Math189R SU17
Homework 5
Monday, June 5, 2017

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

1 (Murphy 12.5 - Deriving the Residual Error for PCA) It may be helpful to reference section 12.2.2 of Murphy.

(a) Prove that

$$\left\| \mathbf{x}_i - \sum_{j=1}^{k} z_{ij} \mathbf{v}_j \right\|^2 = \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j.$$

Hint: first consider the case when $k = 2$. Use the fact that $\mathbf{v}_i^\top \mathbf{v}_j$ is 1 if $i = j$ and 0 otherwise. Recall that $z_{ij} = \mathbf{x}_i^\top \mathbf{v}_j$.

(b) Now show that

$$J_k = \frac{1}{n} \sum_{i=1}^{n} \left(\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j \right) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \lambda_j.$$

Hint: recall that $\mathbf{v}_j^\top \boldsymbol{\Sigma} \mathbf{v}_j = \lambda_j \mathbf{v}_j^\top \mathbf{v}_j = \lambda_j$.

(c) If $k = d$ there is no truncation, so $J_d = 0$. Use this to show that the error from only using $k < d$ terms is given by

$$J_k = \sum_{j=k+1}^{d} \lambda_j.$$

Hint: partition the sum $\sum_{j=1}^{d} \lambda_j$ into $\sum_{j=1}^{k} \lambda_j$ and $\sum_{j=k+1}^{d} \lambda_j$.

■

a) $$\left|\left| x_i - \sum_{j=1}^{k} z_{ij} \vec{v}_j \right|\right|_2^2 = \left(\vec{x}_i - \sum_{j=1}^{k} z_{ij} \vec{v}_j\right)^T \left(x_i - \sum_{j=1}^{k} z_{ij} \vec{v}_j\right)$$

$$= \vec{x}_i^T \vec{x}_i - 2 \sum_{j=1}^{k} z_{ij} \vec{v}_j^T x_i + \left(\sum_{j=1}^{k} z_{ij} \vec{v}_j\right)^T \left(\sum_{j=1}^{k} z_{ij} \vec{v}_j\right)$$

$$= \vec{x}_i^T \vec{x}_i - 2 \sum_{j=1}^{k} z_{ij} v_j^T x_i + \sum_{j=1}^{k} \vec{v}_j^T \vec{x}_i \vec{x}^T \vec{v}_j$$

$$= \vec{x}_i^T \vec{x}_i - \sum_{j=1}^{k} \vec{v}_j^T \vec{x}_i \vec{x}_i^T \vec{v}_j$$

b) $$J_k = \frac{1}{n} \sum_{i=1}^{n} \left(\vec{x}_i^T \vec{x}_i - \sum_{j=1}^{k} v_j^T \vec{x}_i \vec{x}_i^T \vec{v}_j\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \vec{x}_i^T x_i - \sum_{j=1}^{k} \vec{v}_j^T \frac{1}{n} \left(\sum_{i=1}^{n} \vec{x}_i \vec{x}^T\right) \vec{v}_j$$

$$= \frac{1}{n} \sum_{i=1}^{n} \vec{x}_i^T \vec{x}_i - \sum_{j=1}^{k} \vec{v}_j^T \Sigma \vec{v}_j$$

$$= \frac{1}{n} \sum_{i=1}^{n} \vec{x}_i^T \vec{x}_i - \sum_{j=1}^{k} \lambda_i$$

c) $J_0 = 0$, thus $\sum_{j=1}^{d} \lambda_j = \frac{1}{n} \sum_{i=1}^{n} x_i^T x_i$

Thus:

$$J_k = \frac{1}{n} \sum_{i=1}^{n} \vec{x}_i^T \vec{x}_i - \sum_{j=1}^{d} \lambda_j + \sum_{j=k+1}^{d} \lambda_j = \sum_{j=k+1}^{d} \lambda_j$$

reconstruction error = sum of thrown eigenvalues.

2 (ℓ_1-Regularization) Consider the ℓ_1 norm of a vector $\mathbf{x} \in \mathbb{R}^n$:

$$\|\mathbf{x}\|_1 = \sum_i |\mathbf{x}_i|.$$

Draw the norm-ball $B_k = \{\mathbf{x} : \|\mathbf{x}\|_1 \leq k\}$ for $k = 1$. On the same graph, draw the Euclidean norm-ball $A_k = \{\mathbf{x} : \|\mathbf{x}\|_2 \leq k\}$ for $k = 1$ behind the first plot. (Do not need to write any code, draw the graph by hand).

Show that the optimization problem

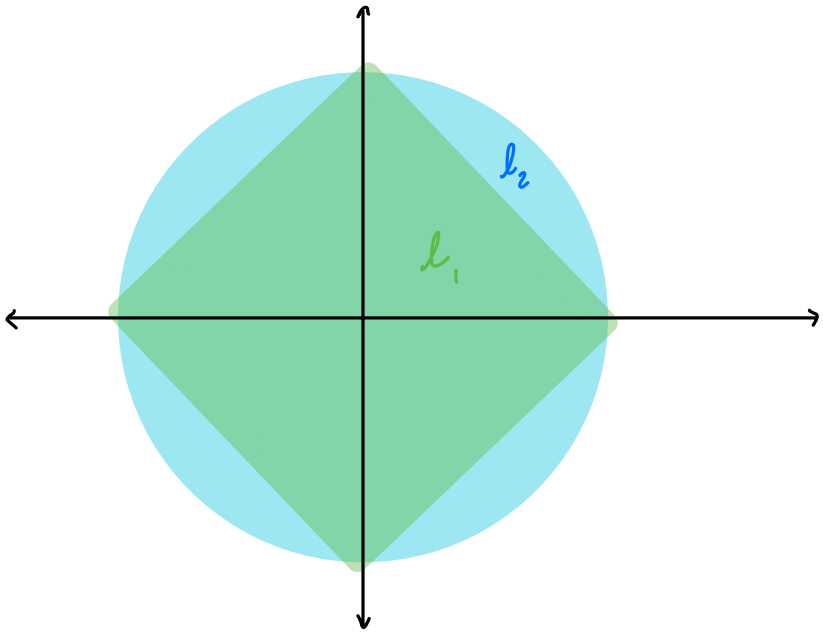
$$\begin{aligned} &\text{minimize: } f(\mathbf{x}) \\ &\quad \text{subj. to: } \|\mathbf{x}\|_p \leq k \end{aligned}$$

is equivalent to

$$\text{minimize: } f(\mathbf{x}) + \lambda\|\mathbf{x}\|_p$$

(hint: create the Lagrangian). With this knowledge, and the plots given above, argue why using ℓ_1 regularization (adding a $\lambda\|\mathbf{x}\|_1$ term to the objective) will give sparser solutions than using ℓ_2 regularization for suitably large λ.

norm-balls :



■

optimization problem: minimize: $f(\vec{x})$

subj to: $||\vec{x}||_p \leq k$

same as: $\inf_{x} \sup_{\lambda \geq 0} \mathcal{L}(\vec{x}, \lambda) = \inf_{x} \sup_{\lambda \geq 0} f(\vec{x}) + \lambda (||\vec{x}||_p - k)$

$\Rightarrow \sup_{\lambda \geq 0} \inf_{x} \underbrace{f(\vec{x}) + \lambda (||\vec{x}||_p - k)}_{-\lambda k \text{ does not depend on } x!} = \sup_{\lambda \geq 0} g(\lambda)$

$\Rightarrow \sup_{\lambda \geq 0} \inf_{x} f(\vec{x}) + \lambda (||\vec{x}||_p)$

same as minimize: $f(\vec{x}) + \lambda (||\vec{x}||_p)$

Extra Credit (Lasso) Show that placing an equal zero-mean Laplace prior on each element of the weights $\boldsymbol{\theta}$ of a model is equivelent to ℓ_1 regularization in the Maximum-a-Posteriori estimate

$$\text{maximize: } \mathbb{P}(\boldsymbol{\theta}|\mathcal{D}) = \frac{\mathbb{P}(\mathcal{D}|\boldsymbol{\theta})\mathbb{P}(\boldsymbol{\theta})}{\mathbb{P}(\mathcal{D})}.$$

Note the form of the Laplace distribution is

$$\text{Lap}(x|\mu, b) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$

where μ is the location parameter and $b > 0$ controls the variance. Draw (by hand) and compare the density $\text{Lap}(x|0,1)$ and the standard normal $\mathcal{N}(x|0,1)$ and suggest why this would lead to sparser solutions than a Gaussian prior on each elements of the weights (which correspond to ℓ_2 regularization).

■