behaves as we rescale the coordinates. This is why the three point function $\langle\zeta^3\rangle$ is equal to the tilt of the spectrum of the two point function in the regime $k_1 \ll k_{2,3}$.

Komatsu and Spergel have performed an analysis of the detectability of non-gaussian features of the temperature fluctuations [8][9]. Their analysis was made for an expected signal which had a slightly different k dependence from the one in $\mathcal{M}_1$ above. This probably would not change their answer too much. Ignoring this point, one would conclude from their analysis that this level of non-gaussianity is not detectable from CMB measurements alone. A more explicit discussion is given below. In some models with more than one field non-gaussianity can be large [10].

Finally we point out that these computations can also be used in investigations of AdS/CFT and dS/CFT. These dualities can be viewed as a statement about the wave-function of the universe. We relate explicitly the computation of stress tensor correlators in the dS and AdS case. They are related by a simple analytic continuation. We also clarify the relation between stress tensor correlators and the spectrum of fluctuations of metric perturbations.

This paper is organized as follows. In section two we review the standard results that follow from the quadratic approximation and give the gaussian answer. In section three we expand the action to third order. In section four we compute the three point functions. In section five we make some remarks on the relationship of these computations to the dS/CFT and AdS/CFT correspondences.

2. Review of the quadratic computation

The computation of primordial fluctuations that arise in inflationary models was first discussed in [11][12][13][14][5][15] and was nicely reviewed in [16].

The starting point is the agrangian of gravity and a scalar field which has the general form

$$S = \frac{1}{2}\int \sqrt{g}[R - (\nabla\phi)^2 - 2V(\phi)] \tag{2.1}$$

up to field redefinitions. We have set ${M_{pl}}^2 \equiv 8\pi G_N = 1$[2], the dependence on G_N is easily reintroduced.

[2] Note that this definition of M_{pl} is different from the definition that some other authors use (including Planck).

The homogeneous solution has the form

$$ds^2 = -dt^2 + e^{2\rho(t)} dx_i dx_i = e^{2\rho}(-d\eta^2 + dx_i dx_i) \tag{2.2}$$

where η is conformal time. The scalar field is a function of time only. ρ and ϕ obey the equations

$$\begin{aligned} 3\dot{\rho}^2 =& \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ \ddot{\rho} =& -\frac{1}{2}\dot{\phi}^2 \\ 0 =& \ddot{\phi} + 3\dot{\rho}\dot{\phi} + V'(\phi) \end{aligned} \tag{2.3}$$

The Hubble parameter is $H \equiv \dot{\rho}$. The third equation follows from the first two. We will make frequent use of these equations.

If the slow roll parameters are small we will have a period of accelerated expansion. The slow roll parameters are defined as

$$\begin{aligned} \epsilon \equiv& \frac{1}{2}\left(\frac{M_{pl}V'}{V}\right)^2 \sim \frac{1}{2}\frac{\dot{\phi}^2}{\dot{\rho}^2}\frac{1}{M_{pl}^2} \\ \eta \equiv& \frac{M_{pl}^2 V''}{V} \sim -\frac{\ddot{\phi}}{\dot{\rho}\dot{\phi}} + \frac{1}{2}\frac{\dot{\phi}^2}{\dot{\rho}^2}\frac{1}{M_{pl}^2} \end{aligned} \tag{2.4}$$

where the approximate relations hold when the slow roll parameters are small.

We now consider small fluctuations around the solution (2.3). We expect to have three physical propagating degrees of freedom, two from gravity and one from the scalar field. The scalar field mixes with other components of the metric which are also scalars under $SO(2)$ (the little group that leaves $\vec{k}$ fixed). There are four scalar modes of the metric which are δg_{00}, δg_{ii}, $\delta g_{0i} \sim \partial_i B$ and $\delta g_{ij} \sim \partial_i \partial_j H$ where B and H are arbitrary functions. Together with a small fluctuation, $\delta\phi$, in the scalar field these total five scalar modes. The action (2.1) has gauge invariances coming from reparametrization invariance. These can be linearized for small fluctuations. The scalar modes are acted upon by two gauge invariances, time reparametrizations and spatial reparametrizations of the form $x^i \to x^i + \epsilon^i(t,x)$ with $\epsilon^i = \partial_i \epsilon$. Other coordinate transformations act on the vector modes[3]. Gauge invariance removes two of the five functions. The constraints in the action remove two others so that we are left with one degree of freedom.

[3] There are no propagating vector modes for this agrangian (2.1). They are removed by gauge invariance and the constraints. Vector modes are present when more fields are included.

In order to proceed it is convenient to work in the ADM formalism. We write the metric as

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2.5}$$

and the action (2.1) becomes

$$S = \frac{1}{2}\int \sqrt{h}\left[NR^{(3)} - 2NV + N^{\ 1}(E_{ij}E^{ij} - E^2) + N^{\ 1}(\dot{\phi} - N^i\partial_i\phi)^2 - Nh^{ij}\partial_i\phi\partial_j\phi\right] \tag{2.6}$$

Where

$$\begin{aligned} E_{ij} &= \frac{1}{2}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i) \\ E &= E^i_i \end{aligned} \tag{2.7}$$

Note that the extrinsic curvature is $K_{ij} = N^{\ 1}E_{ij}$. In the computations we do below it is often convenient to separate the traceless and the trace part of E_{ij}.

In the ADM formulation spatial coordinate reparametrizations are an explicit symmetry while time reparametrizations are not so obviously a symmetry. The ADM formalism is designed so that one can think of h_{ij} and ϕ as the dynamical variables and N and N^i as agrange multipliers. We will choose a gauge for h_{ij} and ϕ that will fix time and spatial reparametrizations. A convenient gauge is

$$\delta\phi = 0\ , \qquad h_{ij} = e^{2\rho}[(1+2\zeta)\delta_{ij} + \gamma_{ij}]\ , \qquad \partial_i\gamma_{ij} = 0\ , \qquad \gamma_{ii} = 0 \tag{2.8}$$

where ζ and γ are first order quantities. ζ and γ are the physical degrees of freedom. ζ parameterizes the scalar fluctuations and γ the tensor fluctuations. The gauge (2.8) fixes the gauge completely at nonzero momentum. In order to find the action for these degrees of freedom we just solve for N and N^i through their equations of motion and plug the result back in the action. This procedure gives the correct answer since N and N^i are agrange multipliers. The gauge (2.8) is very similar to Coulomb gauge in electrodynamics where we set $\partial_i A_i = 0$, solve for A_0 through its equation of motion and plug this back in the action .

The equation of motion for N^i and N are the the momentum and hamiltonian constraints

$$\begin{aligned} &\nabla_i[N^{\ 1}(E^i_j - \delta^i_j E)] = 0 \\ &R^{(3)} - 2V - N^{\ 2}(E_{ij}E^{ij} - E^2) - N^{\ 2}\dot{\phi}^2 = 0 \end{aligned} \tag{2.9}$$

As in electrodynamics in Coulomb gauge we will often find expressions which are not local in the spatial directions. In the linearized theory it is possible to define local gauge invariant observables where these non-local terms disappear.

where we have used that $\delta\phi = 0$ from (2.8). We can solve these equations to first order by setting $N^i = \partial_i \psi + N_T^i$ where $\partial_i N_T^i = 0$ and $N = 1 + N_1$. We find

$$N_1 = \frac{\dot{\zeta}}{\dot{\rho}} \ , \qquad N_T^i = 0 \ , \qquad \psi = -e^{\ -2\rho} \frac{\zeta}{\dot{\rho}} + \chi \ , \qquad \partial^2 \chi = \frac{\dot{\phi}^2}{2\dot{\rho}^2} \dot{\zeta} \tag{2.10}$$

In order to find the quadratic action for ζ we can replace (2.10) in the action and expand the action to second order. For this purpose it is not necessary to compute N or N^i to second order. The reason is that the second order term in N will be multiplying the hamiltonian constraint, $\frac{\partial \mathcal{L}}{\partial N}$ evaluated to zeroth order which vanishes since the zeroth order solution obeys the equations of motion. There is a similar argument for N^i. Direct replacement in the action gives, up to second order,

$$\begin{aligned} S =& \frac{1}{2} \int e^{\rho+\zeta} (1 + \frac{\dot{\zeta}}{\dot{\rho}}) [-4\partial^2 \zeta - 2(\partial\zeta)^2 - 2V e^{2\rho+2\zeta}] + \\ & + e^{3\rho+3\zeta} \frac{1}{(1 + \frac{\dot{\zeta}}{\dot{\rho}})} [-6(\dot{\rho} + \dot{\zeta})^2 + \dot{\phi}^2] \end{aligned} \tag{2.11}$$

where we have neglected a total derivative which is linear in ψ. After integrating by parts some of the terms and using the background equations of motion (2.3) we find the final expression to second order[5]

$$S = \frac{1}{2} \int dt d^3x \frac{\dot{\phi}^2}{\dot{\rho}^2} [e^{3\rho} \dot{\zeta}^2 - e^{\rho} (\partial\zeta)^2] \tag{2.12}$$

No slow roll approximation was made in deriving (2.11). Note that naively the action (2.11) contains terms of the order $\dot{\zeta}^2$, while the final expression contains only terms of the form $\epsilon\dot{\zeta}^2$, so that the action is suppressed by a slow roll parameter. The reason is that the ζ fluctuation would be a pure gauge mode in de-Sitter space and it gets a non-trivial action only to the extent that the slow roll parameter is non-zero. So the leading order terms in slow roll in (2.11) cancel leaving only the terms in (2.12). A simple argument for the dependence of (2.12) on the slow roll parameters is given below.

Since (2.12) is describing a free field we just have a collection of harmonic oscillators. More precisely we expand

$$\zeta(t, x) = \int \frac{d^3k}{(2\pi)^3} \zeta_k(t) e^{i\vec{k}\vec{x}} \tag{2.13}$$

[5] In order to compare this to the expression in [16] set $v = -z\zeta$ in (10.73) of [16].

Each $\zeta_k(t)$ is a harmonic oscillator with time dependent mass and spring constants. The quantization is straightforward [17]. We pick two independent classical solutions $\zeta_k^{cl}(t)$ and $\zeta_k^{cl*}(t)$ of the equations of motion of (2.12)

$$\frac{\delta}{\delta\zeta} = -\frac{d\left(e^{3\rho}\frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}_k\right)}{dt} - \frac{\dot{\phi}^2}{\dot{\rho}^2}e^{\rho}k^2\zeta_k = 0 \tag{2.14}$$

Then we write

$$\zeta_{\vec{k}}(t) = \zeta_k^{cl}(t)a^\dagger_{\vec{k}} + \zeta_k^{cl*}(t)a_{\vec{k}} \tag{2.15}$$

where a and $a^\dagger$ are some operators. Demanding that $a^\dagger$ and a obey the standard creation and annihilation commutation relations we get a normalization condition for ζ_k^{cl}. Different choices of solutions are different choices of vacua for the scalar field. The comoving wavelength of each mode $\lambda_c \sim 1/k$ stays constant but the physical wavelength changes in time. For early times the ratio of the physical wavelength to the Hubble scale is very small and the mode feels it is in almost flat space. We can then use the WKB approximation to solve (2.14) and choose the usual vacuum in Minkowski space. When the physical wavelength is much longer than the Hubble scale

$$\lambda_{phys}H = \frac{\dot{\rho}e^{\rho}}{k} \gg 1 \tag{2.16}$$

the solutions of (2.14) go rapidly to a constant.

A useful example to keep in mind is that of a massless scalar field f in de-Sitter space. In that case the action is $S = \frac{1}{2}\int H^{\ \ 2}\eta^{\ \ 2}[(\partial_\eta f)^2 - (\partial f)^2]$ and the normalized classical solution, analogous to ζ_k^{cl}, corresponding to the standard Bunch Davies vacuum is [17]

$$f_k^{cl} = \frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta} \tag{2.17}$$

where we are using conformal time which runs from $(-\infty, 0)$. Very late times correspond to small $|\eta|$ and we clearly see from (2.17) that f^{cl} goes to a constant. Any solution, including (2.17), approaches a constant at late times as $\eta^2 \sim e^{\ \ 2\rho}$, which is exponentially fast is physical time. In de-Sitter space we can easily compute the two point function for this scalar field and obtain[6]

$$\begin{aligned}\langle f_{\vec{k}}(\eta)f_{\vec{k}'}(\eta)\rangle =&(2\pi)^3\delta^3(\vec{k}+\vec{k}')|f_k^{cl}(\eta)|^2 = (2\pi)^3\delta^3(\vec{k}+\vec{k}')\frac{H^2}{2k^3}(1+k^2\eta^2)\\ \sim&(2\pi)^3\delta^3(\vec{k}+\vec{k}')\frac{H^2}{2k^3} \qquad \text{for} \quad k\eta \ll 1\end{aligned} \tag{2.18}$$

[6] In coordinate space the result for late times is $\langle f(x,t)f(x',t)\rangle sim - \frac{H^2}{(2\pi^{\ \ 2}}\log(|x-x'|/\ \ )$ where is an IR cutoff which is unimportant when we compute differences in f as we do in actual experiments.

We now go back to the inflationary computation. If one knew the classical solution to the equation (2.14) the result for the correlation function of ζ can be simply computed as

$$\langle \zeta_{\vec{k}}(t)\zeta_{\vec{k}'}(t)\rangle = (2\pi)^3\delta^3(\vec{k}+\vec{k}')|\zeta^{cl}_k(t)|^2 \tag{2.19}$$

If the slow roll parameters are small when the comoving scale $\vec{k}$ crosses the horizon then it is possible to estimate the late time behavior of (2.19) by the corresponding result in de-Sitter space (2.18) with a Hubble constant that is the Hubble constant at the moment of horizon crossing. The reason is that at late times ζ is constant while at early times the field is in the vacuum and its wavefunction is accurately given by the WKB approximation. Since the action (2.12) also contains a factor of $\dot\phi/\dot\rho$ we also have to set its value to the value at horizon crossing, this factor only appears in normalizing the classical solution. In other words, near horizon crossing we set $f = \frac{\dot\phi}{\dot\rho}\zeta$ where f is a canonically normalized field in de-Sitter space. This produces the well known result

$$\langle \zeta_{\vec{k}}(t)\zeta_{\vec{k}'}(t)\rangle \sim (2\pi)^3\delta^3(\vec{k}+\vec{k}')\frac{1}{2k^3}\frac{\dot\rho_*^2}{M_{pl}^2}\frac{\dot\rho_*^2}{\dot\phi_*^2} \tag{2.20}$$

where the star means that it is evaluated at the time of horizon crossing, i.e. at time t_* such that

$$\dot\rho(t_*)e^{\rho(t_*)} \sim k\ . \tag{2.21}$$

The dependence of (2.20) on t_* leads to additional momentum dependence. It is conventional to parameterize this dependence by saying that the total correlation function has the form $k^{\ 3+n_s}$ where

$$n_s = k\frac{d}{dk}\log(\frac{\dot\rho_*}{\dot\phi_*^2}) \sim \frac{1}{\dot\rho_*}\frac{d}{dt_*}\log(\frac{\dot\rho_*}{\dot\phi_*^2}) = -2(\frac{\ddot\phi_*}{\dot\rho_*\dot\phi_*}+\frac{\dot\phi_*}{\dot\rho_*}) = 2(\eta-3\epsilon) \tag{2.22}$$

As it has been often discussed, after horizon crossing the mode becomes classical, in the sense that the commutator $[\dot\zeta,\zeta]\to 0$ exponentially fast. So for measurements which only involve ζ or $\dot\zeta$ we can treat the mode as a classical variable.

After the end of inflation the field ϕ ceases to determine the dynamics of the universe and we eventually go over to the usual hot big bang phase. It is possible to prove [5][16] that ζ remains constant outside the horizon as long as no entropy perturbations are generated and a certain condition on the off-diagonal components of the spatial stress tensor is obeyed[7]. These conditions are obeyed if the universe is described by a single fluid or by a

[7] The condition is $\partial_i\partial_j(\delta T_{ij} - \frac{1}{3}\delta_{ij}T_{ll}) = 0$.

single scalar field. We should mention that for a general fluid the variable ζ can be defined in terms of the three metric as above (2.8) in the comoving gauge where $T_i^0 = 0$[8]. In the case of a scalar field this implies that $\delta\phi = 0$. This gauge is convenient conceptually since the variable ζ is directly a function appearing in the metric. We see that the variable ζ tells us how much the spatial directions have expanded in the comoving gauge, so that to linear order ζ determines the curvature of the spatial slices $R^{(3)} = 4k^2\zeta$ [19]. This variable ζ is very useful in order to continue through the end of inflation since it is defined throughout the evolution and it is constant outside the horizon. An intuitive way to understand why ζ is constant is to note that the conditions stated above imply that two observers separated by some distance see the universe undergoing precisely the same history. Outside the horizon (where we can set $k = 0$ in all equations) ζ is just a rescaling of coordinates and this rescaling is a symmetry of the equations.

Other gauges can be more convenient in order to do computations in the slow roll approximation. A gauge that is particularly convenient is

$$\delta\phi \equiv \varphi(t,x) \ , \qquad h_{ij} = e^{2\rho}(\delta_{ij} + \gamma_{ij}) \ , \qquad \partial_i\gamma_{ij} = 0 \ , \qquad \gamma_{ii} = 0 \tag{2.23}$$

where we have denoted the small fluctuation of the scalar field by φ. In order to avoid confusion, from now on ϕ will denote the background value of the scalar field and φ will be its deviation from the background value. We expect that in this gauge the action will be approximately the action of a massless scalar field φ to leading order in slow roll. Indeed, we can check that the first order expressions for N and N^i are

$$N_{1\,\varphi} = \frac{\dot\phi}{2\dot\rho}\varphi \ , \qquad N^i_\varphi = \partial_i\chi \ , \qquad \partial^2\chi = \frac{\dot\phi^2}{2\dot\rho^2}\frac{d}{dt}\left(-\frac{\dot\rho}{\dot\phi}\varphi\right) \tag{2.24}$$

where the φ subindex reminds us that $N_{1\,\varphi}$, N^i_φ are computed in the gauge (2.23). We see that these expressions are subleading in slow roll compared to φ. So in order to compute the quadratic action to lowest order in slow roll it is enough to consider just the $(\nabla\varphi)^2$ term in the action (2.1) since V'' is also of higher order in slow roll. This is just the

[8] For readers who are familiar with Bardeen's classic paper [18], we should mention that the gauge invariant definition of ζ is $\zeta = h + (\quad {}^{1}h' - A) \quad {}^{2}/(\quad {}^{2} - \quad \dot{} \ )$ where $\quad = \rho'$ and primes indicate derivatives with respect to conformal time and $h = H \quad + H_T/3$ with $A,\ H \quad ,\ H_T$ defined in [18]. In circumstances where ζ is conserved then it also reduces to the definition in terms of Bardeen potentials in [5], [16] (actually $\zeta_{here} = -\zeta_{there}$). The gauge choice that makes $h = \zeta$ is $T_i^0 = 0$ or, using the equations of motion, $\dot h = \dot\rho A$.

action of a massless scalar field in the zeroth order background. We can compute the fluctuations in φ in the slow roll approximation and we find a result similar to that of a scalar field in de-Sitter space (2.18) where the Hubble scale is evaluated at horizon crossing. After horizon crossing we can evaluate the gauge invariant quantity ζ. This is most easily done by changing the gauge to the gauge where $\varphi = 0$. This can be achieved by a time reparametrization of the form $\tilde{t} = t + T$ with

$$T = -\frac{\varphi}{\dot{\phi}} \tag{2.25}$$

where t is the time in the gauge (2.8) and $\tilde{t}$ is the time in (2.23). After the gauge transformation (2.25), we find that the metric in (2.23) becomes of the form in (2.8) with

$$\zeta = \dot{\rho} T = -\frac{\dot{\rho}}{\dot{\phi}} \varphi \tag{2.26}$$

Incidentally, this implies that χ in (2.24) is the same as χ in (2.10). So the correlation function for ζ can be computed as the correlation function for φ times the factor in (2.26). In order to get a result as accurate as possible we should perform the gauge transformation (2.26), just after crossing the horizon so that the factor in (2.26), is evaluated at horizon crossing leading finally to (2.20). In principle we could compute ζ from φ at any time. If we were to choose to do it a long time after horizon crossing we would need to take into account that φ changes outside the horizon. This would require evaluating the action (2.1) to higher order in the slow roll parameters. Of course, the dependence for φ outside the horizon is such that it precisely cancels the time dependence of the factor in (2.26) so that ζ is constant.

In summary, the computation is technically simplest if we start with the gauge (2.23) and we compute the two point function of φ after horizon exit and at that time compute the ζ variable which then remains constant. On the other hand the computation in the gauge (2.8) is conceptually simpler since the whole computation always involves the variable of interest which is ζ. In other words, the gauge (2.23) is more useful before and during horizon crossing while the gauge (2.8) is more useful after horizon crossing.

These last few paragraphs are basically simple argument presented in [13]. The computation of fluctuations of φ in de-Sitter produces fluctuations of the order $\varphi = \frac{H}{2\pi}$ and then this leads to a delay in the evolution by $\delta t = -\varphi/\dot{\rho}$ (see (2.25)) which in turn gives an additional expansion of the universe by a factor $\zeta = \dot{\rho}\delta t = -\frac{\dot{\rho}}{\dot{\phi}}\varphi$. This additional expansion is evaluated at horizon crossing in order to minimize the error in the approximation.

We now summarize the discussion of gravitational waves [20]. Inserting (2.8) in the action and focusing on terms quadratic in γ gives

$$S = \frac{1}{8}\int [e^{3\rho}\dot\gamma_{ij}\dot\gamma_{ij} - e^{\rho}\partial_l\gamma_{ij}\partial_l\gamma_{ij}] \tag{2.27}$$

As usual we can expand γ in plane waves with definite polarization tensors

$$\gamma_{ij} = \int \frac{d^3k}{(2\pi)^3}\sum_{s=\pm}\epsilon^s_{ij}(k)\gamma^s_{\vec k}(t)e^{i\vec k\vec x} \tag{2.28}$$

where $\epsilon_{ii} = k^i\epsilon_{ij} = 0$ and $\epsilon^s_{ij}(k)\epsilon^{s'}_{ij}(k) = 2\delta_{ss'}$. So we see that for each polarization mode we have essentially the equation of motion of a massless scalar field. As in our previous discussion, the solutions become constant after crossing the horizon. Computing the correlator just after horizon crossing we get

$$\langle\gamma^s_{\vec k}\gamma^{s'}_{\vec k'}\rangle = (2\pi)^3\delta^3(\vec k + \vec k')\frac{1}{2k^3}\frac{2\dot\rho_*^2}{M_{pl}^2}\delta_{ss'} \tag{2.29}$$

where we reinstated the M_{pl} dependence. We can similarly define the tilt of the gravitational wave spectrum by saying that the correlation function scales as $k^{\;3+n_t}$ where n_t is given by

$$n_t = k\frac{d}{dk}\log\dot\rho_*^2 = -\frac{\dot\phi_*^2}{\dot\rho_*^2} = -2\epsilon \tag{2.30}$$

3. Cubic terms in the agrangian

In this section we compute the cubic terms in the lagrangian in two different gauges. We do this as a check of our computations. The first gauge is similar to (2.8), which is conceptually simpler since one works from the very beginning with the ζ variable in terms of which one wants to compute the answer. We need to fix the gauge to second order in small fluctuations. We achieve this by setting to zero the fluctuations in ϕ and we writing the 3-metric as

$$\begin{aligned}&\delta\phi = 0\\ &h_{ij} = e^{2\rho+2\zeta}\hat h_{ij}\ , \qquad \det\hat h = 1, \qquad \hat h_{ij} = (\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{il}\gamma_{lj} + \cdots)\end{aligned} \tag{3.1}$$

where $\gamma_{ii} = \partial_i\gamma_{ij} = 0$ to second order. The term proportional to γ^2 was introduced with the purpose of simplifying some formulas[9]. Note that it is necessary to define h_{ij} only to second order since any third order term in h_{ij} will not contribute to the action.

[9] We can define the gauge condition as $\partial_i(\log\hat h)_{ij} = 0$.