Graph Cheetsheet

1-B Images, cameras, displays

Dynamic Range: $DR = \frac{L_{max}}{L_{min}}$

$L_{out} = cV^{\gamma} + b$

Output luminance = Contrast.Voltage^Gamma + Brightness

Gamma correction: $\frac{1}{\gamma}$

2 Ray Tracing

$r(t) = o + td$

Computing Eye Rays

$m = e + -w.distance$

$q = m + lu + tv$

$s = q + s_u u - s_v v$

$s_u = (i + 0.5)\frac{r-l}{n_x}$

$s_v = (j + 0.5)\frac{t-b}{n_y}$

$r(t) = e + (s - e)t = e + dt$

Ray Plane Intersection

$(p - a).n = 0$

$(o + td - a).n = 0$

$t = (a - o).n / (d.n)$

Ray Sphere Intersection

$(x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 - R^2 = 0$

$(p - c).(p - c) - R^2 = 0$

$(o + td - c).(o + td - c) - R^2 = 0$

$(d.d)t^2 + 2d.(o - c)t + (o - c).(o - c) - R^2 = 0$

Ray Triangle Intersection - 1

$n = c - b \, x \, (a - b)$

$f(p) = p - a \, .n = 0$

Check if inside for all 3 vertices

$v_p = (p - b) \, x \, (a - b)$

$v_c = (c - b) \, x \, (a - b)$

$v_p . v_c > 0$

Ray Triangle Intersection - 2

$p(\alpha, \beta, \gamma) = \alpha a + \beta b + \gamma c$

$0 < \alpha < 1 \;\; \& \;\; 0 < \beta < 1 \;\; \& \;\; 0 < \gamma < 1$

Computed from Area Ratios

$\alpha = A_a / A$

$\beta = A_b / A$

$\gamma = A_c / A$

$A = A_a + A_b + A_c$

$\alpha + \beta + \gamma = 1$

$\alpha = 1 - \beta - \gamma$

$p(\beta, \gamma) = a + \beta(b - a) + \gamma(c - a)$

Point p is inside triangle iff:

$\beta + \gamma \leq 1$

$0 \leq \beta$ & $0 \leq \gamma$

$o + td = a + \beta(b - a) + \gamma(c - a)$

$o_x + td_x = a_x + \beta(b_x - a_x) + \gamma(c_x - a_x)$

$o_y + td_y = a_y + \beta(b_y - a_y) + \gamma(c_y - a_y)$

$o_z + td_z = a_z + \beta(b_z - a_z) + \gamma(c_z - a_z)$

Cramer's Rule

$$\begin{bmatrix} a_x - b_x & a_x - c_x & d_x \\ a_y - b_y & a_y - c_y & d_y \\ a_z - b_z & a_z - c_z & d_z \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} a_x - o_x \\ a_y - o_y \\ a_z - o_z \end{bmatrix}$$

$$A \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} x - o_x \\ y - o_y \\ a_z - o_z \end{bmatrix}$$

$$\beta = \frac{\begin{bmatrix} a_x - o_x & a_x - c_x & d_x \\ a_y - o_y & a_y - c_y & d_y \\ a_z - o_z & a_z - c_z & d_z \end{bmatrix}}{|A|}$$

$$\gamma = \frac{\begin{bmatrix} a_x - b_x & a_x - o_x & d_x \\ a_y - b_y & a_y - o_y & d_y \\ a_z - b_z & a_z - o_z & d_z \end{bmatrix}}{|A|}$$

$$t = \frac{\begin{bmatrix} a_x - b_x & a_x - c_x & a_x - o_x \\ a_y - b_y & a_y - c_y & a_y - o_y \\ a_z - b_z & a_z - c_z & a_z - o_z \end{bmatrix}}{|A|}$$

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$|A| = a(ei - hf) + b(gf - di) + c(dh - eg)$$

3 Ray Tracing: Shading

$P = \frac{dQ}{dt}$: Power

$I = \frac{dP}{dw}$: Intensity

$L = \frac{dP}{dw\ dAcos\theta} = \frac{dI}{dAcos\theta}$: Radiance

$E = \frac{dP}{dA} = L\ dwcos\theta$: Irradiance

Cosine Law

$$cos\ \theta = \frac{-w_i.n}{|w_i||n|}$$

Surface Normals

$$n = \frac{p - c}{R}$$

$$n = \frac{(b-a)\ x\ (c-a)}{|(b-a)\ x\ (c-a)|}$$

Diffuse Shading

$$L_o{}^d(x,\ wo) = k_d cos\ \theta' E_i(x,\ w_i)\ ,\ cos\ \theta' = max(0,\ w_i.n)$$

$$L_o{}^d(x,\ wo) = k_d cos\ \theta' \frac{I}{r^2}$$

Ambient Shading

$$L_o{}^a(x,\ wo) = k_a I_a$$

Specular Shading

$$h = \frac{w_i + w_o}{|w_i + w_o|}$$

$$L_o{}^s(x, wo) = k_s cos\, \alpha' E_i(x, w_i) \; cos\, \alpha' = max(0, n.h)$$

$$L_o{}^s(x, wo) = k_s (cos\, \alpha')^p E_i(x, w_i)$$

p: phong exponent

$$L_o{}^s(x, wo) = k_s (cos\, \alpha')^p \frac{I}{r^2}$$

Shadows

$$s(t) = x + tw_i$$

$$s(t) = (x + w_i \varepsilon) + tw_i \qquad \varepsilon \approx 0.0001$$

Ideal Specular Reflection

$$w_r = -w_o + 2n\, cos\theta = -w_o + 2n\,(n.w_o)$$

$$L_o{}^m(x, wo) = k_m L_i(x, w_r)$$

4-A Data Structures

// needs more study

$\#V - \#E + \#F = 2$

$F \sim 2V$ because $3F/2 = E$

Face-set (polygon soup) repeated vertices

Indexed Face-Set Data Structure: shared vertex data

4-B Texture Mapping

Only the first 3 steps differ

Spheres

1) $(u, v) \in [0, 1] x [0, 1]$
2) $x = rsin\theta cos\varphi$

   $y = rcos\theta$

   $z = rsin\theta sin\varphi$

   $\theta = arccos(y/r)$

   $\varphi = arctan(z/x)$

   $(\theta, \varphi) \in [0, \pi] x [-\pi, \pi]$

   $u = (-\varphi + \pi) / (2\pi)$

   $v = \theta / \pi$

3) $u = (-\varphi + \pi) / (2\pi)$

   $v = \theta / \pi$
4) $i = u.n_x$

   $j = v.n_y$
5) Interpolate
   a) Nearest Neighbor:
      $Color(x, y, z) = fetch(round(i, j))$
   b) Bilinear Interpolation:
      $Color(x, y, z) =$
      $fetch(p, q).(1 - dx).(1 - dy)$

$$+ fetch(p+1,\ q).(dx).(1-dy)$$
$$+ fetch(p,\ q+1).(1-dx).(dy)$$
$$+ fetch(p+1,\ q+1).(dx).(dy)$$

Triangles

$$p(\beta,\ \gamma) = a + \beta(b-a) + \gamma(c-a)$$

$$u(\beta,\ \gamma) = u_a + \beta(u_b - u_a) + \gamma(u_c - u_a)$$

$$v(\beta,\ \gamma) = v_a + \beta(v_b - v_a) + \gamma(v_c - v_a)$$

5 Modeling Transformations

Translation, Rotation, Scaling

2D Transformations

Translation:

$x' = x + t_x$

$y' = y + t_y$

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$P' = P + T$

Rotation:

a) $(x_r,\ y_r) = (0, 0)$

$$x' = r\,cos(\phi + \theta) = r\,cos\phi\,cos\theta - r\,sin\phi\,sin\theta$$
$$y' = r\,sin(\phi + \theta) = r\,cos\phi\,sin\theta + r\,sin\phi\,cos\theta$$

$$x = r\,cos\phi$$
$$y = r\,sin\phi$$

$$x' = x\,cos\theta - y\,sin\theta$$
$$y' = x\,sin\theta + y\,cos\theta$$

$$R = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$
$$P' = R.P$$

b) arbitrary $(x_r,\ y_r)$

$$x' = x_r + (x - x_r)\,cos\theta - (y - y_r)\,sin\theta$$
$$y' = y_r + (x - x_r)\,sin\theta + (y - y_r)\,cos\theta$$

Scaling:

$$x' = xs_x$$
$$y' = ys_y$$

$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$
$$P' = S.P$$

Homogenous Coordinates

$x = \frac{x_h}{h} \quad y = \frac{y_h}{h}$

$$P = \begin{bmatrix} x_h \\ y_h \\ h \end{bmatrix} = \begin{bmatrix} h.x \\ h.y \\ h \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translation:

$P' = T(t_x, t_y).P$ where

$$T(t_x, t_y) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation:

$P' = R(\theta).P$ where

$$R(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling:

$P' = S(s_x, s_y).P$ where

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Composite Transformations</u>

$$T(t_{2x}, t_{2y}).T(t_{1x}, t_{1y}) = T(t_{1x} + t_{2x}, t_{1y} + t_{2y})$$
$$R(\theta).R(\varphi) = R(\theta + \varphi)$$
$$S(s_{2x}, s_{2y}).S(s_{1x}, s_{1y}) = S(s_{1x}.s_{2x}, s_{1y}.s_{2y})$$

<u>Rotation Around a Pivot Point</u>

$$T(x_r, y_r).R(\theta).T(-x_r, -y_r)$$

<u>Scaling w.r.t. a Fixed Point</u>

$$T(x_f, y_f).S(s_x, s_y).T(-x_f, -y_f)$$

Reflection & Sheer

<u>3D Transformations</u>

Translation:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\theta & 0 & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation Around a Parallel Axis

parallel to the x-axis:

$$P' = T(0,\ y_p, z_p).R_x(\theta).T(0,\ -y_p, -z_p).P$$

Rotation Around an Arbitrary Axis

Reverse are not shown here

1) $v = P_2 - P_1 = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$

$u = \frac{v}{|v|} = (a, b, c)$

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) Align u with z in two steps

a) bring u to xz plane

$u = u_x + u_y + u_z = u_x + u'$

d= $\sqrt{b^2 + c^2}$

$cos\ \alpha = \frac{u_z}{|u'|} = \frac{c}{d}$

$sin\ \alpha = \frac{u_y}{|u'|} = \frac{b}{d}$

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{d} & -\frac{b}{d} & 0 \\ 0 & \frac{b}{d} & \frac{c}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) align with z

$cos\ \beta = \frac{\sqrt{u_y{}^2 + u_z{}^2}}{|u|} = \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}} = \sqrt{b^2 + c^2}$

$sin\ \beta = \frac{u_x}{|u|} = \frac{a}{\sqrt{a^2+b^2+c^2}}$ =a

$$R_y(\beta) = \begin{bmatrix} \sqrt{b^2+c^2} & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & \sqrt{b^2+c^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta) = T(x_1, y_1, z_1).$$
$$R_x(-\alpha).R_y(\beta).$$
$$R_z(\theta)$$
$$.R_y(-\beta).R_x(\alpha)$$
$$.T(-x_1, -y_1, -z_1)$$

Alternative Method

Creating an orthonormal basis uvw

1) using the unit vector u = (a, b, c) if b>c & a>c
   v = (-b, a, 0)
2) **w** = **u** x **b**
3) normalize
4) rotate uvw to align with xyz

$$M^{-1} = \begin{bmatrix} u_x & v_x & w_x & 0 \\ u_y & v_y & w_y & 0 \\ u_z & v_z & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$M^{-1} = M^T$ for orthonormal matrices

$$M = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ w_x & w_y & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Final Rotation: $T^{-1}.M^{-1}.R_x(\theta).M.T$

Transforming Normals

$n.v = 0$
$n'.v' = 0$
$v' = Mv$
$n' = Zn$
$n.v = n^T v$
$n'.v' = n'^T v' = (Zn)^T Mv = n^T Z^T Mv = 0$
$Z^T M$ = I (identity)
$Z = (M^{-1})^T = (M^T)^{-1}$

6 Viewing Transformations

Camera, projection, viewport

Camera Transformations

1) Translate e to world origin

$$T = \begin{bmatrix} 1 & 0 & 0 & -e_x \\ 0 & 1 & 0 & -e_y \\ 0 & 0 & 1 & -e_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) align uvw with xyz

$$M = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ w_x & w_y & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{cam} = \begin{bmatrix} u_x & u_y & u_z & -(u_x e_x + u_y e_y + u_z e_z) \\ v_x & v_y & v_z & -(v_x e_x + v_y e_y + v_z e_z) \\ w_x & w_y & w_z & -(w_x e_x + w_y e_y + w_z e_z) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orthographic Projection

$$M_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective Projection

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} nx \\ ny \\ Az + B \\ -z \end{bmatrix} = \begin{bmatrix} -\frac{nx}{z} \\ -\frac{ny}{z} \\ -A - \frac{B}{z} \\ 1 \end{bmatrix}$$

from: $z' = -A - B/z$

$-n = -A + B/n$

$-f = -A + B/f$

$A = f + n$

$B = fn$

$$M_{p2o} = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & f+n & fn \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$M_{per} = M_{orth} M_{p2o}$

$$M_{per} = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Viewport Transformation

$$M_{vp} = \begin{bmatrix} \frac{n_x}{2} & 0 & 0 & \frac{n_x-1}{2} \\ 0 & \frac{n_y}{2} & 0 & \frac{n_y-1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$