MODELING THE ARMS RACE

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Introduction

Every country is concerned about its national security. Maintaining an inventory of weaponry is one of the priorities in defense, but how a country does it depends on not only its own inventory, but also on other factors, such as technological advances, other countries's weaponry, and the tension among them. In this paper we seek to create a simple differential equations model to investigate the change in weaponry of two countries in relation to one another, and to improve our model by factoring another variable into our equations. Then we apply our model to the Cold War between the United States and the USSR where there was an arm race between the two countries. Due to limited time and resources, as well as techniques and skills in solving differential equations, our model has a lot of room for improvement, considering the number of variables not present in our equations.

1 Assumptions

We define the following variables:

- $x(t) \geq 0$ is the amount of weaponry that country X has at time t,
- $y(t) \geq 0$ is the amount of weaponry that country Y has at time t,
- $a \geq 0$ is the coefficient of new weapon production of country X,
- $b \geq 0$ is the coefficient of new weapon production of country Y.

We restrict this model to a specific time interval in which a and b stay constant, even though in a real-world situation, the production rates might fluctuate.

2 Simple Model

3 A More Realistic Model

Now we consider the situation where both countries decide to update their inventory of weaponry due to new technological advances and expired old weapons. Therefore, we define additional parameters in the differential equations:

$$\begin{cases} \frac{dx}{dt} = & ay - cx \\ \frac{dy}{dt} = & bx - dy \end{cases}$$

where $c \geq 0$ and $d \geq 0$ are constant coefficients of the destruction of old weapons in countries X and Y, respectively.

We can solve this system of differential equations by constructing a coefficient matrix M

from equations (3) and (3) as follows

$$\mathbf{Y}(t) = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} -c & a \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

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$$M = \begin{bmatrix} -c & a \\ b & -d \end{bmatrix}.$$

Our characteristic polynomial is $\lambda^2 + (c+d)\lambda + cd - ab$. Computing the eigenvalues and eigenvectors yield

$$\lambda_1 = -\frac{1}{2}d - \frac{1}{2}c + \frac{1}{2}\sqrt{4ab - 2cd + c^2 + d^2}; \quad \vec{v}_1 = \begin{bmatrix} \frac{a}{\lambda_1} \\ 1 \end{bmatrix}$$

$$\lambda_2 = -\frac{1}{2}d - \frac{1}{2}c - \frac{1}{2}\sqrt{4ab - 2cd + c^2 + d^2}, \quad \vec{v}_2 = \begin{bmatrix} \frac{a}{\lambda_2} \\ 1 \end{bmatrix}$$

We now discuss the different possible cases which can occur during the arms race in terms by using the above eigenvalues.

3.1 Case 1 - Two Real Distinct Eigenvalues

We rewrite the equation for λ as follows

$$\lambda = -\frac{1}{2}(d+c) \pm \frac{1}{2}\sqrt{4ab - 2cd + c^2 + d^2} \tag{1}$$

To have two real distinct eigenvalues, we need the discriminant in equation (1) to be greater than zero

$$4ab - 2cd + c^2 + d^2 > 0, \tag{2}$$

which implies that $4ab + c^2 + d^2 > 2cd$.

Note that $c^2 + d^2 > 2cd$ only when $c \neq d$.

3.2 Case 2 - Repeated Eigenvalues

3.3 Case 2 - Complex Eigenvalues

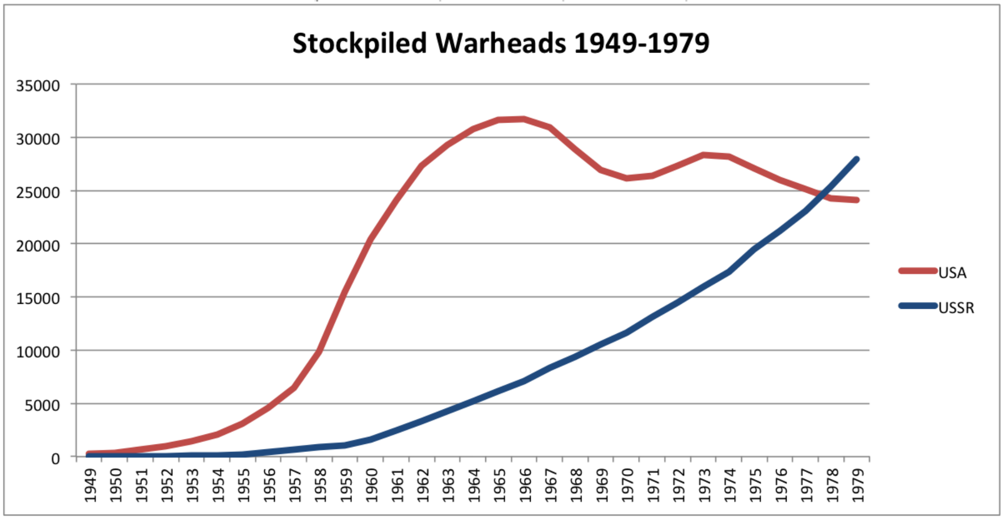
4 Case Study: Cold War

We consider the arm race between the United States and the USSR during the Cold War between 1949 and 1979. Due to limited time and resources, we define the term "weaponry" in this case to only include nuclear warheads. We also exclude other external factors such as internal political conflicts in each country, current wars at the time, and treaties, that would effect the production and destruction rates of weaponry in each country.

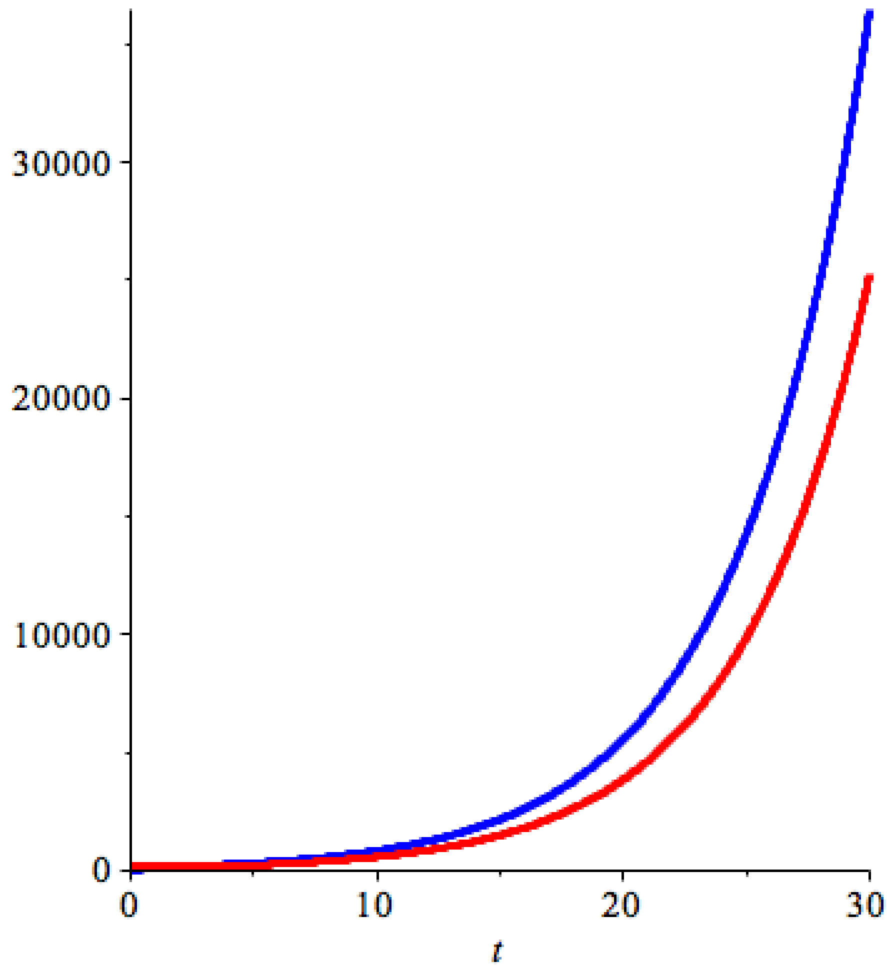
Let the United States be our country X and the USSR be our country Y in the above models. Then $x(t)$ represents the amount of warheads the United States had and $y(t)$ represents the amount of warheads the USSR had.

The number of stockpiled warheads of the United States and the USSR in this period of time is given and illustrated by the following table and graph:

Stockpiled Warheads 1949-1979		
Year	USA	USSR
1949	235	1
1950	369	5
1951	640	25
1952	1005	50
1953	1436	120
1954	2063	150
1955	3057	200
1956	4618	426
1957	6444	660
1958	9822	869
1959	15468	1060
1960	20434	1605
1961	24111	2471
1962	27297	3322
1963	29249	4238
1964	30751	5221
1965	31642	6129
1966	31700	7089
1967	30893	8339
1968	28884	9399
1969	26910	10538
1970	26119	11643
1971	26365	13092
1972	27296	14478
1973	28335	15915
1974	28170	17385
1975	27052	19443
1976	25956	21205
1977	25099	23044
1978	24243	25393
1979	24107	27935



Through a lot of trial-and-error, we find that $a = 10/23$, $b = 14/25$, $c = 11/25$, and $d = 5/25$ give the best approximation of $x(t)$ and $y(t)$ given the actual data. The graphs of theoretical $x(t)$ and $y(t)$ versus t is below, where the red curve is $x(t)$ for the United States, the blue curve is $y(t)$ for the USSR, and the initial condition is $x(0) = 235$ and $y(0) = 1$:



The shape of the actual and theoretical graphs are relatively similar for $y(t)$, but there are definitely some discrepancies for $x(t)$. The gap between theoretical predictions and actual data is understandable because our model only considers the change depending on the two countries' inventories of warheads, while in the real world the number of warheads can also change because of wars, resources, policies, treaties, and so forth.

5 Conclusion

In this paper we consider the change in weaponry inventory of two countries when they are in an arms race, which makes the amount of weaponry that one country has influence the amount that the other has. We also build slightly more realistic model where each country gets rid of a portion of its current weapon inventory. The model does not do very well when compared with actual data from the Cold War because there are so many other factors in the Cold War not considered in our model. We need more time and resources to address these variables to improve our model.

References

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