A Laboratory Exercise to Illustrate the Describing Function Technique

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Abstract—**Many control texts do not provide an adequate insight into the assumptions inherent in the describing function technique for analyzing the stability of nonlinear systems. A laboratory experiment is described which provides this background by investigating the harmonic content of the signals in an analog simulation of a third-order system which incorporates an ideal relay.**

The paper then proceeds to show how the simulation may be extended to investigate the stability of the system when the relay includes a dead zone or hysteresis. It is shown that under these favorable conditions, the describing function gives good agreement with the results of the simulation.

I. Introduction

The describing function technique is well established and is given a thorough treatment in many control engineering texts [1]–[5]. As a test for the existence of continuous oscillations in nonlinear systems (limit cycles), the describing function offers an approach which is readily related to linear system theory. Although the standard text may offer a comprehensive coverage of the analysis of nonlinear systems using the describing function, little attention is paid to the validity of this analysis. Since nonlinear systems belie any form of general treatment, this omission is to some extent understandable. This paper describes a laboratory experiment which has been used widely at both the undergraduate and masters levels to illustrate the significance of the assumptions inherent in the describing function technique before going on to analyze simple relay systems.

II. Equipment

In control engineering courses there is often a choice between practical exercises based on laboratory-scaled models or analog simulation [6]. In this case the emphasis is on the illustration of fundamental concepts and there is a need for an exercise which is uncluttered by practical difficulties. In these circumstances analog simulation is the best choice.

The analog computer used in this study has been developed in the Department of Systems and Control specifically for teaching purposes. It incorporates a number of interesting features: among them a "controler" module which is used in this exercise to simulate an ideal relay to which dead zone and hysteresis are added later. The controller module is shown in Fig. 1. However, a piece of commercially available equipment, the Process Control Simulator PCS 327, manufactured by Feedback Instruments, Crowborough, England, is suitable for this experiment, although it does offer a restricted range of time constants compared to an analog computer.

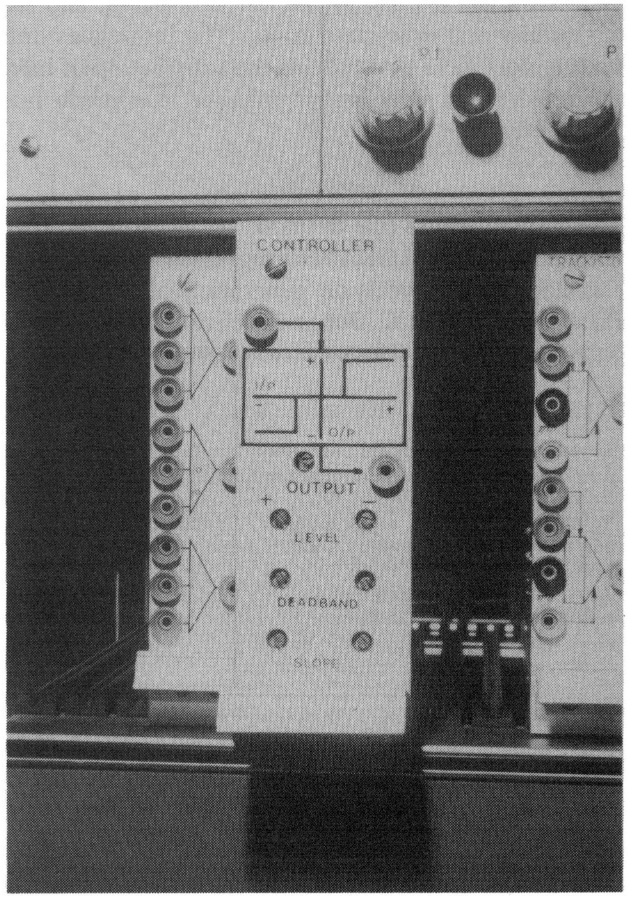

Fig. 1. Controller module.

III. The System

Since the aim of the exercise is to illustrate the describing function and not to test the student's knowledge of analog simulation, the system has been simplified as much as possible. Initially, a linear system with unity feedback is simulated with a forward path transfer function $KG(s)$ where K is the gain and

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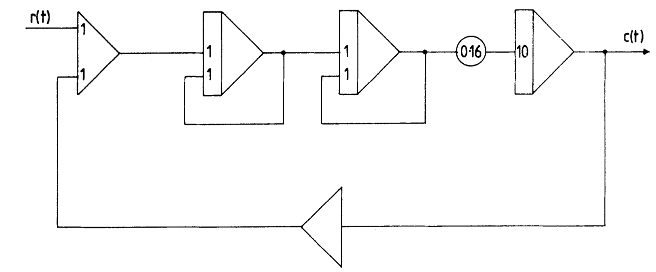


Fig. 2. Simulation of linear system.

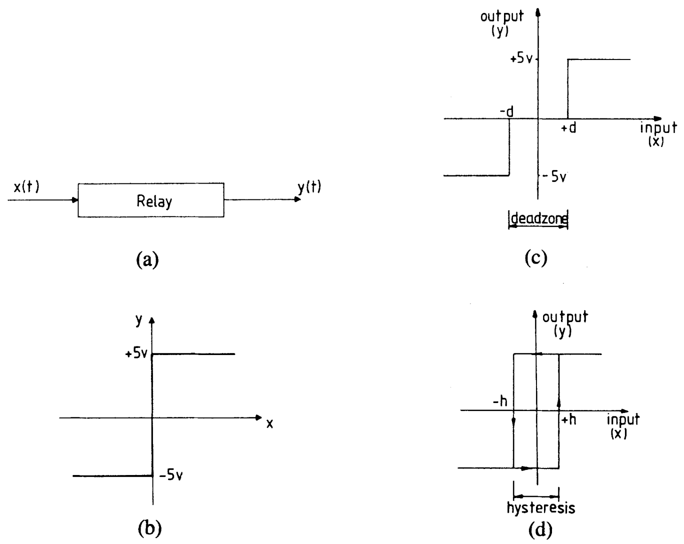


Fig. 3. Relay characteristics.

$$G(s) = \frac{1}{s(1 + s)^2}.$$

The simplicity of the resulting simulation is shown in Fig. 2. Note that saturation is inhibited by placing the pure integrator and the gain after the two first-order lags. For this system it can be shown that marginal stability occurs when the gain K is set to 2 V/V. The gain is normally set to 1.6 V/V for this exercise which gives a stable, but very oscillatory response. The ideal relay with a switching level of ± 5 V is then incorporated in the forward path immediately after the summing amplifier to give the characteristic shown in Fig. 3(b).

IV. Attenuation of Harmonics

The assumptions which are fundamental to the validity of the describing function are the following.

a) The input to the nonlinear component is of a specified form.

b) The harmonics generated by the nonlinearity are attenuated with respect to the fundamental.

In its most common form, the describing function technique assumes that the input to the nonlinearity is sinusoidal.

Consider an input of this form $x(t)$ where

$$x(t) = X \sin wt.$$

Then, the output $y(t)$ will be a square wave whose amplitude is equal to the switching level of the relay M. Fourier analysis of the output waveforms leads to the series

$$y(t) = 4M(\sin wt + \tfrac{1}{3} \sin 3wt + \tfrac{1}{5} \sin 5wt + \cdots).$$

The describing function $N(X)$ for this nonlinearity is derived by neglecting the harmonics and defining

$$N(X) = \frac{\text{Fundamental of } y(t)}{x(t)} = \frac{4M}{\pi X}.$$

The harmonics, with amplitudes of 33 percent ($3wt$) and 20 percent ($5wt$), are invariably dismissed without any further comment. Furthermore, as previously stated, many control texts merely make a passing reference to the capability of the linear components to act as low-pass filters. If the waveforms are recorded after the various stages in the system, it is easy to see the progressive attenuation of the harmonics by the linear components. Fig. 4(a) shows the square wave output of the relay. Fig. 4(b), (c), and (d) shows the gradual progression to a more sinusoidal form with each integration. The system output is fed back to the relay by the comparator (summing amplifier) thus satisfying the condition that the input to the nonlinearity is sinusoidal. This simple but powerful graphical illustration may then be reinforced by analyzing the effect of three stages of integration on the output of the relay $y(t)$.

The output of the relay is

$$y(t) = \frac{4M}{\pi}(\sin wt + \tfrac{1}{3} \sin 3wt + \tfrac{1}{5} \sin 5wt + \cdots).$$

Successive integration with respect to wt gives

First Integration:

$$y(t) = \frac{4M}{\pi}(-\cos wt - \tfrac{1}{9} \cos 3wt - \tfrac{1}{25} \cos 5wt \cdots)$$

Second Integration:

$$y(t) = \frac{4M}{\pi}(-\sin wt - \tfrac{1}{27} \sin 3wt - \tfrac{1}{125} \sin 5wt \cdots)$$

Third Integration:

$$y(t) = \frac{4M}{\pi}(\cos wt + \tfrac{1}{54} \cos 3wt + \tfrac{1}{625} \cos 5wt \cdots).$$

V. System Performance with an Ideal Relay

After establishing the validity of the describing function in this exercise, it is a natural extension to consider the prediction of the existence of a limit cycle and, if such an oscillation exists, determine its amplitude and frequency. For simplicity, the analysis is omitted and only the results are quoted here. A polar plot of the frequency response of the linear components $G(jw)$ and the describing function plotted as $-1/N(X)$ are given in Fig. 5. From this plot a limit cycle may be expected with an amplitude

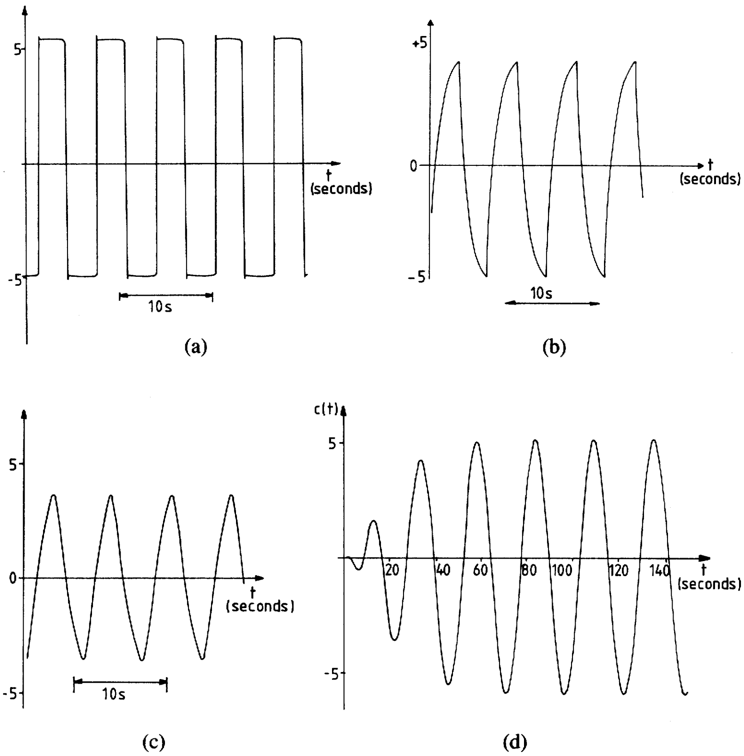


Fig. 4. Signals in the system.

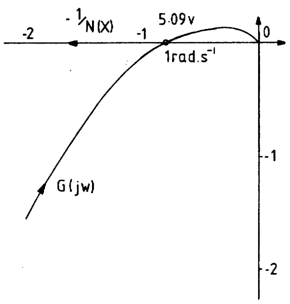


Fig. 5. Polar plot for the ideal relay.

5.09 V and a frequency 1.0 rad · s^{-1}. This differs by only 0.6 percent from the recorded limit cycle shown in Fig. 4(d).

Many students find it intriguing to remove the step input and observe that the limit cycle persists but with a zero dc level. This illustrates that the oscillation is self-sustaining.

More difficult extensions include the prediction of the effect of varying the linear gain K and the switching level M on the limit cycle. Finally, the student is asked to consider the likely effect of reducing the linear part of the system to second-order.

This results in a limit cycle which is ostensibly sinusoidal with an amplitude of 94 mV and a period of 0.6 s when

$$G(s) = \frac{1.6}{s(1 + s)}.$$

The fact that this limit cycle is not predicted by the describing function usually creates considerable interest.

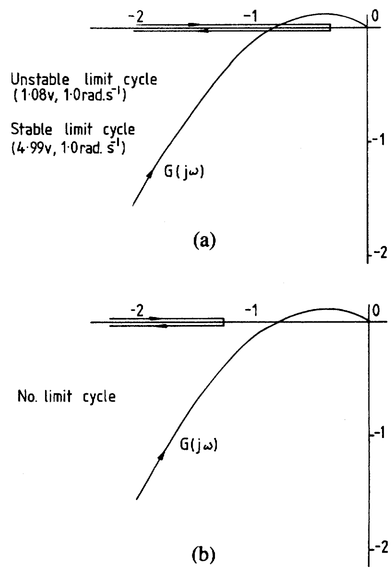


Fig. 6. Polar plots for relay with dead zone.

VI. Performance with Other Relay Characteristics

With simple adjustments to the potentiometers on the control module of the analog computer, the relay characteristic may be modified to include dead zone or hysteresis and give the characteristics shown in Fig. 3(c) and (d).

Again, the analysis has been omitted. For a dead zone of ± 1 V the limit cycle persists and is little different in amplitude (4.99 V) or frequency (1.0 rad · s^{-1}), from the result obtained with the ideal relay. The polar plot in Fig. 6(a) shows that the system is theoretically capable of both

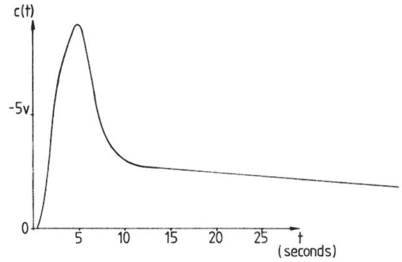
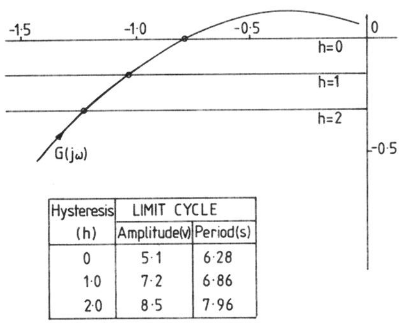


Fig. 7. Output with ±4 V dead zone.



Hysteresis (h)	LIMIT CYCLE Amplitude(v)	Period(s)
0	5·1	6·28
1·0	7·2	6·86
2·0	8·5	7·96

Fig. 8. Polar plot for relay with hysteresis.

unstable and stable limit cycles. Obviously, only the stable one may be recorded. For the larger dead zone of ±4 V, the system does not exhibit a limit cycle at all (Figs. 6(b) and 7).

As increasing amounts of hysteresis are added, the amplitude of the limit cycle increases while the frequency reduces. This is confirmed by the polar plot of Fig. 8.

VII. Conclusion

The laboratory experiment described, involving only a limited knowledge of the techniques of analog simulation, has been shown to give a good insight into the validity of the assumptions inherent in the describing function. In addition, the exercise provides an opportunity to assess the accuracy of the technique in favorable circumstances.

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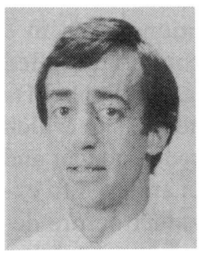

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