A^{-1}A = I = AA^{-1}$$

$$\text{Inverse of } AB: \quad (AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB)$$

4.2 A = LU with a Single Elementary Matrix

As show in L2, we can represent the elimination of A into U (upper triangular matrix) in term of an elementary matrix:

$$\underset{E_{21}}{\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}} \quad \underset{A}{\begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}} \quad = \quad \underset{U}{\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}}$$

Multiply both sides with E_{21}^{-1}:

$$\underset{A}{\begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}} \quad = \quad \underset{E_{21}^{-1}}{\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}} \quad \underset{U}{\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}}$$

We have transformed $EA = U$ to $A = LU$, where L is a lower triangular matrix.

$$L = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

4.3 A = LU with Multiple Elementary Matrices

Assuming no row exchanges took place:

$$E_{32}E_{21}A = U$$

$$(E_{32}E_{21})^{-1} = E_{21}^{-1}E_{32}^{-1}$$

$$A = E_{21}^{-1}E_{32}^{-1}U \Rightarrow L = E_{21}^{-1}E_{32}^{-1}$$

For example,

$$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix} \quad E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{E_{21}^{-1}}{\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -5 & 1\end{pmatrix}} \quad \underset{E_{32}^{-1}}{\begin{pmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{pmatrix}} \quad = \quad \underset{L}{\begin{pmatrix}1 & 0 & 0\\2 & 1 & 0\\0 & 5 & 1\end{pmatrix}}$$

We can also see above that the multipliers, 5 for E_{32} and 2 for E_{21}, go directly into L.

4.4 Runtime of Elimination

Given a $n \times n$ matrix A, how many operations (multiplication and subtraction) does it take to transform A to U? Assume no row exchanges.

It takes about n^2 operations to zero out the first column below the first pivot, $(n-1)^2$ for the second column below the second pivot, and so on...

$$n^2 + (n-1)^2 + ... + 1^2 \approx \frac{1}{3}n^3$$

5 Transposes, Permutations, Spaces R^n (L5)

5.1 Permutations

Row exchanges can be represented as matrix operations with permutation matrix P.

2×2 matrix for example:

$$\underset{P}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

P is an identity matrix with reordered rows. Below are all possible 3×3 permutation matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

There are 3! possible orderings for a 3×3 identity matrix, thus 6 different 3×3 permutation matrices. In general, there are $n!$ possible orderings for a $n \times n$ identity matrix, thus $n!$ different $n \times n$ permutation matrices.

To account for the row exchanges, we can rewrite $A = LU$ as $PA = LU$.

5.2 Transposes

$$(A^T)_{ij} = Aji$$

$$\text{For example:} \quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{pmatrix}$$

Transposes of permutation matrices

$$P^T = P^{-1} \quad \text{i.e.} \quad P^T P = I$$

Inverse of transposes

$$A^T (A^{-1})^T = I = (A^{-1})^T A^T$$

5.2.1 Symmetric Matrices

A is symmetric if $A^T = A$.

A^TA always produces a symmetric matrix because $(A^TA)^T = A^TA^{TT} = A^TA$

$$\begin{pmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{pmatrix}$$

5.3 Vector Spaces

Vector space R^n represents all the n-dimensional real vectors i.e. all column vectors with n real components.

For example, R^2 represents all the 2-dimensional real vectors i.e. x-y plane. All column vectors that consist of 2 components such as $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \pi \\ e \end{pmatrix}$ are all part of R^2.

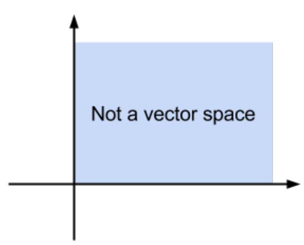
5.3.1 Vector Subspaces

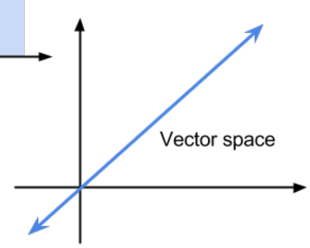
Subspaces are vector spaces inside of R^n.

If a vector space contains vectors v and w, then it must also contain $cv + dw$ for any combinations of c and d. This means that every vector space must contain the zero vector $(c = 0, d = 0)$.

For example, a subspace of R^2 is either:

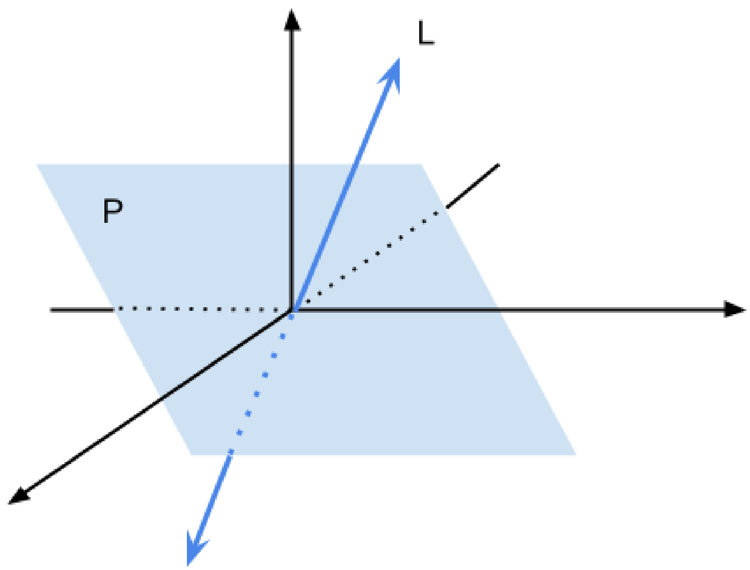
- All of R^2
- Line through the origin
- Only the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. Z





Combination of Subspaces

Given P is a plane through the origin and L is a line through the origin, P and L are both subspaces of R^3.



$P \cup L$ is not a subspace of R^3, but $P \cap L$ i.e. the zero vector is a subspace of R^3.

In general, the intersection of two subspaces $S \cap T$ is also a subspace.