MODELING THE ARMS RACE

Jonathan Ledesma, Alishan Premani, Khoa Nguyen

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1 Assumptions

We define the following variables:

- $x(t) \geq 0$ is the amount of weaponry that country X has at time t,
- $y(t) \geq 0$ is the amount of weaponry that country Y has at time t,
- $a \geq 0$ is the coefficient of weapon production of country X,
- $b \geq 0$ is the coefficient of weapon production of country Y.

We restrict this model to a specific time interval in which a and b stay constant, even though in a real-world situation, the production rates might fluctuate.

2 Simple Model

3 A More Realistic Model

Now we consider the situation when both countries decide to update their inventory of weaponry due to new technological advances and old weapons being expired. Therefore, we define additional parameters in the differential equations:

$$\frac{dx}{dt} = ay - cx \tag{1}$$

$$\frac{dy}{dt} = bx - dy \tag{2}$$

where $c \geq 0$ and $d \geq 0$ are constant coefficients of the destruction of old weapons in countries X and Y, respectively.

We can solve this system of differential equations by constructing a coefficient matrix M from equations (1) and (2) as follows

$$\mathbf{Y}(t) = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} -c & a \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$M = \begin{bmatrix} -c & a \\ b & d \end{bmatrix}.$$

Our characteristic polynomial is $\lambda^2 + (d+c)\lambda + cd - ab$. Computing the eigenvalues and eigenvectors yields

$$\lambda_1 = -\frac{1}{2}d - \frac{1}{2}c + \frac{1}{2}\sqrt{4ab - 2cd + c^2 + d^2}, \quad \vec{v}_1 = \begin{bmatrix} \frac{a}{\lambda_1} \\ 1 \end{bmatrix} \tag{3}$$

$$\lambda_2 = -\frac{1}{2}d - \frac{1}{2}c - \frac{1}{2}\sqrt{4ab - 2cd + c^2 + d^2}, \quad \vec{v}_2 = \begin{bmatrix} \frac{a}{\lambda_2} \\ 1 \end{bmatrix} \tag{4}$$

We now discuss the different possible cases which can occur during the arms race in terms by using the above eigenvalues.

3.1 Case 1 - Two Real Distinct Eigenvalues

We rewrite the equation for λ as follows

$$\lambda = -\frac{1}{2}(d+c) \pm \frac{1}{2}\sqrt{4ab - 2cd + c^2 + d^2} \tag{5}$$

To have two real distinct eigenvalues, we need the discriminant in equation (5) to be greater than zero

$$4ab - 2cd + c^2 + d^2 > 0, \tag{6}$$

which implies that $4ab + c^2 + d^2 > 2cd$.

Note that $c^2 + d^2 > 2cd$ only when $c \neq d$.

3.2 Case 2 - Repeated Eigenvalues

3.3 Case 2 - Complex Eigenvalues

4 Case Study: Cold War

We consider the arm race between the United States and the USSR during the Cold War. Due to limited time and resources, we define the term “weaponry” in this case to only include nuclear warheads. We also exclude other external factors such as internal political conflicts in each country, current wars at the time, and treaties, that would effect the production rate of weaponry in each country.

5 Reference