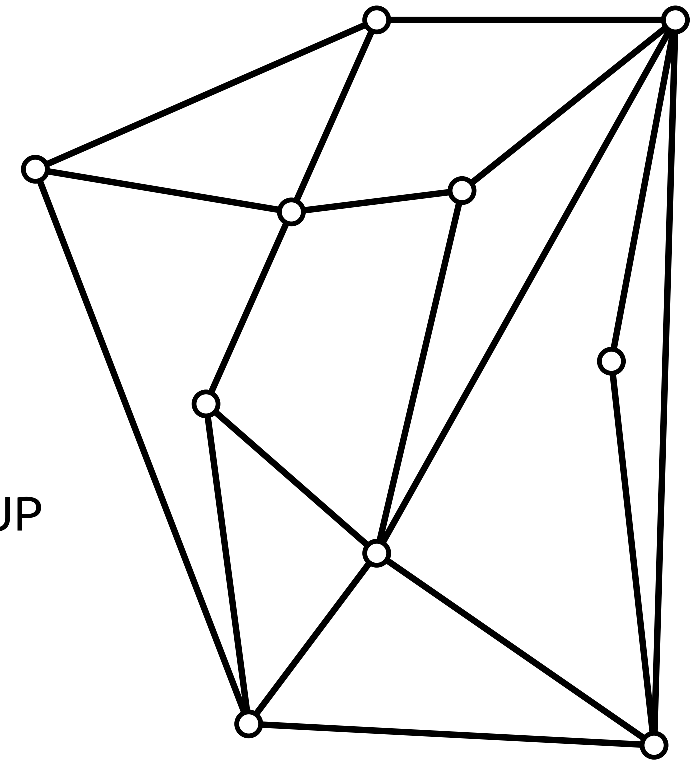
Straight-Line Planar Graph Drawing – Part 2

Lecture Graph Drawing Algorithms · 192.053

Martin Nöllenburg

24.04.2018

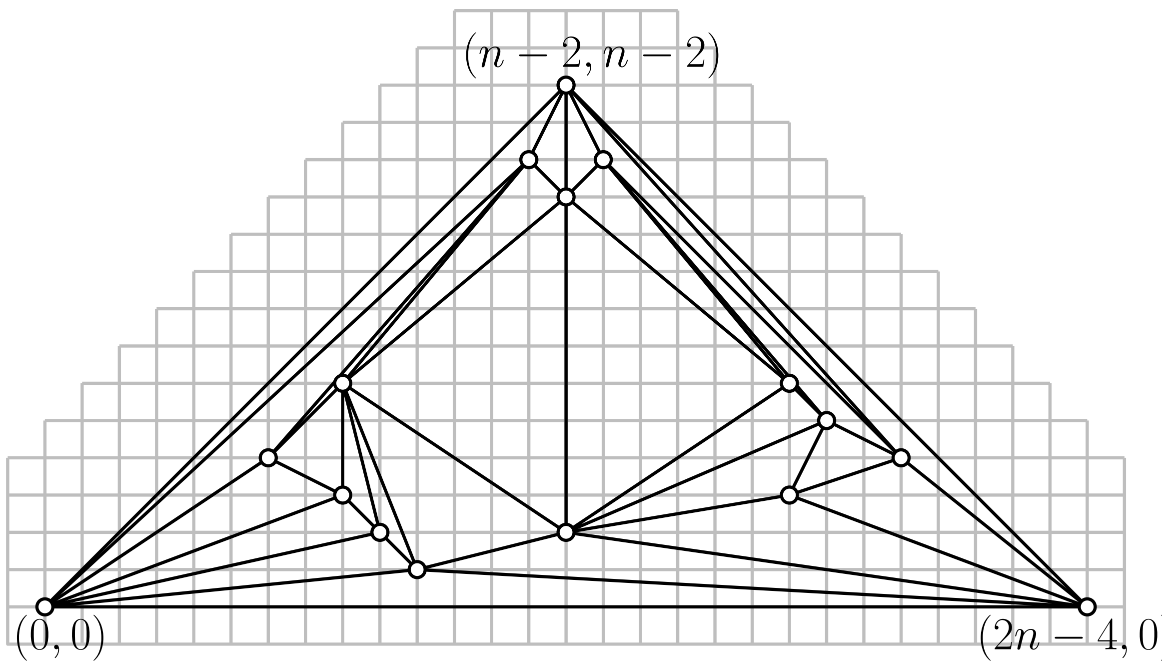
ALGORITHMS AND
COMPLEXITY GROUP



Last Week

Theorem: Every n-vertex embedded planar graph $G = (V, E)$ has a straight-line planar drawing on a grid of size $(2n - 4) \times (n - 2)$. [de Fraysseix, Pach, Pollack 1988]

Theorem: The corresponding shift algorithm can be implemented to run in $O(n)$ time. [Chrobak, Payne 1995]



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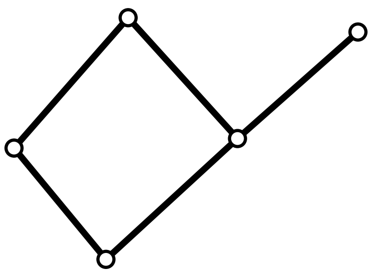

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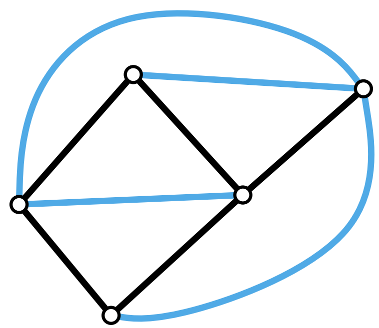

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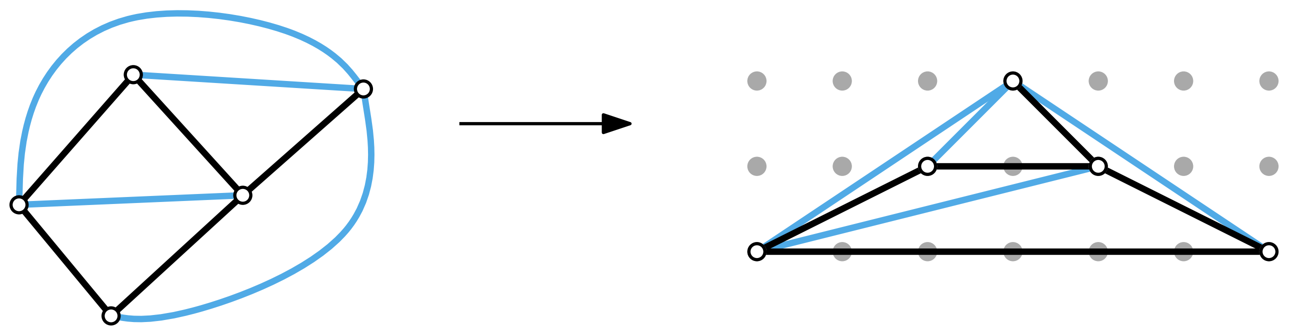

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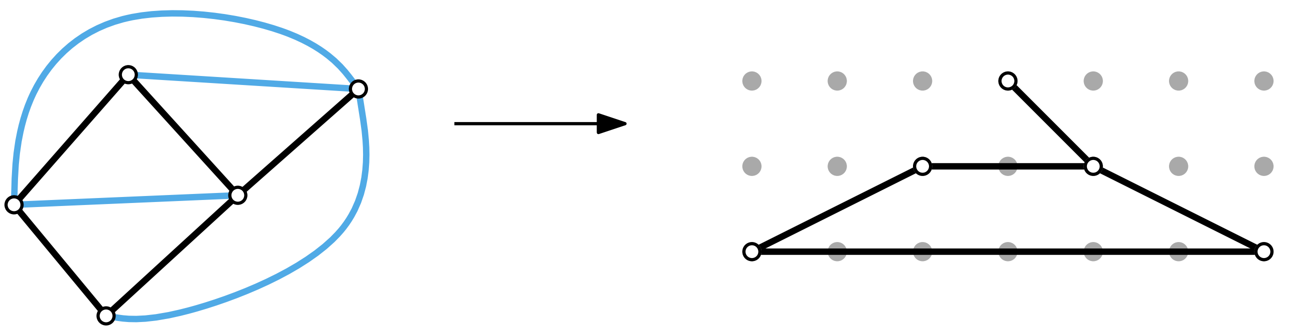

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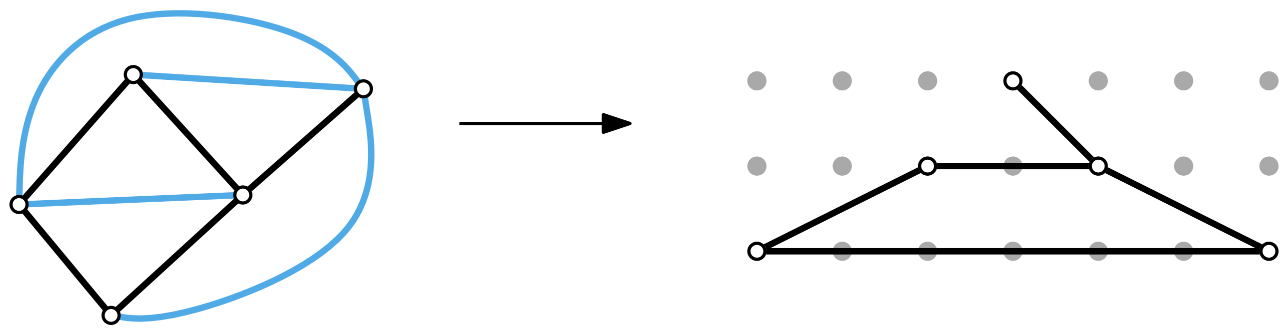

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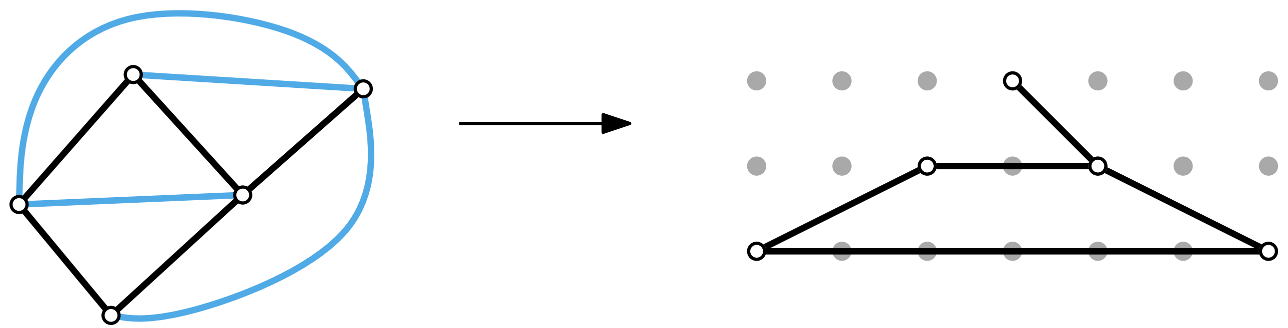

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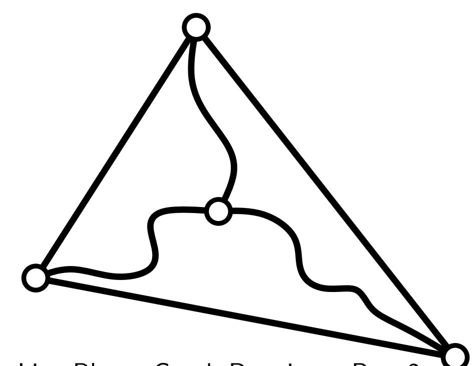
Today: direct assignment of coordinates based on combinatorial properties of planar drawings



Overview

Barycentric representations

Schnyder labeling

Schnyder realizer

Planar straight-line drawings

Barycentric Coordinates

Def: For three points $A, B, C \in \mathbb{R}^2$ and a point P in the triangle $\triangle ABC$, a triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3_{\geq 0}$ with

- $\alpha + \beta + \gamma = 1$
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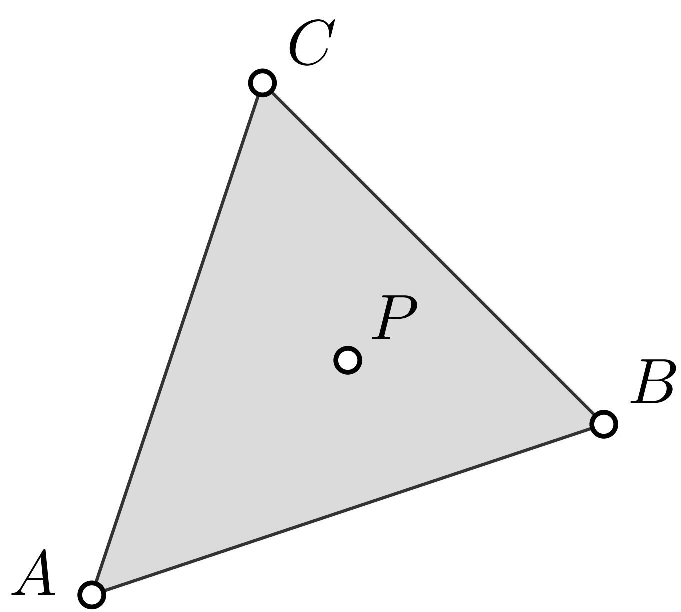

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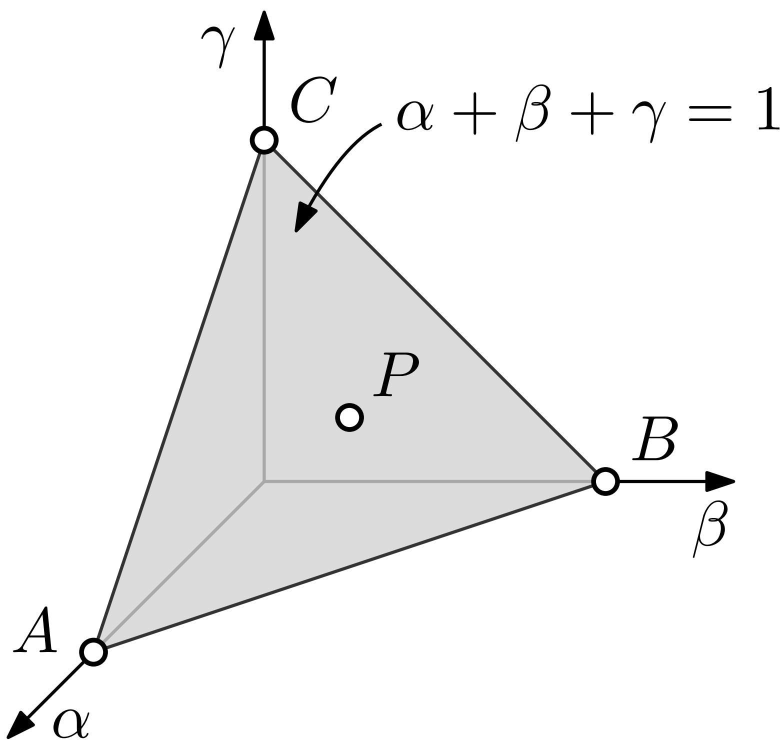

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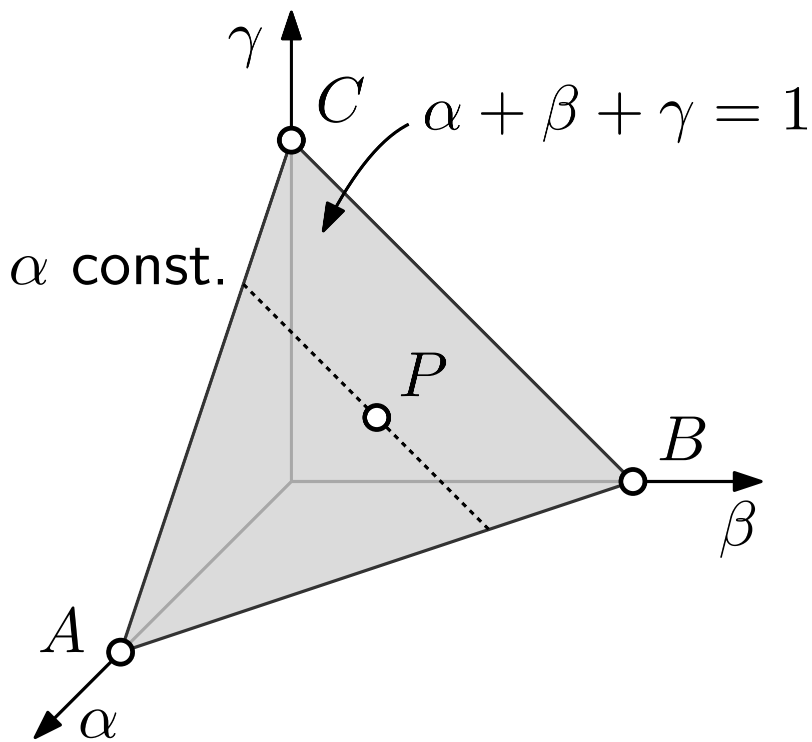

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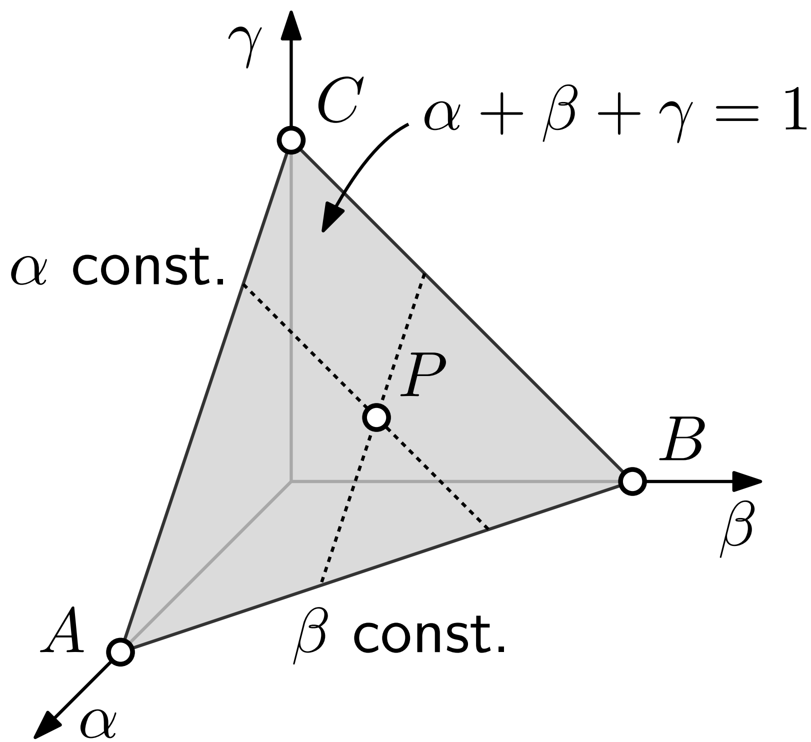

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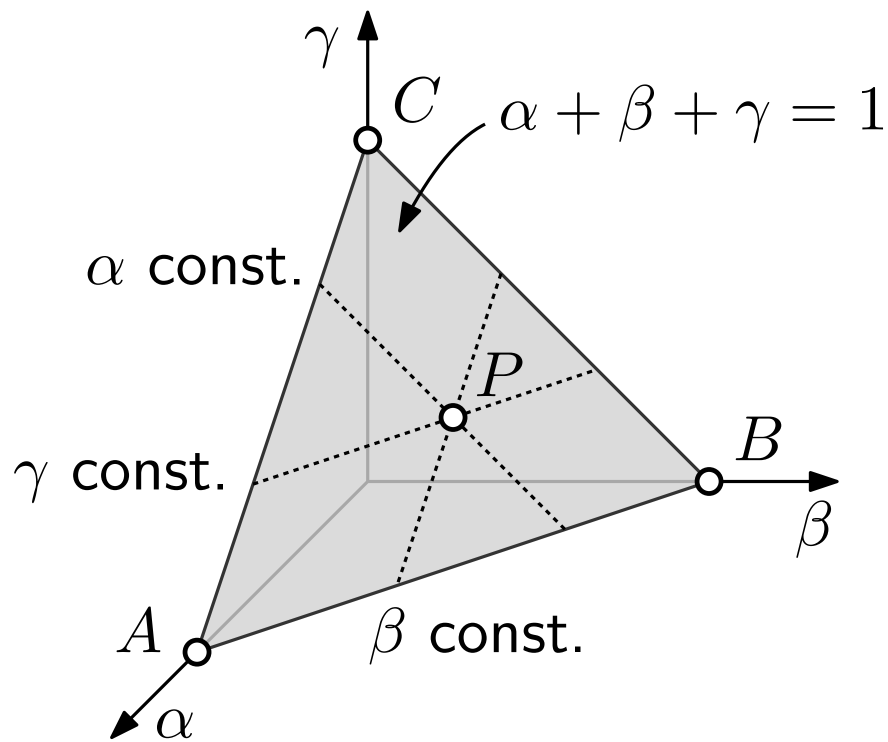

Barycentric Representation

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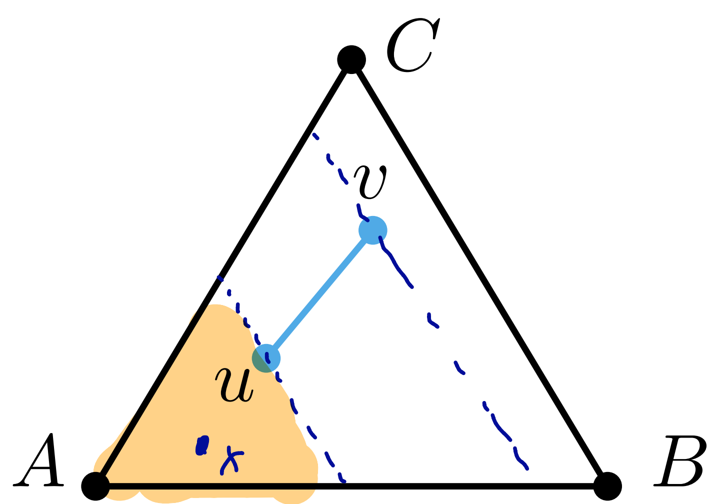

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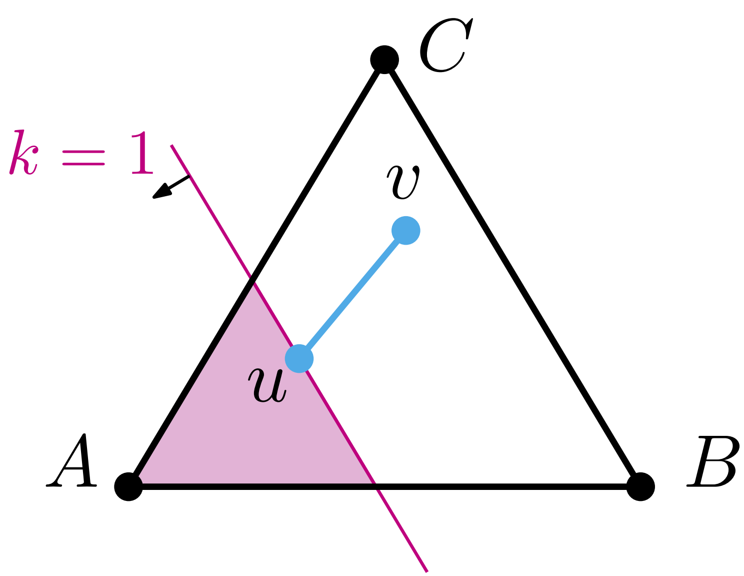

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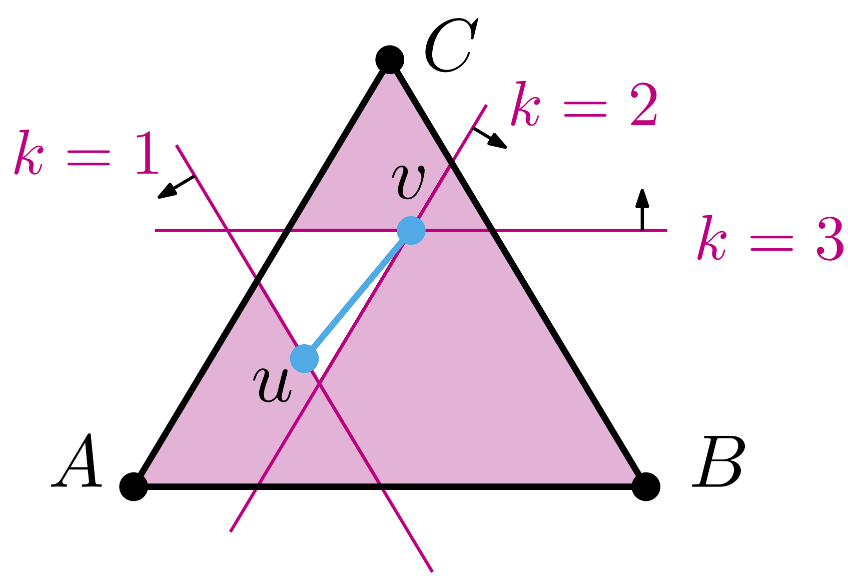

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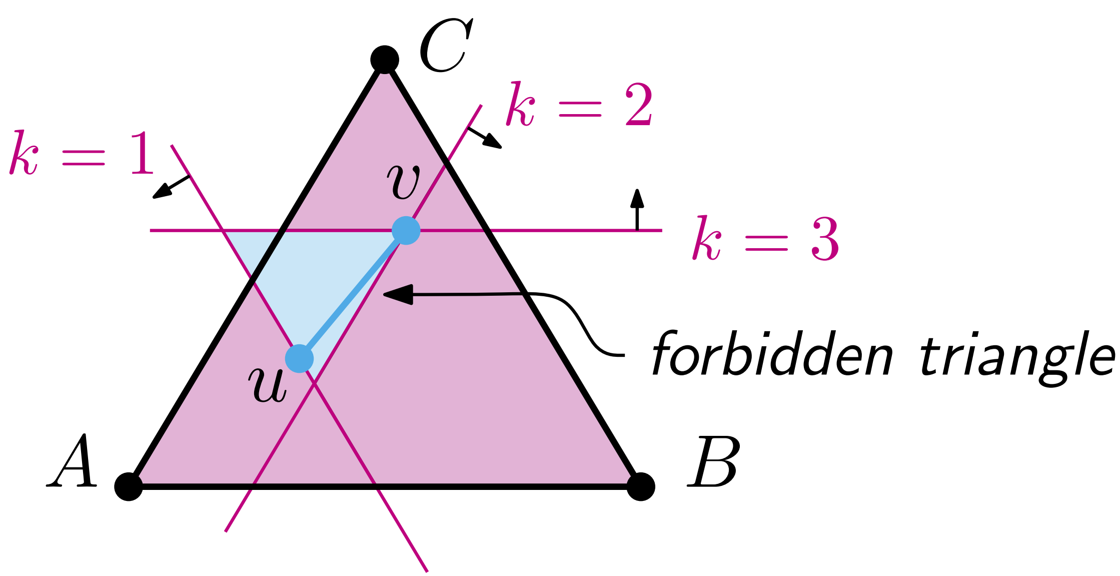

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Variation

Def: A **weak barycentric representation** of a graph $G = (V, E)$ is an *injective* map $v \in V \mapsto (v_1, v_2, v_3) \in \mathbb{R}^3_{\geq 0}$ with

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- for every edge $(u, v) \in E$ and every $x \in V \setminus \{u, v\}$ there exists $k \in \{1, 2, 3\}$ with
  $(u_k < x_k$ or $u_k = x_k \wedge u_{k+1} < x_{k+1})$ and
  $(v_k < x_k$ or $v_k = x_k \wedge v_{k+1} < x_{k+1})$.

lexicographic order

Planar Drawings

Lemma: Let $v \in V \mapsto (v_1, v_2, v_3) \in \mathbb{R}^3_{\geq 0}$ be a barycentric representation of a graph $G = (V, E)$ and let $A, B, C \in \mathbb{R}^2$ in general position. Then the mapping $f\colon v \in V \mapsto v_1 A + v_2 B + v_3 C$ yields a planar straight-line drawing of G inside $\triangle(A, B, C)$.

not collinear

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Proof:

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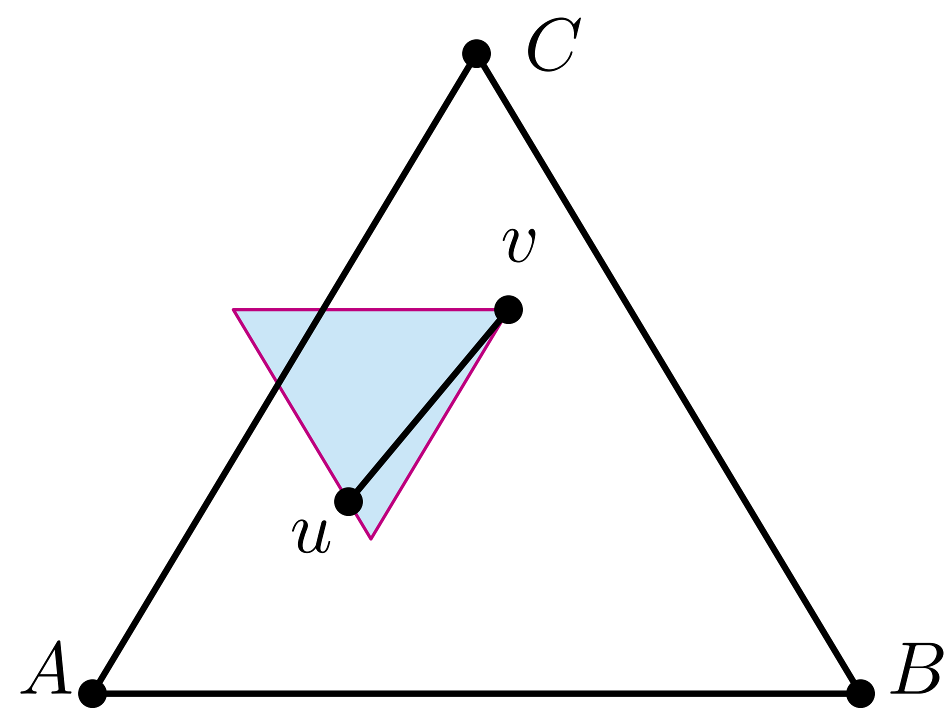

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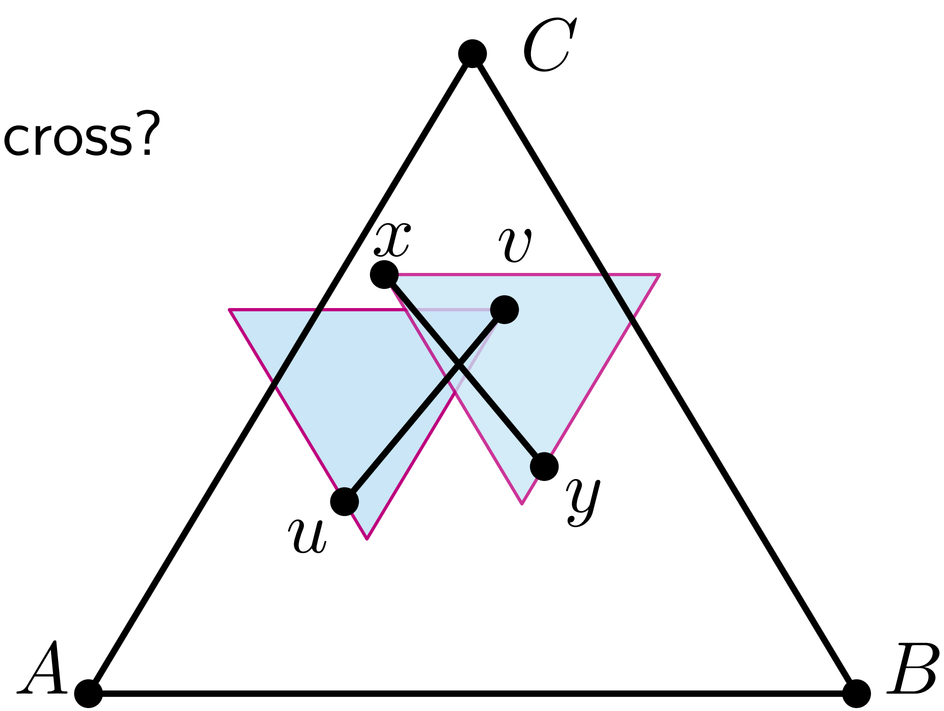

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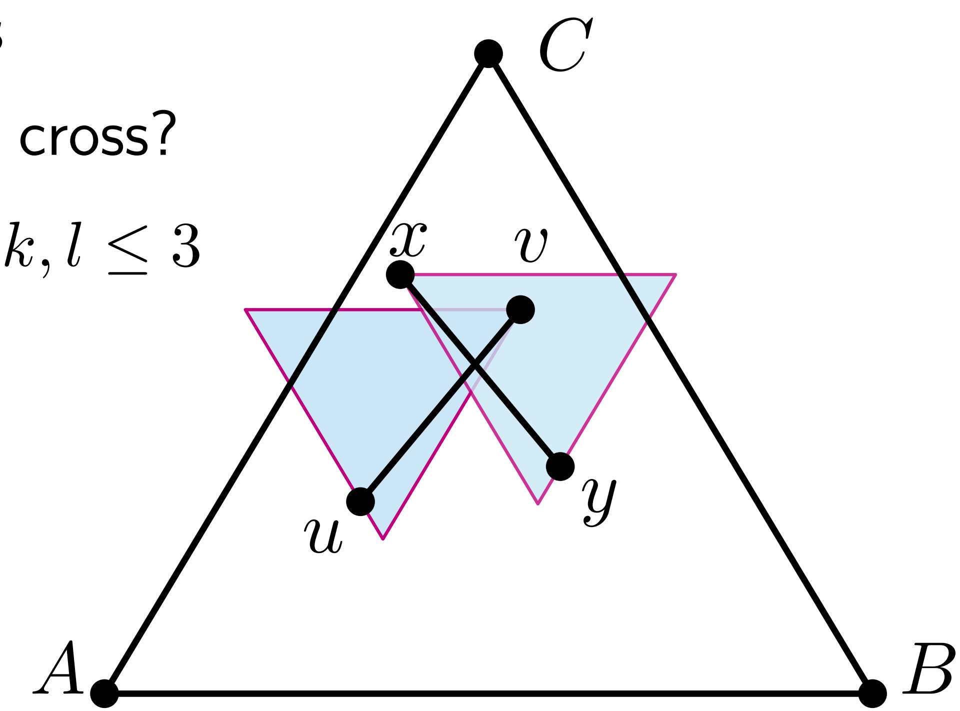

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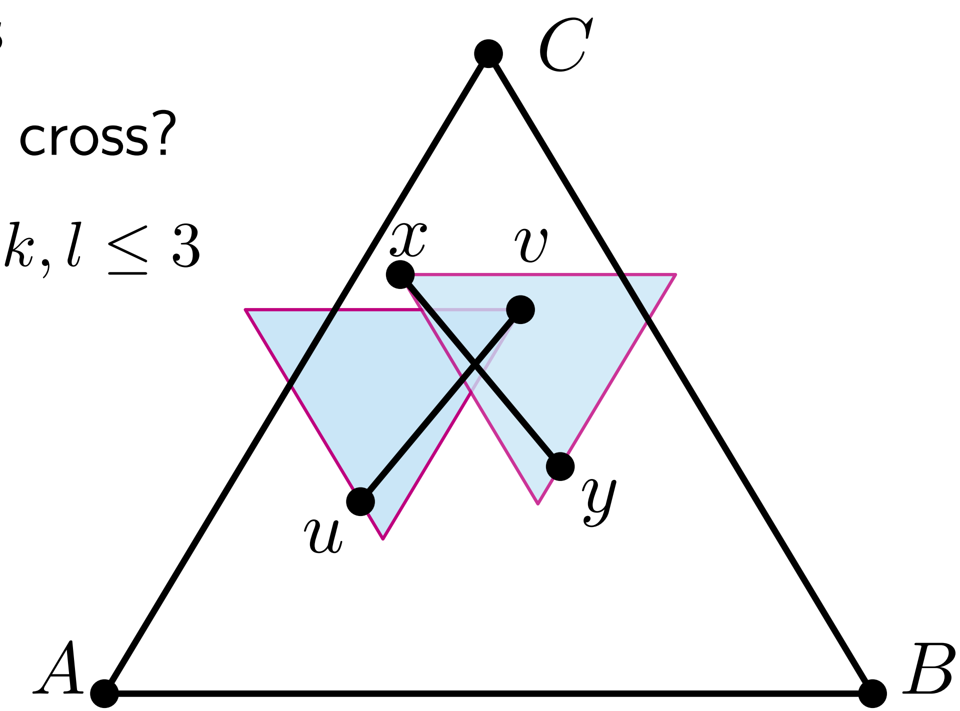

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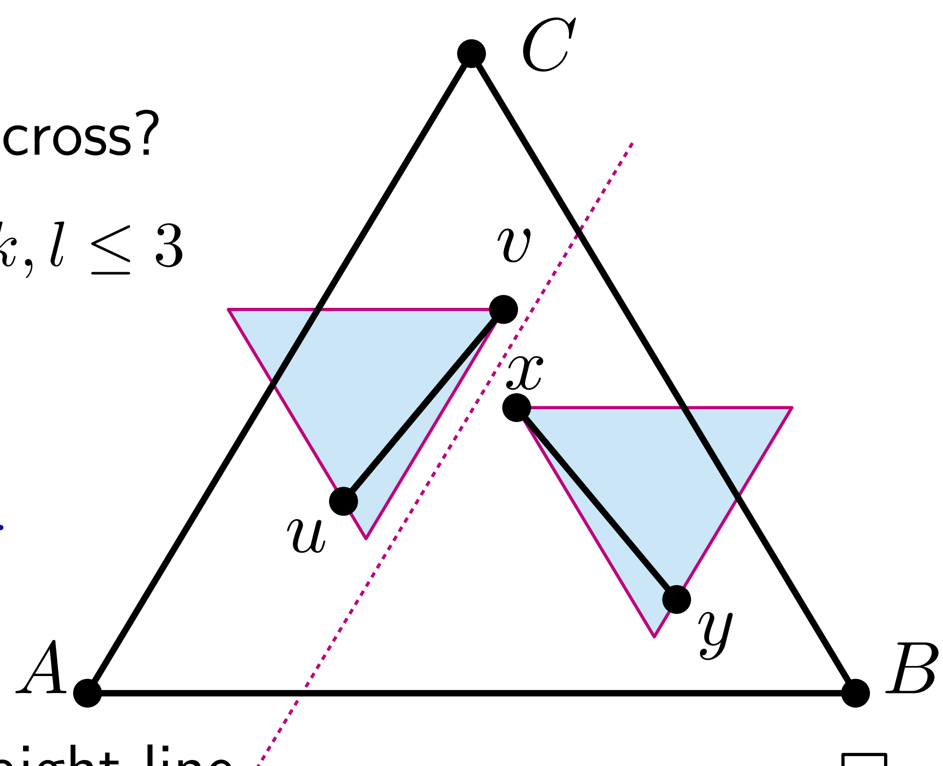
other cases are analogous

assume $i = j = 2 \Rightarrow x_2, y_2 > u_2, v_2$

$\Rightarrow (u, v)$ and (x, y) are separated by straight line



$\square$

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What do we need to apply the lemmas for our purpose?

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Schnyder realizer

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Schnyder Labeling

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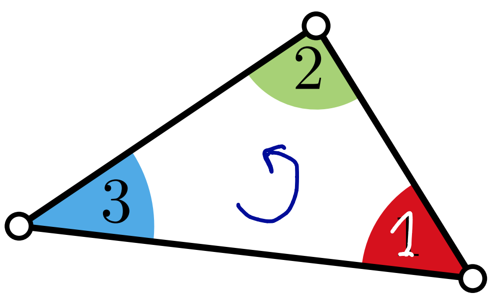
- f a**ce** each triangle contains all three labels 1, 2, 3 in counterclockwise order



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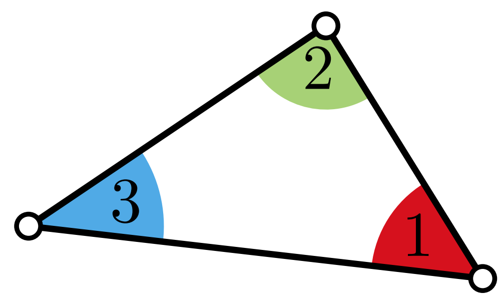
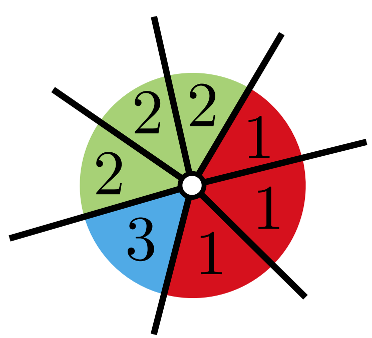
- vertex around each internal vertex labels 1, 2, 3 form non-empty contiguous intervals in counterclockwise order





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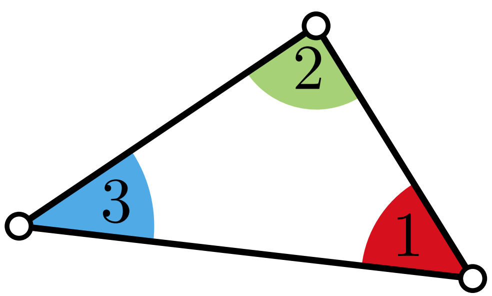

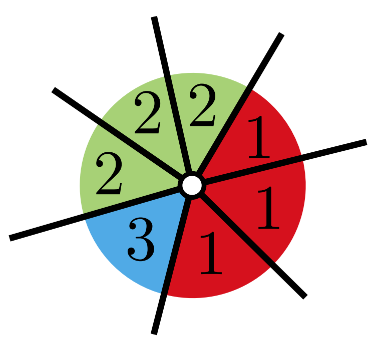

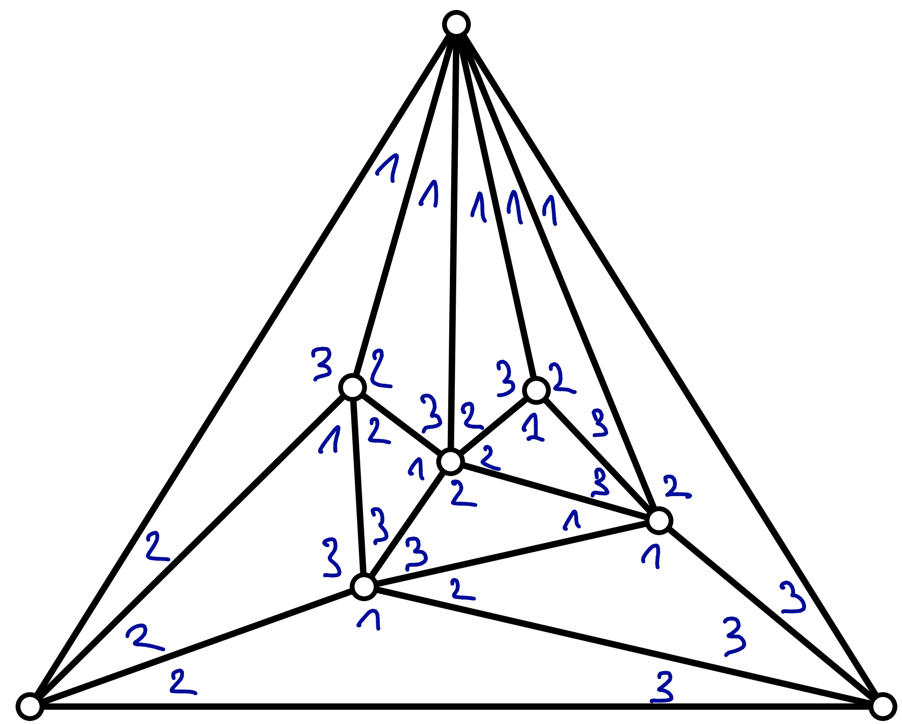

Edge Contractions

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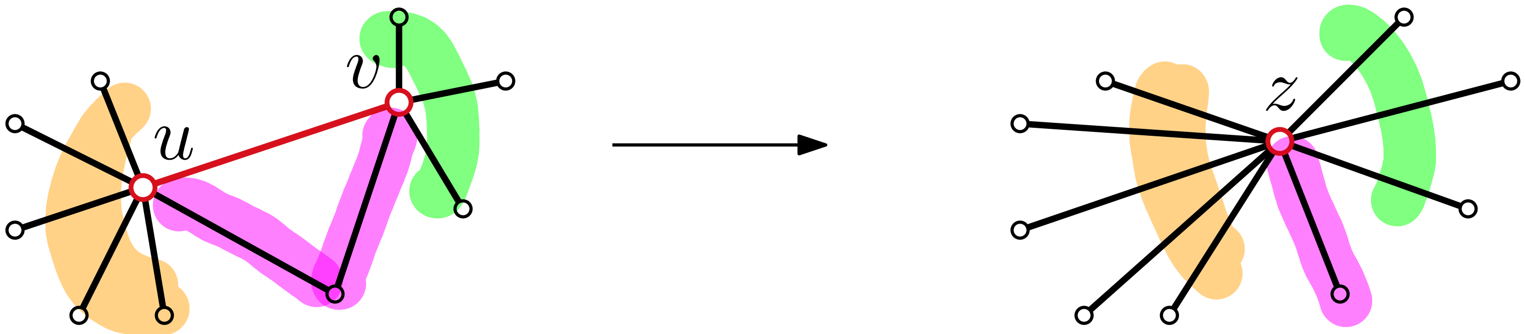

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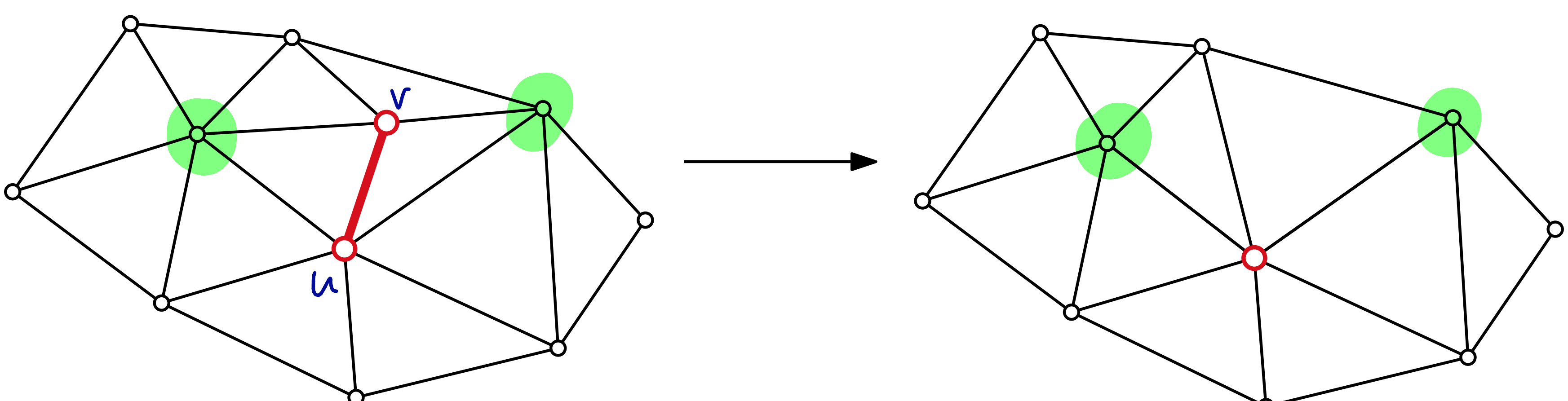
An edge (u, v) in a plane triangulated graph G is **contractible** if u and v have exactly two common neighbors. Contracting a contractible edge leaves $G/(u, v)$ plane and triangulated.



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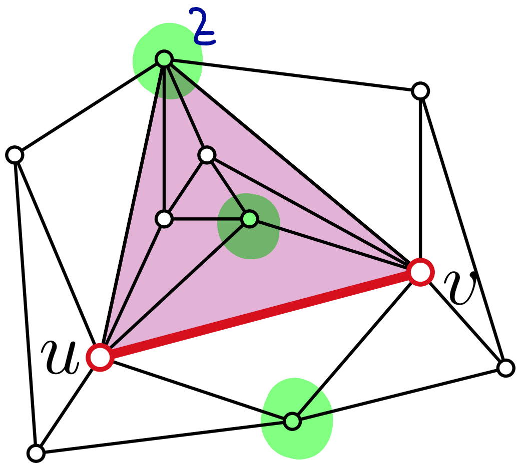
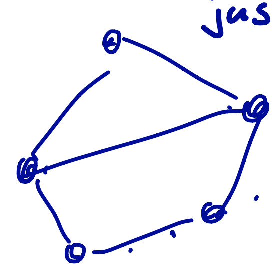
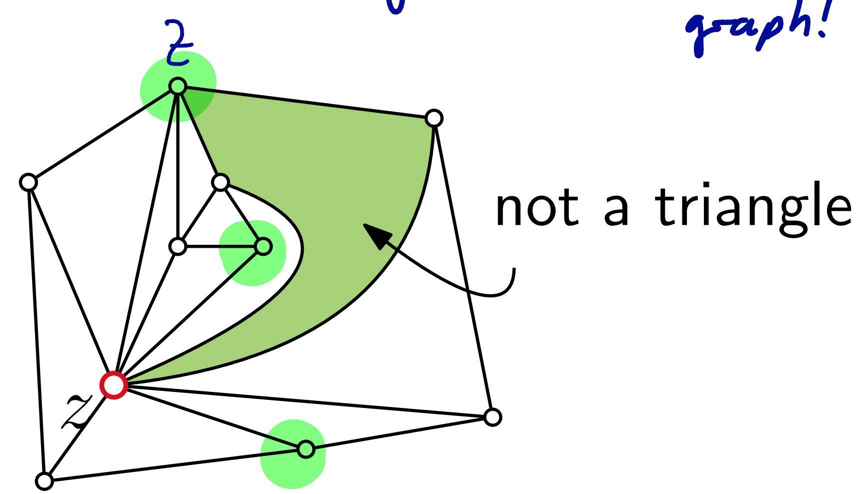
An edge (u,v) is contractible if and only if it is not part of a **separating triangle**.

just one common neighbor? Not in a triangul. graph!







Existence of Schnyder Labelings

Lemma: Let G be a plane triangulated graph with $n \geq 4$ vertices and let a, b, c be the vertices of its outerface. Then there is a neighbor x of a such that $x \notin \{b, c\}$, and $\{a, x\}$ is contractible.

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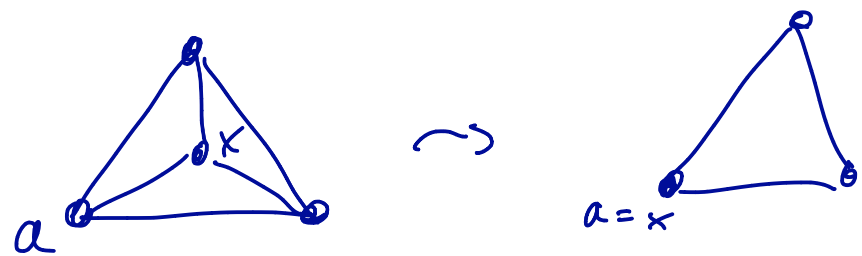
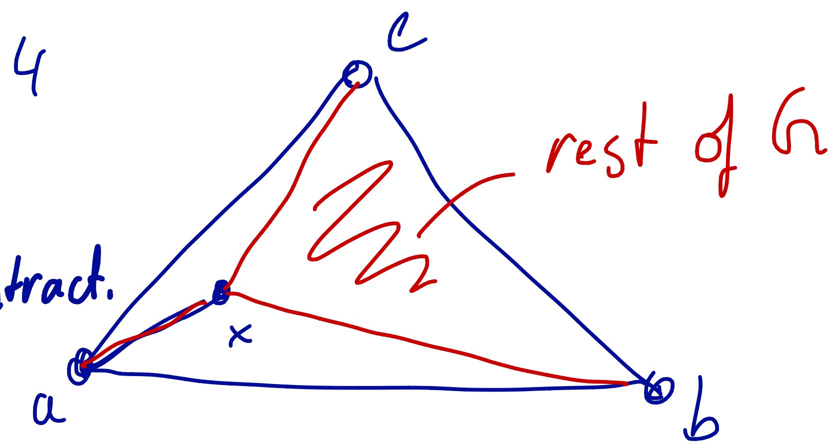
Proof: by induction on n

$n = 4$

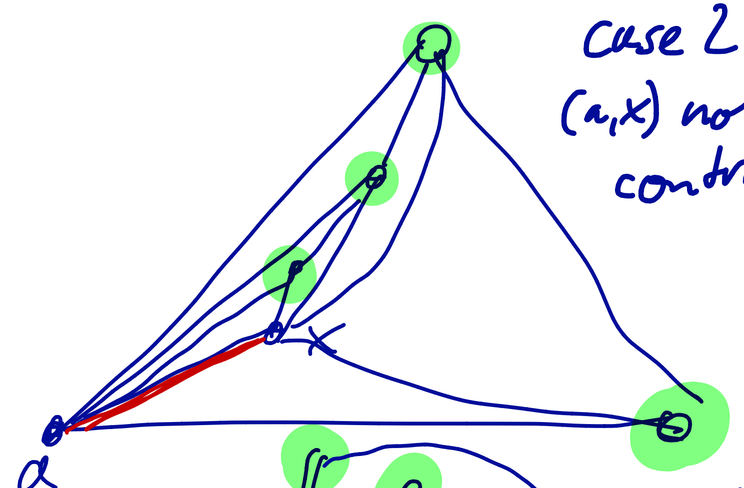


$n > 4$

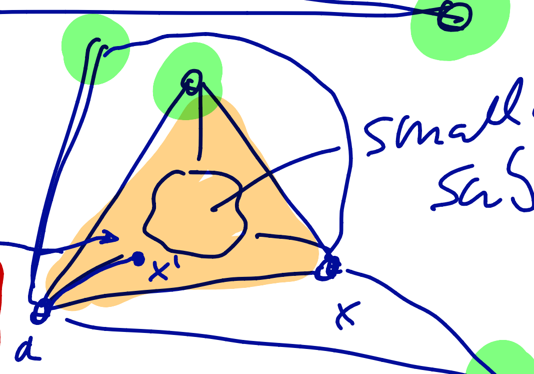
case 1:
(a,x) contract.



case 2:
(a,x) not contractible



smaller subgraph



apply induction hyp. on graph inside sep. triangle ⇒ ∃ contractible neighbor x' of a

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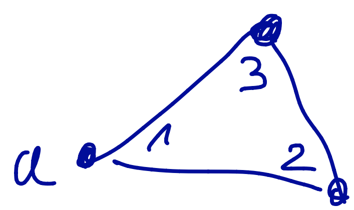
- Let a be an outer vertex. Show that there is a Schnyder labeling with all angles at a having label 1.
- obviously true for $n = 3$



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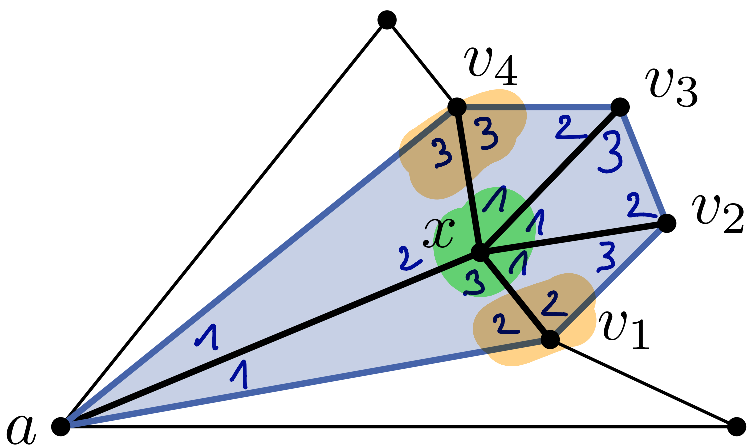
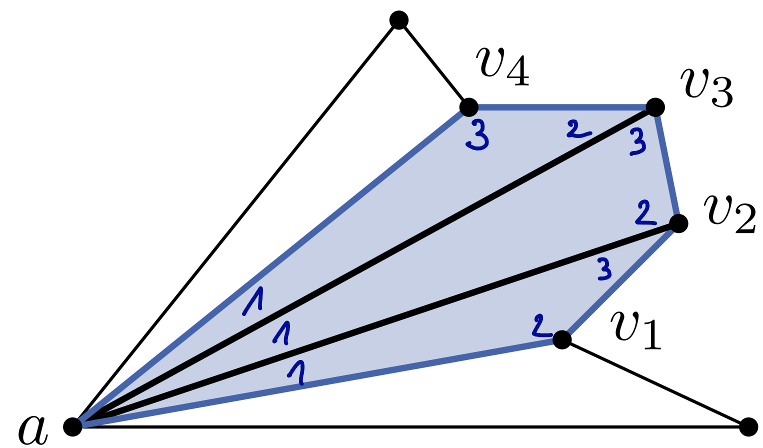
- Let a be an outer vertex. Show that there is a Schnyder labeling with all angles at a having label 1.
- obviously true for $n = 3$
- Let (a, x) be a contractible edge incident to a (exists by above Lemma).
- Take Schnyder labeling for $G/(a, x)$ (induction hypothesis) and extend.





Overview

Barycentric representations

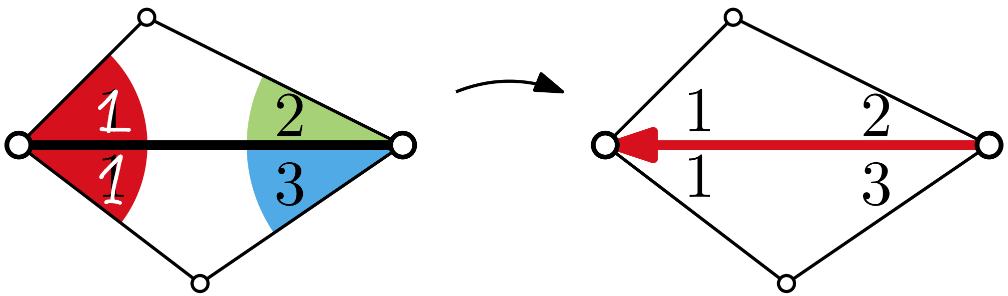
Schnyder labeling

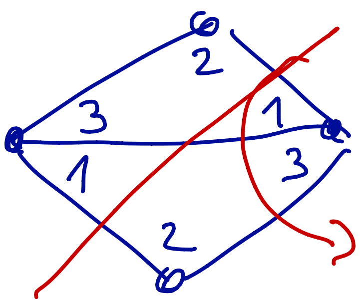
Schnyder realizer

Planar straight-line drawings

Schnyder Realizer

From a Schnyder labeling of a plane triangulated graph G we can obtain edge orientations and labelings for G.





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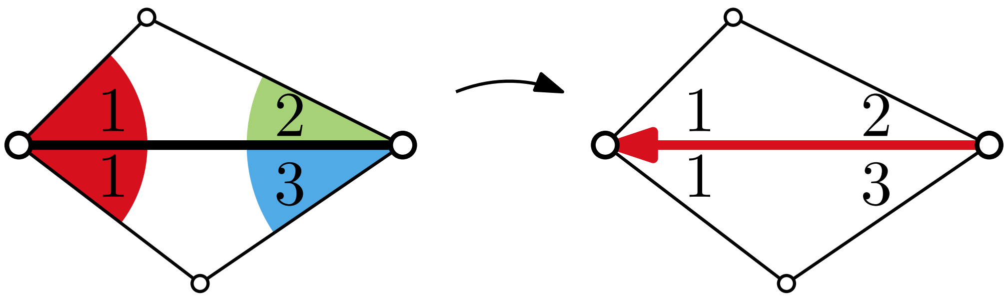

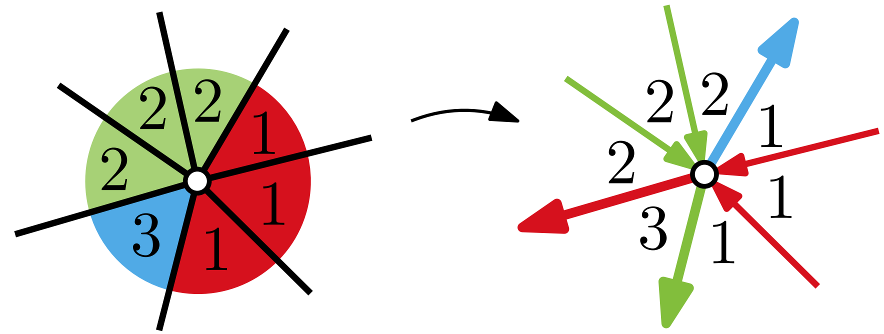

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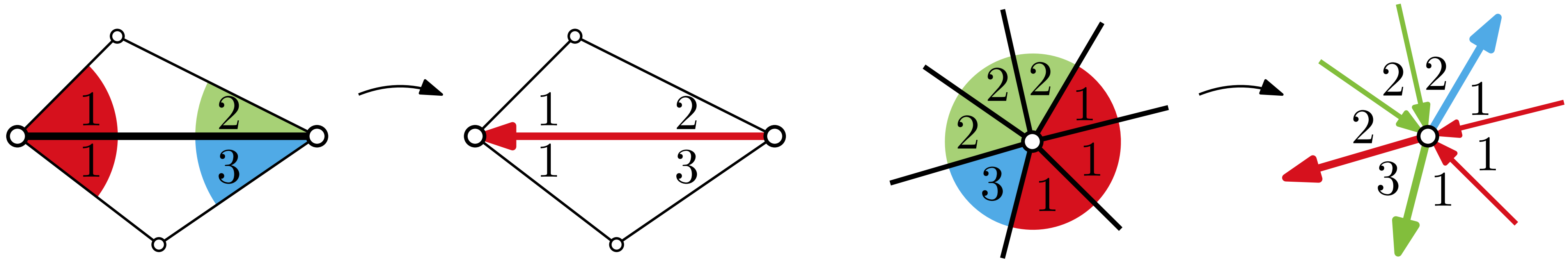

Def: A **Schnyder realizer** of a plane triangulated graph $G = (V, E)$ is a partition and orientation of its edge set E in three sets T_1, T_2, T_3 of directed edges, so that for each internal vertex $v \in V$:

- v has out-degree 1 in each of T_1, T_2, and T_3.
- The counterclockwise order of edges around v is: outgoing T_1, incoming T_3, outgoing T_2, incoming T_1, outgoing T_3, incoming T_2.

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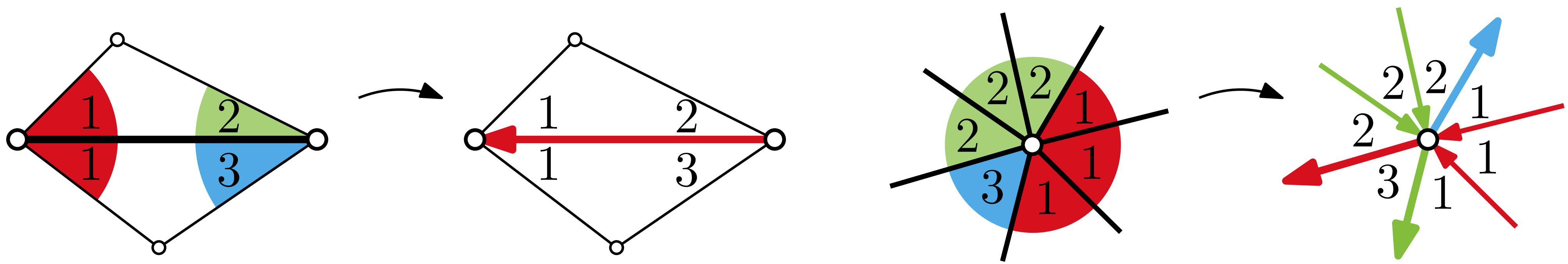

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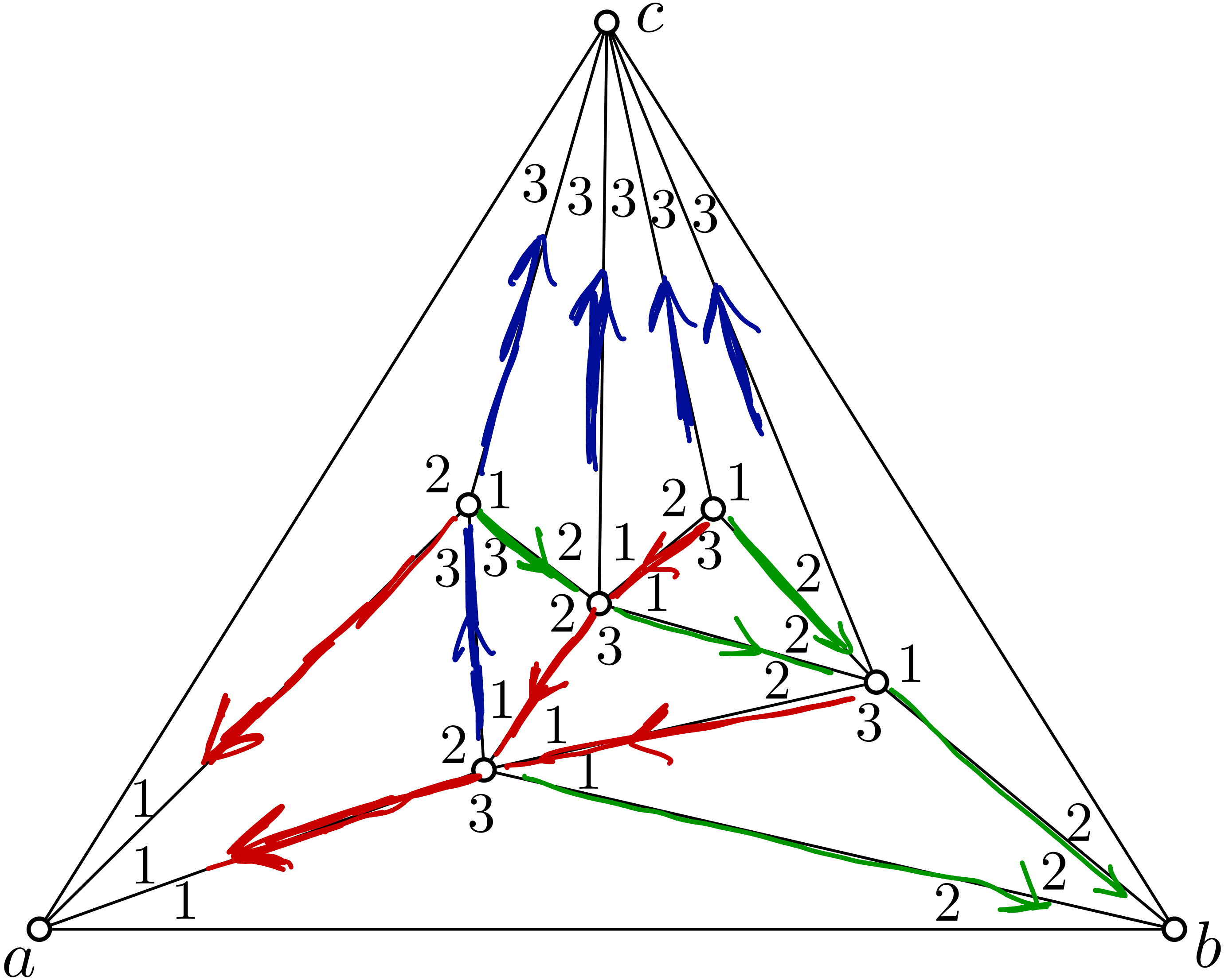
We know that every plane triangulated graph has a Schnyder labeling, hence by the above construction also a Schnyder realizer.

Properties of Schnyder Realizers



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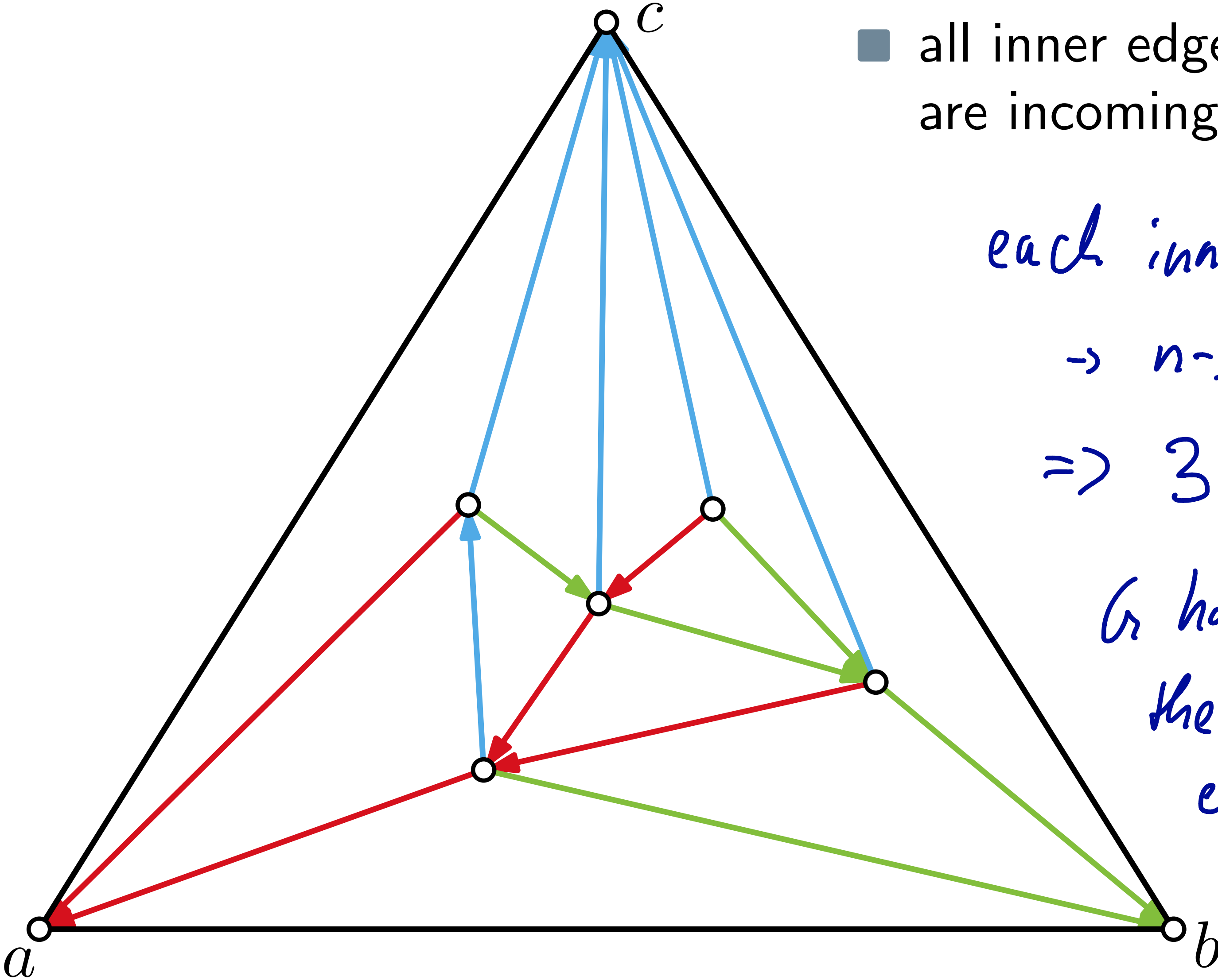


- all inner edges incident to a, b, and c are incoming in the same color

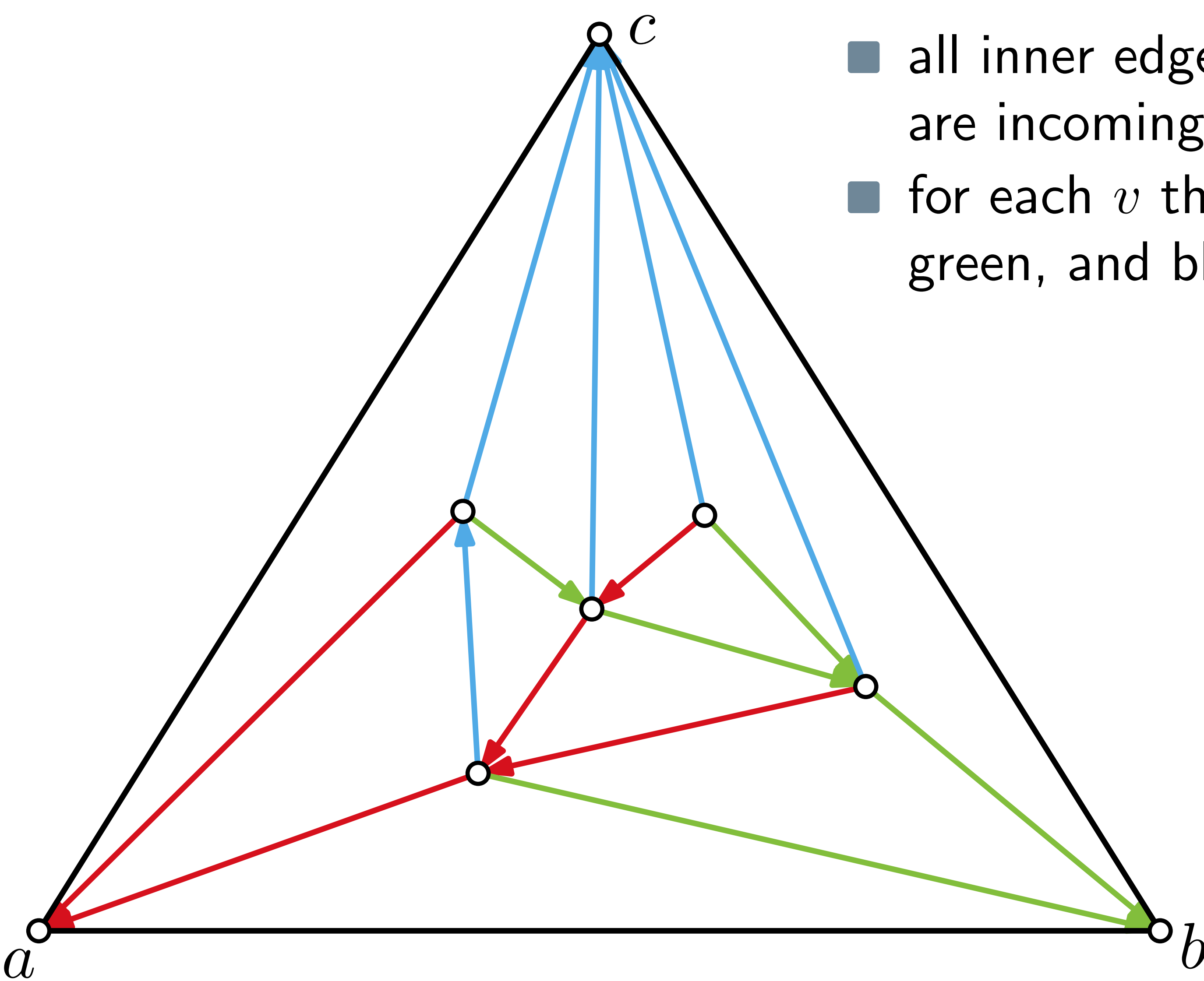
each inner vertex has 3 outgoing

$\rightarrow$ $n-3$ inner vtcs

$\Rightarrow$ $3n-9$ outgoing edges

G has $3n-6$ edges, 3 on the outer face, so all edges covered

Properties of Schnyder Realizers



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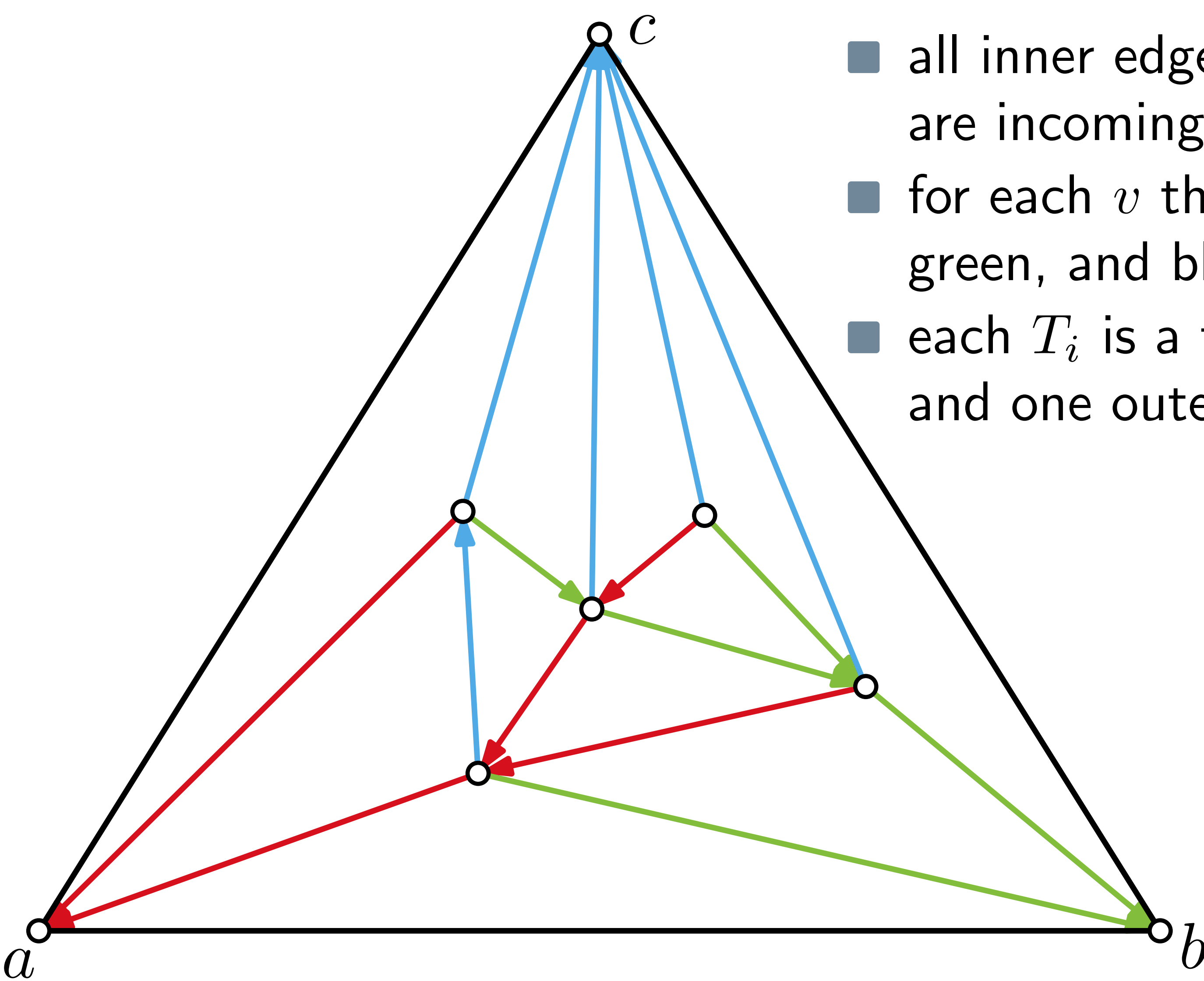

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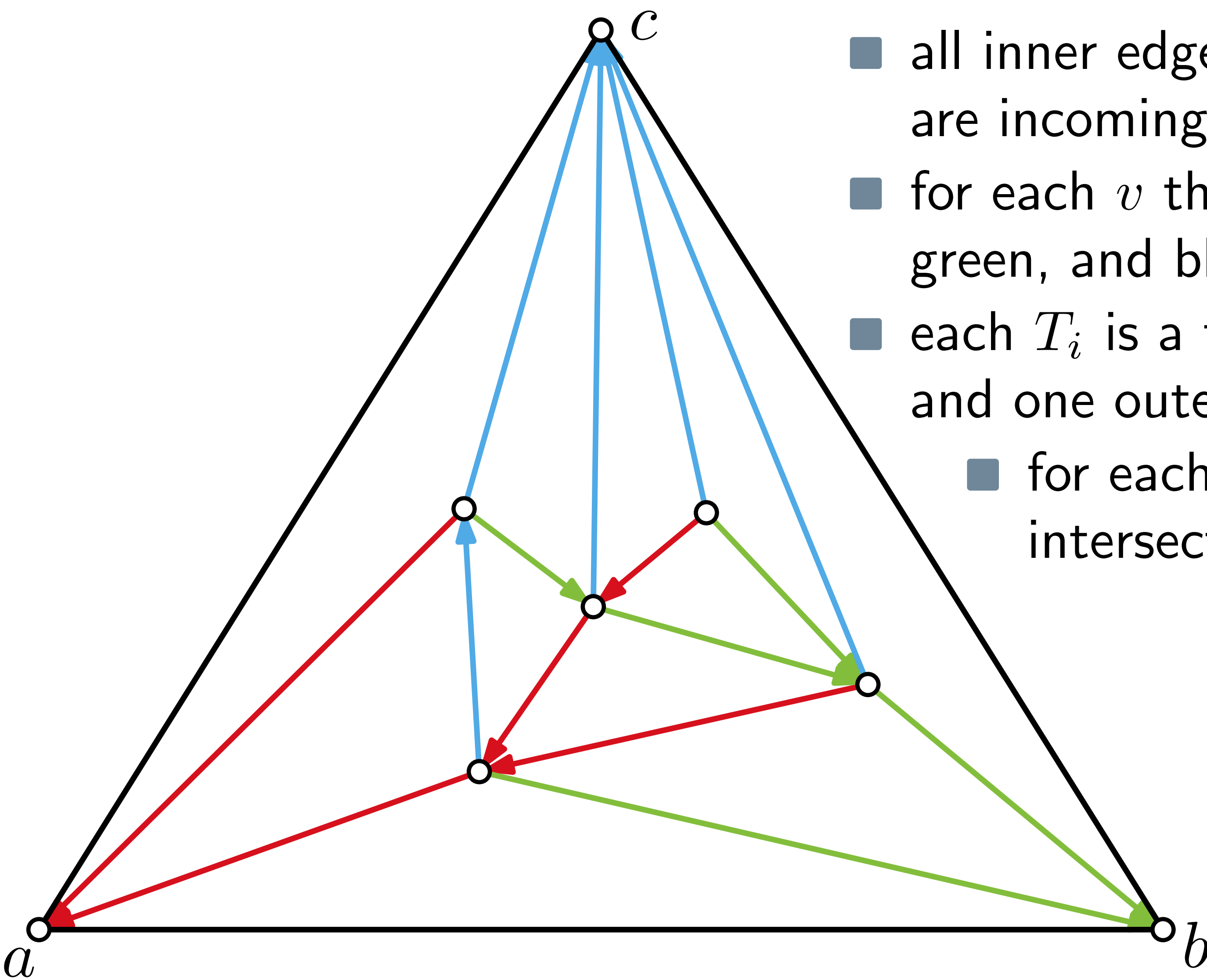

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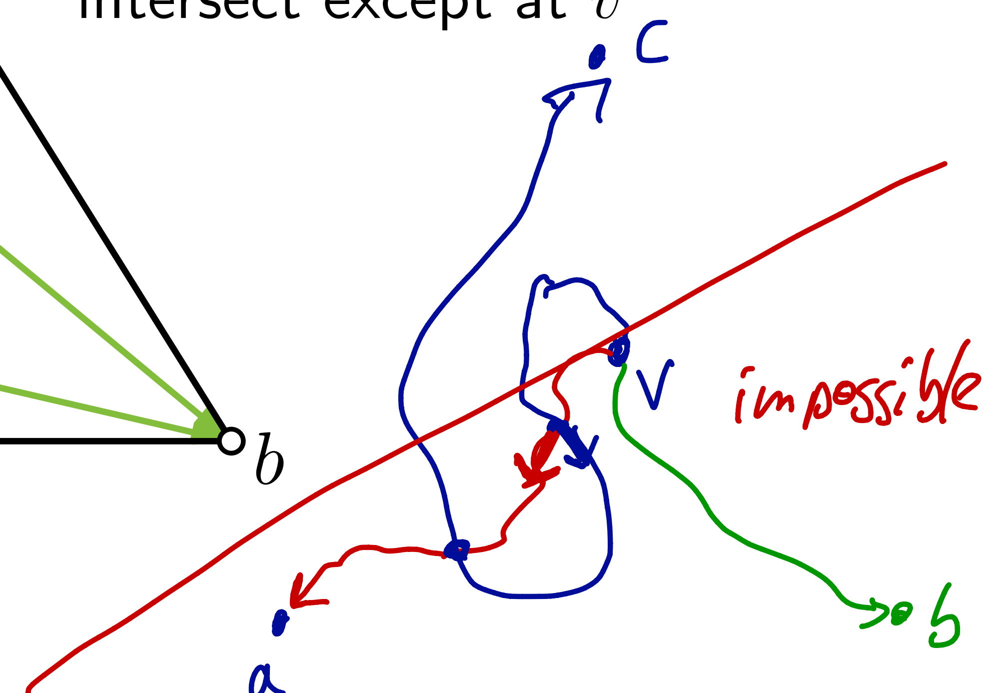

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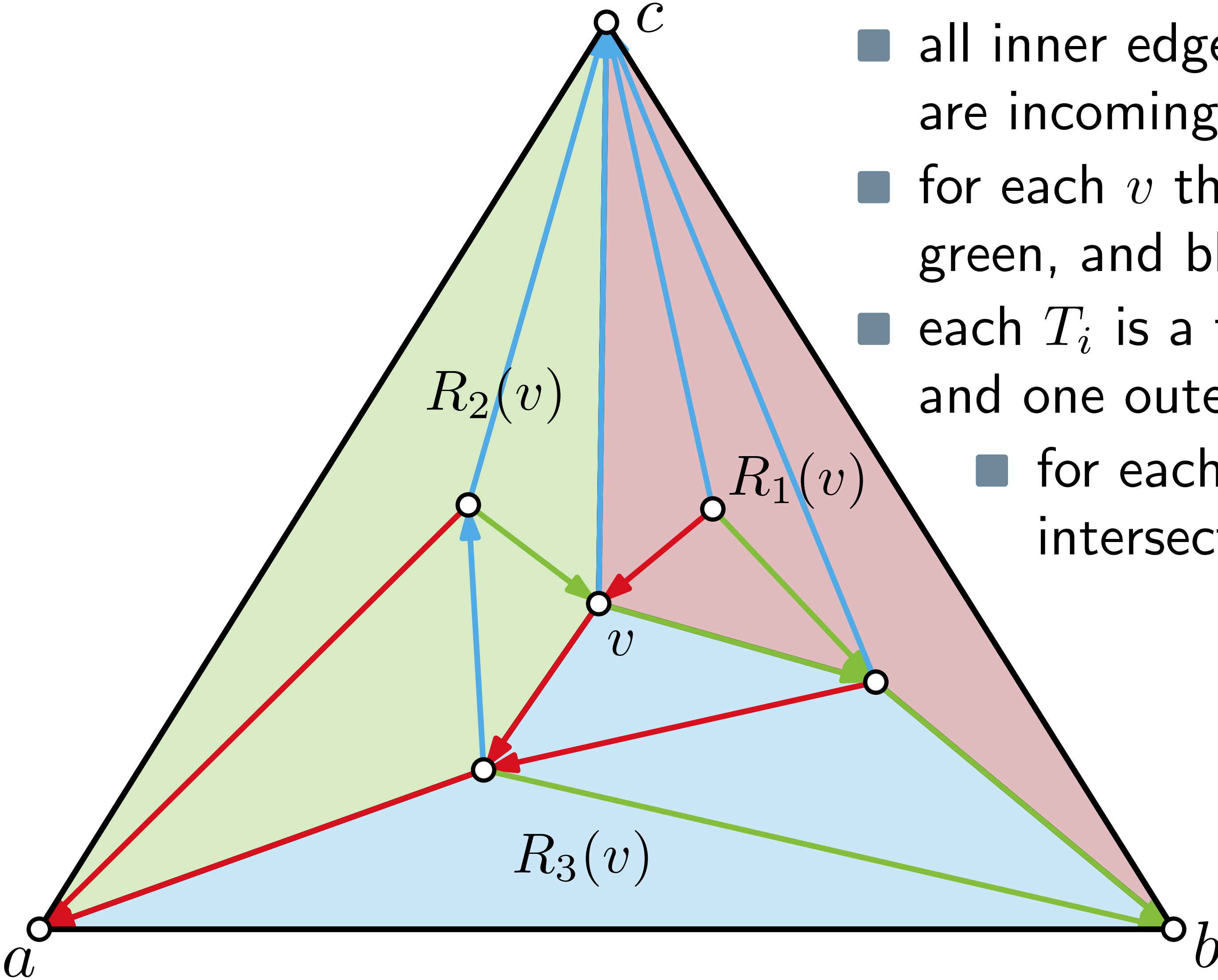

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- for each v there exist directed red, green, and blue paths from v to a, b, c
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- for each v the three paths $P_1(v)$, $P_2(v)$, and $P_3(v)$ to the root in T_1, T_2, T_3 divide G into three regions $R_1(v)$, $R_2(v)$, and $R_3(v)$

More Properties of Schnyder Realizers

Lemma: Let u, v be two distinct inner vertices of a plane triangulated graph G with a Schnyder realizer and $i \in \{1, 2, 3\}$. If $u \in R_i(v)$ then $R_i(u) \subsetneq R_i(v)$.

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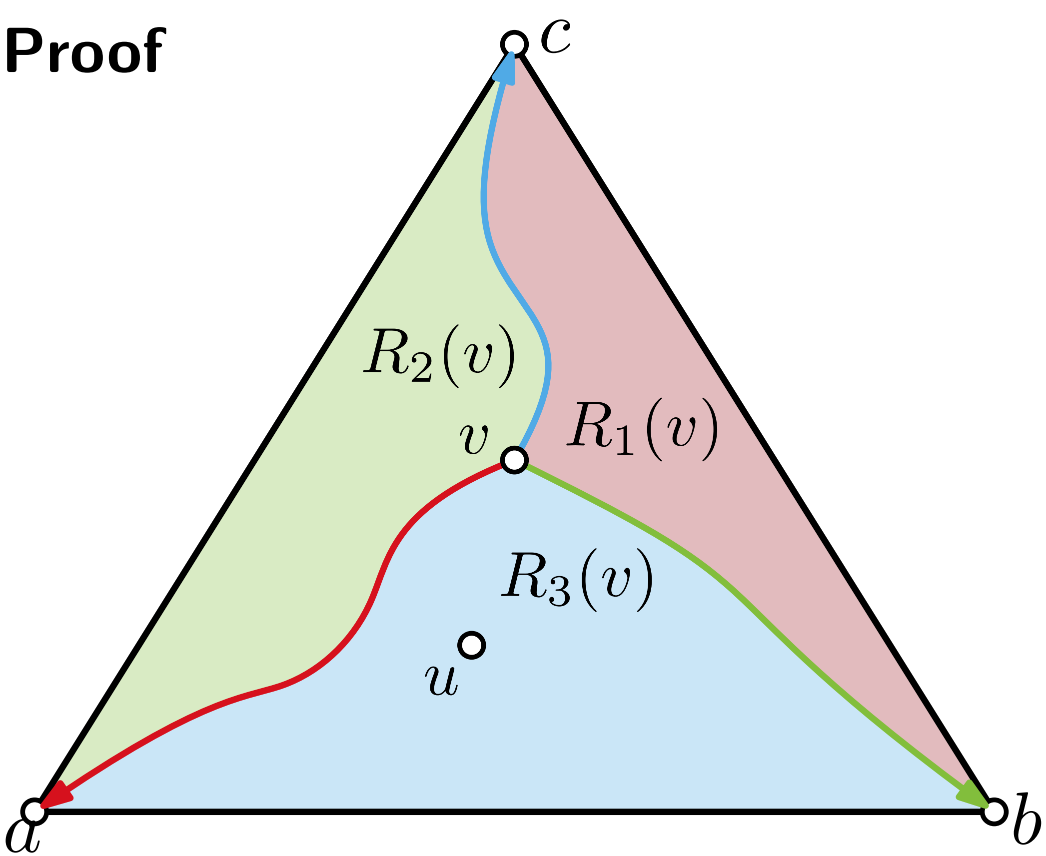

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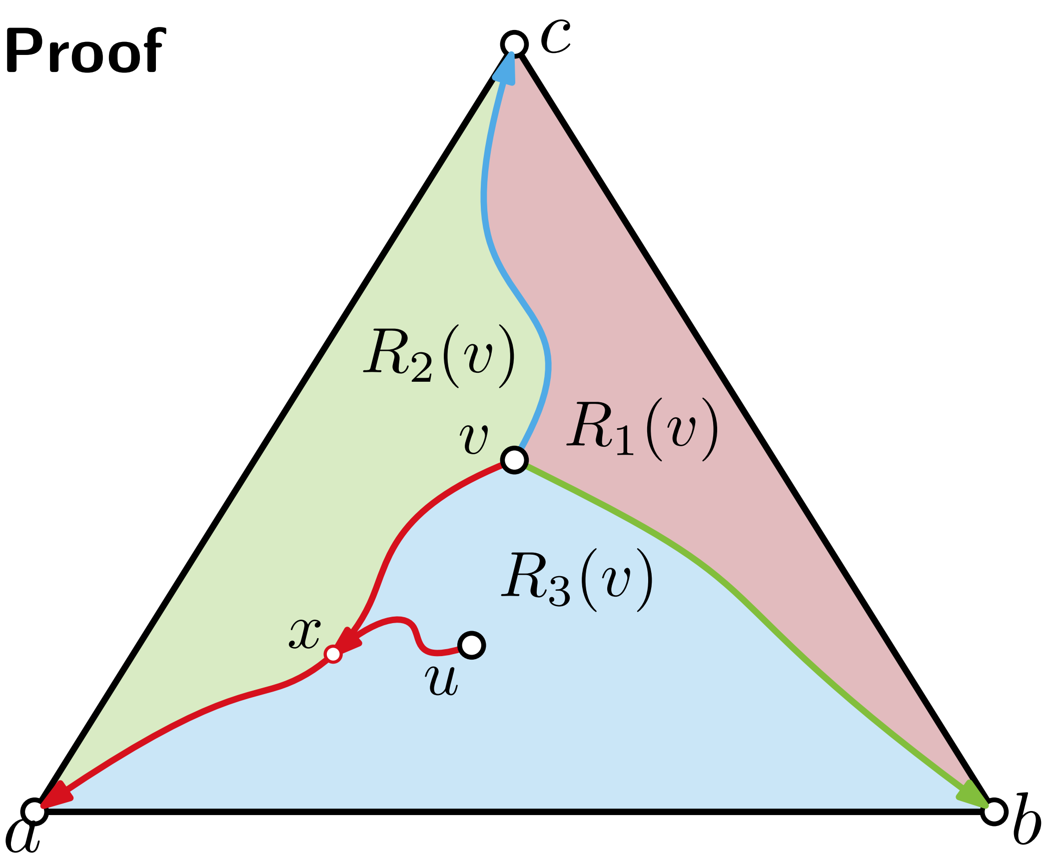

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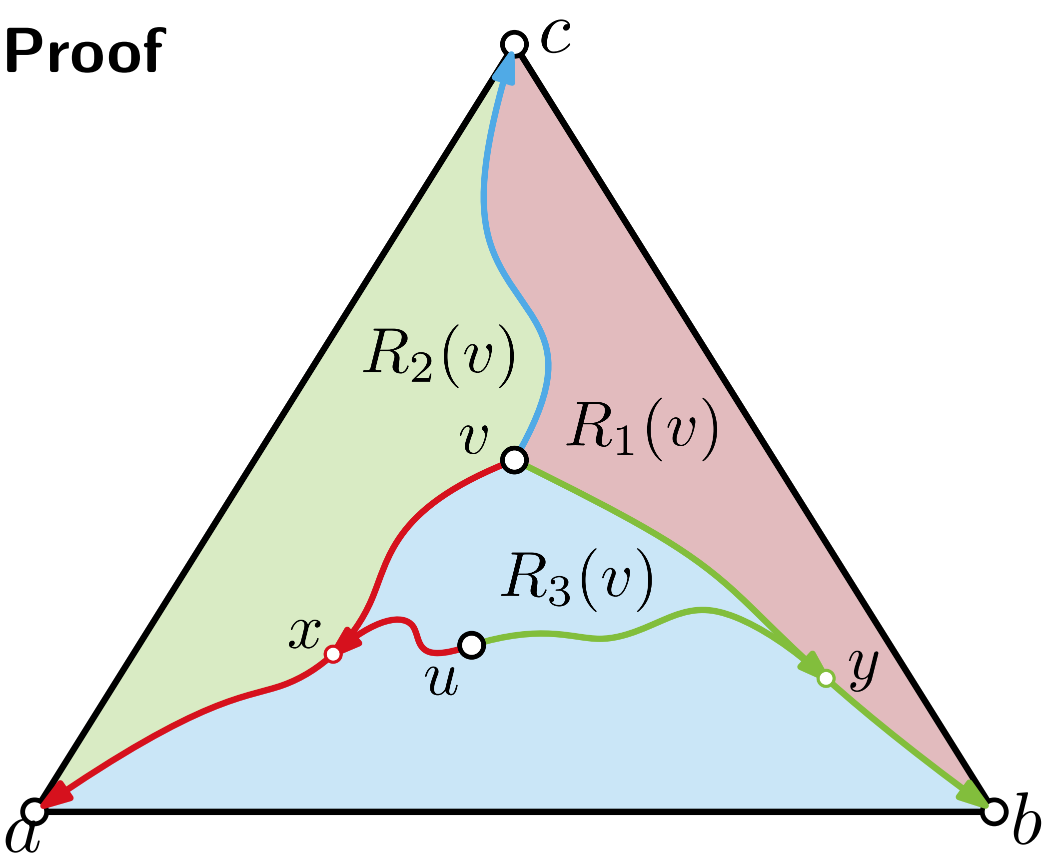

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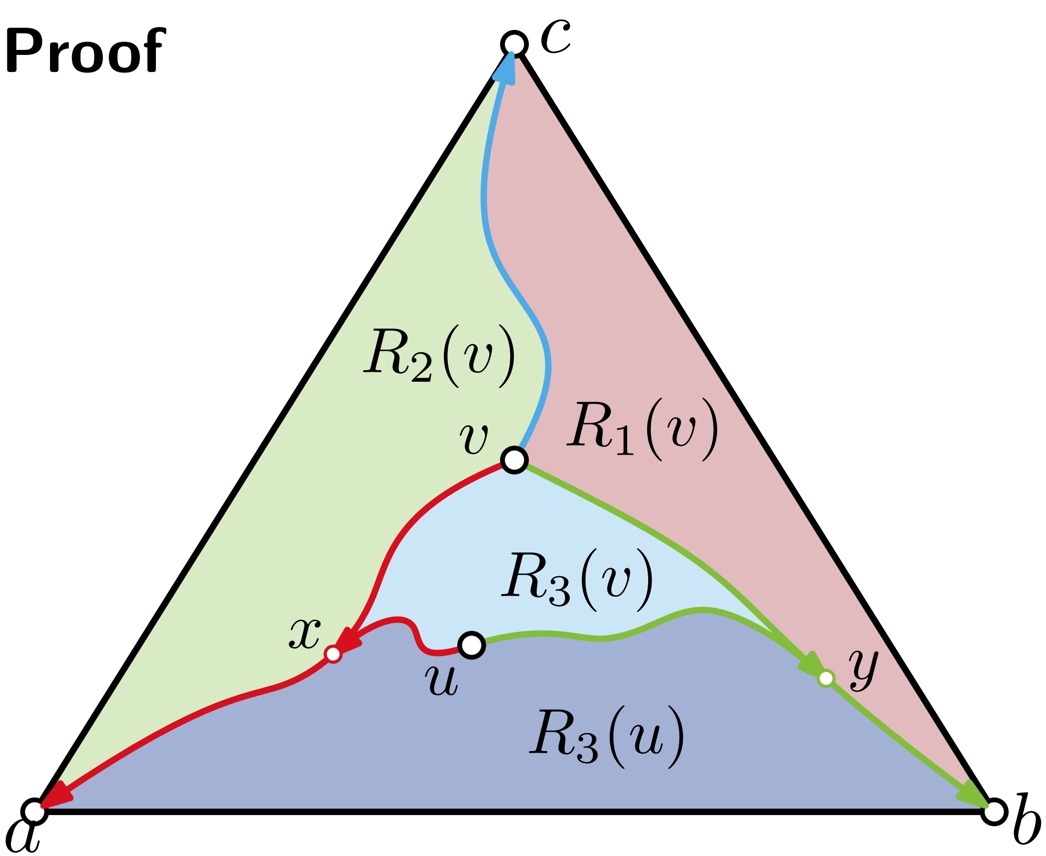

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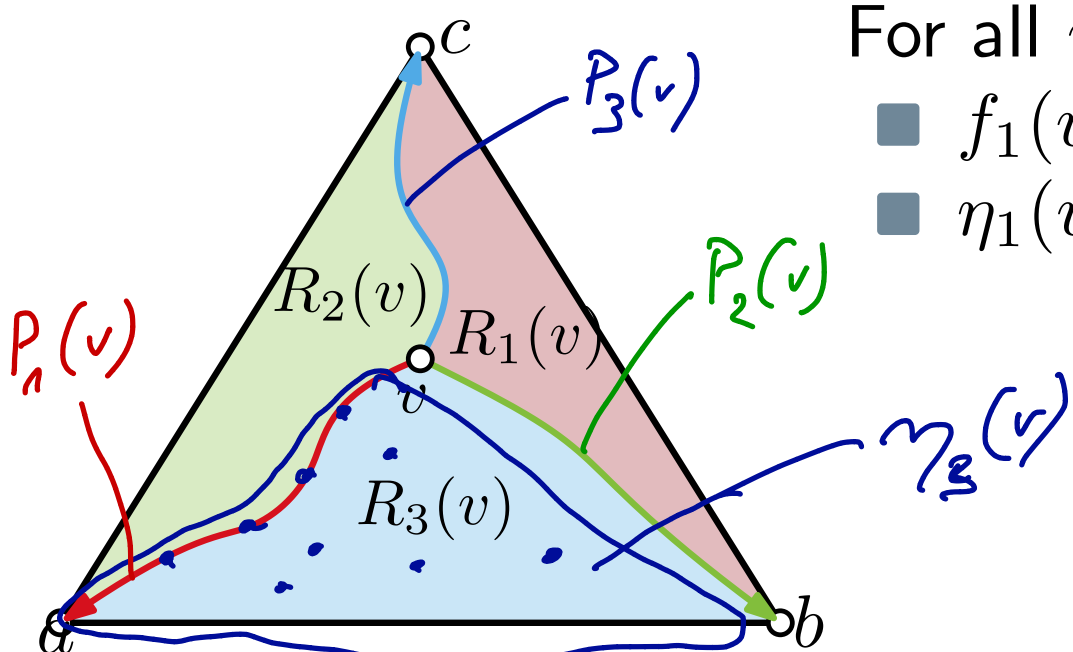
For each region $R_i(v)$ let $f_i(v)$ be the number of faces in $R_i(v)$.

For each region $R_i(v)$ let $\eta_i(v)$ be the number of vertices in $R_i(v) - P_{i-1}(v)$.



For all v it holds:

- $f_1(v) + f_2(v) + f_3(v) = 2n - 5$
- $\eta_1(v) + \eta_2(v) + \eta_3(v) = n - 1$

Overview

Barycentric representations

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Planar Straight-Line Drawing

Finally, let's put it all together!

Theorem: For a plane triangulated graph G the mapping

$$f: v \mapsto \tfrac{1}{2n-5}(f_1(v), f_2(v), f_3(v))$$

is a barycentric representation of G.

Planar Straight-Line Drawing

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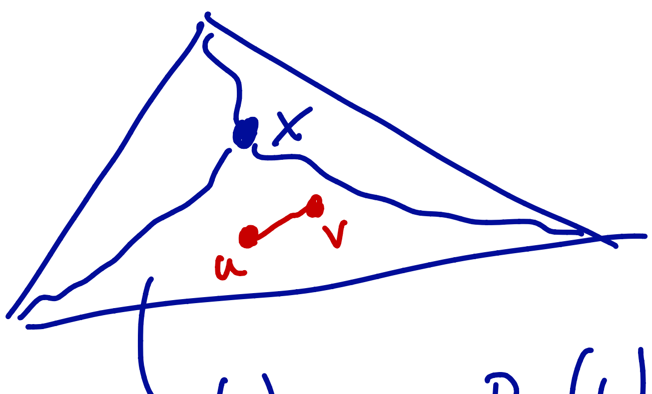
Proof:

- $v_1 + v_2 + v_3 = 1$: $\frac{f_1(v)}{2n-5} + \frac{f_2(v)}{2n-5} + \frac{f_3(v)}{2n-5} = 1$ ✓
- edge (u,v), vtx $x \notin \{u,v\}$



$\Rightarrow f_3(u), f_3(v) < f_3(x)$

$R_3(x)$ $\Rightarrow$ $R_3(u) \subsetneq R_3(x)$

$R_3(v) \subsetneq R_3(x)$

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Set $A = (2n-5, 0, 0)$, $B = (0, 2n-5, 0)$, and $C = (0, 0, 0)$. Then the resulting drawing is planar and can be projected to a $(2n-5) \times (2n-5)$-grid.

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Variation

Theorem: For a plane triangulated graph G the mapping
$$g\colon v \mapsto \tfrac{1}{n-1}(\eta_1(v), \eta_2(v), \eta_3(v))$$
is a **weak** barycentric representation of G.

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Set $A = (n-1, 0, 0)$, $B = (0, n-1, 0)$, and $C = (0, 0, 0)$. Then the resulting drawing is planar and can be projected to an $(n-2) \times (n-2)$-grid.

Running Times

Steps for computing straight-line drawings:

- Schnyder labeling $O(n)$ time, related to canonical ordering
- Schnyder realizer → consider each edge once → $O(n)$
- barycentric representation $O(n)$ by tree traversals [details in literature]

Running Times

Steps for computing straight-line drawings:

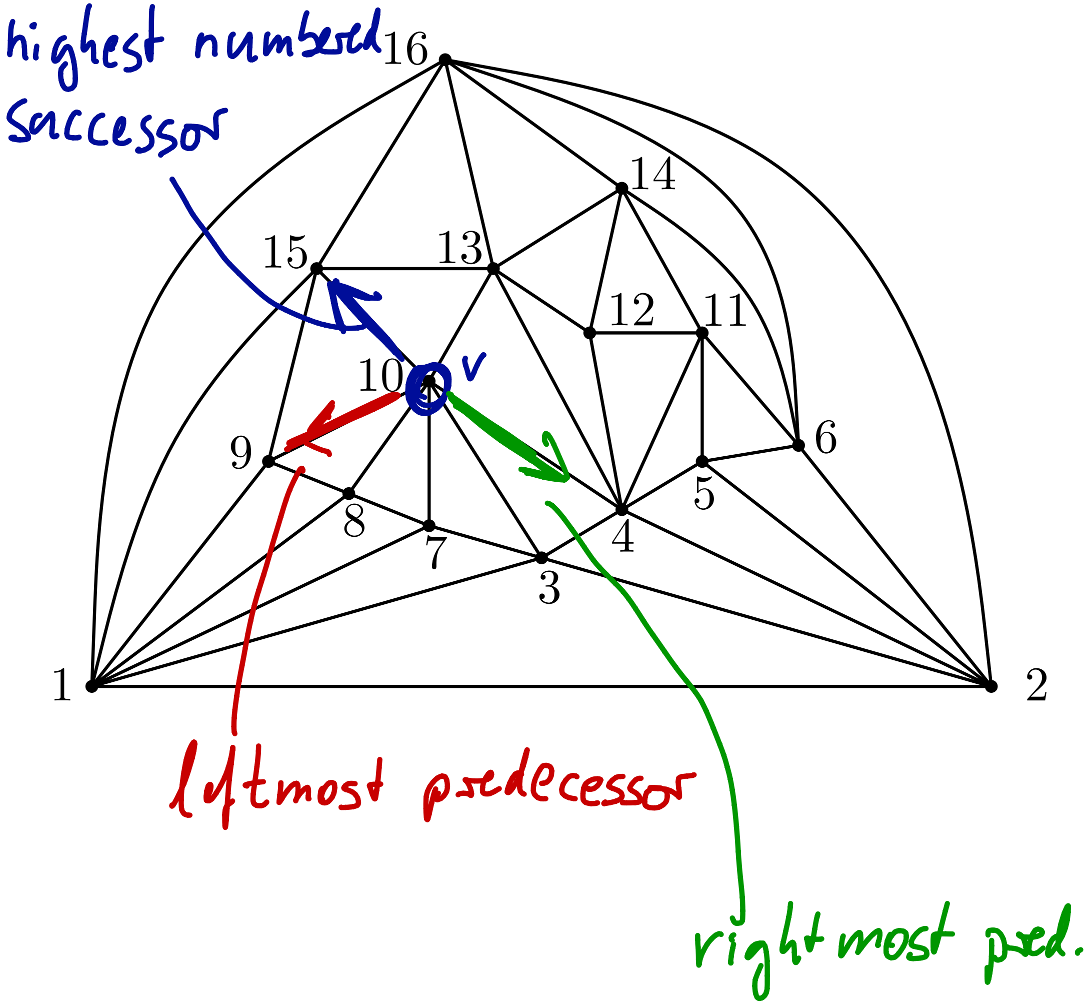
- Schnyder labeling
- Schnyder realizer
- barycentric representation

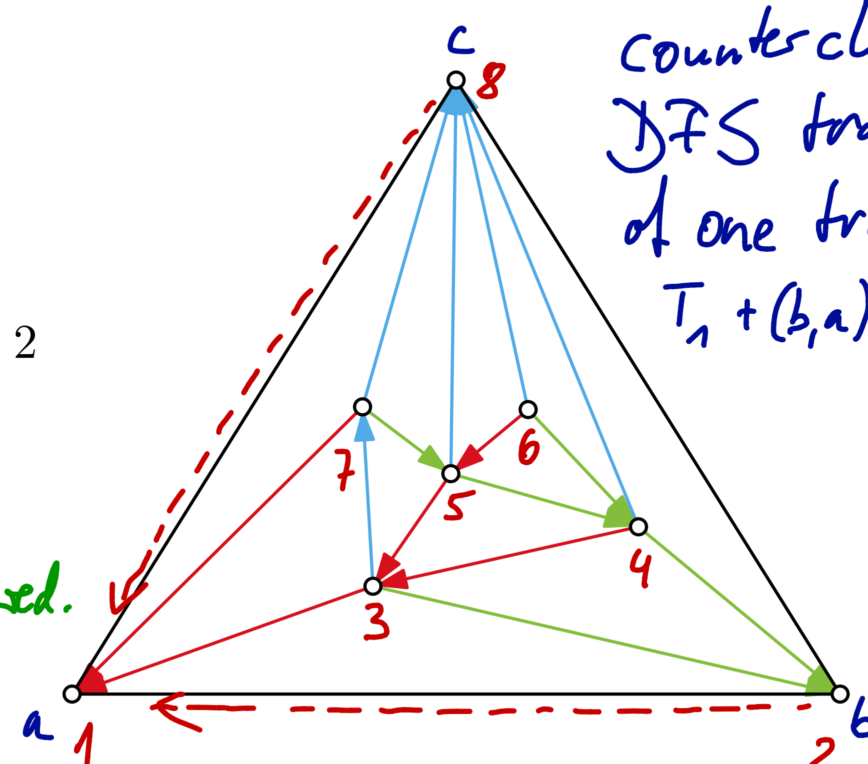
Theorem: Every n-vertex embedded planar graph $G = (V, E)$ has a straight-line planar drawing on a grid of size $(n-2) \times (n-2)$. It can be computed in $O(n)$ time.

[Schnyder 1990]

Canonical Ordering and Schnyder Realizers

In fact, canonical orderings and Schnyder realizers can be transformed into each other!





counterclockwise DFS traversal of one tree, say $T_1 + (b,a) + (c,a)$

Summary

Last week:

Theorem: Every n-vertex embedded planar graph $G = (V, E)$ has a straight-line planar drawing on a grid of size $(2n-4) \times (n-2)$. It can be computed in $O(n)$ time.

[de Fraysseix, Pach, Pollack 1988], [Chrobak, Payne 1995]

Today:

Theorem: Every n-vertex embedded planar graph $G = (V, E)$ has a straight-line planar drawing on a grid of size $(n-2) \times (n-2)$. It can be computed in $O(n)$ time.

[Schnyder 1990]