MODELING AN ARMS RACE

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Introduction

Every country is concerned about its national security. Maintaining an inventory of weaponry is one of the priorities in defense, but how a country does it depends on not only its own inventory, but also on other factors, such as technological advances, other countries' inventories of weaponry, and the tension between them. In this paper we seek to create a simple model of differential equations to investigate the change in weaponry of two countries in relation to one another, and to improve our model by factoring another variable into our equations. Then we apply our model to the Cold War between the United States and the USSR where there was an arms race between the two countries. Due to limited time and resources, as well as techniques and skills in solving differential equations, our model has a lot of room for improvement, considering the number of variables not present in our equations.

1 Assumptions

We define the following variables:

- $x(t) \geq 0$ is the amount of weaponry that country X has at time t,
- $y(t) \geq 0$ is the amount of weaponry that country Y has at time t,
- $a \geq 0$ is the coefficient of new weapon production of country X,
- $b \geq 0$ is the coefficient of new weapon production of country Y.

We restrict this model to a specific time interval in which a and b stay constant, even though in a real-world situation, the production rates might fluctuate.

2 Simple Model

The simple model looks at the production of weaponry for each country solely dependent on the other country's amount of weapons. Without considering other factors, a country will only look at the other country's rate of production and number of weapons at time t. The simple model can be defined as:

$$\begin{cases} \frac{dx}{dt} = & ay \\ \frac{dy}{dt} = & bx. \end{cases}$$

In this model a and b are the rates of production and both x and y are the amount of weapons the other country holds. From this system of equations we create the matrix,

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}.$$

From this matrix we obtain the characteristic polynomial and our eigenvalues,

$$\lambda = \pm\sqrt{ab}. \tag{1}$$

Thus we get the general solution for y,

$$y(t) = k_1 e^{\sqrt{ab}t} + k_2 e^{-\sqrt{ab}t}. \tag{2}$$

Finally plugging in our general solution for y into $\frac{dx}{dt}$ we obtain the general solution for x,

$$x(t) = \sqrt{\frac{a}{b}}(k_1 e^{\sqrt{ab}t} - k_2 e^{-\sqrt{ab}t}). \tag{3}$$

We now plug in the initial conditions $x(0) = 235$ and $y(0) = 1$ with rates of production $a = \frac{10}{23}$ and $b = \frac{14}{25}$. Section 4 will explain the choice of this particular initial condition and these parameters. Solving for the constants k_1 and k_2, we get $k_1 \approx 104$ and $k_2 \approx -103$. Thus the solution for this initial value problem is:

$$\begin{cases} x(t) = \sqrt{\frac{125}{161}}(104e^{\sqrt{\frac{28}{115}}t} + 103e^{-\sqrt{\frac{28}{115}}t}) \\ y(t) = 104e^{\sqrt{\frac{28}{115}}t} - 103e^{-\sqrt{\frac{28}{115}}t}. \end{cases}$$

In the next section, we observe a model that accounts for more factors.

3 A More Realistic Model

Now we consider the situation where both countries decide to update their inventory of weaponry due to new technological advances and expired old weapons. Therefore, we define additional parameters in the differential equations:

$$\begin{cases} \frac{dx}{dt} = & ay - cx \\ \frac{dy}{dt} = & bx - dy \end{cases}$$

where $c \geq 0$ and $d \geq 0$ are constant coefficients of the destruction of old weapons in countries X and Y, respectively.

We can solve this system of differential equations by constructing a coefficient matrix M as follows

$$\mathbf{Y}(t) = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} -c & a \\ b & -d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad M = \begin{bmatrix} -c & a \\ b & -d \end{bmatrix}.$$

where $c \geq 0$ and $d \geq 0$ are constant coefficients of the destruction of expired weapons in countries X and Y, respectively. Our equilibrium solution is (0,0), when both the countries' amounts of weaponry are zero. When $Det(M) = cd - ab$, we have infinitely many equilibrium points along a straight line. Note that the trace $Tr(M) \leq 0$ because of our assumptions that $c \geq 0$ and $d \geq 0$.

Our characteristic polynomial is $\lambda^2 + (d + c)\lambda + cd - ab$. We use Maple to compute the eigenvalues and eigenvectors, yielding

$$\lambda_1 = -\frac{1}{2}d - \frac{1}{2}c + \frac{1}{2}\sqrt{4ab - 2cd + c^2 + d^2}, \quad \vec{v}_1 = \begin{bmatrix} \frac{a}{\lambda_1} \\ 1 \end{bmatrix}, \tag{4}$$

$$\lambda_2 = -\frac{1}{2}d - \frac{1}{2}c - \frac{1}{2}\sqrt{4ab - 2cd + c^2 + d^2}, \quad \vec{v}_2 = \begin{bmatrix} \frac{a}{\lambda_2} \\ 1 \end{bmatrix}. \tag{5}$$

We now discuss the different possible cases which can occur during the arms race by using the above eigenvalues and eigenvectors. These cases are in terms of various combinations of rates of production and destruction of weaponry and potential relations between the two countries.

3.1 Case 1 - Two Real Distinct Eigenvalues

We rewrite the equation for λ as follows

$$\lambda = -\frac{1}{2}(d + c) \pm \frac{1}{2}\sqrt{4ab - 2cd + c^2 + d^2}. \tag{6}$$

Since $Tr(M) \leq 0$, we have

- a saddle when $Det(M) = cd - ab < 0$, implying that $cd < ab$. This is when one country has higher and lower production and destruction rates respectively than the other, leading it to win the race.

- a sink when $Det(M) = cd - ab > 0$, implying that $cd > ab$. This is when the product of both the countries' rates of destruction of weapons is greater than the product of their rates of production. This might happen when, for instance, both countries decide to either end the arms race for whatever reason, or shift the race to another advanced class of new weaponry.

To have two real distinct eigenvalues, we need the discriminant in equation (6) to be greater than zero

$$4ab - 2cd + c^2 + d^2 > 0, \tag{7}$$

which implies that $4ab + c^2 + d^2 > 2cd$.

We break this down further into separate cases.

1. The first sub-case is where $c = d$, which gives us $4ab > 0$. This implies that $a \neq 0$ and $b \neq 0$ for the inequality to hold true. Therefore, when both countries destroy their own

arms at equal rates, they both must have non-zero production rates for arms. This is because we assume both countries to be competing to have more arms than the other. So if both countries destroy their weapons at equal rates, then these countries must produce at least some weapons in order to stay competitive.

2. The second sub-case is where $c \neq d$ (both countries destroy their weapons at different rates), which implies that $c^2 + d^2 > 2cd$. Thus, it must hold true that $a \geq 0$ and $b \geq 0$, which is our assumption. We can infer that in this case, we can expect both countries' rates of production to be flexible in that either one or both of them may not be producing any weapons at any given time. Since it must be true that one of the countries rates of destruction is less than the others' when $c \neq d$, then that country would not be as worried as the other country to produce more weapons to stay competitive.

The general solution for this case is

$$\mathbf{Y}(t) = k_1 e^{\lambda_1 t} \vec{v}_1 + k_2 e^{\lambda_2 t} \vec{v}_2, \tag{8}$$

where k_1 and k_2 are constants depending on the initial condition.

3.2 Case 2 - Repeated Eigenvalues

To have repeated eigenvalues, we must have our eigenvalue be in the form $\lambda_r = -\frac{1}{2}(d+c)$ from equation (6). Note that we can have only non-positive repeated eigenvalues since $a \geq 0$ and $b \geq 0$. We can only have a sink, where both countries tend to have a net decrease in the amount of weaponry they possess.

The discriminant in equation (6) must be equal to zero

$$4ab - 2cd + c^2 + d^2 = 0, \tag{9}$$

which implies that $4ab + c^2 + d^2 = 2cd$. Again, we break this into further branches.

1. The first case is where $c = d$, which implies that $c^2 + d^2 = 2cd$. Thus at least one of a and b must be zero, which means that at least one country's production must be zero. Since both countries have a net decrease in the amount of weaponry they have, it is logical to predict at least one country to not produce any weaponry at all.

2. The second sub-case where either $c = 0$ or $d = 0$ but not both, since $c \neq d$. For example, if $c = 0$, then $4ab = -d^2$, which is not possible because of our assumptions.

Therefore, it is not possible to have this case because both countries must have a non-zero rate of destruction, since there is a net decrease in their respective amounts of weaponry, as noted earlier.

The general solution for this case is

$$\mathbf{Y}(t) = e^{\lambda_r t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{\lambda_r t} \begin{bmatrix} -c - \lambda_r & a \\ b & -d - \lambda_r \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \tag{10}$$

where x_0 and y_0 are the initial conditions.

3.3 Case 3 - Complex Eigenvalues

For complex eigenvalues to occur, we must have the discriminant to be less than zero

$$4ab - 2cd + c^2 + d^2 < 0, \tag{11}$$

implying that $4ab + (c - d)^2 < 0$. However, due to our assumptions that $a, b, c, d \geq 0$, this can never occur.

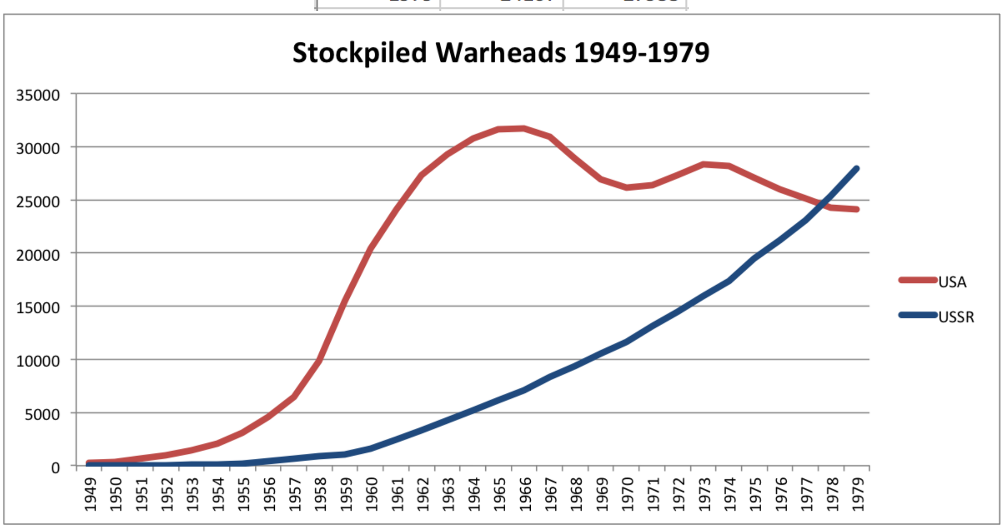
4 Case Study: Cold War

We consider the arms race between the United States and the USSR during the Cold War between 1949 and 1979. Due to limited time and resources, we define the term "weaponry" in this case to only include nuclear warheads. We also exclude other external factors such as internal political conflicts in each country, current wars at the time, and treaties, that would effect the production and destruction rates of weaponry in each country.

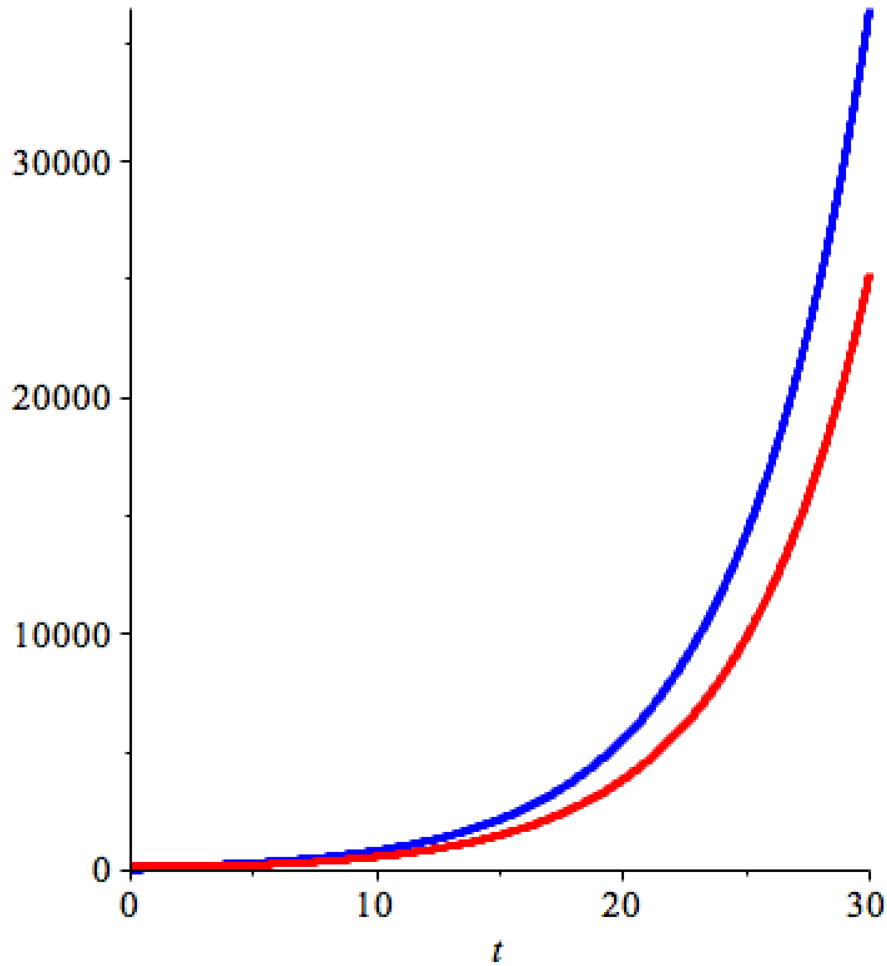
Let the United States be our country X and the USSR be our country Y in the above models. Then $x(t)$ represents the amount of warheads the United States had and $y(t)$ represents the amount of warheads the USSR had.

The number of stockpiled warheads of the United States and the USSR in this period of time is given and illustrated by the following table and graph:

Stockpiled Warheads 1949-1979		
Year	USA	USSR
1949	235	1
1950	369	5
1951	640	25
1952	1005	50
1953	1436	120
1954	2063	150
1955	3057	200
1956	4618	426
1957	6444	660
1958	9822	869
1959	15468	1060
1960	20434	1605
1961	24111	2471
1962	27297	3322
1963	29249	4238
1964	30751	5221
1965	31642	6129
1966	31700	7089
1967	30893	8339
1968	28884	9399
1969	26910	10538
1970	26119	11643
1971	26365	13092
1972	27296	14478
1973	28335	15915
1974	28170	17385
1975	27052	19443
1976	25956	21205
1977	25099	23044
1978	24243	25393
1979	24107	27935



Through a lot of trial-and-error, we find that $a = 10/23$, $b = 14/25$, $c = 11/25$, and $d = 1/5$ give the best approximation of $x(t)$ and $y(t)$ given the actual data. The graphs of theoretical $x(t)$ and $y(t)$ versus t are below, where the red curve is $x(t)$ for the United States, the blue curve is $y(t)$ for the USSR, and the initial condition is $x(0) = 235$ and $y(0) = 1$:



The shape of the actual and theoretical graphs are relatively similar for $y(t)$, but there are definitely some discrepancies for $x(t)$. The gap between theoretical predictions and actual data is understandable because our model only considers the change depending on the two countries' inventories of warheads, while in the real world the number of warheads can also change because of wars, resources, policies, treaties, and so forth.

5 Conclusion

In this paper we consider the change in weaponry inventory of two countries when they are in an arms race, which makes the amount of weaponry that one country has influence the amount that the other has. We also build slightly more realistic model where each country gets rid of a portion of its current weapon inventory. The model does not do very well when compared with actual data from the Cold War because there are so many other factors in the Cold War not considered in our model. We need more time and resources to address these issues in our model.

References

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