Math 320 Final Study Guide

Alisa Liu

December 2019

1 The Real Number System

- An upper bound b of $S \subseteq \mathbb{R}$ is the **supremum** of S iff for any $\varepsilon > 0$, there is $x \in S$ such that $b - \varepsilon < x$.
- A lower bound ℓ of $S \subseteq \mathbb{R}$ is the **infimum** of S iff for any $\varepsilon > 0$, there is $x \in S$ such that $x < \ell + \varepsilon$.
- **The completeness axiom**: If $S \subseteq \mathbb{R}$ is non-empty and bounded above, then S has a supremum. If $S \subseteq R$ is non-empty and bounded below, then S has an infimum.
- Problem: Show $\sup S = b$.
  - Let $x \in S$. Then $x \leq b$, so b is an upper bound.
  - Let $\varepsilon > 0$. Pick x and show that $x \in S$ and $b - \varepsilon < x$.

2 Sequences in R

- A sequence (x_n) **converges** to $L \in \mathbb{R}$ iff for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon \quad \text{for all } n \geq N$$

- A sequence (x_n) is **Cauchy** iff for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$|x_m - x_n| < \varepsilon \quad \text{for all } n, m \geq N$$

- Theorems
  - Every convergent sequence is bounded.
  - Suppose b is an upper bound of $S \subset \mathbb{R}$. Then b is the supremum of S if and only if there is a sequence (x_n) in S converging to b.
  - **Monotone convergence**: If (x_n) is increasing and bounded above, or if (x_n) is decreasing and bounded below, then (x_n) converges.
  - Every sequence (x_n) has a monotone subsequence (x_{n_k}).
  - **Bolzano-Wierstrass**: Every bounded subsequence has a convergence subsequence.
  - A sequence (x_n) is Cauchy if and only if it converges.
- Problem: Show that a recursively defined sequence (x_n) converges.
  - Show that it's monotone (induction) and bounded (induction). So it converges by the monotone convergence theorem.
- Problem: Suppose (a_n) converges. Then (b_n) converges.
  - Since (a_n) converges, (a_n) is Cauchy. Express $|b_{n+k} - b_n|$ in terms of $|a_{n+k} - a_n|$ to show that (b_n) is Cauchy as well.

3 Functions on R

- The **limit** of f as x approaches a is $L \iff$

$$\lim_{x \to a} f(x) = L \iff \text{if } x_n \to a, \text{ then } f(x_n) \to L$$
$$\iff \text{for all } \varepsilon > 0, \text{ there is } \delta > 0 \text{ such that } 0 < |x-a| < \delta \text{ implies } |f(x) - L| < \varepsilon$$

- f is **continuous** at $a \iff$

$$\lim_{x \to a} f(x) = f(a) \iff \text{if } x_n \to a, \text{ then } f(x_n) \to f(a)$$
$$\iff \text{for all } \varepsilon > 0, \text{ there is } \delta > 0 \text{ such that } |x-a| < \delta \text{ implies } |f(x) - f(a)| < \varepsilon$$

- f is **uniformly continuous** on $[a,b]$ iff for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$|x-y| < \delta \text{ implies } |f(x) - f(y)| < \varepsilon \quad \text{for all } x, y \in [a,b]$$

- Theorems
  - **Squeeze theorem**: Suppose f, g, h are functions $\mathbb{R} \to \mathbb{R}$ with $f(x) \leq g(x) \leq h(x)$ for all $x \in \mathbb{R}$, and suppose $a \in \mathbb{R}$.
  $$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x) \quad \implies \quad \lim_{x \to a} g(x) = L$$
  - Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and $f(r) = g(r)$ for all $r \in \mathbb{R}$ for all $r \in \mathbb{Q}$. Then $f(x) = g(x)$ for all $x \in \mathbb{R}$.
  - **Extreme value**: If $f : [a,b] \to \mathbb{R}$ is continuous, then f is bounded and attains a maximum and a minimum.
  - **Intermediate value**: If f is a continuous function and a, b, y are such that $f(a) < y < f(b)$, then there is $x \in [a,b]$ such that $f(x) = y$.
  - If $f : E \to \mathbb{R}$ is uniformly continuous and (x_n) is a Cauchy sequence, then $f(x_n)$ is Cauchy as well.
  - If $f : (a,b) \to \mathbb{R}$ is uniformly continuous, then f can be extended to a continuous function $f : [a,b] \to \mathbb{R}$.
  - Any continuous function $f : [a,b] \to \mathbb{R}$ is uniformly continuous.

- Problem: Show $\lim_{x \to a} f(x) = L$.
  - To get an upper bound for x
  $$|x-a| < 1 \implies |x| - |a| \leq |x-a| < 1 \implies |x| < |a| + 1$$
  - To get a lower bound for x
  $$|x-a| < 1 \implies |a| - |x| \leq |a-x| < 1 \implies |x| > |a| - 1$$
  $$|x-a| < \frac{|a|}{2} \implies |x| > \frac{|a|}{2}$$

- Problem: Show that $\lim_{x \to a} f(x)$ does not exist.
  (1) An example of $x_n \to a$ such that $f(x_n)$ diverges
  (2) An example of $x_n \to a$, $y_n \to a$ such that $f(x_n) \to L, f(y_n) \to K$ with $L \neq K$
  (3) There is $\varepsilon > 0$ such that for all $\delta > 0$ and L, there is x satisfying $0 < |x-a| < \delta$ but $|f(x) - L| \geq \varepsilon$

- Problem: Show that $f(x)$ is not continuous at a.
  (1) An example of $x_n \to a$ such that $f(x_n) \not\to f(a)$
  (2) An example of f, g, with g continuous, such that $f(r) = g(r)$ but $f(x) \neq g(x)$ for $x \notin \mathbb{Q}$

- Problem: Show that $f(x)$ is not uniformly continuous.
  (1) An example of a Cauchy sequence (x_n) such that $f(x_n)$ is not Cauchy
  (2) Show it cannot be extended continuously to $[a, b]$
- Example: Function continuous at only a, b, c

$$f(x) = \begin{cases} (x-a)(x-b)(x-c) & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- Example: Function continuous at only the integers

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

4 Differentiability on R

- $f : E \to \mathbb{R}$ is **differentiable** at a $a \in E$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists, in which case we call it the **derivative** and denote it $f'(a)$
- The n-th order **Taylor polynomial** of f centered at a is the polynomial $P_n(x)$ given by

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The corresponding Taylor remainder is

$$R(x) = f(x) - P_n(x)$$

- Theorems
  - If f is differentiable at a, then it is continuous at a
  - If f is differentiable and has a local maximum or minimum at a, then $f'(a) = 0$
  - If f, g are differentiable at a, then
    - ◇ $f \cdot g$ is differentiable at a and

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$

    - ◇ $\frac{f}{g}$ is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}$$

    - ◇ $f \circ g$ is differentiable at a and

$$(f \circ g)'(a) = g'(f(a))f'(a)$$

  - Suppose f is differentiable and $f'(a) < y < f'(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = y$.
  - **Rolle's**: Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then if $f(a) = f(b)$, there exists $c \in (a, b)$ such that $f'(c) = 0$.
  - **Mean value**: Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \iff f(b) - f(a) = f'(c)(b - a)$$

  - **2nd derivative test**: Suppose f is differentiable and $f'(a) = 0$. If $f''(a) > 0$, then f has a local minimum at a. If $f''(a) < 0$, then f has a local maximum at a.
  - Suppose f is continuous on $\mathbb{R}$ and differentiable at all $x \neq a$, and that $\lim_{x \to a} f'(x) = L$ exists. Then f is differentiable at a as well and $f'(a) = L$.

– Suppose f is differentiable and its derivative is bounded. Then f is uniformly continuous.

– **Taylor's theorem**: Suppose that f is $(n+1)$-times differentiable. Then for any x and a there exists c between x and a such that

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)(c)}}{(n+1)!}(x-a)^{n+1}$$

- Example: The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not differentiable at 0. The function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable but g' is not continuous at 0.

- Example: Function differentiable at only a, b, c

$$f(x) = \begin{cases} (x-a)^2(x-b)^2(x-c)^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- Example: Function differentiable everywhere except a, b, c

$$f(x) = |x-a| + |x-b| + |x-c|$$

- Example: Function continuous at one point and not differentiable there

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{cases}$$

5 Integrability on R

- For a bounded function $f : [a, b] \to \mathbb{R}$ and partition P of $[a, b]$
  - the **upper Darboux sum** is

$$U(f, P) = \sum_{I_k} (\sup f \text{ over } I_k)(\text{length of } I_k)$$

  - the **lower Darboux sum** is

$$L(f, P) = \sum_{I_k} (\inf f \text{ over } I_k)(\text{length of } I_k)$$

- For a bounded function $f : [a, b] \to \mathbb{R}$
  - The **upper Darboux integral** is

$$(U) \int_a^b f(x)dx = \inf \left\{ U(f, P) \mid P \text{ is a partition of } [a, b] \right\}$$

  - The **lower Darboux integral** is

$$(L) \int_a^b f(x)dx = \sup \left\{ L(f, P) \mid P \text{ is a partition of } [a, b] \right\}$$

- A bounded function f is **Darboux integrable** over $[a, b]$ if

$$(U) \int_a^b f(x)\,dx = (L) \int_a^b f(x)\,dx$$

  in which case we call the common value the **integral** and denote it $\int_a^b f(x)\,dx$

- A bounded function f is **integrable** on $[a, b]$ if for any $\varepsilon > 0$, there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$

- Let f be a function on $[a, b]$, P be a partition of $[a, b]$, and $\{t_k\}$ be a collection of sample points with each $t_k \in I_k$. The **Riemann sum** is

$$S(f, P, t_k) = \sum_{k=1}^{n} f(t_k)(\text{length of } I_k)$$

- A function f is **Riemann integrable** with integral I if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any partition P with $||P_n|| < \delta$ and any sample points $\{t_k\}$,

$$|S(f, P, t_k) - I| < \varepsilon$$

- Theorems
  - For any refinement P' of a partition P,

$$U(f, P) \geq U(f, P') \qquad L(f, P) \leq L(f, P')$$

  - For any partitions P and Q, $L(f, P) \leq U(f, Q)$.
  - If f is piecewise continuous, then f is integrable.
  - A continuous function f on a closed interval $[a, b]$ is integrable.
  - Suppose f, g are integrable with $g(x) \leq f(x)$ for all $x \in [a, b]$. Then

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$$

  - An integrable function f is bounded.
  - Suppose f is integrable on $[a, b]$ with integral I. Let P_n be a sequence of partitions such that $||P_n|| \to 0$, and take any collection of sample points $\{t_{n,k}\}$ for each P_n. Then $S(f, P_n, t_{n,k}) \to I$.
  - **Fundamental Theorem of Calculus (I)**: Suppose that f is differentiable and f' is integrable on $[a, b]$. Then

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

  - Integrals are always continuous.
  - **Fundamental Theorem of Calculus (II)**: Suppose f is continuous on $[a, b]$ and define the function F on $[a, b]$ by

$$F(x) = \int_a^x f(t)dt$$

    Then F is differentiable and $F'(x) = f(x)$.
  - **Mean value theorem for integrals**: Suppose f is continuous on $[a, b]$. Then there exists $c \in (a, b)$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)\,dt$$

- Problem: Show f is integrable.
  (1) Show for all partitions P, $U(f, P) = L(f, P)$, so the upper and lower Darboux integrals agree
  (2) Show for any $\varepsilon > 0$, there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$

(3) Show f is continuous on a closed interval

(4) Show f is a linear combination of integrable functions

- Problem: Given that f is integrable, find $\int_a^b f(x)\,dx$.

  (1) If $U(f,P)$ or $L(f,P)$ is constant for every P, then that gives the integral

  (2) Show that for any P,

  $$L(f,P) \le I \le U(f,P) \implies (U)\int_a^b f(x)\,dx \le I \le (L)\int_a^b f(x)\,dx \implies \int_a^b f(x)\,dx = I$$

  (3) For a sequence of partitions P_n, show $L(f,P_n) \to I$ and $U(f,P_n) \to I$. Since

  $$L(f,P_n) \le \int_0^b f(x)\,dx \le U(f,P_n)$$

  It follows from the squeeze theorem that $\int_a^b f(x)\,dx = I$.

  (4) For a sequence of partitions P_n with $||P_n|| \to 0$ and any collection of sample points $t_{n,k}$ for each P_n, show $S(f,P_n,t_k) \to I$. Then $\int_a^b f(x)\,dx = I$.

- Problem: Show f is not integrable.

  (1) Show for all partitions P, $U(f,P) = c$ and $L(f,P) = d$ with $c \ne d$, so

  $$(U)\int_a^b f(x)\,dx \ne (L)\int_a^b f(x)\,dx$$

  (2) Show that for any partition P, $U(f,P) - L(f,P)$ cannot be made smaller than an arbitrary $\varepsilon > 0$