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Exact sequences on Powell–Sabin splits

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Abstract
We construct smooth finite elements spaces on Powell–Sabin triangulations that form an exact sequence. The first space of the sequence coincides with the classical C^1 Powell–Sabin space, while the others form stable and divergence-free yielding pairs for the Stokes problem. We develop degrees of freedom for these spaces that induce projections that commute with the differential operators.

Keywords Finite elements · Exact sequences · Commuting diagrams · Powell–Sabin triangulations

Mathematics Subject Classification 65N30

1 Introduction

In the finite element exterior calculus [3, 4], sequences of discrete spaces that conform to the continuous de Rham complex are used to approximate solutions of the Hodge–Laplacian. While this framework has been successfully applied to the de Rham complex with minimal L^2 smoothness, recent progress has extended this methodology to higher order Sobolev spaces, i.e., spaces with greater smoothness. Such constructions naturally lead to structure-preserving discretizations for the Stokes/Navier–Stokes problem as well as problems in linear elasticity. For example, in recent work [5, 7] specific mesh refinements were used to build spaces of continuous piecewise polynomial k-forms with continuous exterior derivative. In particular, it is shown in [7] that *locally*, smooth finite element

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spaces form an exact sequence on so-called Alfeld splits in any spatial dimension and for any polynomial degree. Global spaces in three dimensions are also constructed in Fu et al. [7], leading to stable finite element pairs for the Stokes problem (also see [16]). On the other hand, Christiansen and Hu [5] considered low-order approximations in any dimension. However, they use different splits as they move along the de Rham sequence. For zero forms they use the finest split (e.g., in two dimensions it is the Powell–Sabin split). For n forms, where n is the dimension, they use the Alfeld split if the n forms are assumed to be continuous. If the n forms may be discontinuous, they do not use a splitting. Furthermore, in two dimensions, they define a de Rham sequence with arbitrarily high polynomial order, where each split in the sequence is Clough-Tocher.

In this paper we construct smooth finite element spaces on Powell–Sabin splits that form an exact sequence. In the lowest order case, the first space in the sequences coincides with the piecewise quadratic C^1 Powell–Sabin space [12, 14]. However, we construct these spaces for any polynomial degree which appears to be new (cf. [8, 9]). We also define smooth spaces on Powell–Sabin splits for vector-valued polynomial spaces, define commuting projections onto the finite element spaces, and characterize the range and kernel of differential operators acting on the finite element spaces. The last two spaces in the sequence form stable finite element pairs for the Stokes problem that enforce the incompressibility constraint exactly; see [11].

A potential advantage of the use of Powell–Sabin splits is that the minimal polynomial degree of the global spaces is not expected to increase with respect to the spacial dimension. For example, the lowest polynomial degree of C^1 spaces on Powell–Sabin splits is two in both two and three dimensions. In contrast, the polynomial degree of smooth piecewise polynomials must necessarily increase with dimension on Alfeld splits. In two dimensions, C^1 piecewise polynomials have degree of at least three, whereas in three dimensions the minimal polynomial degree is five [1, 12]. These degree restrictions for C^1 conforming spaces also dictate the polynomial degrees of other finite element spaces on Alfeld splits. For example, finite element spaces that approximate the velocity in the Stokes problem must have degree of at least the spatial dimension [2, 10, 16].

Let us describe the Powell–Sabin split here. Let $\Omega \subset \mathbb{R}^2$ be a polyhedral domain, and let $\mathcal{T}_h$ be a simplicial, shape-regular triangulation of Ω. Then the Powell–Sabin triangulation $\mathcal{T}_h^{\mathrm{ps}}$ is obtained as follows. We select an interior point of each triangle $T \in \mathcal{T}_h$ and adjoin this point with each vertex of T. Next, the interior points of each adjacent pair of triangles are connected with an edge. For any T that shares an edge with the boundary of Ω, an arbitrary point on the boundary edge is selected to connect with the interior point of T, so that each $T \in \mathcal{T}_h$ is split into six triangles. See Fig. 1. In order for the resulting refinement $\mathcal{T}_h^{\mathrm{ps}}$ to be well-defined, the interior points must be selected such that their adjoining edge intersects the edge shared by their respective triangles in $\mathcal{T}_h$, in which case $\mathcal{T}_h^{\mathrm{ps}}$ is the Powell–Sabin refinement of $\mathcal{T}_h$. One common choice of interior points that produces a well-defined triangulation is the incenter of each $T \in \mathcal{T}_h$, i.e., the center point of the largest circle that fits within T [12]. We define the set $\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}})$ to be the points of intersection of the edges of $\mathcal{T}_h$ with the edges that adjoin interior points. An interesting fact about the meshes constructed is that the points in $\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}})$ are singular vertices of the mesh $\mathcal{T}_h^{\mathrm{ps}}$; see [15].

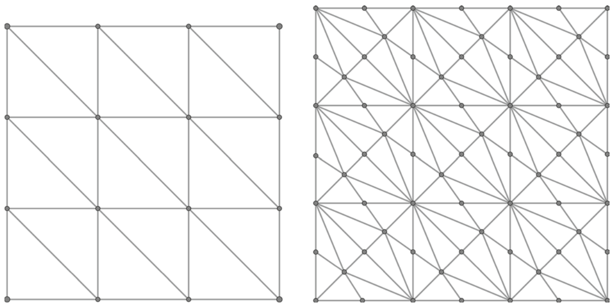

Fig. 1 (*left*) A triangulation of the unit square, and (*right*) its Powell–Sabin refinement

Hence, the last space in our sequence has to be modified accordingly; see the global space $\mathcal{V}_r^2(\mathcal{T}_h^{\mathrm{ps}})$ below.

Related to the current work is [17, 18], where conforming finite element pairs are proposed and studied for the Stokes problem on Powell–Sabin meshes. There it is shown that if the discrete velocity space is the linear Lagrange finite element space, and if the pressure space is the image of the divergence operator acting on the discrete velocity space, then the resulting pair is inf-sup stable.

Note that, by design, the discrete pressure spaces in [17, 18], and correspondingly the range of the divergence operator, is not explicitly given. Practically, this issue is bypassed by using the iterative penalty method to solve the finite element method without explicitly constructing a basis of the discrete pressure space. In this paper we explicitly construct the discrete pressure space and characterize the space of divergence-free functions for any polynomial degree.

The rest of the paper is organized as follows. In the next section we state some preliminary definitions and results on a single macro-triangle. In Sect. 3 we show that the smooth finite element spaces form an exact sequence on macro-triangles, and in Sect. 4 we develop degrees of freedom and projections for these spaces, and prove commutative properties of these projections. We extend these results to the global setting in Sect. 5 and derive similar results. We end the paper in Sect. 6 with some concluding remarks.

2 Spaces on one macro-triangle

Let T be a triangle with vertices z_1, z_2, and z_3, labelled counter-clockwise, and let z_0 be an interior point of T. Denote the edges of T by $\{e_i\}_{i=1}^3$, labelled such that z_i is not a vertex of e_i, i.e., $e_i = [z_{i+1}, z_{i+2}]$. We denote the outward unit normal of ∂T restricted to e_i as n_i and the tangent vector by t_i. Let z_{3+i} be an interior point of edge e_i. We then construct the triangulation $T^{\mathrm{ps}} = \{T_1, \dots, T_6\}$ by connecting each z_i to z_0 for $1 \le i \le 6$; see Fig. 2. We let $\mathcal{E}^b(T^{\mathrm{ps}})$ be the set containing the six boundary edges of T^{ps}. We also let $\mathcal{M}(T^{\mathrm{ps}}) = \{z_4, z_5, z_6\}$ and use the notation for $z \in \mathcal{M}(T^{\mathrm{ps}})$, $\mathcal{T}(z) = \{K_1, K_2\}$, where $K_i \in T^{\mathrm{ps}}$ have z as a vertex. We also set $T(z) = K_1 \cup K_2$. Let

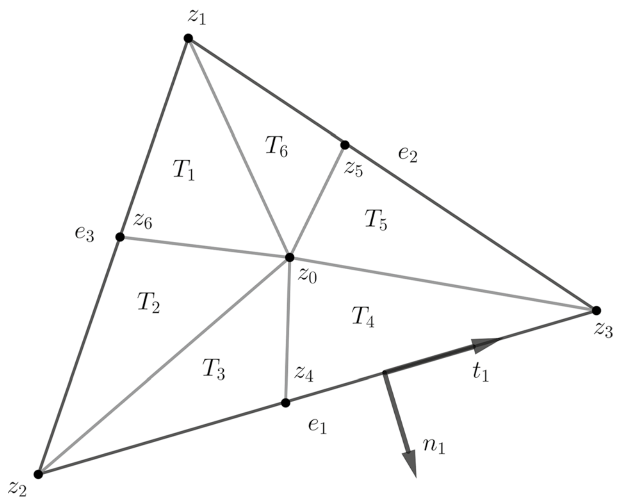

Fig. 2 A pictorial description of a Powell–Sabin split of a triangle

$z \in \mathcal{M}(T^{\mathrm{ps}})$ and suppose that $\mathcal{T}(z) = \{K_1, K_2\}$ with common edge e. Then we define the jump as follows

$$[\![p]\!](z) = p_1(z)m_1 + p_2(z)m_2,$$

where $p_i = p|_{K_i}$ and m_i is the outward pointing normal to K_i perpendicular to e. We see then that $[\![p]\!](z) = (p_1(z) - p_2(z))m_1 = -(p_1(z) - p_2(z))m_2$.

Let μ be the unique piecewise linear function on the mesh T^{ps} such that $\mu(z_0) = 1$ and $\mu = 0$ on ∂T. We use the notation $\nabla\mu_i := \nabla\mu|_{e_i} = \nabla\mu|_{T(z_{3+i})}$ and note that

$$\frac{1}{|\nabla\mu_i|}\nabla\mu_i = -n_i \quad (i = 1, 2, 3), \tag{2.1}$$

and hence

$$\nabla\mu_i \cdot t_i = 0 \quad (i = 1, 2, 3). \tag{2.2}$$

2.1 Local finite element spaces

In this section we consider three classes of finite element spaces each with varying smoothness on T^{ps}. First we define the differential operators

$$\mathrm{rot}\, q = \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1}\right)^{\intercal}, \qquad \mathrm{div}\, v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2},$$

and corresponding spaces, for an open bounded domain $S \subset \mathbb{R}^2$,

$$H(\mathrm{rot};S)=\{q\in L^2(S):\ \mathrm{rot}\, q\in L^2(S)\},\qquad H(\mathrm{div};S)=\{v\in[L^2(S)]^2:\ \mathrm{div}\, v\in L^2(S)\},$$
$$\mathring{H}(\mathrm{rot};S)=\{q\in H(\mathrm{rot};S):\ q|_{\partial S}=0\},\qquad \mathring{H}(\mathrm{div};S)=\{v\in H(\mathrm{div};S):\ v\cdot n_S|_{\partial S}=0\},$$

where n_S denotes the outward unit normal of S. We also denote by $\mathring{L}^2(S)$ the space of square integrable functions on S with vanishing mean.

For $r\in\mathbb{N}$, let $\mathcal{P}_r(S)$ denote the space of polynomials of degree $\le r$ with domain S, and we use the convention $\mathcal{P}_r(S)=\{0\}$ for $r<0$. Define the piecewise polynomial space on the Powell–Sabin split as

$$\mathcal{P}_r(T^{\mathrm{ps}})=\{q\in L^2(T):\ q|_S\in\mathcal{P}_r(S),\quad \forall S\in T^{\mathrm{ps}}\}.$$

Remark 1 For any $q\in\mathcal{P}_r(T^{\mathrm{ps}})$ satisfying $q|_{\partial T}=0$, there exists $p\in\mathcal{P}_{r-1}(T^{\mathrm{ps}})$ such that $q=\mu p$.

Definition 1 Let $r\in\mathbb{N}$. The *Nédélec spaces* (of the second-kind) with and without boundary conditions are given by [13]

$$\begin{aligned}
V_r^0(T^{\mathrm{ps}})&=\mathcal{P}_r(T^{\mathrm{ps}})\cap H(\mathrm{rot};T), & \mathring{V}_r^0(T^{\mathrm{ps}})&=\mathcal{P}_r(T^{\mathrm{ps}})\cap \mathring{H}(\mathrm{rot};T),\\
V_r^1(T^{\mathrm{ps}})&=\mathcal{P}_r(T^{\mathrm{ps}})\cap H(\mathrm{div};T), & \mathring{V}_r^1(T^{\mathrm{ps}})&=\mathcal{P}_r(T^{\mathrm{ps}})\cap \mathring{H}(\mathrm{div};T),\\
V_r^2(T^{\mathrm{ps}})&=\mathcal{P}_r(T^{\mathrm{ps}}), & \mathring{V}_r^2(T^{\mathrm{ps}})&=\mathcal{P}_r(T^{\mathrm{ps}})\cap \mathring{L}^2(T).
\end{aligned}$$

Definition 2 The *Lagrange space* $L_r^k(T^{\mathrm{ps}})$ (resp., $\mathring{L}_r^k(T^{\mathrm{ps}})$) is the subspace of $V_r^k(T^{\mathrm{ps}})$ (resp., $\mathring{V}_r^k(T^{\mathrm{ps}})$) consisting of continuous piecewise polynomials, i.e.,

$$\begin{aligned}
L_r^0(T^{\mathrm{ps}})&=\mathcal{P}_r(T^{\mathrm{ps}})\cap C(T), & \mathring{L}_r^0(T^{\mathrm{ps}})&=L_r^0(T^{\mathrm{ps}})\cap \mathring{H}(\mathrm{rot};T),\\
L_r^1(T^{\mathrm{ps}})&=[L_r^0(T^{\mathrm{ps}})]^2, & \mathring{L}_r^1(T^{\mathrm{ps}})&=[\mathring{L}_r^0(T^{\mathrm{ps}})]^2,\\
L_r^2(T^{\mathrm{ps}})&=L_r^0(T^{\mathrm{ps}}), & \mathring{L}_r^2(T^{\mathrm{ps}})&=\mathring{L}_r^0(T^{\mathrm{ps}})\cap\mathring{V}_r^2(T^{\mathrm{ps}}).
\end{aligned}$$

Remark 2 Note the redundancies in notation, $L_r^0(T^{\mathrm{ps}})=V_r^0(T^{\mathrm{ps}})$ and $\mathring{L}_r^0(T^{\mathrm{ps}})=\mathring{V}_r^0(T^{\mathrm{ps}})$.

Definition 3 We define the *smooth spaces* with and without boundary conditions as

$$\begin{aligned}
S_r^0(T^{\mathrm{ps}})&=\{v\in L_r^0(T^{\mathrm{ps}}):\ \mathrm{rot}\, v\in[C(T)]^2\}, & \mathring{S}_r^0(T^{\mathrm{ps}})&=\{v\in S_r^0(T^{\mathrm{ps}}):\ v=0 \text{ and } \mathrm{rot}\, v=0 \text{ on } \partial T\},\\
S_r^1(T^{\mathrm{ps}})&=\{v\in L_r^1(T^{\mathrm{ps}}):\ \mathrm{div}\, v\in C(T)\}, & \mathring{S}_r^1(T^{\mathrm{ps}})&=\{v\in S_r^1(T^{\mathrm{ps}}):\ v=0 \text{ and } \mathrm{div}\, v=0 \text{ on } \partial T\},\\
S_r^2(T^{\mathrm{ps}})&=L_r^2(T^{\mathrm{ps}}), & \mathring{S}_r^2(T^{\mathrm{ps}})&=\mathring{L}_r^2(T^{\mathrm{ps}}).
\end{aligned}$$

3 Exact sequences on a macro triangle

The goal of this section is to derive exact sequences consisting of the piecewise polynomial spaces defined in the previous section. As a first step, we state a well-known result, that the Nédélec spaces form exact sequences [3, 4].

Proposition 1 *The following sequences are exact, i.e., the range of each map is the kernel of the succeeding map*

$$\mathbb{R} \to V_r^0(\mathcal{T}^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} V_{r-1}^1(\mathcal{T}^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} V_{r-2}^2(\mathcal{T}^{\mathrm{ps}}) \to 0,$$
$$0 \to \mathring{V}_r^0(\mathcal{T}^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} \mathring{V}_{r-1}^1(\mathcal{T}^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} \mathring{V}_{r-2}^2(\mathcal{T}^{\mathrm{ps}}) \to 0.$$

The goal now is to extend Proposition 1 to incorporate smooth spaces. An integral component of this extension is a characterization of the range of the divergence operator acting on the (vector-valued) Lagrange space. For example, it is known [15, Proposition 2.1] that if $v \in \mathring{L}_r^1(\mathcal{T}^{\mathrm{ps}})$ then $\mathrm{div}\, v$ is continuous at the vertices z_4, z_5, z_6. In particular, this is because each of these vertices is a *singular vertex*, i.e., the edges meeting at the vertex fall on exactly two straight lines. Hence, in order to extend Proposition 1 and to characterize the range of $\mathrm{div}\, \mathring{L}_r^1(\mathcal{T}^{\mathrm{ps}})$, we will consider the spaces

$$\mathcal{V}_r^2(\mathcal{T}^{\mathrm{ps}}) = \{q \in V_r^2(\mathcal{T}^{\mathrm{ps}}) : q \text{ is continuous at } z_4, z_5, z_6\},$$
$$\mathring{\mathcal{V}}_r^2(\mathcal{T}^{\mathrm{ps}}) = \mathcal{V}_r^2(\mathcal{T}^{\mathrm{ps}}) \cap \mathring{L}^2(T).$$

We then have that $\mathrm{div}\, \mathring{L}_r^1(\mathcal{T}^{\mathrm{ps}}) \subset \mathring{\mathcal{V}}_{r-1}^2(\mathcal{T}^{\mathrm{ps}})$. In this section we show that $\mathrm{div} : \mathring{L}_r^1(\mathcal{T}^{\mathrm{ps}}) \to \mathring{\mathcal{V}}_{r-1}^2(\mathcal{T}^{\mathrm{ps}})$ is surjective, i.e., $\mathrm{div}\, \mathring{L}_r^1(\mathcal{T}^{\mathrm{ps}}) = \mathring{\mathcal{V}}_{r-1}^2(\mathcal{T}^{\mathrm{ps}})$.

The proof of this result is based on several preliminary lemmas. As a first step, we state the canonical degrees of freedom for the lowest order Nédélec $H(\mathrm{div})$ -conforming finite element space on the unrefined triangulation [13].

Lemma 1 *Any $w \in [\mathcal{P}_1(T)]^2$ is uniquely determined by the values*

$$\int_{e_i} (w \cdot n_i)\kappa \quad \forall \kappa \in \mathcal{P}_1(e_i).$$

Lemma 2 *Let $q \in \mathcal{V}_r^2(\mathcal{T}^{\mathrm{ps}})$ and $r \geq 1$, then there exists $w \in L_r^1(\mathcal{T}^{\mathrm{ps}})$ and $g \in V_{r-1}^2(\mathcal{T}^{\mathrm{ps}})$ such that $\mu^s q = \mathrm{div}\,(\mu^{s+1} w) + \mu^{s+1} g$ for any $s \geq 0$.*

Proof Let $b_i \in \mathcal{P}_1(e_i)$ be the linear function such that $q|_{e_i} - b_i$ vanishes at the end points of e_i. Because $q - b_i$ vanishes at the endpoints and q is continuous at z_{3+i}, there exists $a_i \in L_r^0(\mathcal{T}^{\mathrm{ps}})$ such that $a_i|_{e_i} = (q - b_i)|_{e_i}$ and $\mathrm{supp}\, a_i \in T(z_{3+i})$. Note that $a_i|_{e_j} = 0$ for $i \neq j$.

Next, using (2.1) and the Nédélec degrees of freedom stated in Lemma 1, we construct a unique function $w_1 \in [\mathcal{P}_1(T)]^2$ such that

$$(s+1) w_1 \cdot \nabla \mu_i = b_i \quad \text{on } e_i, \quad i = 1, 2, 3.$$

We set $\ell_i = \frac{\nabla \mu_i}{|\nabla \mu_i|^2}$,

$$w_2 = \frac{1}{s+1}(a_1 \ell_1 + a_2 \ell_2 + a_3 \ell_3), \quad \text{and} \quad w = w_1 + w_2.$$

We then see that, on e_i,

$$(s+1)w\cdot\nabla\mu_i=(s+1)w_1\cdot\nabla\mu_i+(s+1)w_2\cdot\nabla\mu_i=b_i+a_i=q.$$

Therefore the function $(s+1)w\cdot\nabla\mu-q$ vanishes on ∂T, which implies that $\mu v=(s+1)w\cdot\nabla\mu-q$ for some $v\in V^2_{r-1}(T^{\mathrm{ps}})$; see Remark 1.

Finally we compute

$$\mu^s q=\mu^s q+\mathrm{div}\,(\mu^{s+1}w)-\mu^{s+1}\mathrm{div}\,(w)-\mu^s(s+1)w\cdot\nabla\mu=\mathrm{div}\,(\mu^{s+1}w)-\mu^{s+1}(\mathrm{div}\,(w)+v).$$

The proof is complete upon setting $g=-(\mathrm{div}\,w+v)$. □

Lemma 3 *For any $\theta\in V^2_r(T^{\mathrm{ps}})$ with $r\ge 0$, there exists $\psi\in L^1_1(T^{\mathrm{ps}})$ and $\gamma\in\mathcal{V}^2_r(T^{\mathrm{ps}})$ such that*

$$\mu^s\theta=\mathrm{div}\,(\mu^s\psi)+\mu^s\gamma\qquad\text{for any } s\ge 0. \tag{3.1}$$

Proof Given $\theta\in V^2_r(T^{\mathrm{ps}})$, we define $a_i\in L^0_1(T^{\mathrm{ps}})$ uniquely by the conditions

$$a_i(z_j)=0,\ \ j=0,1,2,3,\qquad a_i(z_{3+j})=0,\ \ j\neq i,\qquad [\![\nabla a_i\cdot t_i]\!](z_{3+i})=[\![\theta]\!](z_{3+i}).$$

We clearly have $\mathrm{supp}\,a_i\in T(z_{3+i})$. Setting $\psi=a_1t_1+a_2t_2+a_3t_3$ we have

$$\mathrm{div}\,\psi|_{e_i}=\nabla a_i\cdot t_i,$$

and therefore, by the construction of a_i, $\gamma:=\theta-\mathrm{div}\,\psi\in\mathcal{V}^2_r(T^{\mathrm{ps}})$. Furthermore, we have $\psi\cdot\nabla\mu|_{T(z_{3+i})}=a_it_i\cdot\nabla\mu|_{T(z_{3+i})}=0$ for $i=1,2,3$ by (2.2), and so $\psi\cdot\nabla\mu=0$ in T. It then follows that

$$\mu^s\theta-\mathrm{div}\,(\mu^s\psi)=\mu^s(\theta-\mathrm{div}\,\psi)-s\mu^{s-1}\nabla\mu\cdot\psi=\mu^s\gamma.$$

□

We combine the previous two lemmas to obtain the following.

Lemma 4 *Let $q\in\mathcal{V}^2_r(T^{\mathrm{ps}})$ and $r\ge 1$. Then there exists $v\in L^1_r(T^{\mathrm{ps}})$ and $Q\in\mathcal{V}^2_{r-1}(T^{\mathrm{ps}})$ such that $\mu^s q=\mathrm{div}\,(\mu^{s+1}v)+\mu^{s+1}Q$ for any $s\ge 0$.*

Our last lemma handles the lowest order case which follows from [7, Lemma 3.11].

Lemma 5 *Let $q\in\mathcal{V}^2_0(T^{\mathrm{ps}})$ with $\int_T\mu^s q=0$. Then there exists $w\in L^1_0(T^{\mathrm{ps}})$ such that $\mu^s q=\mathrm{div}\,(\mu^{s+1}w)$ for any $s\ge 0$.*

We can now state and prove the main result.

Theorem 1 *For each $p\in\mathring{\mathcal{V}}^2_r(T^{\mathrm{ps}})$, with $r\ge 0$, there exists a $v\in\mathring{L}^1_{r+1}(T^{\mathrm{ps}})$ such that $\mathrm{div}\,v=p$.*

Proof We adopt similar arguments to those given in [10]. Let $p_r = p$ and suppose we have found $w_{r-j} \in L^1_{r-j}(T^{\mathrm{ps}})$ for $0 \le j \le \ell - 1$ and $p_{r-j} \in \mathcal{V}^2_{r-j}(T^{\mathrm{ps}})$ for $0 \le j \le \ell$ such that

$$\operatorname{div}(\mu^{j+1} w_{r-j}) = \mu^j p_{r-j} - \mu^{j+1} p_{r-(j+1)} \qquad \text{for all } 0 \le j \le \ell - 1. \tag{3.2}$$

We can then apply Lemma 4 to find $w_{r-\ell} \in L^1_{r-\ell}(T^{\mathrm{ps}})$ and $p_{r-(\ell+1)} \in \mathcal{V}^2_{r-(\ell+1)}(T^{\mathrm{ps}})$ such that

$$\operatorname{div}(\mu^{\ell+1} w_{r-\ell}) = \mu^{\ell} p_{r-\ell} - \mu^{\ell+1} p_{r-(\ell+1)}. \tag{3.3}$$

Hence, by induction we can find $w_{r-j} \in L^1_{r-j}(T^{\mathrm{ps}})$ for $0 \le j \le r - 1$ and $p_{r-j} \in \mathcal{V}^2_{r-j}(T^{\mathrm{ps}})$ for $0 \le j \le r$ such that (3.3) holds. Therefore,

$$\operatorname{div}(\mu w_r + \mu^2 w_{r-1} + \cdots + \mu^r w_1) = p - \mu^r p_0.$$

We have that $\int_T \mu^r p_0 = 0$ and hence by Lemma 5 we can find $w_0 \in L^1_0(T^{\mathrm{ps}})$ such that $\operatorname{div}(\mu^{r+1} w_0) = \mu^r p_0$. The result follows after setting $v = \mu w_r + \mu^2 w_{r-1} + \cdots \mu^r w_1 + \mu^{r+1} w_0$. □

We have several corollaries that follow from Theorem 1. First we show that the analogous result without boundary conditions is satisfied.

Corollary 1 *For each $p \in V^2_r(T^{\mathrm{ps}})$ there exists a $v \in L^1_{r+1}(T^{\mathrm{ps}})$ such that* $\operatorname{div} v = p$.

Proof Let $p \in V^2_r(T^{\mathrm{ps}})$. By Lemma 3 there exists $w \in L^1_1(T^{\mathrm{ps}})$ and $g \in \mathcal{V}^2_r(T^{\mathrm{ps}})$ with

$$p = \operatorname{div} w + g.$$

We let $\psi = \left(\frac{1}{|T|}\int_T g\right)\frac{1}{2}x \in L^1_1(T^{\mathrm{ps}})$ and hence $\int_T \operatorname{div}\psi = \int_T g$. We then have

$$p = \operatorname{div}(w + \psi) + (g - \operatorname{div}\psi).$$

By Theorem 1 there exists a $\theta \in \mathring{L}^1_{r+1}(T^{\mathrm{ps}})$ such that $\operatorname{div}\theta = g - \operatorname{div}\psi$. Therefore, we have

$$p = \operatorname{div}(w + \psi + \theta).$$

The proof is complete after we set $v = w + \psi + \theta$. □

Corollary 2 *For each $p \in \mathring{L}^2_r(T^{\mathrm{ps}})$ (resp., $p \in L^2_r(T^{\mathrm{ps}})$) there exists a $v \in \mathring{S}^1_{r+1}(T^{\mathrm{ps}})$ (resp., $v \in S^1_{r+1}(T^{\mathrm{ps}})$) such that* $\operatorname{div} v = p$. *Likewise for each $v \in \mathring{L}^1_r(T^{\mathrm{ps}})$ (resp., $v \in L^1_r(T^{\mathrm{ps}})$) with* $\operatorname{div} v = 0$ *there exists a $z \in \mathring{S}^0_{r+1}(T^{\mathrm{ps}})$ (resp., $z \in S^0_{r+1}(T^{\mathrm{ps}})$) such that* $\operatorname{rot} z = v$.

Proof Let $p \in \mathring{L}^2_r(T^{\mathrm{ps}}) \subset \mathring{\mathcal{V}}^2_r(T^{\mathrm{ps}})$ and we can apply Theorem 1 to find $v \in \mathring{L}^1_{r+1}(T^{\mathrm{ps}})$ such that $\operatorname{div} v = p$. However, clearly $v \in \mathring{S}^1_{r+1}(T^{\mathrm{ps}})$.

Next, let $v \in \mathring{L}^1_r(T^{\mathrm{ps}}) \subset V^1_r(T^{\mathrm{ps}})$ be divergence-free. Proposition 1 shows that there exists $z \in \mathring{V}^0_r(T^{\mathrm{ps}})$ such that $\operatorname{rot} z = v$. Since v is continuous and vanishes on the boundary, we have rot $z \in [C(T)]^2$ and $z|_{\partial T} = 0$, rot $z|_{\partial T} = 0$. Thus $z \in \mathring{S}_r^0(T^{\mathrm{ps}})$ by definition.

This proof applies *mutatis mutandis* to the statements without boundary conditions. □

Remark 3 To summarize, Proposition 1, Theorem 1, and Corollaries 1 and 2 show that the following two sets of sequences are exact:

$$\begin{aligned}
\mathbb{R} &\longrightarrow L_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} V_{r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} V_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0,\\
\mathbb{R} &\longrightarrow S_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} L_{r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} V_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0,\\
\mathbb{R} &\longrightarrow S_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} S_{r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} L_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
0 &\longrightarrow \mathring{L}_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} \mathring{V}_{r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} \mathring{V}_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0,\\
0 &\longrightarrow \mathring{S}_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} \mathring{L}_{r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} \mathring{V}_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0,\\
0 &\longrightarrow \mathring{S}_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} \mathring{S}_{r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} \mathring{L}_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0.
\end{aligned}$$

3.1 Dimension counting

We can easily count the dimensions of the smooth spaces $S_r^k(T^{\mathrm{ps}})$ via the rank-nullity theorem and the exactness of sequences ($k = 0, 1$):

$$\begin{aligned}
\dim S_r^k(T^{\mathrm{ps}}) &= \dim \operatorname{range} S_r^k(T^{\mathrm{ps}}) + \dim \ker S_r^k(T^{\mathrm{ps}})\\
&= \dim \ker L_{r-1}^{k+1}(T^{\mathrm{ps}}) + \dim \ker L_r^k(T^{\mathrm{ps}})\\
&= \dim L_{r-1}^{k+1}(T^{\mathrm{ps}}) - \dim \operatorname{range} L_{r-1}^{k+1}(T^{\mathrm{ps}}) + \dim L_r^k(T^{\mathrm{ps}}) - \operatorname{range} L_r^k(T^{\mathrm{ps}})\\
&= \dim L_{r-1}^{k+1}(T^{\mathrm{ps}}) + \dim L_r^k(T^{\mathrm{ps}}) - \dim \ker V_{r-2}^{k+2}(T^{\mathrm{ps}}) - \dim \ker V_{r-1}^{k+1}(T^{\mathrm{ps}})\\
&= \dim L_{r-1}^{k+1}(T^{\mathrm{ps}}) + \dim L_r^k(T^{\mathrm{ps}}) - \dim V_{r-1}^{k+1}(T^{\mathrm{ps}}).
\end{aligned}$$

Now we easily find

$$\dim L_r^k(T^{\mathrm{ps}}) = \binom{2}{k}\left[3r^2 + 3r + 1\right], \quad \dim V_r^k(T^{\mathrm{ps}}) = \begin{cases} 3r^2 + 3r + 1 & k = 0,\\ 6r^2 + 12r + 6 & k = 1,\\ 3r^2 + 9r + 6 & k = 2. \end{cases}$$

Thus, we have

$$\dim S_r^k(T^{\mathrm{ps}}) = \begin{cases} 3r^2 - 3r + 3 & k = 0, \\ 6r^2 + 3 & k = 1, \\ 3r^2 + 3r + 1 & k = 2. \end{cases}$$

Similar calculations also show that

$$\dim \mathring{S}_r^k(T^{\mathrm{ps}}) = \begin{cases} 3(r-2)(r-3) & k = 0, \\ 6(r-1)(r-2) & k = 1, \\ 3r(r-1) & k = 2. \end{cases}$$

4 Commuting projections on a macro triangle

In this section we define commuting projections. In order to do so, we give the degrees of freedom for C^1 polynomials on a line segment. Let $a < m < b$, and define the space

$$W_r(\{a, m, b\}) = \{v \in C^1([a, b]) \,:\, v|_{[a,m]} \in \mathcal{P}_r([a, m]) \text{ on } v|_{[m,b]} \in \mathcal{P}_r([m, b])\}.$$

The classical degrees of freedom for $W_r(\{a, m, b\})$ is given in the next result.

Lemma 6 *Let $r \geq 1$. A function $z \in W_r(\{a, m, b\})$ is uniquely determined by the following degrees of freedom.*

$$\begin{aligned} z(a), z(b) & \\ z'(a), z'(b) & \quad \text{if } r \geq 2, \\ z(m), z'(m) & \quad \text{if } r \geq 3, \\ \int_a^m z(x)q(x) & \quad \text{for all } q \in \mathcal{P}_{r-4}([a, m]), \\ \int_m^b z(x)q(x) & \quad \text{for all } q \in \mathcal{P}_{r-4}([m, b]). \end{aligned}$$

Other degrees of freedom are given in the next lemma. Its proof is found in the Appendix.

Lemma 7 *Let $r \geq 1$. A function $z \in W_r(\{a, m, b\})$ is uniquely determined by the following degrees of freedom.*

$$z(a), z(b) \tag{4.1a}$$

$$\int_a^m z(x)q(x) \quad \text{for all } q \in \mathcal{P}_{r-2}([a, m]), \tag{4.1b}$$

$$\int_m^b z(x)q(x) \quad \text{for all } q \in \mathcal{P}_{r-2}([m,b]). \tag{4.1c}$$

Lemma 8 *Suppose that $q \in S_r^0(T^{\mathrm{ps}})$ with $q|_{e_i} = 0$ for some $i \in \{1,2,3\}$. Then $q|_{T(z_{3+i})} = \mu p|_{T(z_{3+i})}$ for some $p \in L_{r-1}^0(T^{\mathrm{ps}})|_{T(z_{3+i})}$, and $p \in C^1(T(z_{3+i}))$. In particular, if $q|_{\partial T} = 0$, then $q = \mu p$ for some $p \in L_{r-1}^0(T^{\mathrm{ps}})$ and $p|_{T(z_{3+i})} \in C^1(T(z_{3+i}))$ for $i = 1,2,3$.*

Proof The statement $q|_{T(z_{3+i})} = \mu p|_{T(z_{3+i})}$ is a consequence of Remark 1. Because q and μ are continuous, it follows that p is continuous, i.e., $p \in L_{r-1}^0(T^{\mathrm{ps}})|_{T(z_{3+i})}$. We also have $\nabla q = \mu \nabla p + p \nabla \mu$, and therefore

$$\mu \nabla p|_{T(z_{3+i})} = \big(\nabla q - p \nabla \mu\big)|_{T(z_{3+i})}.$$

Since $\nabla \mu$ is constant on $T(z_{3+i})$, we find that $\mu \nabla p|_{T(z_{3+i})}$ is continuous. Because μ is positive in the interior of $T(z_{3+i})$, we conclude that ∇p is continuous on $T(z_{3+i})$. □

We are now ready to give degrees of freedom (DOFs) for functions in $S_r^0(T^{\mathrm{ps}})$.

Lemma 9 *A function $q \in S_r^0(T^{\mathrm{ps}})$, with $r \geq 2$, is uniquely determined by*

$$q(z_i), \nabla q(z_i) \qquad 1 \leq i \leq 3, \quad \text{(9 DOFs)} \tag{4.2a}$$

$$q(z_{3+i}), \partial_t q(z_{3+i}) \qquad 1 \leq i \leq 3, \ \text{if } r \geq 3, \quad \text{(6 DOFs)} \tag{4.2b}$$

$$\int_e \partial_n q\, p \qquad \forall p \in \mathcal{P}_{r-3}(e),\ e \in \mathcal{E}^b(T^{\mathrm{ps}}), \quad \text{(6(r-2) DOFs)} \tag{4.2c}$$

$$\int_e qp \qquad \forall p \in \mathcal{P}_{r-4}(e),\ e \in \mathcal{E}^b(T^{\mathrm{ps}}), \quad \text{(6(r-3) DOFs)} \tag{4.2d}$$

$$\int_T \operatorname{rot} q \cdot \operatorname{rot} p \qquad \forall p \in \mathring{S}_r^0(T^{\mathrm{ps}}), \quad \text{(3(r-2)(r-3) DOFs)} \tag{4.2e}$$

Proof The number of DOFs given is $3r^2 - 3r + 3 = \dim S_r^0(T^{\mathrm{ps}})$. We will show that the only function q for which (4.2a)–(4.2e) are equal to zero must be zero on T.

Suppose that q vanishes on (4.2a)–(4.2d) restricted to a single edge e_i. Then q satisfies all conditions of Lemma 6 on each edge of T, so $q \equiv 0$ on e_i. It then follows from Lemma 8 that $q|_{T(z_{3+i})} = \mu p|_{T(z_{3+i})}$, where $p \in C^1(T(z_{3+i}))$ is a piecewise polynomial of degree $(r-1)$. We then have $\nabla q|_{e_i} = p \nabla \mu_i|_{e_i}$, and so by (4.2a), $p = 0$ on the endpoints of e_i. Also (4.2c) yields $\int_e pw\partial_n \mu = 0$ for all $w \in \mathcal{P}_{r-3}(e)$ and for all $e \in \mathcal{E}^b(T^{\mathrm{ps}})$ with $e \subset e_i$. Since $\partial_n \mu$ is constant on each edge $e \in \mathcal{E}^b(T^{\mathrm{ps}})$, we have $\int_e pw = 0$ for all $w \in \mathcal{P}_{r-3}(e)$ and $e \subset e_i$. Using Lemma 7, it follows that $p \equiv 0$ on e_i. Thus $\nabla q|_{e_i} = 0$.

We conclude that if q vanishes on (4.2), then $q \in \mathring{S}_r^0(T^{\mathrm{ps}})$. Finally, condition (4.2e) yields rot $q = 0$ on T, and hence $q \equiv 0$ on T.

□

Lemma 10 *A function $v \in L_r^1(T^{\mathrm{ps}})$ is uniquely determined by*

$$v(z_i), \qquad 1 \le i \le 3, \quad (6 \text{ DOFs}), \tag{4.3a}$$

$$\int_{e_i} (v \cdot n_i) \qquad \text{if } r = 1, \tag{4.3b}$$

$$[\![\operatorname{div} v]\!](z_{3+i}) \qquad 1 \le i \le 3, \quad (3 \text{ DOFs}), \tag{4.3c}$$

$$v(z_{3+i}) \cdot n_i \qquad 1 \le i \le 3, \text{ if } r \ge 2, \quad (3 \text{ DOFs}), \tag{4.3d}$$

$$\int_e v \cdot w \qquad \forall w \in [\mathcal{P}_{r-2}(e)]^2, \text{ for all } e \in \mathcal{E}^b(T^{\mathrm{ps}}), \quad (12(\text{r-1}) \text{ DOFs}), \tag{4.3e}$$

$$\int_T v \cdot \operatorname{rot} w \qquad \forall w \in \mathring{S}_{r+1}^0(T^{\mathrm{ps}}), \quad (3(\text{r-1})(\text{r-2}) \text{ DOFs}), \tag{4.3f}$$

$$\int_T \operatorname{div} v\, w \qquad \forall w \in \mathring{\mathcal{V}}_{r-1}^2(T^{\mathrm{ps}}), \quad (3\text{r}(\text{r+1})\text{-4 DOFs}). \tag{4.3g}$$

Proof The number of degrees of freedom given is $6r^2 + 6r + 2$ which equals the dimension of $L_r^1(T^{\mathrm{ps}})$. We show that if $v \in L_r^1(T^{\mathrm{ps}})$ vanishes on (4.3), then v is identically zero.

Suppose that v vanishes on (4.3a)–(4.3e) restricted to a single edge e_i. Recall that $T(z_{3+i}) = T_{2i+1} \cup T_{2i+2}$ is the union of two triangles that have z_{3+i} as a vertex, and n_i and t_i are, respectively, the outward normal and unit tangent vectors of the edge $e_i = \partial T \cap \partial T(z_{3+i})$. Let s_i be a unit vector that is tangent to the interior edge $[z_0, z_{3+i}]$, which is necessarily linearly independent of t_i. Thus we may write

$$v|_{T(z_{3+i})} = a_i t_i + b_i s_i$$

for some $a_i, b_i \in L_r^0(T^{\mathrm{ps}})|_{T(z_{3+i})}$. We then see that

$$\operatorname{div} v|_{T(z_{i+3})} = \partial_{t_i} a_i + \partial_{s_i} b_i.$$

Because b_i is continuous on $T(z_{3+i})$ we have that $[\![\partial_{s_i} b_i]\!](z_{i+3}) = 0$ and hence $0 = [\![\operatorname{div} v]\!](z_{3+i}) = [\partial_{t_i} a_i](z_{3+i})$. Therefore $a_i|_{e_i}$ is C^1 on e_i. To continue, we split the proof into two step.

Case $r = 1$:

By the first set of DOFs (4.3a), there holds $a_i(z_j) = b_i(z_j) = 0$ for $j \in \{1,2,3\}\backslash\{i\}$. Because $a_i|_{e_i}$ is piecewise linear and C^1, we conclude that $a_i \equiv 0$ on e_i. Next, using (4.3b) yields

$$\int_{e_i} b_i(s_i \cdot n_i) = 0.$$

Because $s_i \cdot n_i \neq 0$, we conclude that $\int_{e_i} b_i = 0$. Since b_i vanishes at the endpoints of e_i, and since b_i is piecewise linear on e_i, we conclude that $b_i = 0$ on e_i, and therefore $v|_{e_i} = 0$.

Case $r \geq 2$:

Again, there holds $a_i(z_j) = b_i(z_j) = 0$ by the first set of DOFs (4.3a). Combining Lemma 7 with the DOFs (4.3e), noting that a_i is C^1 on e_i, then yields $a_i = 0$ on e_i. Likewise the DOFs (4.3a), (4.3e), and (4.3d) show that $b_i = 0$ on e_i. We conclude that $v|_{e_i} = 0$.

Thus, if v vanishes on (4.3) then $v \in \mathring{L}_1^1(T^{\rm ps})$. The DOFs (4.3g) then show that $\operatorname{div} v = 0$, and therefore, by Corollary 2, $v = \operatorname{rot} z$ for some $z \in \mathring{S}_{r+1}(T^{\rm ps})$. Finally, by (4.3f), we conclude that $v \equiv 0$. ☐

Lemma 11 *A function $q \in V_r^2(T^{\rm ps})$ is uniquely determined by*

$$[\![q]\!](z_{3+i}) \qquad 1 \leq i \leq 3, \qquad \text{(3 DOFs)}, \tag{4.4a}$$

$$\int_T q \qquad \text{(1 DOF)}, \tag{4.4b}$$

$$\int_T qp \qquad \forall p \in \mathring{\mathcal{V}}_r^2(T^{\rm ps}), \qquad \text{(3(r+1)(r+2)-4 DOFs)}, \tag{4.4c}$$

Proof If $q \in V_r^2(T^{\rm ps})$ is such that (4.4a) are zero then q is continuous at z_{3+i} for $1 \leq i \leq 3$. Then (4.4b) yields that $q \in \mathring{\mathcal{V}}_r^2(T^{\rm ps})$, and it follows from (4.4c) that $q \equiv 0$ on T. ☐

Lemma 12 *A function $v \in S_r^1(T^{\rm ps})$ is uniquely determined by the following degrees of freedom.*

$$v(z_i),\ \operatorname{div} v(z_i) \qquad 1 \leq i \leq 3, \qquad \text{(9 DOFs)}, \tag{4.5a}$$

$$\int_{e_i} v \cdot n_i \qquad 1 \leq i \leq 3, \ \text{ if } r = 1, \tag{4.5b}$$

$$v(z_{3+i}) \cdot n,\ \operatorname{div} v(z_{3+i}) \qquad 1 \leq i \leq 3, \ \text{ if } r \geq 2 \qquad \text{(6 DOFs)}, \tag{4.5c}$$

$$\int_e v \cdot w \qquad \forall w \in [\mathcal{P}_{r-2}(e)]^2,\ e \in \mathcal{E}^b(T^{\mathrm{ps}}), \quad \text{(12(r-1) DOFs)}, \tag{4.5d}$$

$$\int_e (\operatorname{div} v) q \qquad \forall q \in \mathcal{P}_{r-3}(e),\ e \in \mathcal{E}^b(T^{\mathrm{ps}}), \quad \text{(6(r-2) DOFs)}, \tag{4.5e}$$

$$\int_T v \cdot \operatorname{rot} q, \qquad \forall q \in \mathring{S}^0_{r+1}(T^{\mathrm{ps}}) \qquad \text{(3(r-1)(r-2) DOFs)}, \tag{4.5f}$$

$$\int_T (\operatorname{div} v) q, \qquad \forall q \in \mathring{L}^2_{r-1}(T^{\mathrm{ps}}) \qquad \text{(3(r-1)(r-2) DOFs)}. \tag{4.5g}$$

Proof If v vanishes at the DOFs, then $v \in S^1_r(T^{\mathrm{ps}}) \subset L^1_r(T^{\mathrm{ps}})$ vanishes on (4.3a)–(4.3e). The proof of Lemma 10 then shows that $v|_{\partial T} = 0$, and therefore $\int_T \operatorname{div} v = 0$. Using (4.5a),(4.5c), and (4.5e), we also find that $\operatorname{div} v|_{\partial T} = 0$, i.e., $\operatorname{div} v \in \mathring{L}^2_{r-1}(T^{\mathrm{ps}})$. The DOFs (4.5g) yield $\operatorname{div} v = 0$ in T, and therefore $v = \operatorname{rot} q$ for some $q \in \mathring{S}^0_{r+1}(T^{\mathrm{ps}})$ by Corollary 2. Finally (4.5f) gives $v \equiv 0$. Noting that the number of DOFs is $6r^2 + 3$, the dimension of $S^1_r(T^{\mathrm{ps}})$, we conclude that (4.5) form a unisolvent set over $S^1_r(T^{\mathrm{ps}})$. $\square$

Lemma 13 *Let $q \in L^2_r(T^{\mathrm{ps}})$ with $r \geq 1$. Then q is uniquely determined by the following degrees of freedom.*

$$q(z_i) \qquad 1 \leq i \leq 3, \qquad \text{(3 DOFs)}, \tag{4.6a}$$

$$q(z_{3+i}) \qquad 1 \leq i \leq 3, \qquad \text{(3 DOFs)}, \tag{4.6b}$$

$$\int_e qp \qquad \forall p \in \mathcal{P}_{r-2}(e), \quad e \in \mathcal{E}^b(T^{\mathrm{ps}}), \qquad \text{(6(r-1) DOFs)}, \tag{4.6c}$$

$$\int_T q \qquad \text{(1 DOF)}, \tag{4.6d}$$

$$\int_T qp \qquad \forall p \in \mathring{L}^2_r(T^{\mathrm{ps}}), \qquad \text{(3r(r-1) DOFs)}. \tag{4.6e}$$

Proof Let $q \in L^2_r(T^{\mathrm{ps}})$ such that all DOFs (4.6) are equal to zero. The conditions (4.6a)–(4.6c) yield that $q \equiv 0$ on ∂T. Therefore, using (4.6d), $q \in \mathring{L}^2_r(T^{\mathrm{ps}})$, and by (4.6e), $q \equiv 0$ on T. $\square$

The next two theorems show that projections induced by the degrees of freedom given in Lemmas 9–13 commute.

Theorem 2 *Let* $\Pi_0^r : C^\infty(T) \to S_r^0(T^{\rm ps})$ *be the projection induced by the DOFs* (4.2), *that is*,

$$\phi(\Pi_0^r p) = \phi(p), \quad \forall \phi \in \text{DOFs in (4.2)}.$$

Likewise, let $\Pi_1^{r-1} : [C^\infty(T)]^2 \to L_{r-1}^1(T^{\rm ps})$ *be the projection induced by the DOFs* (4.3), *and let* $\Pi_2^{r-2} : C^\infty(T) \to V_{r-2}^2(T^{\rm ps})$ *be the projection induced by the DOFs* (4.4). *Then for* $r \geq 2$, *the following diagram commutes*

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(T) \arrow[r, "\mathrm{rot}"] \arrow[d, "\Pi_0^r"] & {[C^\infty(T)]^2} \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi_1^{r-1}"] & C^\infty(T) \arrow[r] \arrow[d, "\Pi_2^{r-2}"] & 0 \\
\mathbb{R} \arrow[r] & S_r^0(T^{\rm ps}) \arrow[r, "\mathrm{rot}"] & L_{r-1}^1(T^{\rm ps}) \arrow[r, "\mathrm{div}"] & V_{r-2}^2(T^{\rm ps}) \arrow[r] & 0.
\end{tikzcd}$$

In other words, we have for $r \geq 2$

$$\mathrm{div}\, \Pi_1^{r-1} v = \Pi_2^{r-2} \mathrm{div}\, v, \qquad \forall v \in [C^\infty(T)]^2, \tag{4.7a}$$

$$\mathrm{rot}\, \Pi_0^r p = \Pi_1^{r-1} \mathrm{rot}\, p, \qquad \forall p \in C^\infty(T). \tag{4.7b}$$

Proof (1) *Proof of* (4.7a). We take $v \in [C^\infty(T)]^2$. Since $\rho := \mathrm{div}\, \Pi_1^{r-1} v - \Pi_2^{r-2} \mathrm{div}\, v \in V_{r-2}^2(T^{\rm ps})$, we only need to prove that ρ vanishes at the DOFs (4.4). For the jump condition at points z_{3+i} for $1 \leq i \leq 3$, we have

$$[\![\rho]\!](z_{3+i}) = [\![\mathrm{div}\, \Pi_1^{r-1} v - \Pi_2^{r-2} \mathrm{div}\, v]\!](z_{3+i}) = [\![\mathrm{div}\, \Pi_1^{r-1} v - \mathrm{div}\, v]\!](z_{3+i}) = 0,$$

where we have used the definitions of Π_2^{r-2} and Π_1^{r-1} along with the DOFs (4.4a) and (4.3c).

For the interior DOFs, we have,

$$\int_T \rho = \int_T \left(\mathrm{div}\, \Pi_1^{r-1} v - \mathrm{div}\, v\right) = \int_{\partial T} \left(\Pi_1^{r-1} v - v\right) \cdot n = 0,$$

where we have used the definitions of Π_1^{r-1} and Π_2^{r-2} and DOFs (4.4b) and either (4.3b) if $r = 2$ or (4.3e) if $r \geq 3$. Finally, for any $p \in \mathring{V}_{r-2}^2(T^{\rm ps})$,

$$\int_T \rho p = \int_T \left(\mathrm{div}\, \Pi_1^{r-1} v - \Pi_2^{r-1} \mathrm{div}\, v\right) p = 0$$

by the definitions of Π_1^{r-1} and Π_2^{r-2} along with DOFs (4.4c) and (4.3g). By Lemma 11, ρ is exactly zero on T, and the projections in (4.7a) commute.

(2) *Proof of* (4.7b). Let $p \in C^\infty(T)$ and set $\rho := \mathrm{rot}\, \Pi_0^r p - \Pi_1^{r-1} \mathrm{rot}\, p \in L_{r-1}^1(T^{\rm ps})$. We will show that ρ vanishes for all DOFs (4.3).

First, for each vertex z_i with $1 \leq i \leq 3$,

$$\rho(z_i) = \mathrm{rot}\, \Pi_0^r p(z_i) - \Pi_1^{r-1} \mathrm{rot}\, p(z_i) = \mathrm{rot}\, p(z_i) - \Pi_1^{r-1} \mathrm{rot}\, p(z_i) = 0, \tag{4.8}$$

by (4.2a) and (4.3a). Furthermore, at nodes z_{3+i}, we have by (4.3c)

$$\begin{aligned}[\![\mathrm{div}\, \rho]\!](z_{3+i}) &= [\![\mathrm{div}\, \mathrm{rot}\, \Pi_0^r p - \mathrm{div}\, \Pi_1^{r-1} \mathrm{rot}\, p]\!](z_{3+i}) \\ &= -[\![\mathrm{div}\, \Pi_1^{r-1} \mathrm{rot}\, p]\!](z_{3+i}) \\ &= -[\![\mathrm{div}\, \mathrm{rot}\, p]\!](z_{3+i}) = 0,\end{aligned}$$

For the DOFs on each edge $e \in \mathcal{E}^b(T^{\mathrm{ps}})$, we will use that $\mathrm{rot}\, \varphi \cdot n = \partial_t \varphi$ and $\mathrm{rot}\, \varphi \cdot t = -\partial_n \varphi$. Then we have, for $r \geq 3$,

$$\begin{aligned}\rho(z_{3+i}) \cdot n_i &= \left(\mathrm{rot}\, \Pi_0^r p(z_{3+i})\right) \cdot n_i - \left(\Pi_1^{r-1} \mathrm{rot}\, p(z_{3+i})\right) \cdot n_i \\ &= \partial_t p(z_{3+i}) - \left(\Pi_1^{r-1} \mathrm{rot}\, p(z_{3+i})\right) \cdot n_i \\ &= \partial_t p(z_{3+i}) - \mathrm{rot}\, p(z_{3+i}) \cdot n_i = 0\end{aligned} \tag{4.9}$$

by (4.2b) and (4.3d). If $r = 2$ (so that $\rho \in L_1^1(T^{\mathrm{ps}})$),

$$\int_{e_i} \rho \cdot n_i = \int_{e_i} \left(\mathrm{rot}\, \Pi_r^0 p - \Pi_{r-1}^1 \mathrm{rot}\, p\right) \cdot n_i = \int_{e_i} \partial_{t_i} \left(\Pi_r^0 p - p\right) = 0$$

by (4.3b) and (4.2a), so (4.7b) is proved.

Now let $r \geq 3$. We have, for all $q \in \mathcal{P}_{r-3}(e)$ and for all $e \in \mathcal{E}^b(T^{\mathrm{ps}})$,

$$\begin{aligned}\int_e (\rho \cdot n) q &= \int_e \left(\mathrm{rot}\, \left(\Pi_0^r p - p\right) \cdot n\right) q \\ &= \int_e \partial_t \left(\Pi_0^r p - p\right) q \\ &= -\int_e \left(\Pi_0^r p - p\right) \partial_t q = 0,\end{aligned}$$

by (4.3e), (4.2b) and (4.2d). Likewise, for $q \in \mathcal{P}_{r-3}(e)$,

$$\begin{aligned}\int_e (\rho \cdot t) q &= \int_e \left(\left(\mathrm{rot}\, \Pi_0^r p - \Pi_1^{r-1} \mathrm{rot}\, p\right) \cdot t\right) q \\ &= \int_e \left(\mathrm{rot}\, (\Pi_0^r p - p) \cdot t\right) q \\ &= \int_e -\partial_n \left(\Pi_0^r p - p\right) q = 0\end{aligned}$$

by (4.3e) and (4.2c). For the interior DOFs, for any $w \in \mathring{S}_{r-1}^0(T^{\mathrm{ps}})$, we have

$$\int_T \rho \cdot \mathrm{rot}\, w = \int_T \left(\mathrm{rot}\, \Pi_0^r p - \Pi_1^{r-1} \mathrm{rot}\, p\right) \cdot \mathrm{rot}\, w = 0$$

by (4.2e) and (4.3f). Finally, for any $w \in \mathring{\mathcal{V}}_{r-2}^2(T^{\mathrm{ps}})$,

$$\int_T \operatorname{div} \rho\, w = \int_T \operatorname{div}\left(\operatorname{rot} \Pi_0^r p - \Pi_1^{r-1} \operatorname{rot} p\right) w$$
$$= \int_T -\operatorname{div}(\operatorname{rot} p) w = 0$$

where we used the DOF (4.3g). Therefore ρ is equal to zero on T, and the identity (4.7b) is proved. □

The proof of the following result can be found in the Appendix.

Theorem 3 *Let* $\Pi_0^r : C^\infty(T) \to S_r^0(T^{\mathrm{ps}})$ *be the projection induced by the DOFs* (4.2), *that is,*

$$\phi(\Pi_0^r p) = \phi(p), \quad \forall \phi \in \text{DOFs in (4.2)}.$$

Likewise, let $\varpi_1^{r-1} : [C^\infty(T)]^2 \to S_{r-1}^1(T^{\mathrm{ps}})$ *be the projection induced by the DOFs* (4.5), *and let* $\varpi_2^{r-2} : C^\infty(T) \to L_{r-2}^2(T^{\mathrm{ps}})$ *be the projection induced by the DOFs* (4.6). *Then for* $r \geq 2$, *the following diagram commutes*

$$\begin{tikzcd}
\mathbb{R} \arrow[r] & C^\infty(T) \arrow[r, "\mathrm{rot}"] \arrow[d, "\Pi_0^r"] & {[C^\infty(T)]^2} \arrow[r, "\mathrm{div}"] \arrow[d, "\varpi_1^{r-1}"] & C^\infty(T) \arrow[r] \arrow[d, "\varpi_2^{r-2}"] & 0 \\
\mathbb{R} \arrow[r] & S_r^0(T^{\mathrm{ps}}) \arrow[r, "\mathrm{rot}"] & S_{r-1}^1(T^{\mathrm{ps}}) \arrow[r, "\mathrm{div}"] & L_{r-2}^2(T^{\mathrm{ps}}) \arrow[r] & 0.
\end{tikzcd}$$

In other words, we have for $r \geq 2$

$$\operatorname{rot} \Pi_0^r p = \varpi_1^{r-1} \operatorname{rot} p, \qquad \forall p \in C^\infty(T), \tag{4.10a}$$

$$\operatorname{div} \varpi_1^{r-1} v = \varpi_2^{r-2} \operatorname{div} v, \qquad \forall v \in [C^\infty(T)]^2. \tag{4.10b}$$

5 Global spaces

In this section, we study the global finite element spaces induced by the degrees of freedom in Sect. 4. We let $\mathcal{T}_h$ represent the simplicial triangulation of the polygonal domain $\Omega \subset \mathbb{R}^2$, and $\mathcal{T}_h^{\mathrm{ps}}$ represents the Powell–Sabin refinement of $\mathcal{T}_h$, as discussed in the introduction. We define the set $\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}})$ to be the points of intersection of the edges of $\mathcal{T}_h$ with the edges that adjoin interior points. We also let $\mathcal{E}^b(\mathcal{T}_h^{\mathrm{ps}})$ be the collection of all the new edges of $\mathcal{T}_h^{\mathrm{ps}}$ that were obtained by sub-dividing edges of $\mathcal{T}_h$. We let $\mathcal{E}(\mathcal{T}_h)$ be the edges of $\mathcal{T}_h$. By the construction of $\mathcal{T}_h^{\mathrm{ps}}$ every $x \in \mathcal{M}(\mathcal{T}_h^{\mathrm{ps}})$ belongs to four edges that lie on two straight lines. Therefore, these vertices are singular vertices [15]. It is important to note that to make our global spaces to have the correct continuity it is essential to construct the meshes in such a way [12, 14]. Furthermore, as previously mentioned, the divergence of continuous, piecewise polynomials have a weak continuity property at singular vertices, i.e., at the vertices in $\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}})$. In detail, let $z \in \mathcal{M}(\mathcal{T}_h^{\mathrm{ps}})$ and suppose that z is an interior vertex. Then it is a vertex of four triangles $K_1, \dots, K_4 \in \mathcal{T}_h^{\mathrm{ps}}$. For a function q we define

$$\theta_z(q) := |q|_{K_1}(z) - q|_{K_2}(z) + q|_{K_3}(z) - q|_{K_4}(z)|.$$

Then, if v is a continuous piecewise polynomial with respect to $\mathcal{T}_h^{\mathrm{ps}}$, there holds $\theta_z(\operatorname{div} v) = 0$ [15].

The degrees of freedom stated in Lemmas 9–13 induce the following spaces

$$\begin{aligned}
S_r^0(\mathcal{T}_h^{\mathrm{ps}}) &= \{q \in C^1(\Omega) : q|_T \in S_r^0(T^{\mathrm{ps}}) \quad \forall T \in \mathcal{T}_h\},\\
S_r^1(\mathcal{T}_h^{\mathrm{ps}}) &= \{v \in [C(\Omega)]^2 : \operatorname{div} v \in C(\Omega), v|_T \in S_r^1(T^{\mathrm{ps}}) \quad \forall T \in \mathcal{T}_h\},\\
L_r^1(\mathcal{T}_h^{\mathrm{ps}}) &= \{v \in [C(\Omega)]^2 : v|_T \in L_r^1(T^{\mathrm{ps}}) \quad \forall T \in \mathcal{T}_h\},\\
L_r^2(\mathcal{T}_h^{\mathrm{ps}}) &= \{p \in C(\Omega) : p|_T \in L_r^2(T^{\mathrm{ps}}) \quad \forall T \in \mathcal{T}_h\},\\
\mathcal{V}_r^2(\mathcal{T}_h^{\mathrm{ps}}) &= \{p \in L^2(\Omega) : p|_T \in V_r^2(T^{\mathrm{ps}}) \quad \forall T \in \mathcal{T}_h,\ \theta_z(p) = 0, \quad \forall z \in \mathcal{M}(\mathcal{T}_h^{\mathrm{ps}}) \text{ and } z \text{ an interior node}\}.
\end{aligned}$$

Remark 4 Let $z \in \mathcal{M}(\mathcal{T}_h^{\mathrm{ps}})$ be an interior vertex and $T_1, T_2 \in \mathcal{T}_h$ share a common edge where z lies. Then $\theta_z(q) = 0$ if and only if $[\![q_1]\!](z) = [\![q_2]\!](z)$ where $q_i = q|_{T_i}$. Therefore, the local degrees of freedom for $V_r^2(T^{\mathrm{ps}})$ with the jump condition (4.4a) do indeed induce the global space $\mathcal{V}_r^2(\mathcal{T}_h^{\mathrm{ps}})$ above.

We list the degrees of freedom of these spaces. The global DOF come directly from the local DOF. We list them here to be precise.

It follows from Lemma 9 that a function $q \in S_r^0(\mathcal{T}_h^{\mathrm{ps}})$, with $r \geq 2$, is uniquely determined by

$$\begin{aligned}
q(z),\ \nabla q(z) &\qquad \text{for every vertex } z \text{ of } \mathcal{T}_h,\\
q(z), \partial_t q(z) &\qquad \forall z \in \mathcal{M}(\mathcal{T}_h^{\mathrm{PS}}),\ \text{if } r \geq 3,\\
\int_e \partial_n q\, p &\qquad \forall p \in \mathcal{P}_{r-3}(e),\ \text{for all } e \in \mathcal{E}^b(\mathcal{T}_h^{\mathrm{PS}})\\
\int_e qp &\qquad \forall p \in \mathcal{P}_{r-4}(e),\ \text{for all } e \in \mathcal{E}^b(\mathcal{T}_h^{\mathrm{PS}}),\\
\int_T \operatorname{rot} q \cdot \operatorname{rot} p &\qquad \forall p \in \mathring{S}_r^0(T^{\mathrm{ps}}),\ \text{for all } T \in \mathcal{T}_h.
\end{aligned}$$

Remark 5 The degrees of freedom for $r = 2$ coincide with the known degrees of freedom of Powell–Sabin [12, 14]. Recently, results for polynomial degrees $r = 3, 4$ have appeared [8, 9].

Lemma 10 shows that a function $v \in L_r^1(\mathcal{T}_h^{\mathrm{ps}})$ is uniquely determined by the values

$$
\begin{aligned}
&v(z), && \text{for every vertex } z \text{ of } \mathcal{T}_h,\\
&\int_e (v\cdot n), && \forall e\in\mathcal{E}(\mathcal{T}_h),\ \text{if } r=1,\\
&[\![\operatorname{div} v]\!](z), && \forall z\in\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}}),\\
&v(z)\cdot n, && \forall z\in\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}}),\ \text{if } r\geq 2,\\
&\int_e v\cdot w, && \forall w\in[\mathcal{P}_{r-2}(e)]^2,\quad \forall e\in\mathcal{E}^b(\mathcal{T}_h^{\mathrm{ps}}),\\
&\int_T v\cdot \operatorname{rot} w, && \forall w\in\mathring{S}^0_{r+1}(T^{\mathrm{ps}}),\quad \forall T\in\mathcal{T}_h,\\
&\int_T \operatorname{div} v\, w, && \forall w\in\mathring{\mathcal{V}}^2_{r-1}(T^{\mathrm{ps}}),\quad \forall T\in\mathcal{T}_h.
\end{aligned}
$$

A function $q\in\mathcal{V}_r^2(\mathcal{T}_h^{\mathrm{ps}})$, for $r\geq 0$, is uniquely determined by

$$
\begin{aligned}
&[\![q]\!](z), && \forall z\in\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}}),\\
&\int_T q=0, && \forall T\in\mathcal{T}_h,\\
&\int_T qp && \forall p\in\mathring{\mathcal{V}}^2_r(\mathcal{T}_h^{\mathrm{ps}}),\quad \forall T\in\mathcal{T}_h.
\end{aligned}
$$

A function $v\in S_r^1(\mathcal{T}_h^{\mathrm{ps}})$ is determined by the following degrees of freedom.

$$
\begin{aligned}
&v(z),\ \operatorname{div} v(z) && \text{for every vertex } z \text{ of } \mathcal{T}_h,\\
&\int_e (v\cdot n_i), && \forall e\in\mathcal{E}(\mathcal{T}_h), \text{if } r=1,\\
&v(z)\cdot n,\ \operatorname{div} v(z) && \forall z\in\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}}),\ \text{if } r\geq 2,\\
&\int_e v\cdot w && \forall w\in[\mathcal{P}_{r-2}(e)]^2,\ e\in\mathcal{E}^b(\mathcal{T}_h^{\mathrm{ps}}),\\
&\int_e (\operatorname{div} v)q && \forall q\in\mathcal{P}_{r-3}(e),\ e\in\mathcal{E}^b(\mathcal{T}_h^{\mathrm{ps}}),\\
&\int_T v\cdot\operatorname{rot} w && \forall w\in\mathring{S}^0_{r+1}(T^{\mathrm{ps}}) \text{ for all } T\in\mathcal{T}_h,\\
&\int_T \operatorname{div} v\, w && \forall w\in\mathring{L}^2_{r-1}(T^{\mathrm{ps}}) \text{ for all } T\in\mathcal{T}_h.
\end{aligned}
$$

A function $q\in L_r^2(\mathcal{T}_h^{\mathrm{ps}})$, if $r\geq 1$, is determined by the degrees of freedom

$$
\begin{aligned}
&q(z)\quad 1\leq i\leq 3,\quad \text{for every vertex } z \text{ of } \mathcal{T}_h,\\
&q(z)\quad 1\leq i\leq 3,\quad \forall z\in\mathcal{M}(\mathcal{T}_h^{\mathrm{ps}}),\\
&\textstyle\int_e qp\ \ \forall p\in\mathcal{P}_{r-2}(e),\quad \forall e\in\mathcal{E}^b(T^{\mathrm{ps}}),\\
&\textstyle\int_T q\\
&\textstyle\int_T qp\ \ \forall p\in\mathring{L}^2_r(\mathcal{T}_h^{\mathrm{ps}}).
\end{aligned}
$$

Each of the following sequences of spaces forms a complex.

$$\mathbb{R} \longrightarrow S_r^0(\mathcal{T}_h^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} L_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} \mathcal{V}_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}) \longrightarrow 0, \quad r \geq 2, \tag{5.2a}$$

$$\mathbb{R} \longrightarrow S_r^0(\mathcal{T}_h^{\mathrm{ps}}) \xrightarrow{\mathrm{rot}} S_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}}) \xrightarrow{\mathrm{div}} L_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}) \longrightarrow 0, \quad r \geq 3. \tag{5.2b}$$

Remark 6 The spaces $L_1^1(\mathcal{T}_h^{\mathrm{ps}})$ and $\mathrm{div}\, L_1^1(\mathcal{T}_h^{\mathrm{ps}})$ were considered by Zhang [17] for approximating incompressible flows. In particular, he proved inf-sup stability of this pair. However, he did not explicitly write the relationship $\mathcal{V}_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}) = \mathrm{div}\, L_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}})$, which we know holds.

Additionally, we can define commuting projections. For example, for the sequences (5.2a) and (5.2b), we define π_i^r such that, for $0 \leq i \leq 2$, $\pi_i^r v|_T = \Pi_i^r(v|_T)$ for all $T \in \mathcal{T}_h$. By using Theorem 2, we find that following diagram commutes:

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(S) & \xrightarrow{\mathrm{rot}} & [C^\infty(S)]^2 & \xrightarrow{\mathrm{div}} & C^\infty(S) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_0^r} & & \downarrow{\scriptstyle \pi_1^{r-1}} & & \downarrow{\scriptstyle \pi_2^{r-2}} & & \\
\mathbb{R} & \longrightarrow & S_r^0(\mathcal{T}_h^{\mathrm{ps}}) & \xrightarrow{\mathrm{rot}} & L_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}}) & \xrightarrow{\mathrm{div}} & \mathcal{V}_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}) & \longrightarrow & 0.
\end{array}$$

Similarly, defining the projections $\chi_i^r v|_T = \varpi_i^r(v|_T)$ for $i = 1, 2$, it follows from Theorem 3 that the following diagram commutes:

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(S) & \xrightarrow{\mathrm{rot}} & [C^\infty(S)]^2 & \xrightarrow{\mathrm{div}} & C^\infty(S) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_0^r} & & \downarrow{\scriptstyle \chi_1^{r-1}} & & \downarrow{\scriptstyle \chi_2^{r-2}} & & \\
\mathbb{R} & \longrightarrow & S_r^0(\mathcal{T}_h^{\mathrm{ps}}) & \xrightarrow{\mathrm{rot}} & S_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}}) & \xrightarrow{\mathrm{div}} & L_{r-2}^2(\mathcal{T}_h^{\mathrm{ps}}) & \longrightarrow & 0.
\end{array}$$

The proofs that these projections commute are similar to the local cases. The top sequences (the non-discrete spaces) are exact if S is simply connected [6]. In the next result, we will show that the bottom sequences (the discrete spaces) are also exact on simply connected domains.

Theorem 4 *Suppose that Ω is simply connected. Then the sequence (5.2a) is exact for $r \geq 2$, and the sequence (5.2b) is exact for $r \geq 3$.*

Proof Suppose that $v \in L_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}})$ satisfies $\mathrm{div}\, v = 0$. Using the inclusion $S_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}}) \subset H(\mathrm{div};\Omega)$ and standard results, there exists $q \in H(\mathrm{rot};\Omega)$ such that $v = \mathrm{rot}\, q$. Because v is a piecewise polynomial of degree $r-1$, it follows that q is a piecewise polynomial of degree r. Moreover, v is continuous and therefore $q \in C^1(S)$. Thus it follows that $q \in S_r^0(\mathcal{T}_h^{\mathrm{ps}})$. Note that this result shows that if $v \in L_{r-1}^1(\mathcal{T}_h^{\mathrm{ps}})$ satisfies $\mathrm{div}\, v = 0$, then $v = \mathrm{rot}\, q$ for some $q \in S_r^0(\mathcal{T}_h^{\mathrm{ps}})$.

Thus to prove the result, it suffices to show that the mappings $\operatorname{div}: L^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}}) \to V^2_{r-2}(\mathcal{T}_h^{\mathrm{ps}})$ and $\operatorname{div}: S^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}}) \to L^2_{r-2}(\mathcal{T}_h^{\mathrm{ps}})$ are surjections. This will be accomplished by showing that $\dim(\operatorname{div} L^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}})) = \dim V^2_{r-2}(\mathcal{T}_h^{\mathrm{ps}})$ and $\dim(\operatorname{div} S^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}})) = \dim L^2_{r-2}(\mathcal{T}_h^{\mathrm{ps}})$.

Denote by $\mathbb{V}$, $\mathbb{E}$, and $\mathbb{T}$ the number of vertices, edges, and triangles in $\mathcal{T}_h$, respectively. The degrees of freedom given above show that, for $r \geq 2$,

$$\begin{aligned}
\dim S^0_r(\mathcal{T}_h^{\mathrm{ps}}) &= 3\mathbb{V} + (4r-8)\mathbb{E} + 3(r-2)(r-3)\mathbb{T},\\
\dim L^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}}) &= 2\mathbb{V} + (4r-6)\mathbb{E} + 3(r-2)(r-3)\mathbb{T} + \big(3(r-1)r-4\big)\mathbb{T},\\
\dim V^2_{r-2}(\mathcal{T}_h^{\mathrm{ps}}) &= \mathbb{E} + \mathbb{T} + \big(3(r-1)r-4\big)\mathbb{T}.
\end{aligned}$$

We then find, by the rank-nullity theorem and the Euler relation $\mathbb{V} - \mathbb{E} + \mathbb{T} = 1$ that

$$\begin{aligned}
\dim(\operatorname{div} L^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}})) &= \dim L^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}}) - \dim(\operatorname{rot} S^0_r(\mathcal{T}_h^{\mathrm{ps}}))\\
&= \dim L^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}}) - \dim S^0_r(\mathcal{T}_h^{\mathrm{ps}}) + 1\\
&= \dim L^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}}) - \dim S^0_r(\mathcal{T}_h^{\mathrm{ps}}) + (\mathbb{V} - \mathbb{E} + \mathbb{T})\\
&= 2\mathbb{V} + (4r-6)\mathbb{E} + 3(r-2)(r-3)\mathbb{T} + \big(3(r-1)r-4\big)\mathbb{T}\\
&\qquad - \big(3\mathbb{V} + (4r-8)\mathbb{E} + 3(r-2)(r-3)\mathbb{T}\big) + (\mathbb{V} - \mathbb{E} + \mathbb{T})\\
&= \mathbb{E} + \mathbb{T} + \big(3(r-1)r-4\big)\mathbb{T} = \dim V^2_{r-2}(\mathcal{T}_h^{\mathrm{ps}}).
\end{aligned}$$

Likewise, we have for $r \geq 3$,

$$\begin{aligned}
\dim S^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}}) &= 3\mathbb{V} + (6r-12)\mathbb{E} + 3(r-2)(r-3)\mathbb{T} + 3(r-2)(r-3)\mathbb{T},\\
\dim L^2_{r-2}(\mathcal{T}_h^{\mathrm{ps}}) &= \mathbb{V} + (2r-5)\mathbb{E} + \mathbb{T} + 3(r-2)(r-3)\mathbb{T},
\end{aligned}$$

and therefore

$$\begin{aligned}
\dim(\operatorname{div} S^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}})) &= \dim S^1_{r-1}(\mathcal{T}_h^{\mathrm{ps}}) - \dim S^0_r(\mathcal{T}_h^{\mathrm{ps}}) + (\mathbb{V} - \mathbb{E} + \mathbb{T})\\
&= 3\mathbb{V} + (6r-12)\mathbb{E} + 6(r-2)(r-3)\mathbb{T}\\
&\qquad - \big(3\mathbb{V} + (4r-8)\mathbb{E} + 3(r-2)(r-3)\mathbb{T}\big) + (\mathbb{V} - \mathbb{E} + \mathbb{T})\\
&= \mathbb{V} + (2r-5)\mathbb{E} + 3(r-2)(r-3)\mathbb{T} + \mathbb{T} = \dim L^2_{r-2}(\mathcal{T}_h^{\mathrm{ps}}).
\end{aligned}$$

□

6 Conclusion

We have developed smooth finite element spaces on Powell–Sabin splits that form exact sequences in two dimensions. We plan to investigate the extension to higher-dimensions in the near future. Another interesting question is whether smoother finite element spaces (e.g., C^2) fit an exact sequence on Powell–Sabin triangulations.

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Appendix

Proof of Lemma 7

Proof Suppose $z \in W_r(\{a, m, b\})$ is such that (4.1a)–(4.1c) are all zero. We will show that z must be identically zero on $[a, b]$. Let $\psi(x)$ be a degree r polynomial on the interval [0, 1] satisfying

$$\begin{aligned} \psi(0) = 1, \quad \psi(1) = 0, \\ \int_0^1 \psi(x)p(x) = 0 \quad \forall p \in \mathcal{P}_{r-2}([0, 1]). \end{aligned} \tag{A.1}$$

We note that these conditions uniquely determine ψ. Since z is continuous at m and equal to zero at a and b, and in view of (4.1b)–(4.1c), it follows that z may be represented by

$$z(y) = z(m)\begin{cases} \psi\left(\frac{y-m}{a-m}\right) & y \in [a, m], \\ \psi\left(\frac{m-y}{m-b}\right) & y \in [m, b]. \end{cases}$$

Since $z'(y)$ is continuous at m, it must hold that

$$\frac{-1}{m-b}\psi'(0) = \frac{1}{a-m}\psi'(0).$$

Furthermore, given the conditions (A.1) on ψ, we can show that $\psi'(0) \neq 0$. Suppose that $\psi'(0) = 0$ in addition to (A.1). Then for any $p \in \mathcal{P}_{r-1}([0, 1])$ with $p(0) = 0$,

$$\int_0^1 \psi'(x)p(x) = -\int_0^1 \psi(x)p'(x) + \psi(1)p(1) - \psi(0)p(0) = -\int_0^1 \psi(x)p'(x) = 0$$

since $p'(x) \in \mathcal{P}_{r-2}([0, 1])$. But $\psi'(x)$ is itself such a function $p(x)$, so it follows that

$$\int_0^1 |\psi'(x)|^2 = 0.$$

Then $\psi'(x) = 0$, and ψ is constant on [0, 1]. This contradicts (A.1), so $\psi'(0) \neq 0$. Furthermore, since $1/(b-m) \neq 1/(a-m)$, it follows that $z(m) = 0$. Therefore $z = 0$ on $[a, b]$. □

Proof of Theorem 3

Proof (1) *Proof of* (4.10a). Let $p \in C^\infty(T)$ and $\rho := \operatorname{rot} \Pi_0^r p - \varpi_1^{r-1} \operatorname{rot} p \in S_{r-1}^1(T^{\mathrm{ps}})$. We show that ρ vanishes on (4.5).

First,

$$\rho(z_i) = \operatorname{rot} \Pi_0^r p(z_i) - \varpi_1^{r-1} \operatorname{rot} p(z_i) = 0,$$
$$\operatorname{div} \rho(z_i) = -\operatorname{div} \varphi_1^{r-1} \operatorname{rot} p(z_i) = -\operatorname{div} \operatorname{rot} p(z_i) = 0,$$

by the definitions of Π_0^r and ϖ_1^{r-1} along with DOFs (4.2a) and (4.5a).

Next, if $r = 2$,

$$\int_{e_i} \rho \cdot n_i = \int_{e_i} \left(\operatorname{rot} \Pi_0^r p - \varpi_1^{r-1} \operatorname{rot} p\right) \cdot n_i$$
$$= \int_{e_i} \left(\operatorname{rot} \Pi_0^r p - \Pi_1^{r-1} \operatorname{rot} p\right) \cdot n_i = 0,$$

using (4.5b), (4.3b) and (4.7b). Similar arguments show that, for $r \geq 3$,

$$\rho(z_{3+i}) \cdot n_i = (\operatorname{rot} \Pi_0^r p(z_{3+i}) - \Pi_1^{r-1} \operatorname{rot} p(z_{3+i})) \cdot n_i = 0,$$
$$\int_e \rho \cdot w = \int_e (\operatorname{rot} \Pi_0^r p - \varpi_1^{r-1} \operatorname{rot} p) \cdot w = \int_e (\operatorname{rot} \Pi_0^r p - \Pi_1^{r-1} \operatorname{rot} p) \cdot w = 0,$$

and

$$\int_T \rho \cdot \operatorname{rot} w = \int_T (\operatorname{rot} \Pi_0^r p - \Pi_1^{r-1} \operatorname{rot} p) \cdot w = 0.$$

Next using (4.5c) gives

$$\operatorname{div} \rho(z_{3+i}) = -\operatorname{div} \varpi_1^{r-1} \operatorname{rot} p(z_{3+i}) = -\operatorname{div} \operatorname{rot} p(z_{3+i}) = 0,$$

and (4.5e) yields

$$\int_e (\operatorname{div} \rho) q = -\int_e (\operatorname{div} \varpi_1^{r-1} \operatorname{rot} p) q = -\int_e (\operatorname{div} \operatorname{rot} p) q = 0$$

for all $q \in \mathcal{P}_{r-4}(e)$ and $e \in \mathcal{E}^b(T^{\mathrm{ps}})$. The same arguments, but using (4.5g), gives

$$\int_T (\operatorname{div} \rho) q = 0 \qquad \forall q \in \mathring{L}_{r-1}^2(T^{\mathrm{ps}}).$$

Applying Lemma 12 shows that $\rho \equiv 0$, and so (4.10a) holds.

(2) *Proof of* (4.10b). For some $v \in [C^\infty(T)]^2$, we define $\rho := \operatorname{div} \varpi_1^{r-1} v - \varpi_2^{r-2} \operatorname{div} v \in L_{r-2}^2(T^{\mathrm{ps}})$. Then we need only show that ρ is zero for all DOFs in (4.6). For the vertex DOFs, we have for each z_i,

$$\rho(z_i) = \operatorname{div} \varpi_1^{r-1} v(z_i) - \varpi_2^{r-2} \operatorname{div} v(z_i) = 0,$$

by (4.5a) and (4.6a). Next, for each $i = 1, 2, 3$,

$$\rho(z_{3+i}) = \operatorname{div} \varpi_1^{r-1} v(z_{3+i}) - \varpi_2^{r-2} \operatorname{div} v(z_{3+i}) = 0,$$

where we have used (4.5a) and (4.6b). Similar arguments show that

$$\int_e \rho q = 0 \qquad \forall q \in \mathcal{P}_{r-4}(e),\ e \in \mathcal{E}^b(T^{\mathrm{ps}}),$$

by (4.5e) and (4.6c), and that

$$\int_T \rho q = 0 \qquad \forall q \in \mathring{L}^2_{r-2}(T^{\mathrm{ps}})$$

by (4.5g) and (4.6e). Using (4.6d) and (4.5b) if $r = 2$ or (4.5d) if $r > 2$,

$$\int_T \rho = \int_T \operatorname{div} \varpi_1^{r-1} v - \varpi_2^{r-2} \operatorname{div} v = \int_T \operatorname{div} (\varpi_1^{r-1} v - v) = \int_{\partial T} (\varpi_1^{r-1} v - v) \cdot n = 0.$$

Therefore, $\rho \equiv 0$ on T by Lemma 13, and (4.10b) is proved. □

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