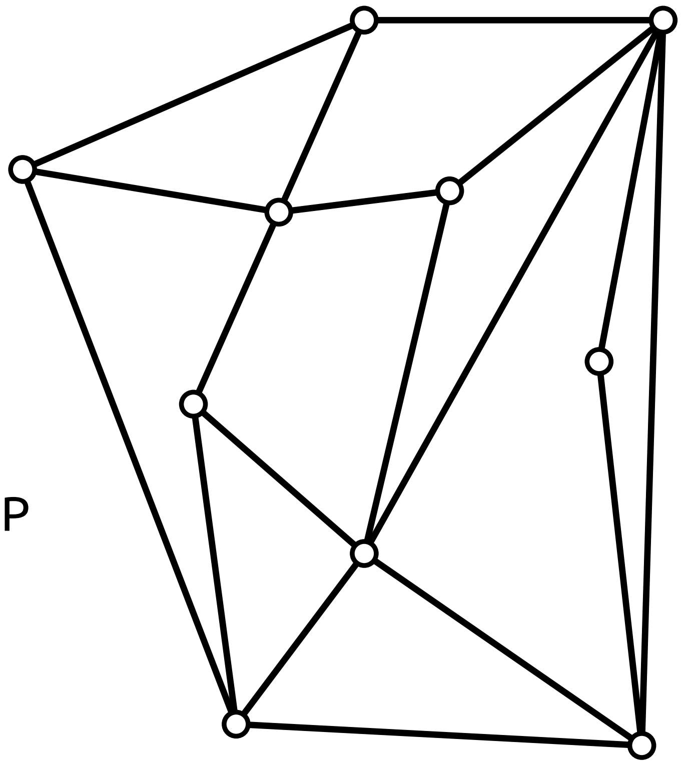
Straight-Line Planar Graph Drawing – Part 1

Lecture Graph Drawing Algorithms · 192.053

Martin Nöllenburg

17.04.2018



ALGORITHMS AND COMPLEXITY GROUP

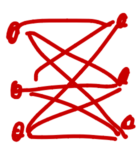
Planar Graphs

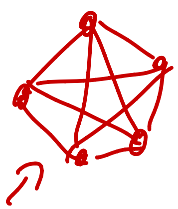
Planar graphs are an important graph class in graph drawing and graph theory.

What do you know about planar graphs?

- average degree ≤ 6
- Euler characteristic: $|V| - |E| + |F| = 2$

$|E| \leq 3|V| - 6$





- no $K_{3,3}$ or K_5 as a minor
- 4 colorable

Planar Graphs

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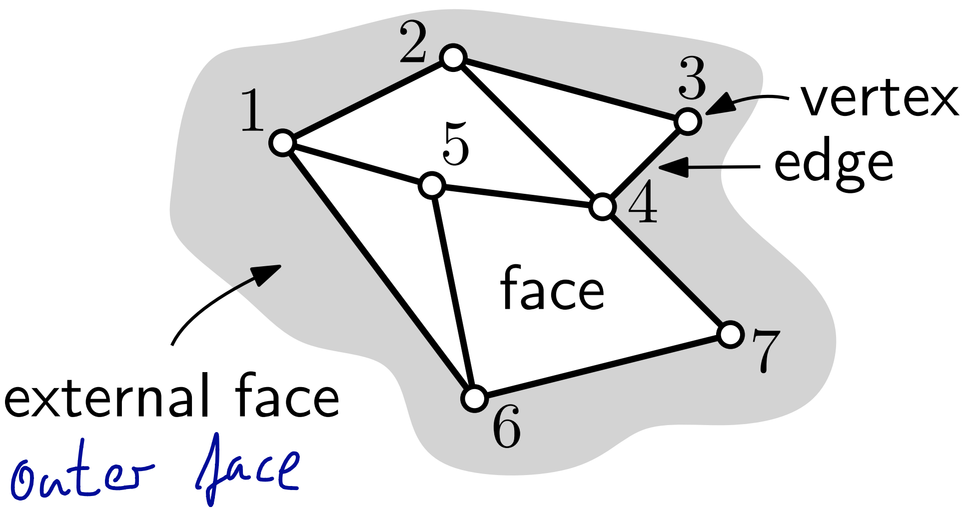
Def: A **planar graph** G is a simple graph that can be drawn/embedded in the plane $\mathbb{R}^2$ without edge crossings.



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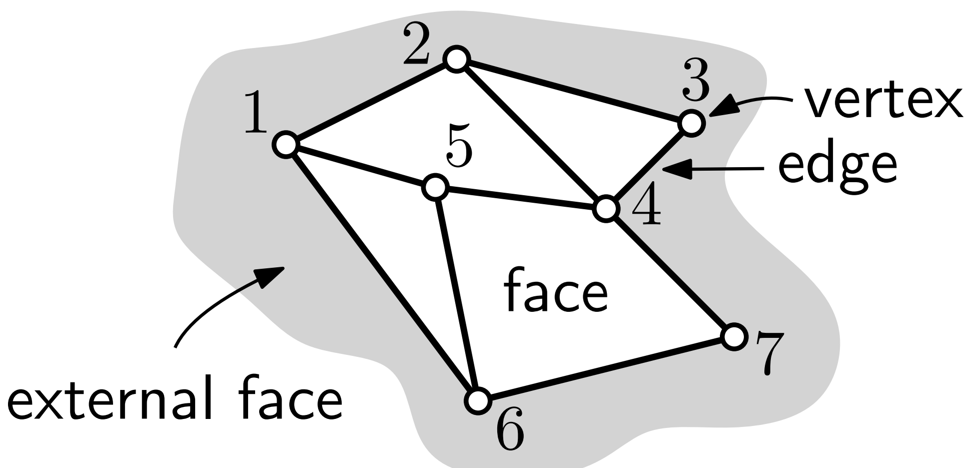
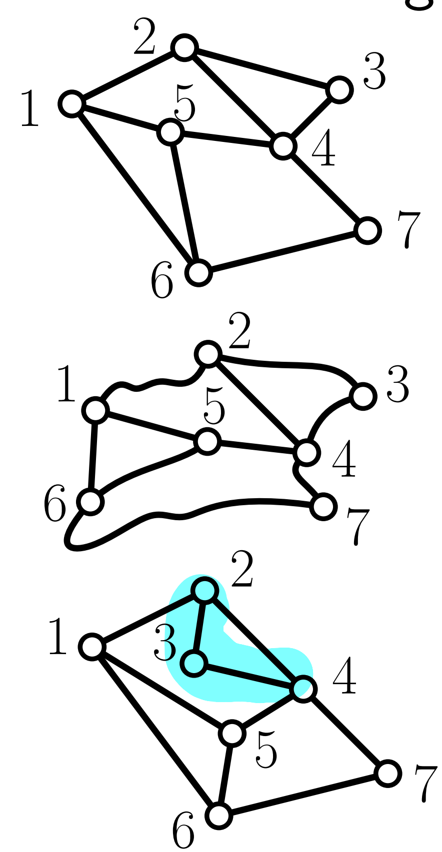


different drawings   same embedding?



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plane graph = planar graph + embedding

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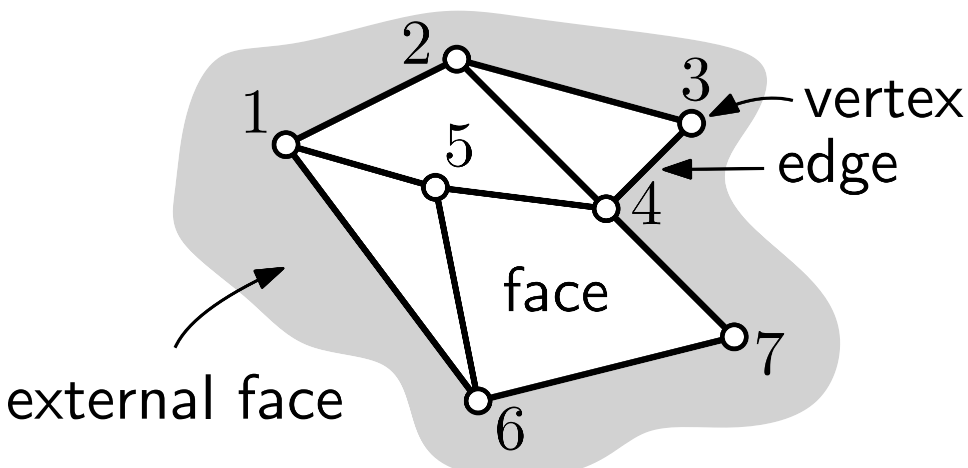
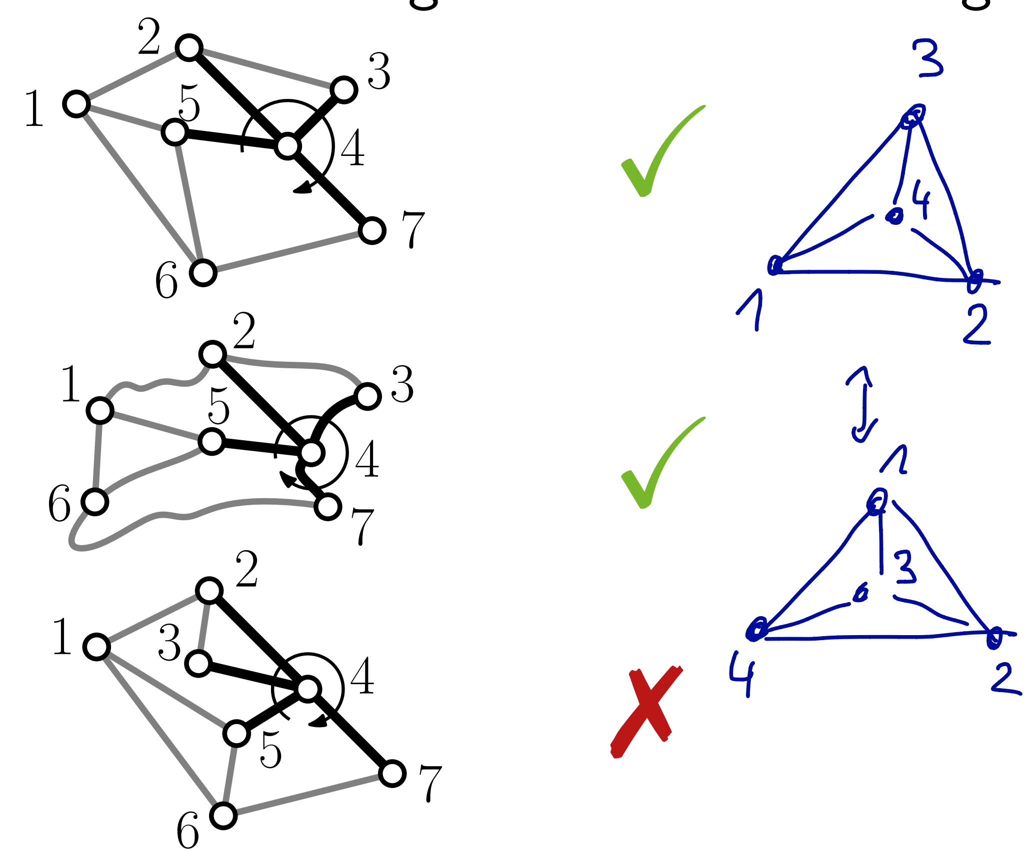


Def: The **rotation scheme** of a planar drawing is the circular ordering of the edges incident to each vertex.

Two planar drawings have the same embedding if they have the same rotation scheme and the same external face.

different drawings same embedding?



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Theorem: G is planar $\Leftrightarrow$ G has no K_5 or $K_{3,3}$ minor.

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Theorem: $G = (V, E)$ is planar $\Rightarrow |V| - |E| + |F| = 2$
[Euler 1707-1783]

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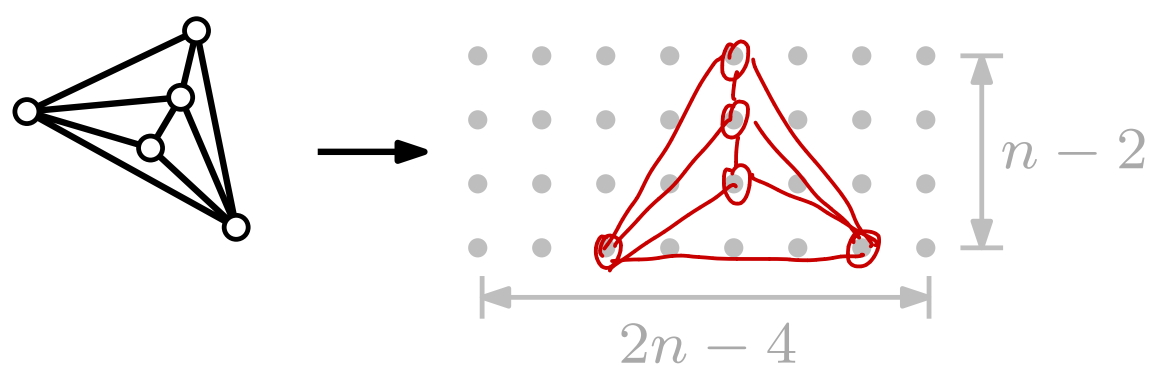
Theorem: Every n-vertex embedded planar graph has a planar straight-line drawing on an $(2n-4) \times (n-2)$ grid.

[de Fraysseix, Pach, Pollack 1988]



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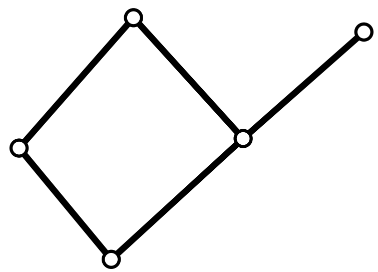

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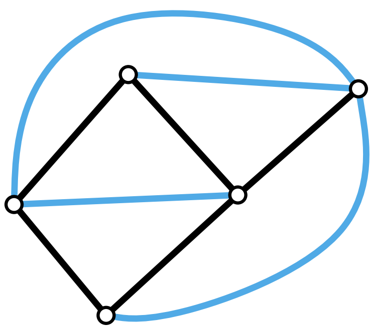

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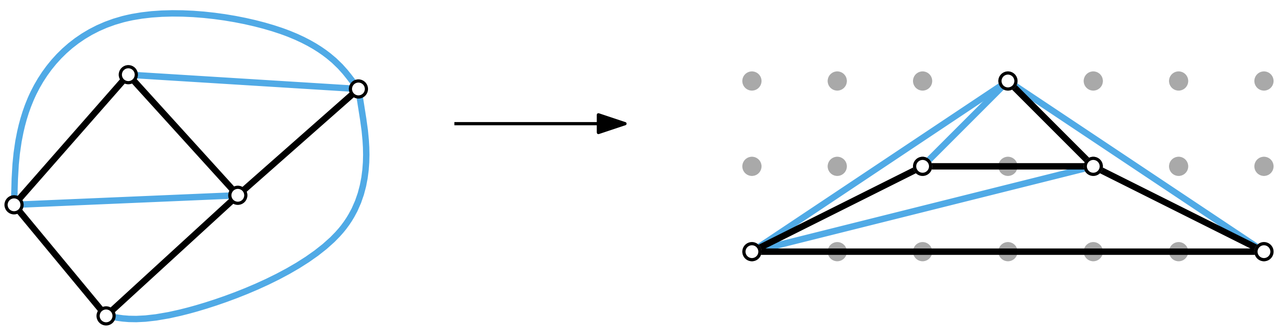

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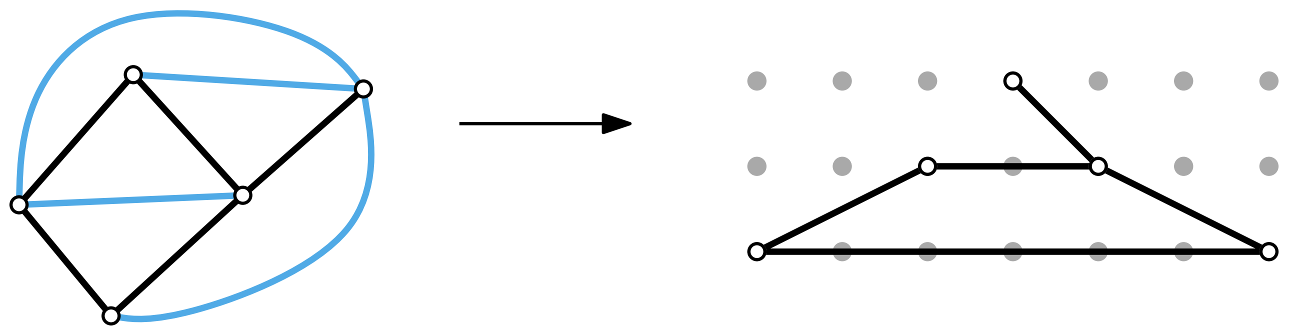

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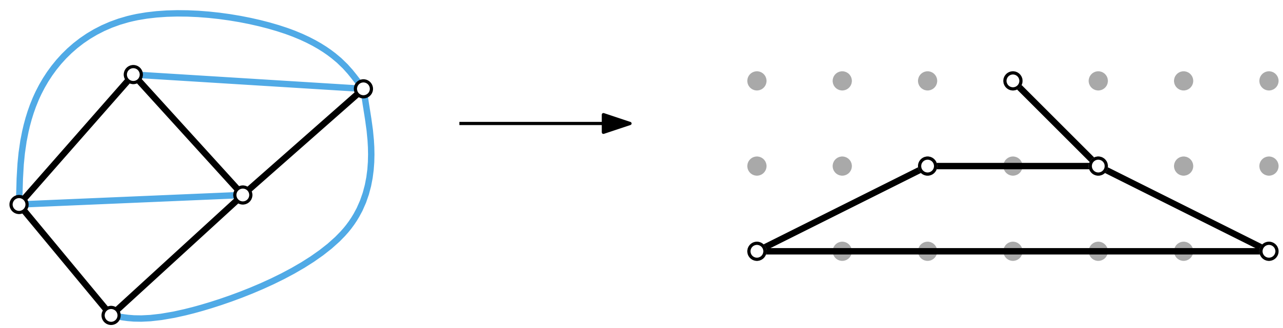

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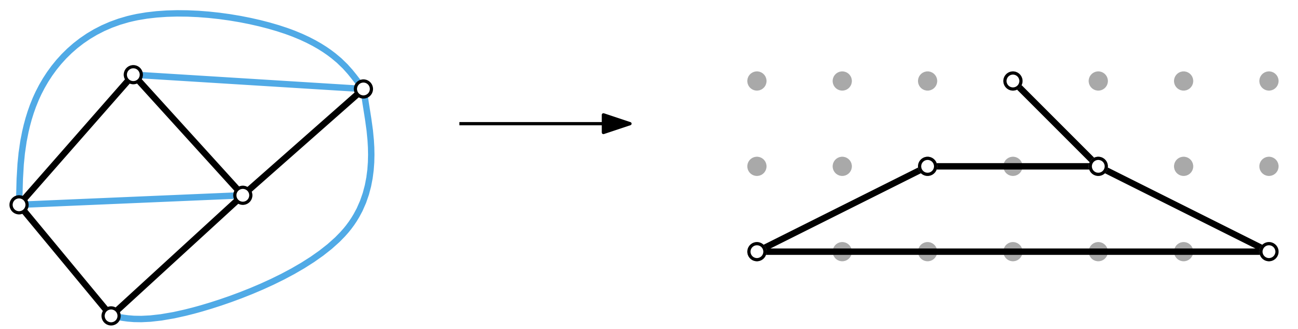

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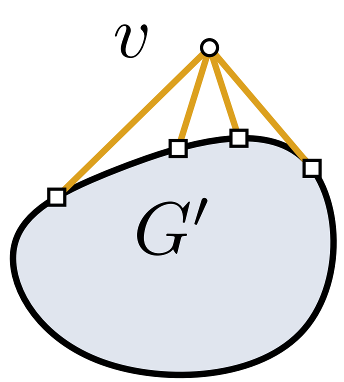

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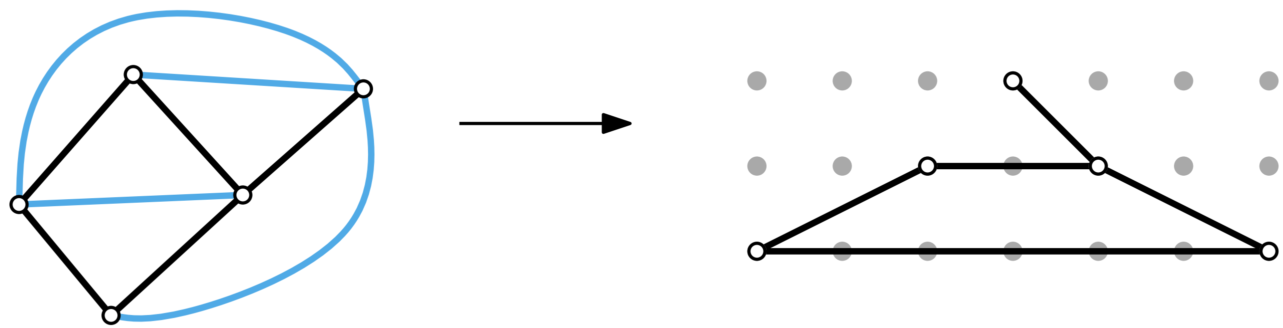

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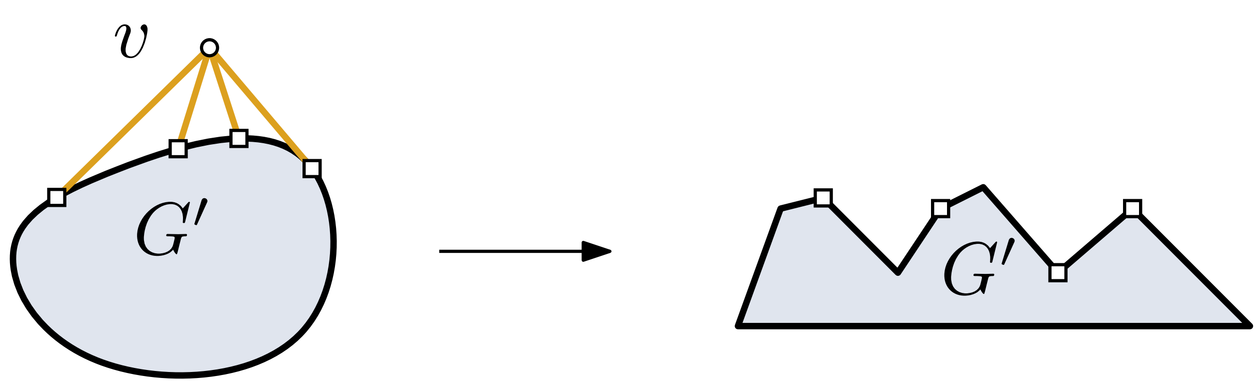

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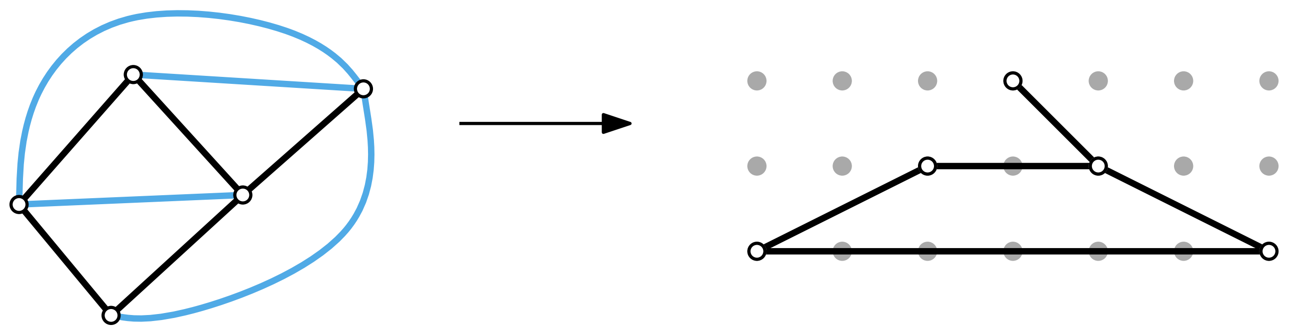

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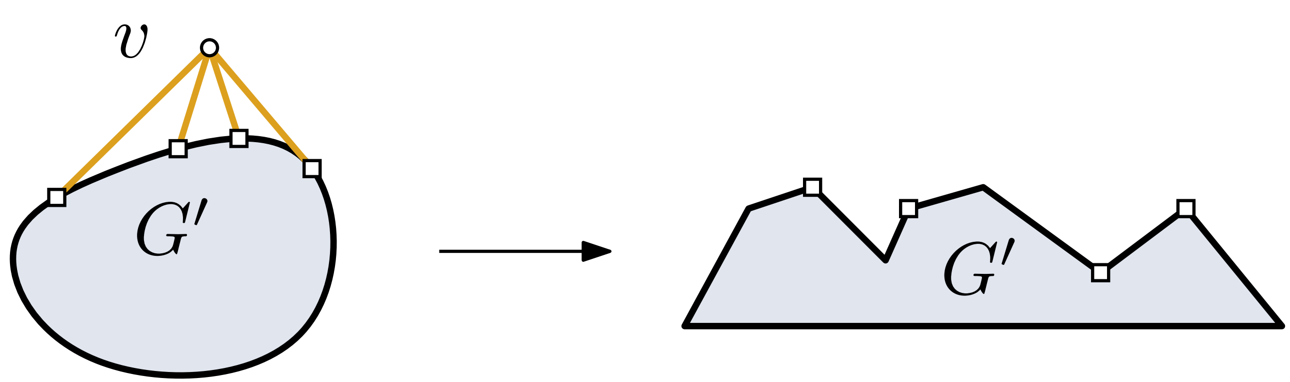

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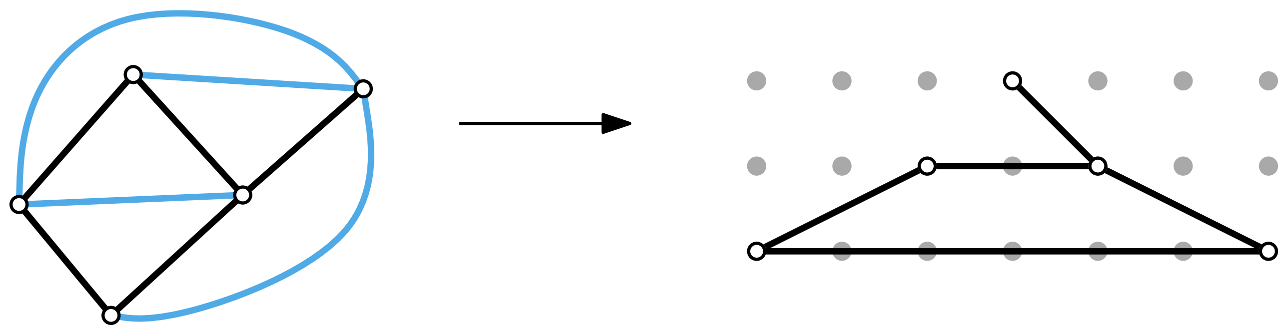

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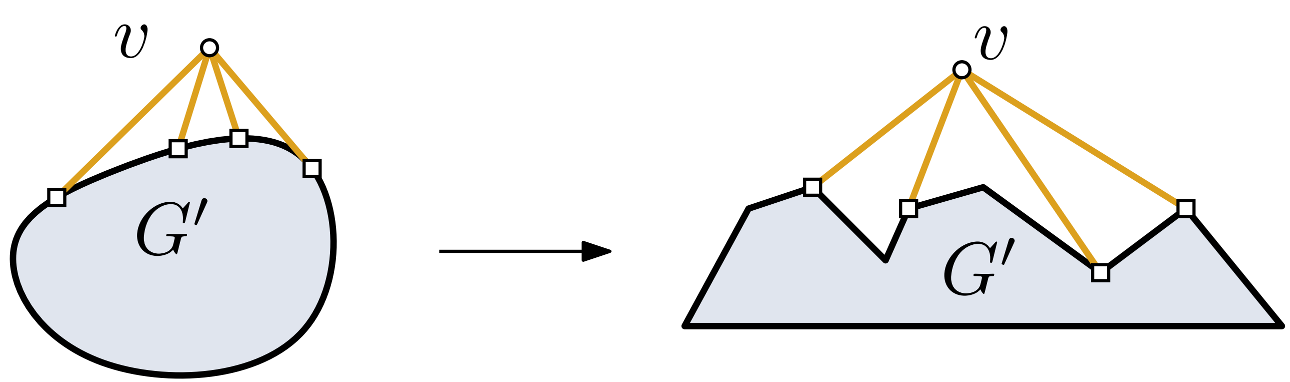


Overview

Canonical ordering

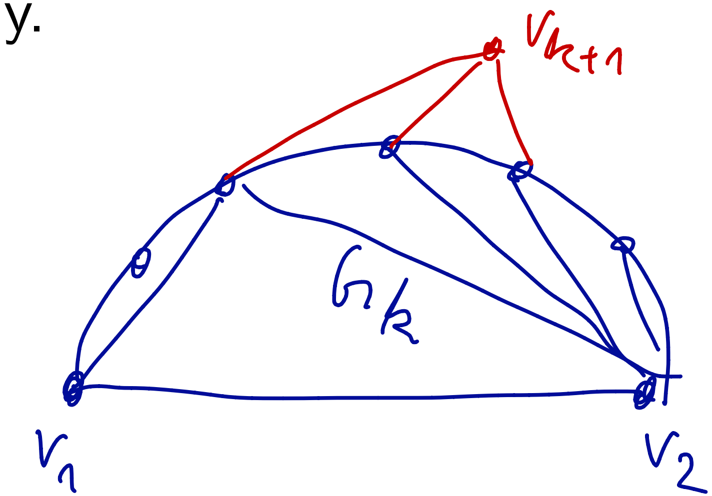
Shift algorithm

Implementation

Canonical Ordering

Def: Let $G = (V, E)$ be a **triangulated planar embedded graph** with $n \geq 3$ vertices. An ordering $\pi = (v_1, v_2, \ldots, v_n)$ is called a **canonical ordering**, if the following conditions hold for each k, $3 \leq k \leq n$

- $\{v_1, \ldots v_k\}$ induce 2-connected internally triangulated graph G_k,
- edge (v_1, v_2) belongs to the outer face of each G_k,
- for all $k < n$ vertex v_{k+1} lies in the external face of G_k, and all neighbors of v_{k+1} in G_k appear on the boundary C_k of G_k consecutively.



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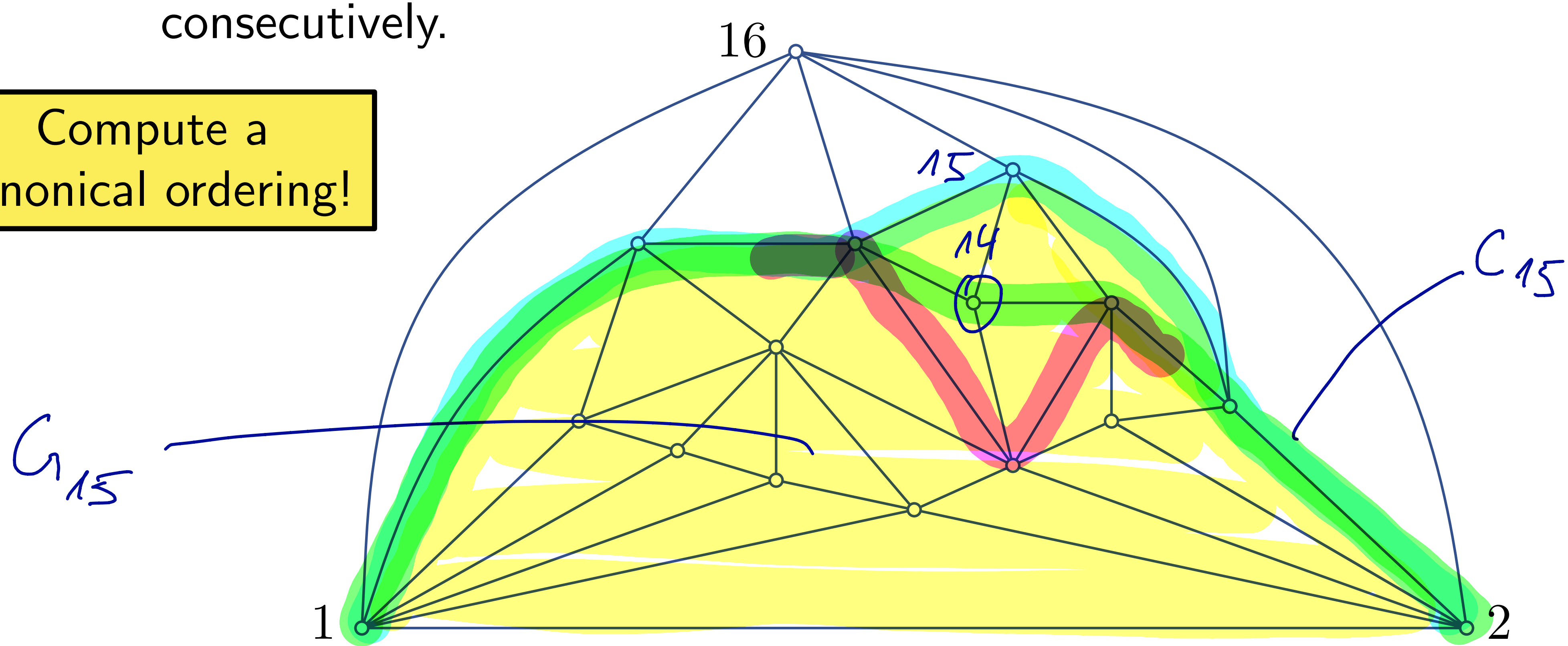
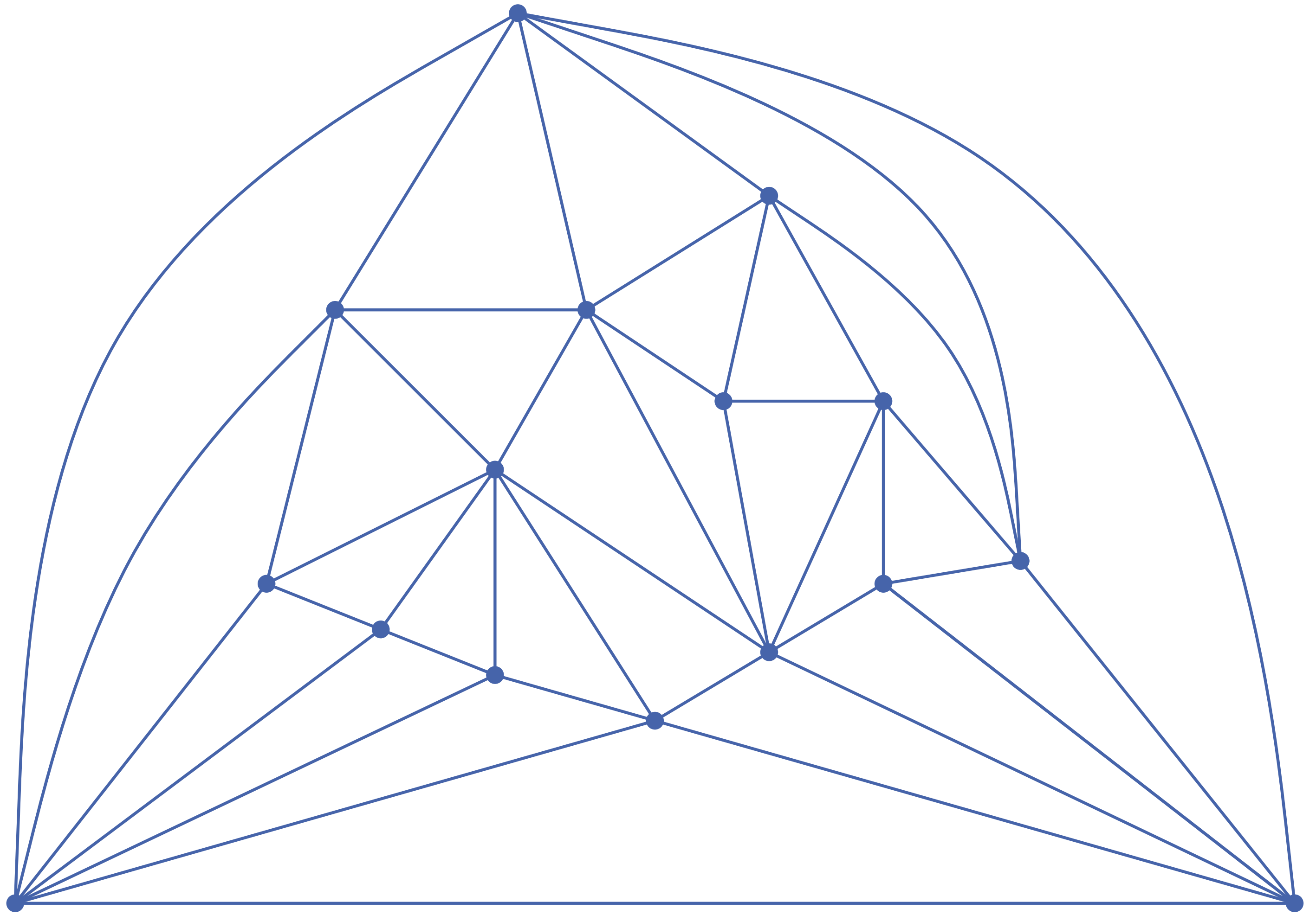
Compute a canonical ordering!



Example: Canonical Ordering



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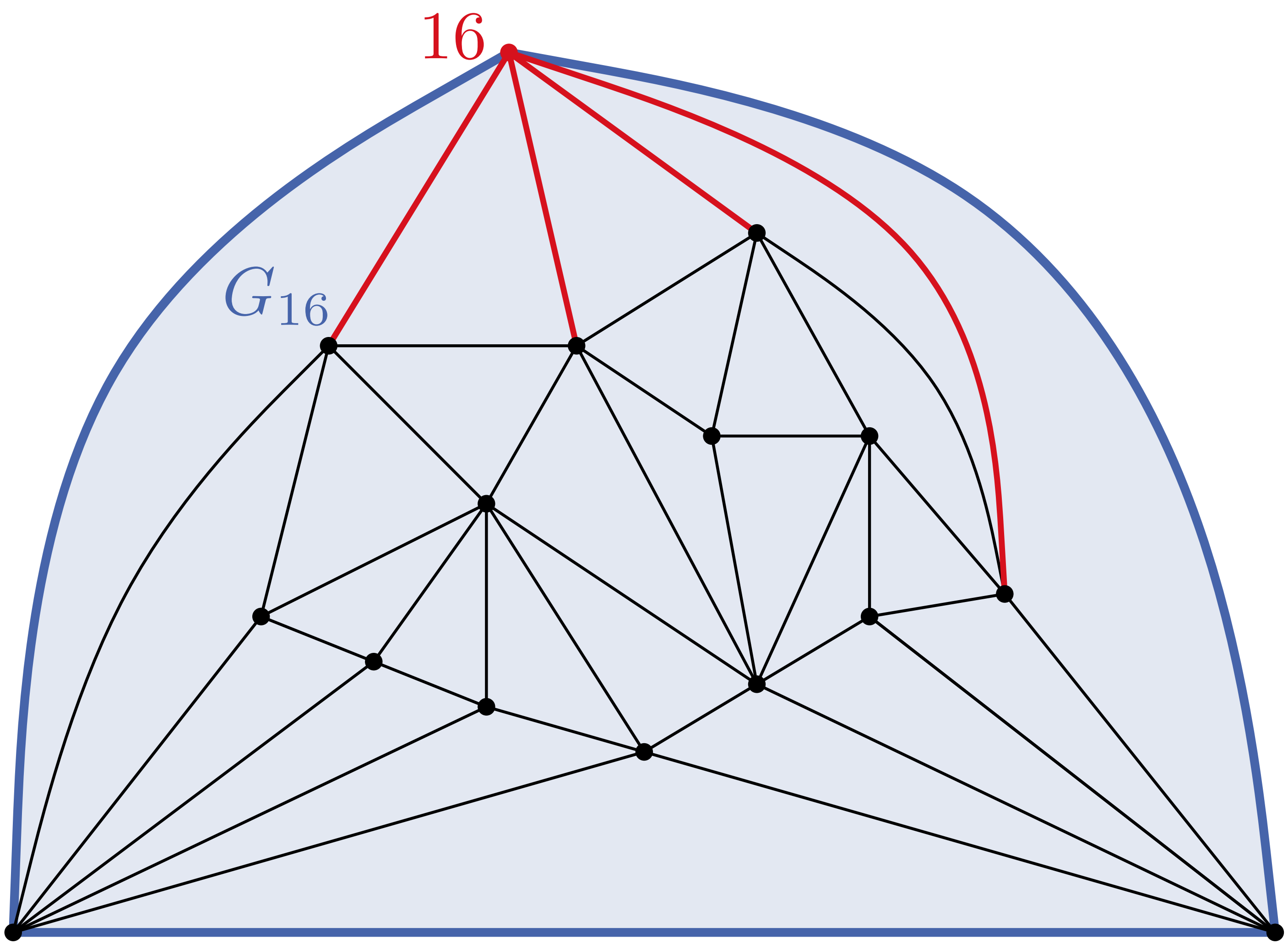

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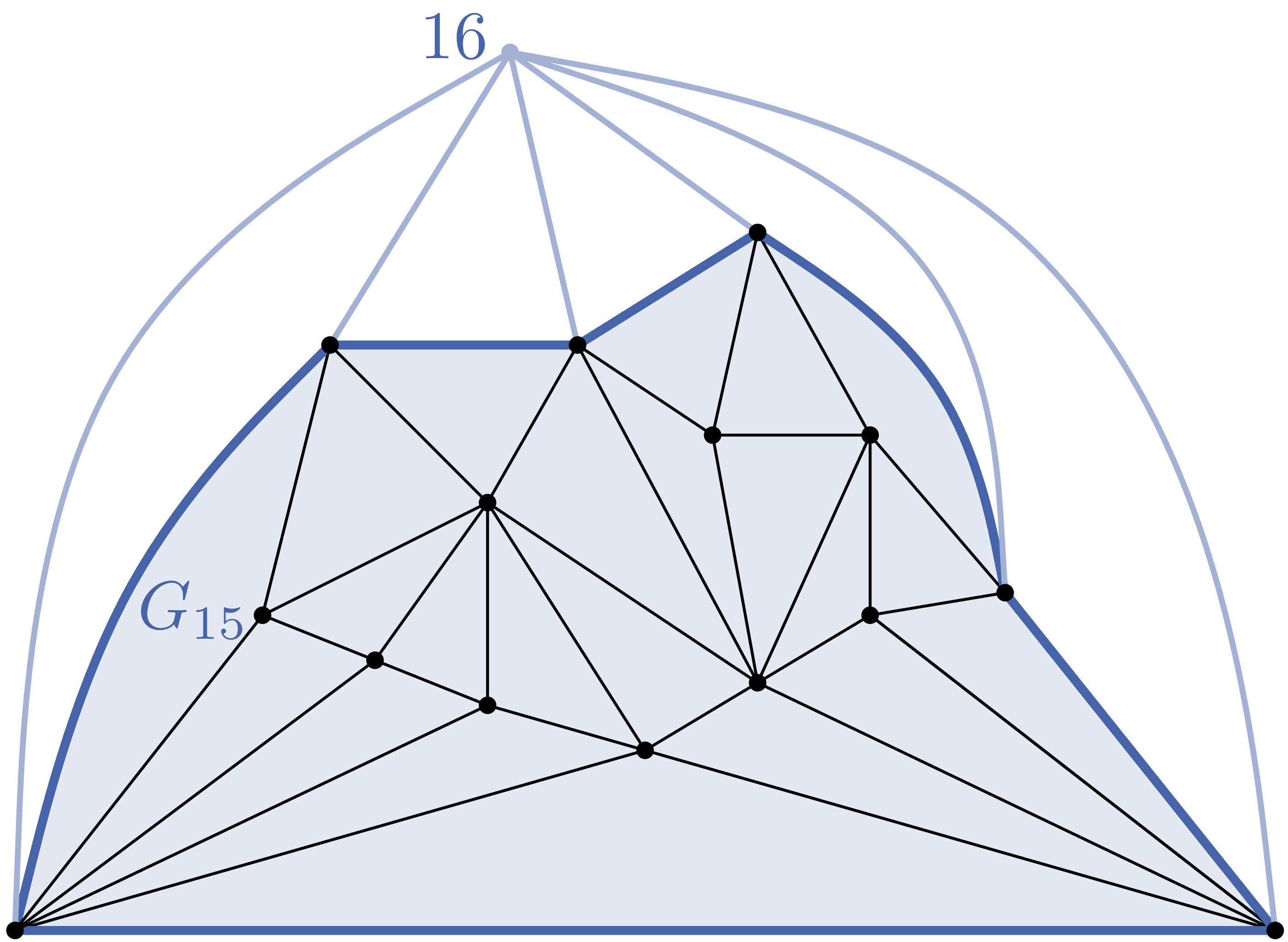

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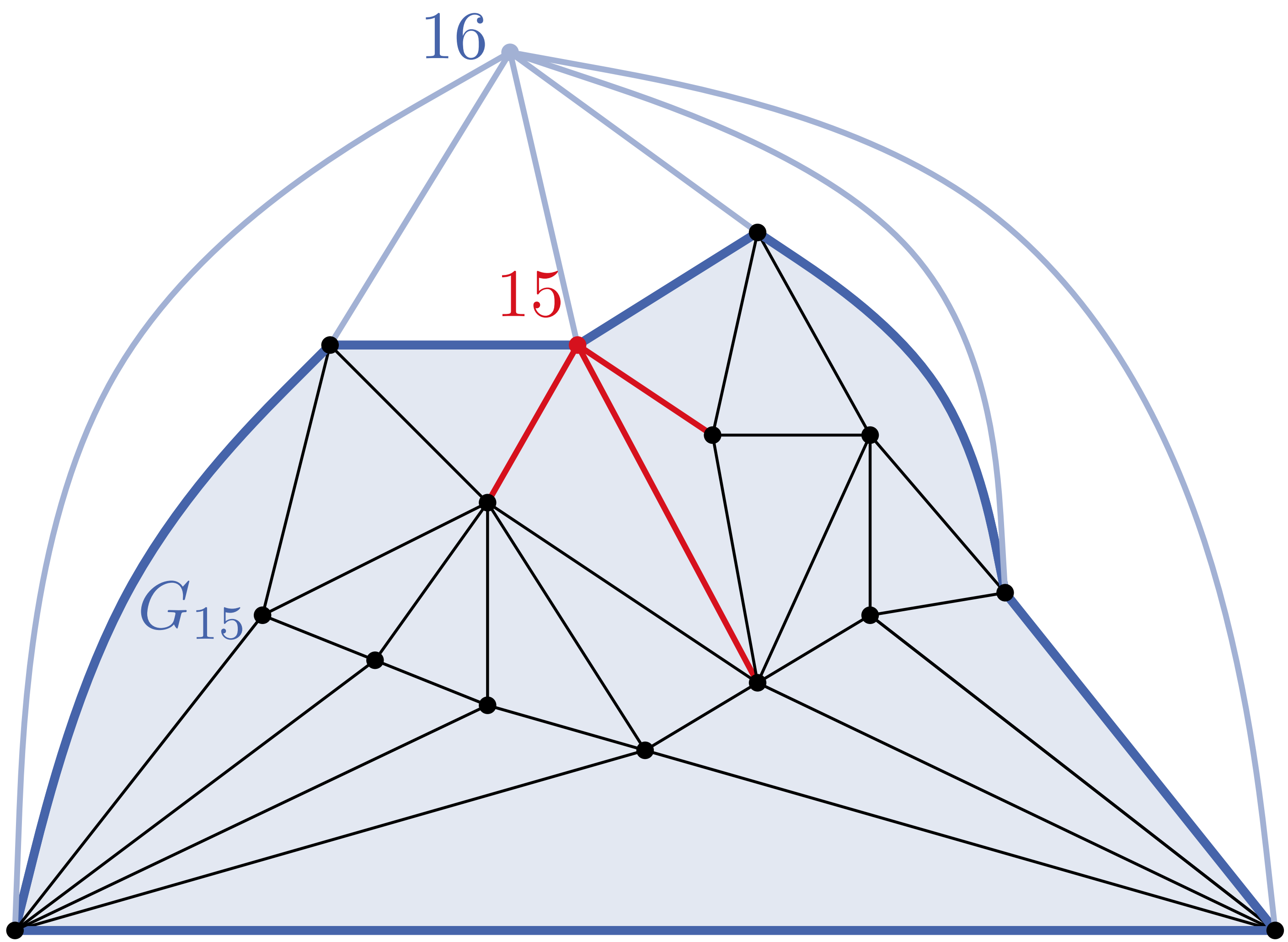

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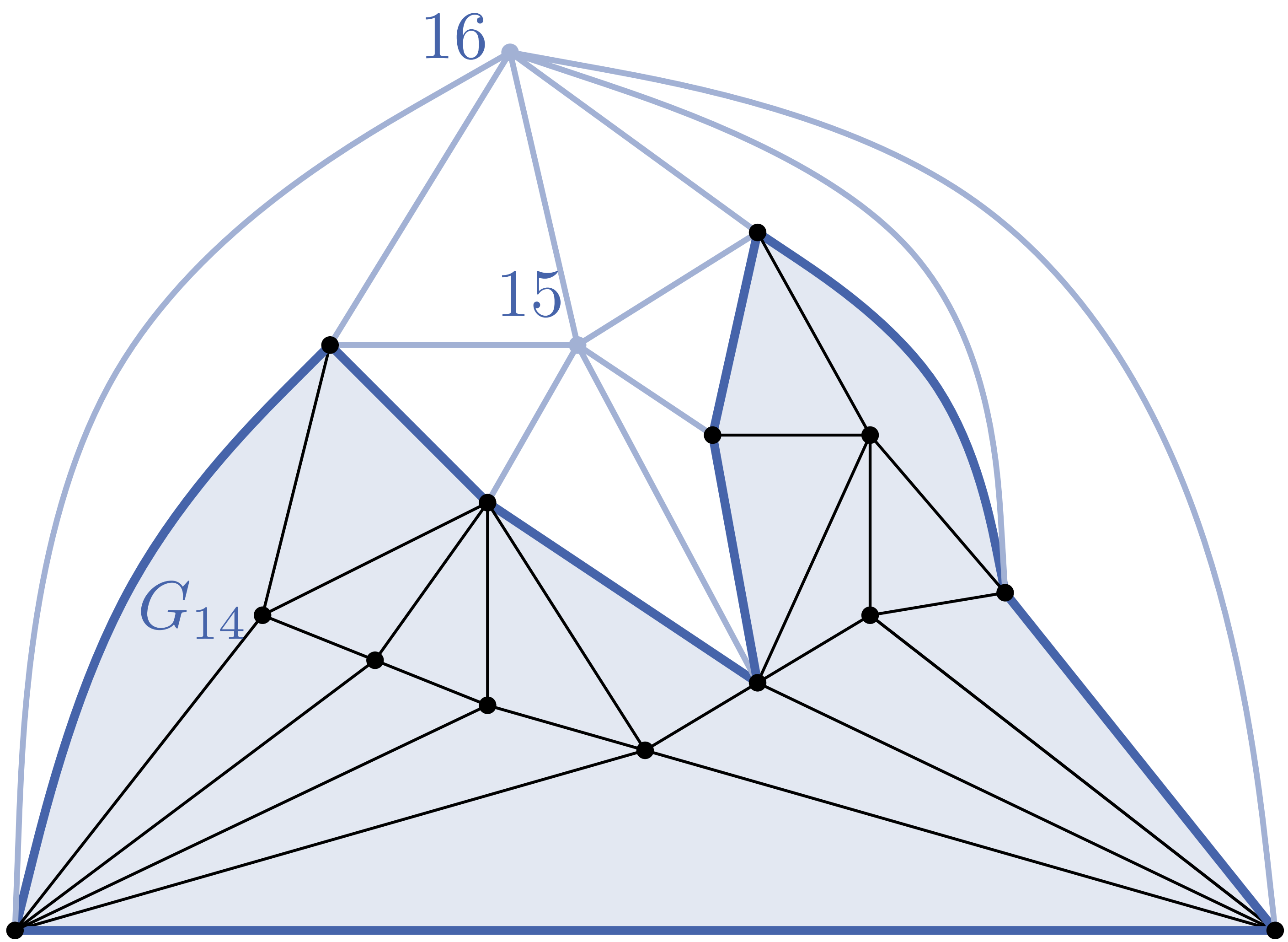

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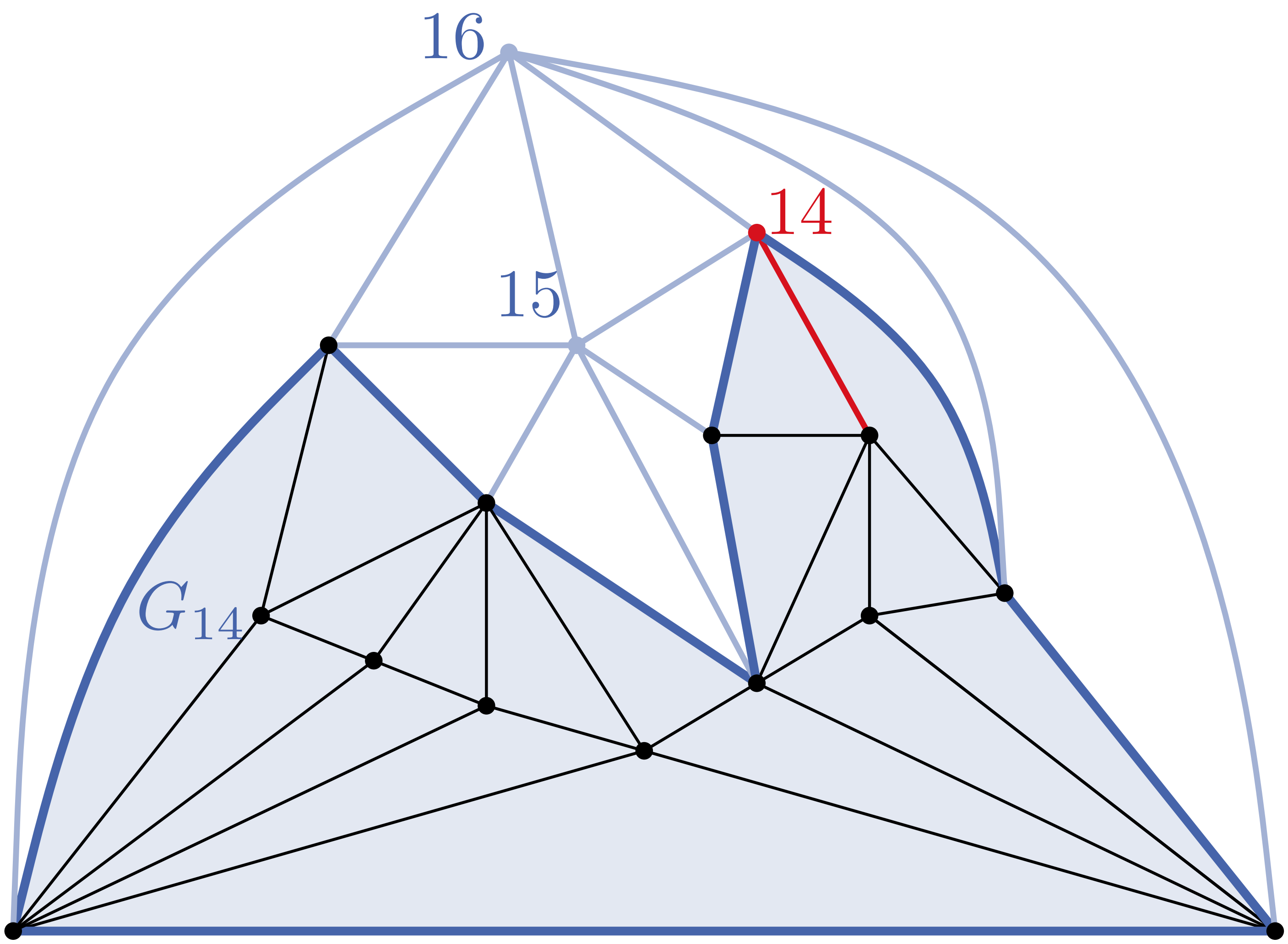

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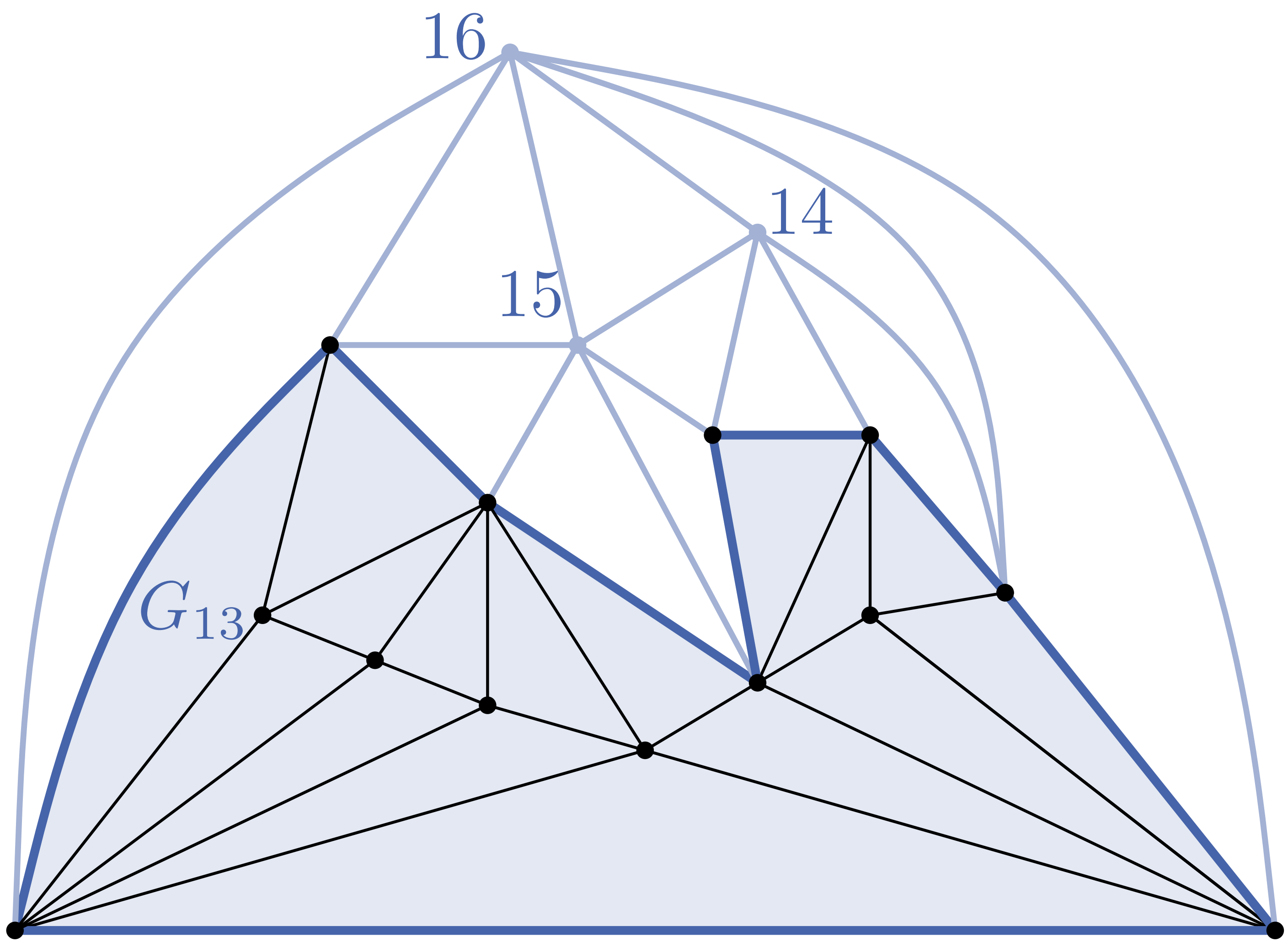

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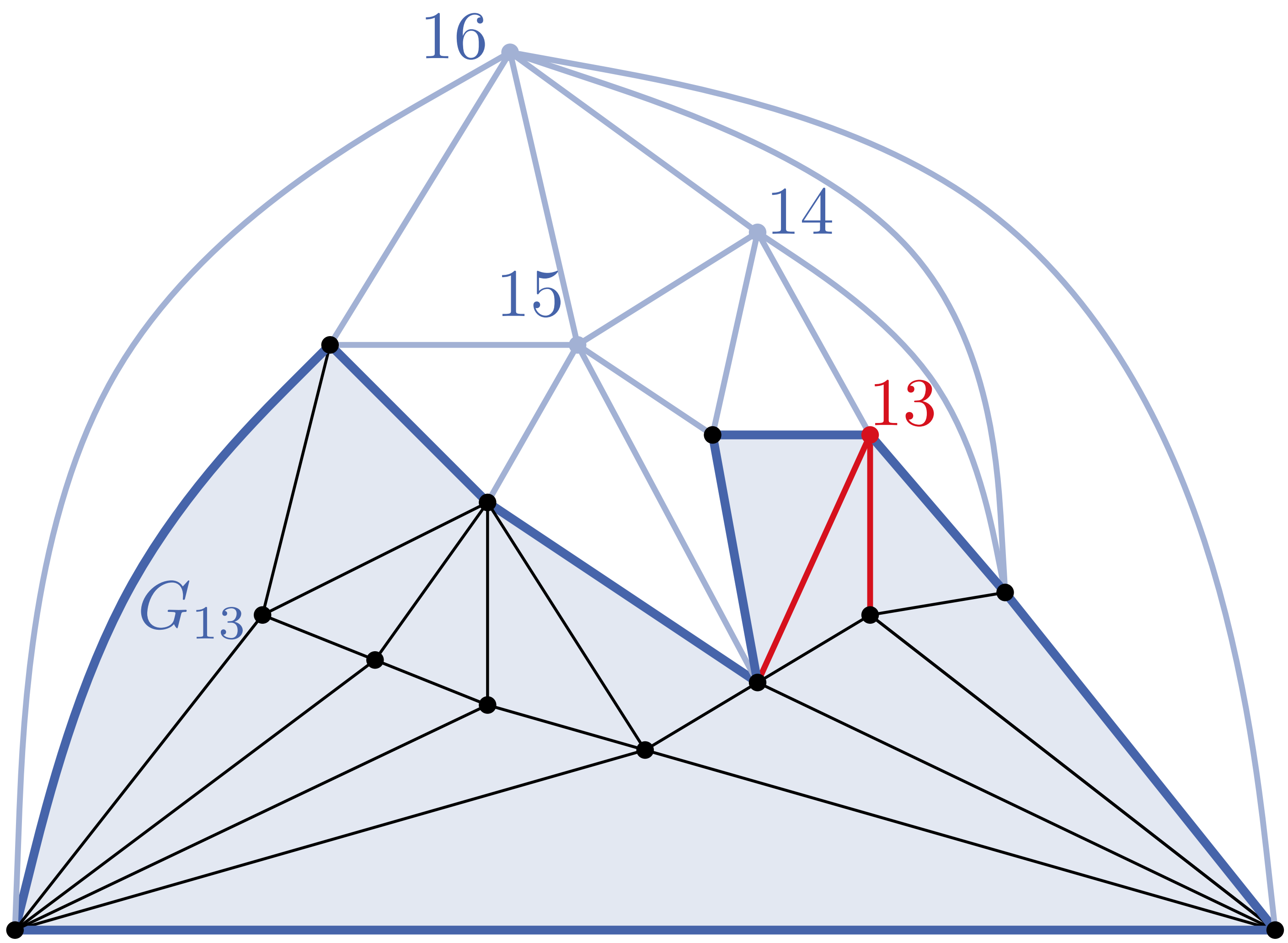

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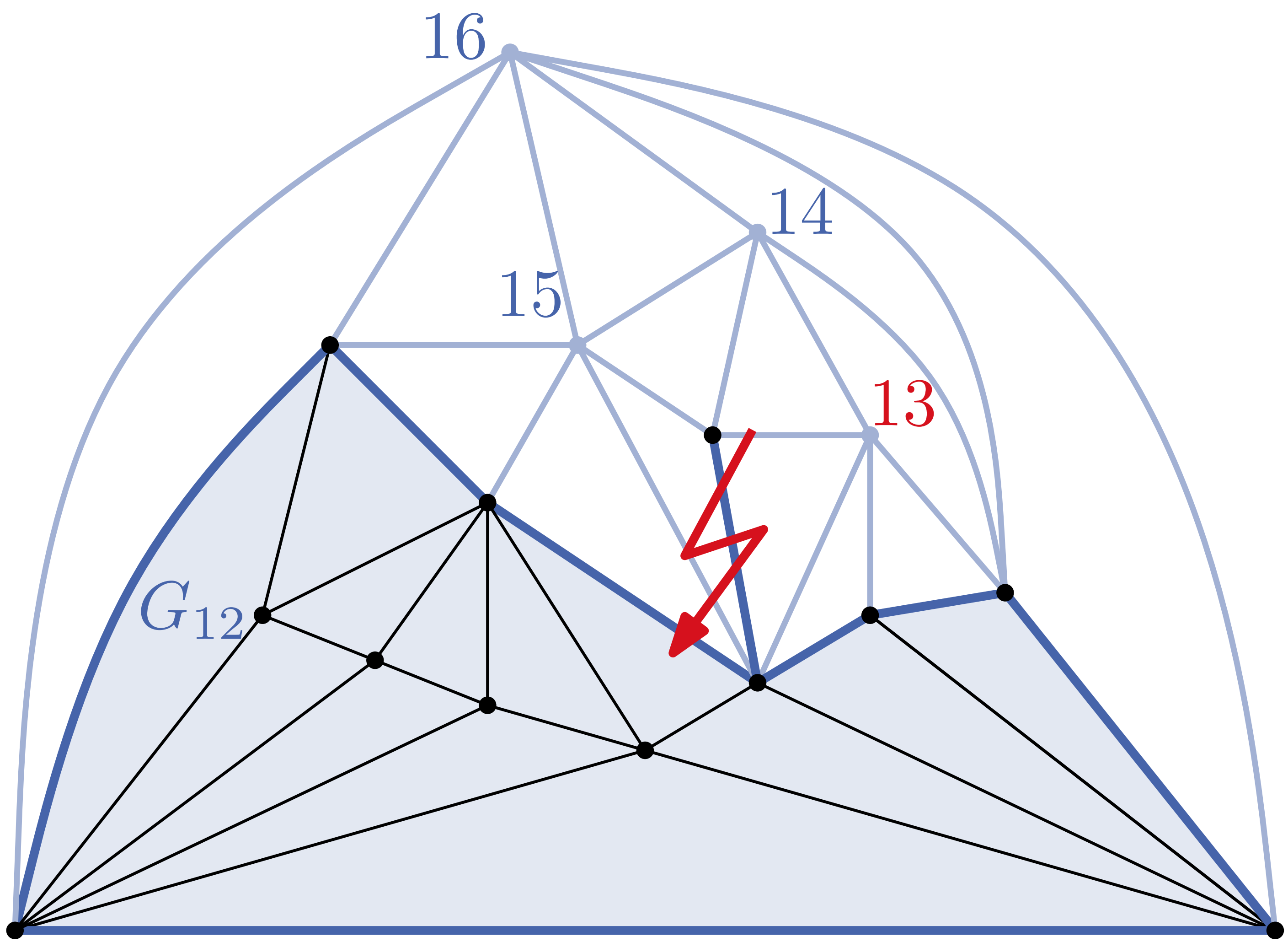

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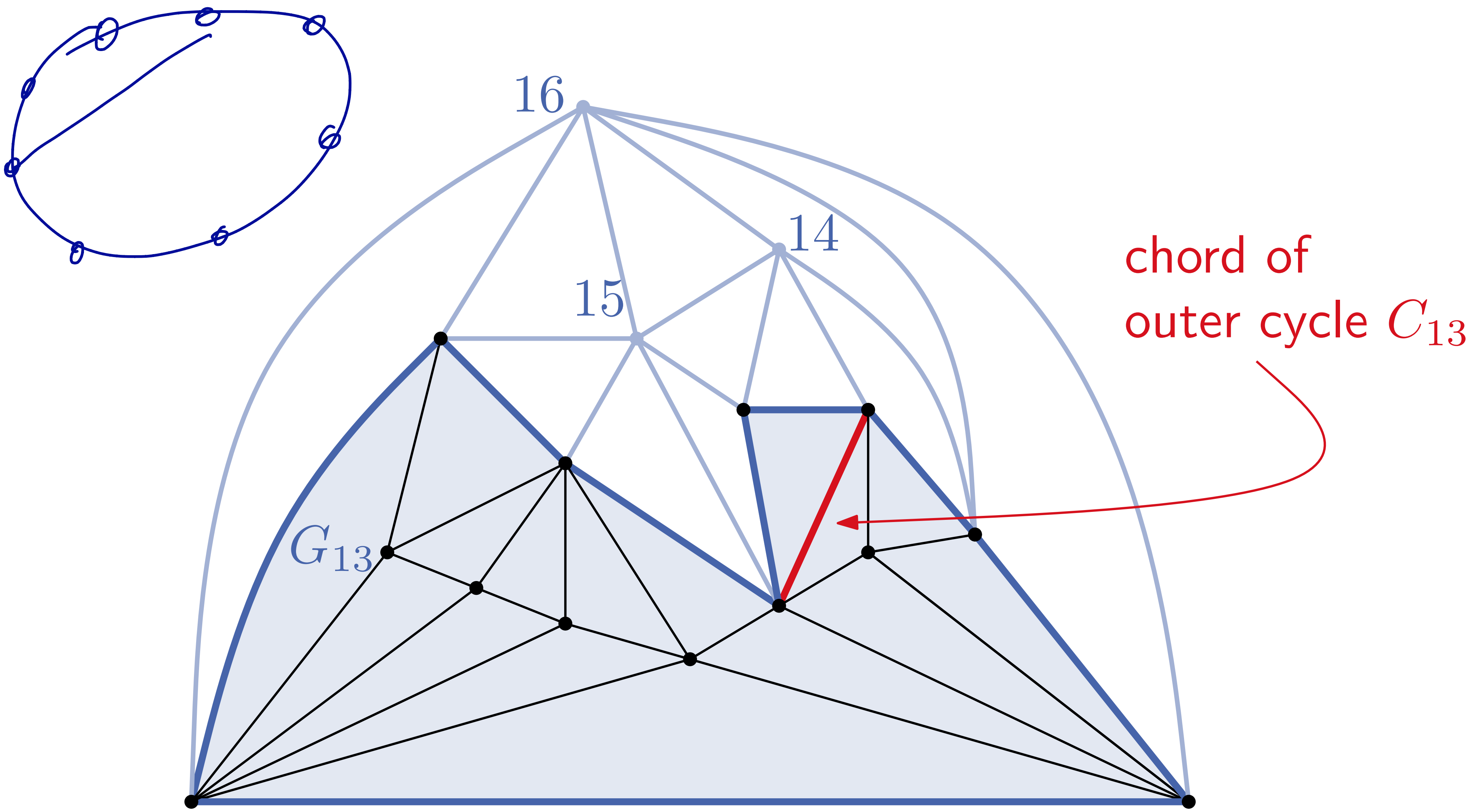

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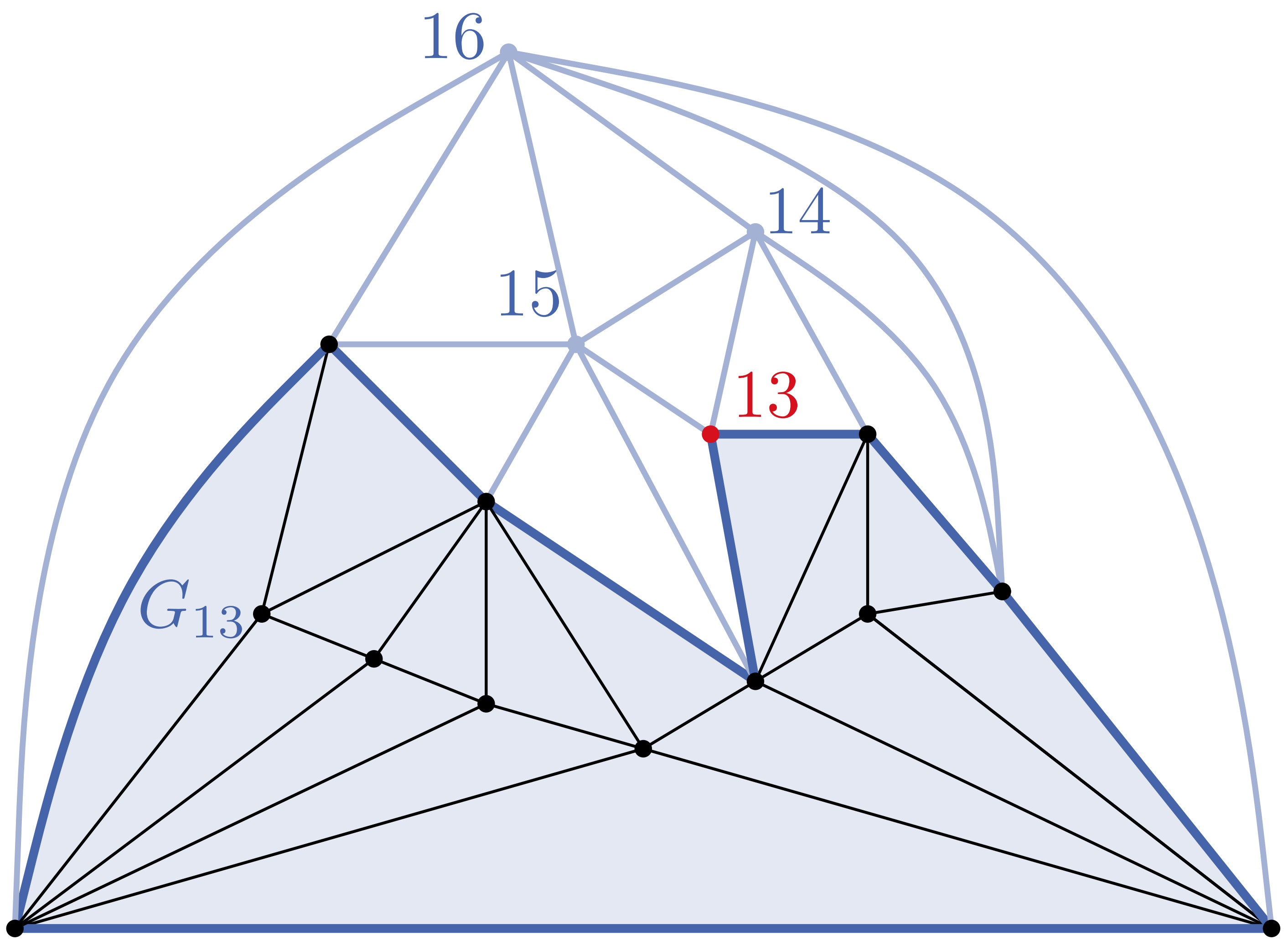

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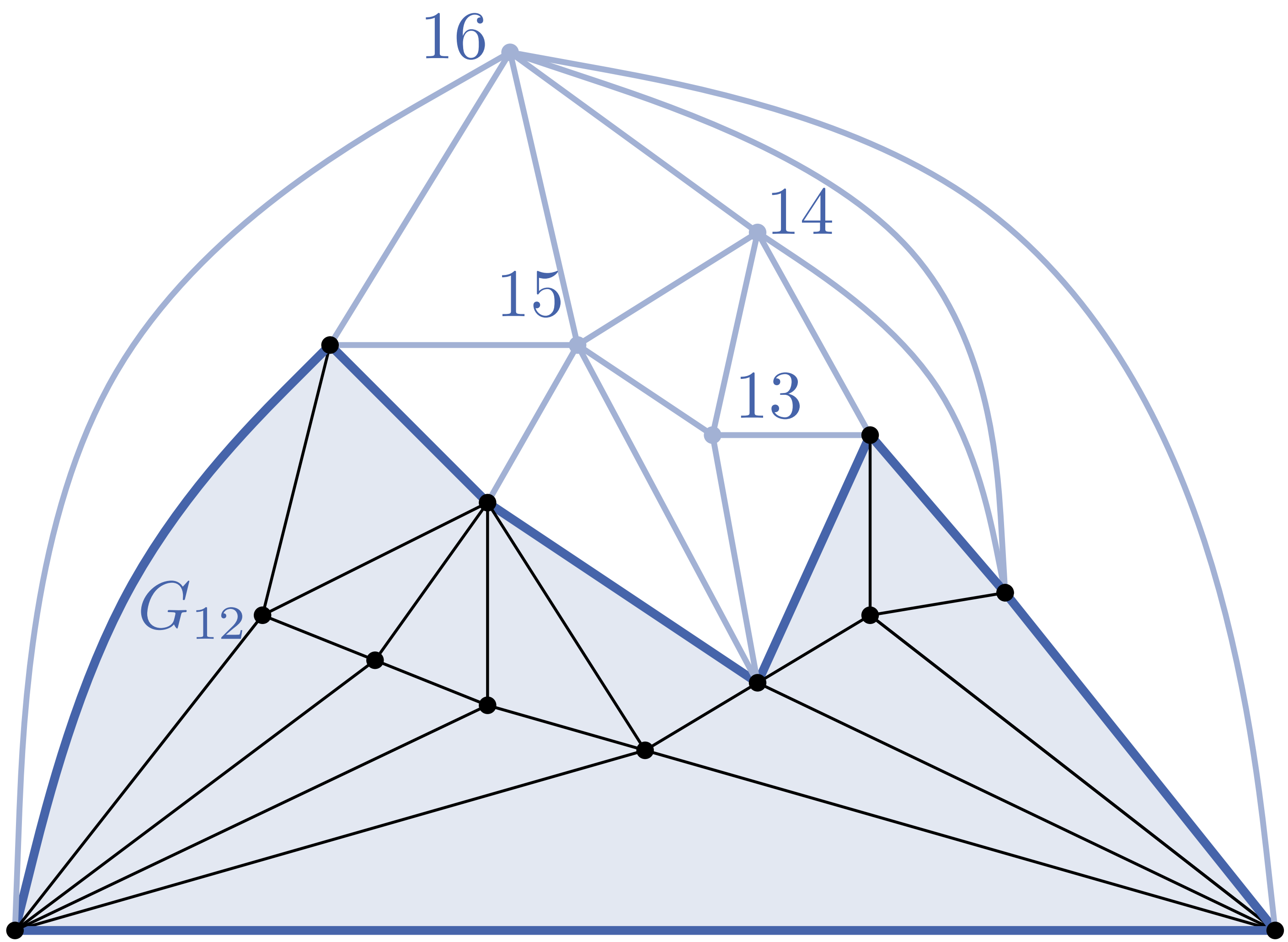

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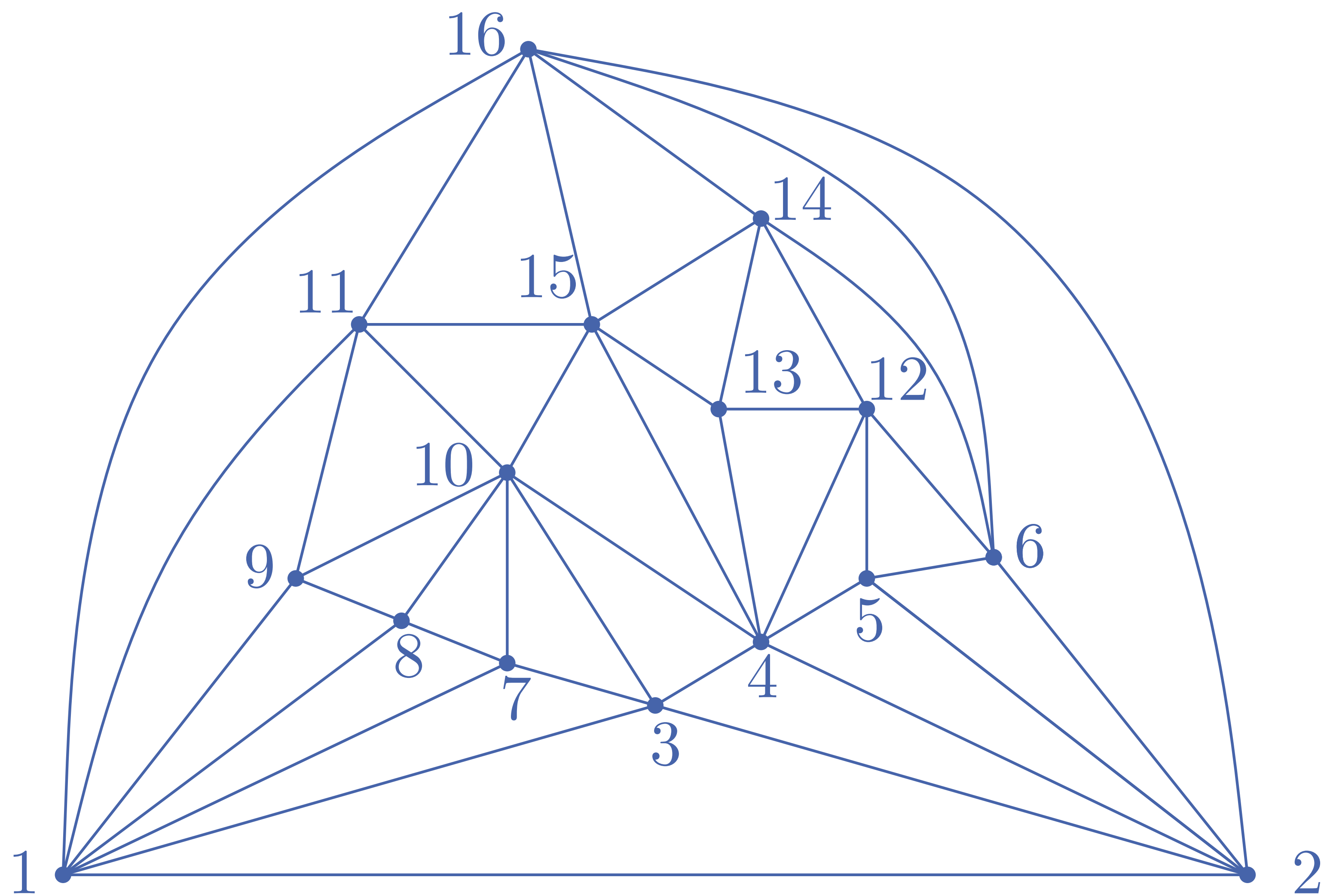

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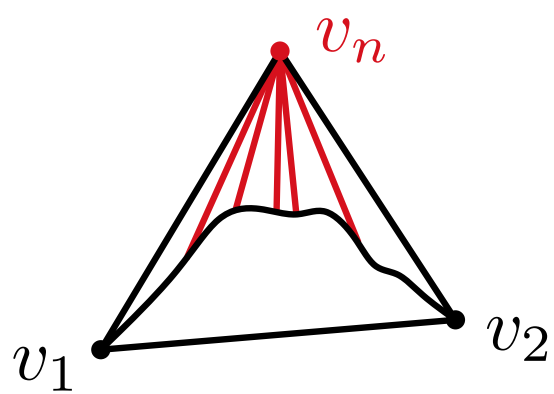

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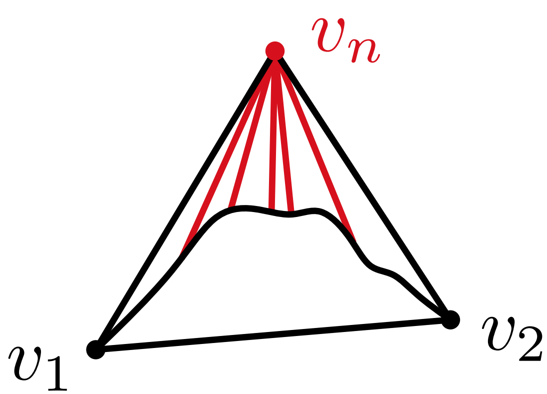

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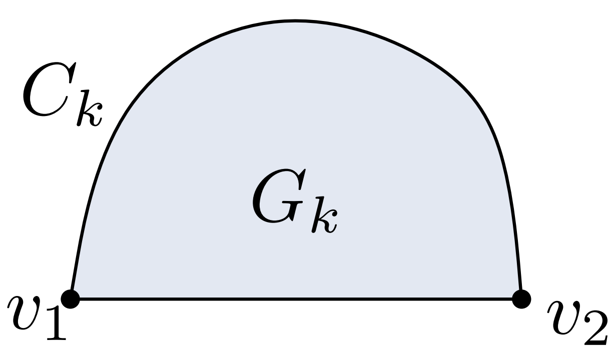

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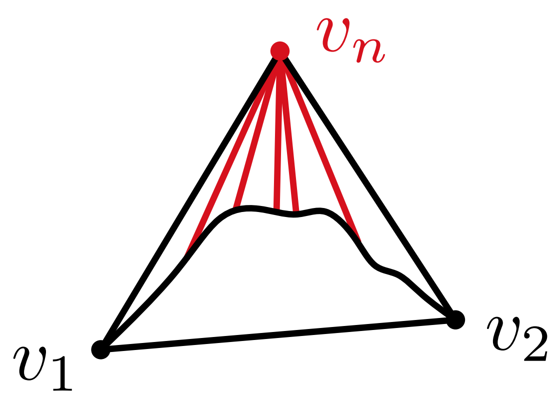

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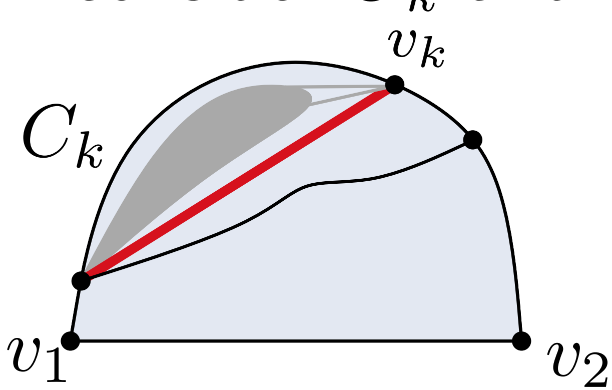

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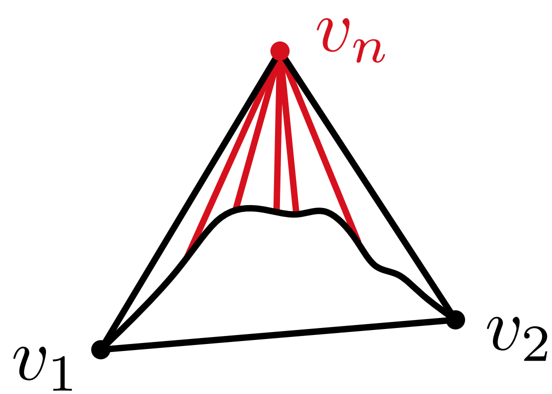

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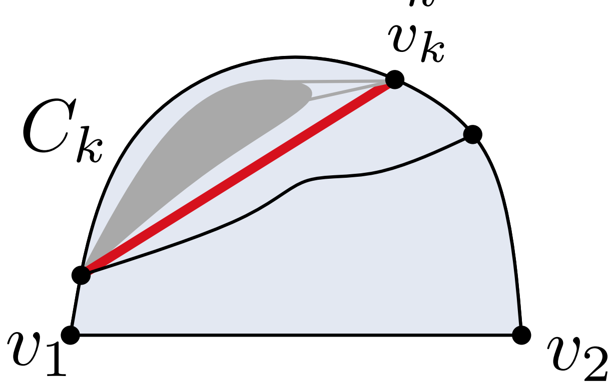

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Is this sufficient?

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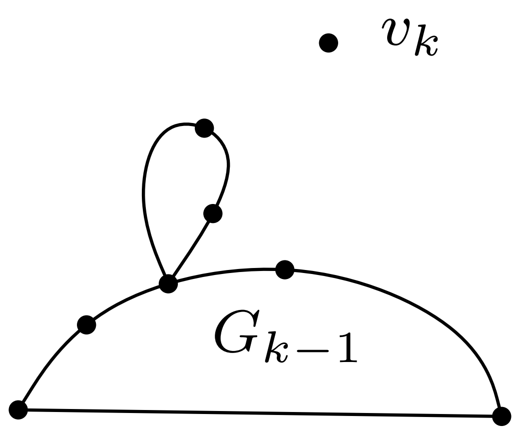

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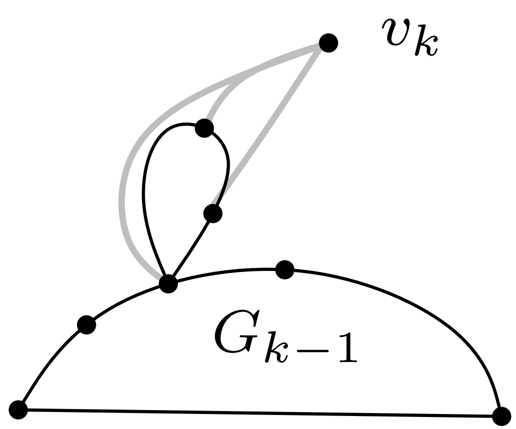

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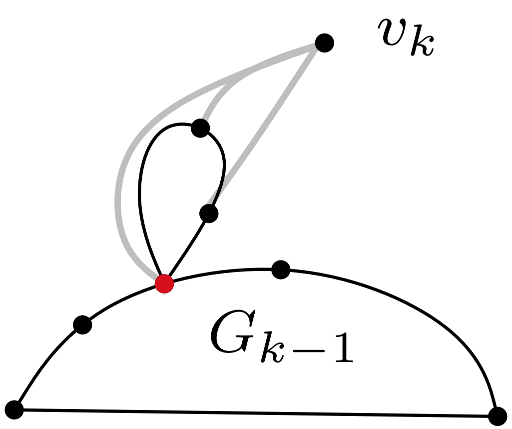

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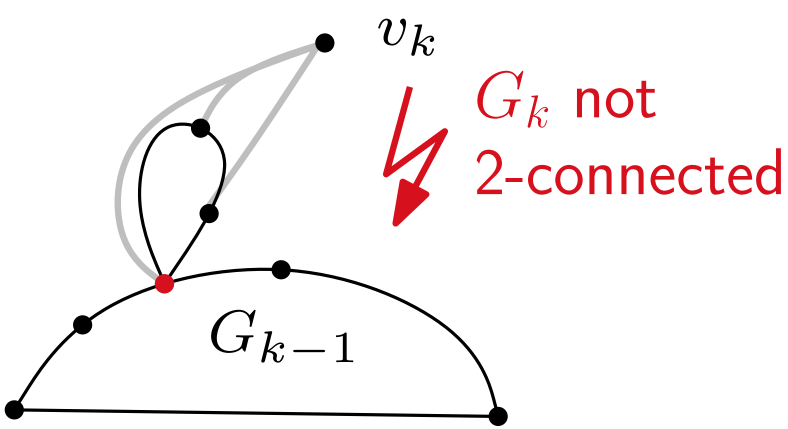

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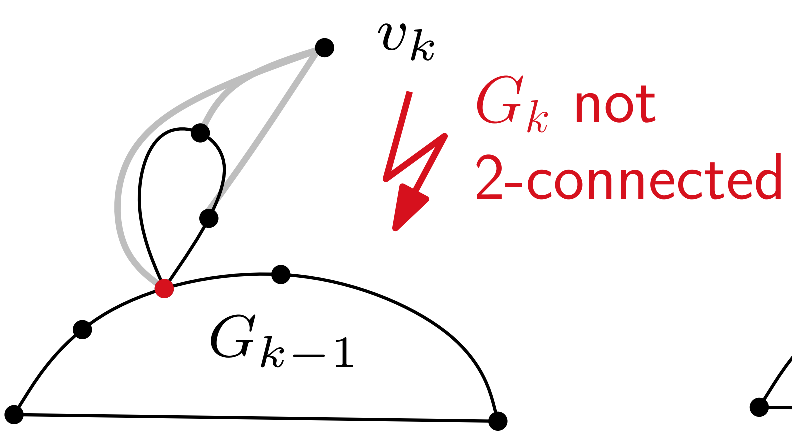

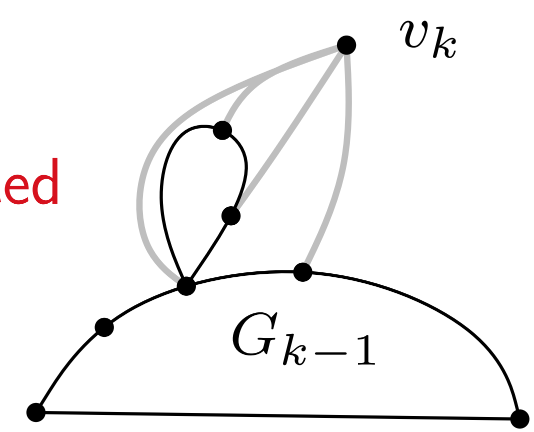

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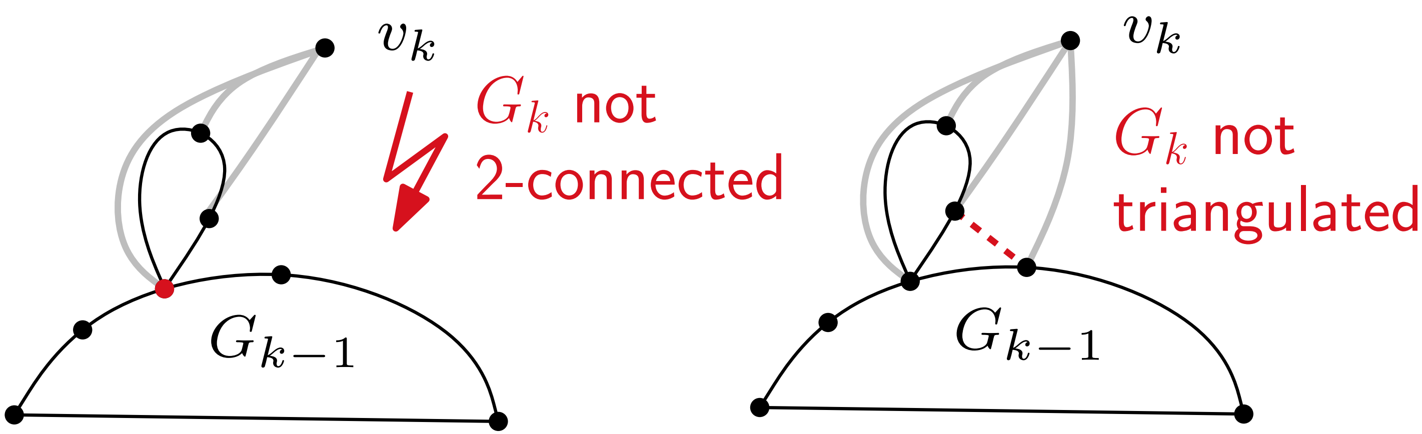

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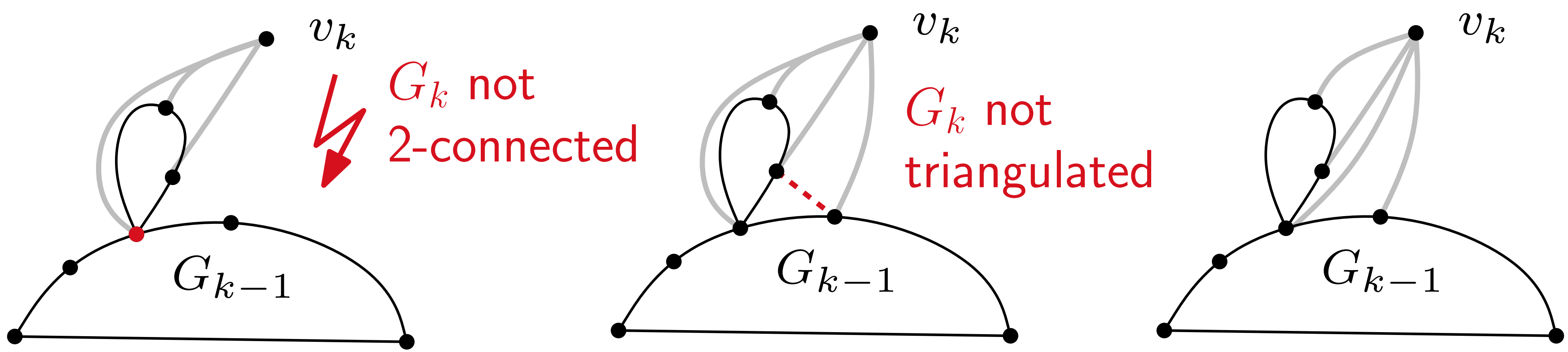

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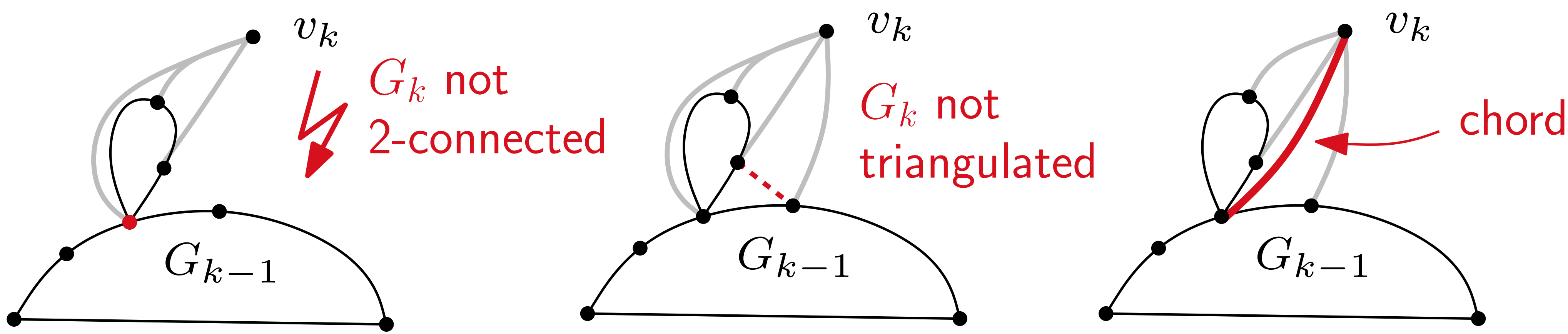

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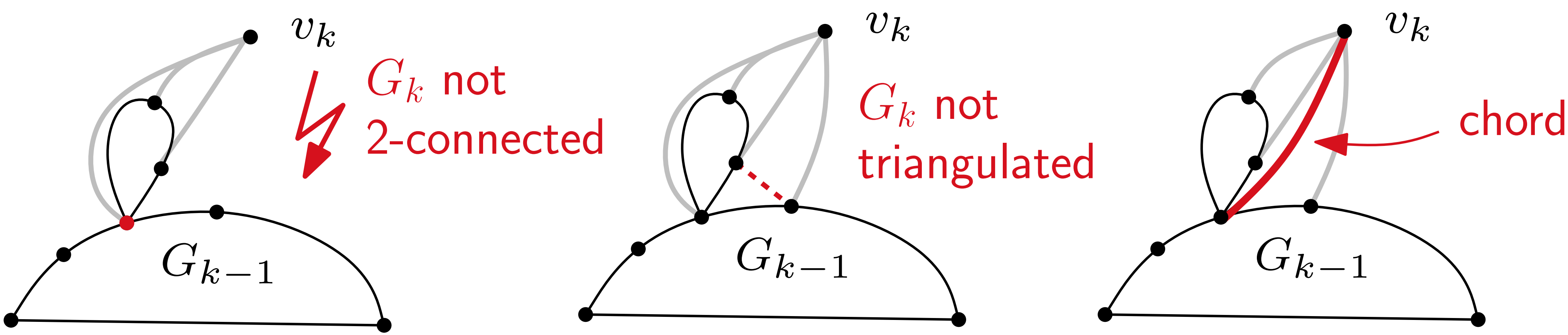

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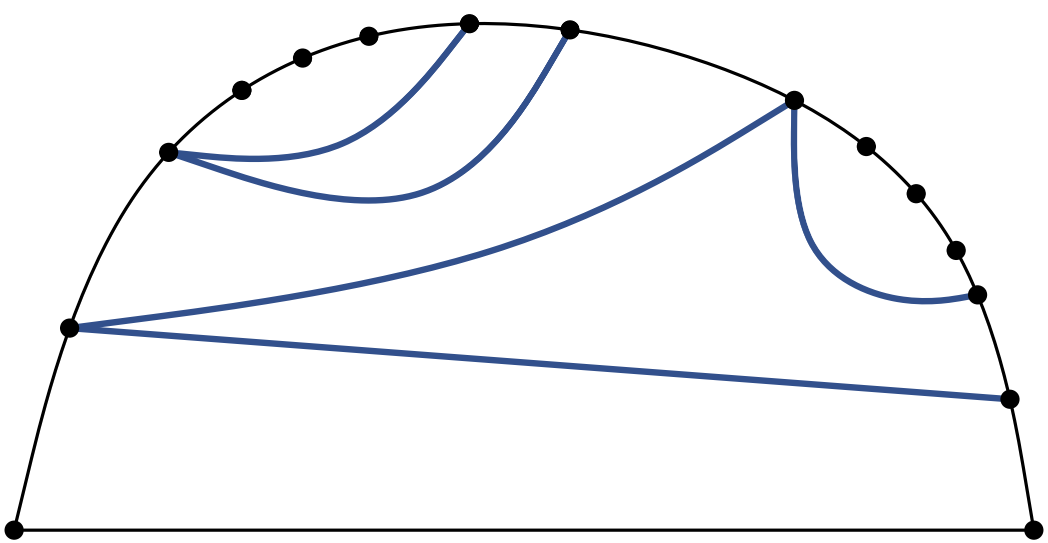

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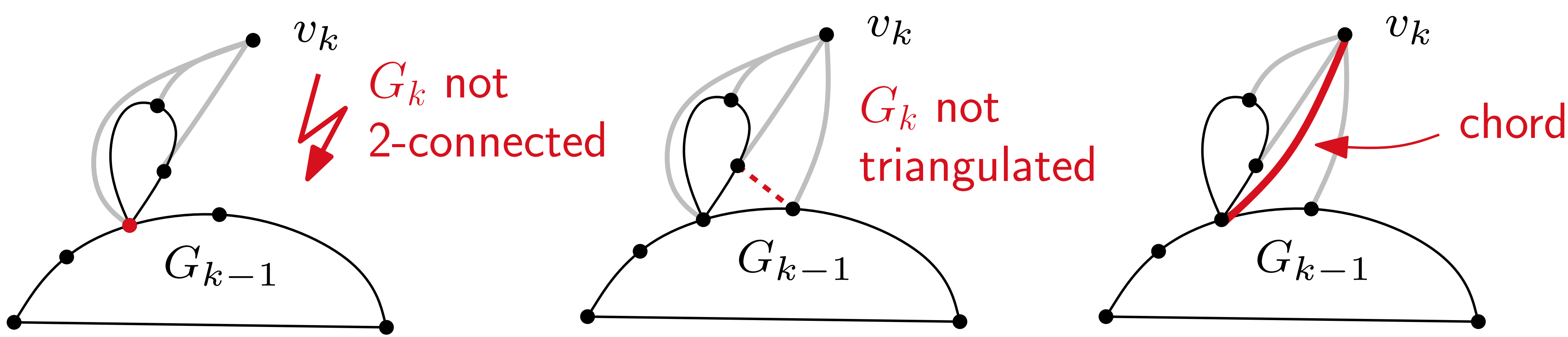

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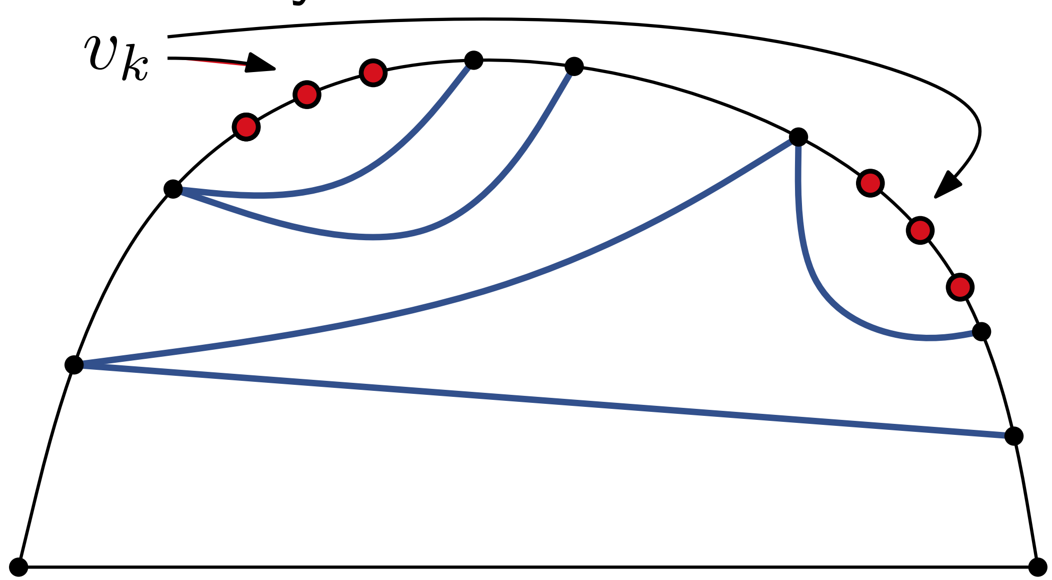


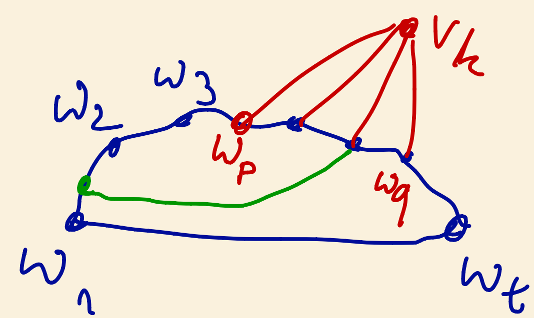
- chords form proper nesting
- innermost chords span at least one such vertex

□

Computing a Canonical Ordering

`CanonicalOrdering`(plane graph $G = (V, E)$)

forall $v \in V$ **do**
  chords(v) $\leftarrow 0$; out(v) $\leftarrow$ false; mark(v) $\leftarrow$ false;
out(v_1), out(v_2), out(v_n) $\leftarrow$ true
for $k = n$ **to** 3 **do**
  choose $v \neq v_1, v_2$ with mark(v) = false, out(v) = true, chords(v) = 0
  $v_k \leftarrow v$; mark(v) $\leftarrow$ true
  $(w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2) \leftarrow C_{k-1}$
  $(w_p, \ldots, w_q) \leftarrow$ unmarked neighbors of v_k
  out(w_i) $\leftarrow$ true for all $p < i < q$
  update chords($\cdot$) for these w_i and their neighbors



- chord(v) = number of chords incident to v
- mark(v) = true iff vertex v was numbered
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Lemma: Algorithm `CanonicalOrdering` computes a canonical ordering of G in $O(n)$ time.

Overview

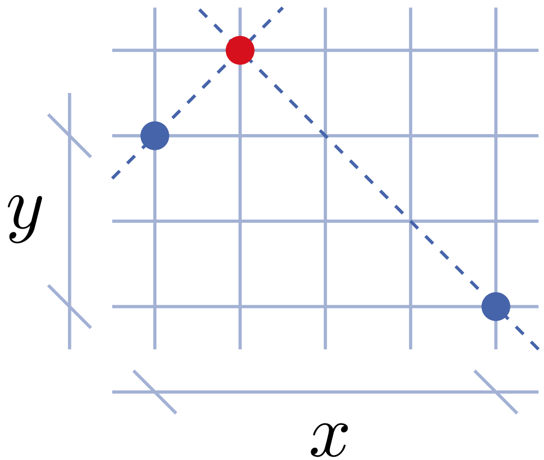
Canonical ordering

Shift algorithm

Implementation

Shift Algorithm [de Fraysseix, Pach, Pollack 1988]

even L_1-distance



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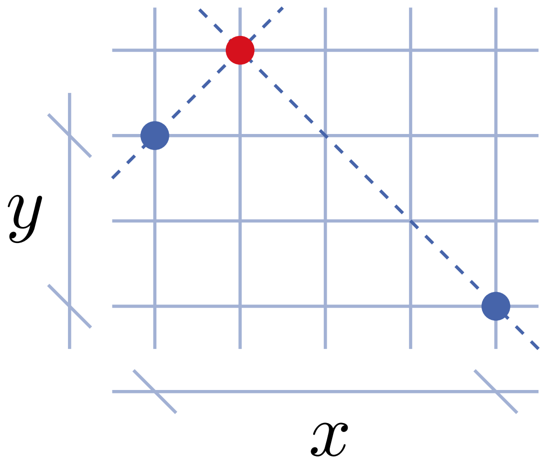

Algorithm invariants: G_{k-1} is drawn such that

- v_1 is on $(0, 0)$, v_2 is on $(2k - 6, 0)$
- boundary C_{k-1} of G_{k-1} (minus edge (v_1, v_2)) is drawn x-monotone
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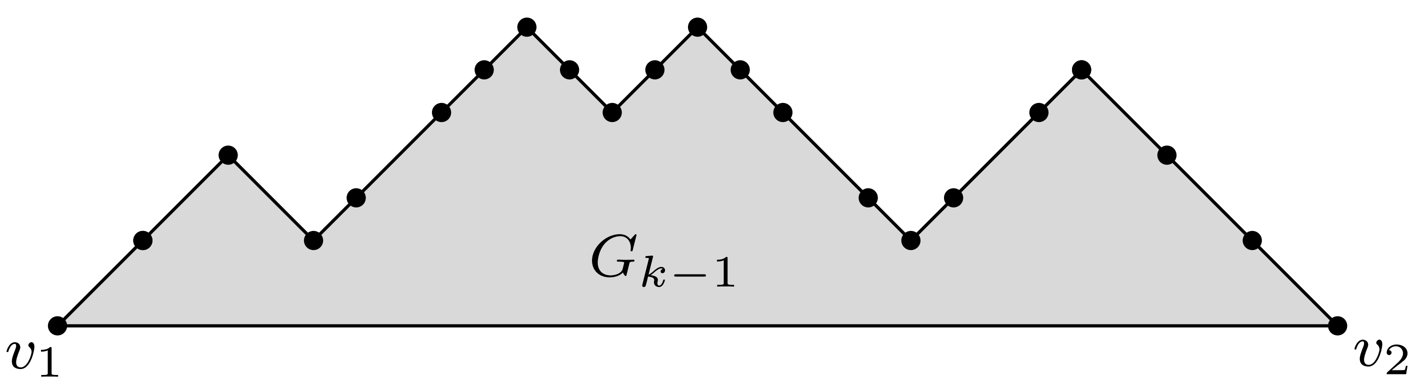

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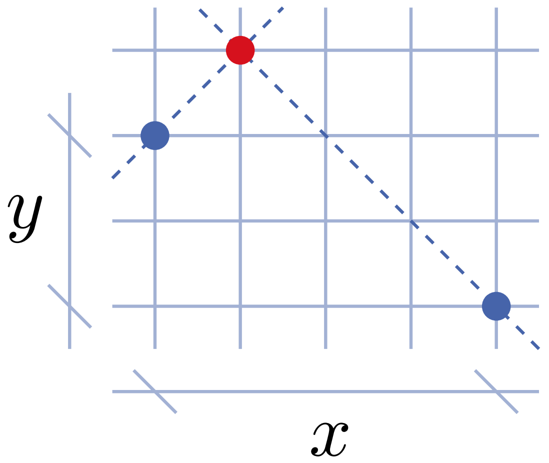

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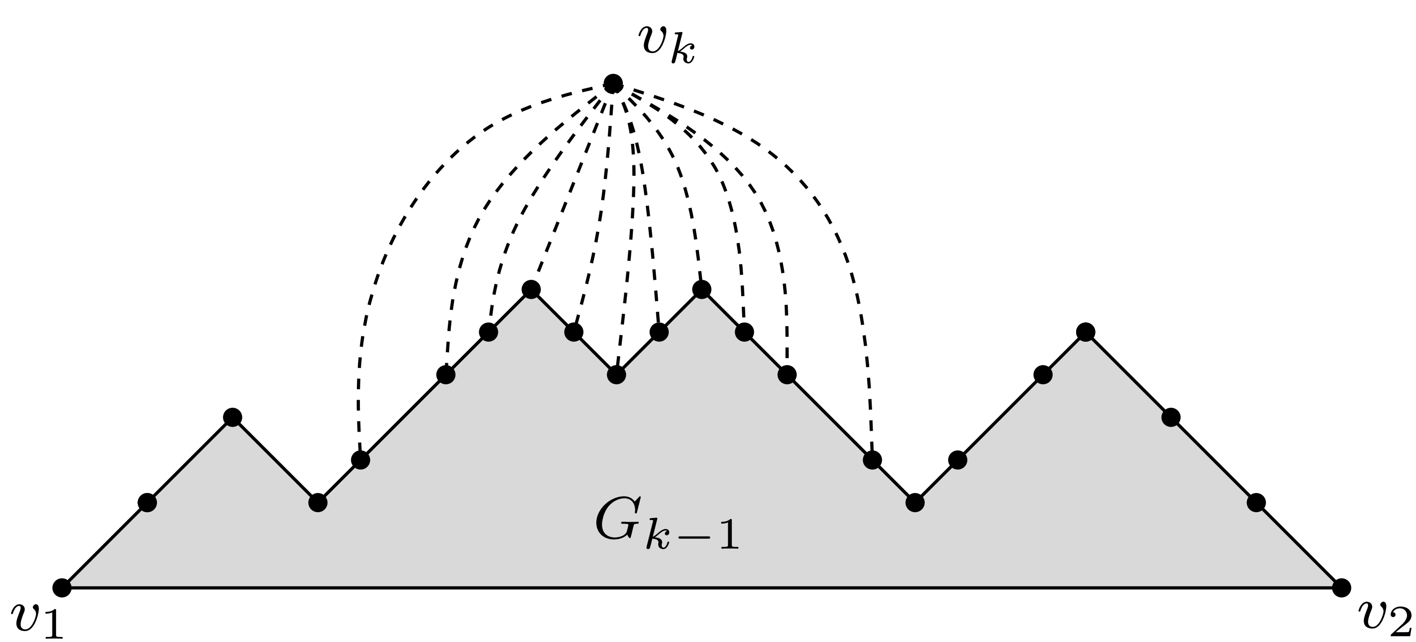

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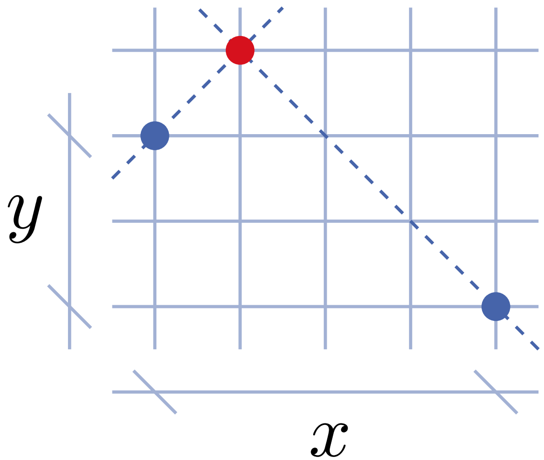

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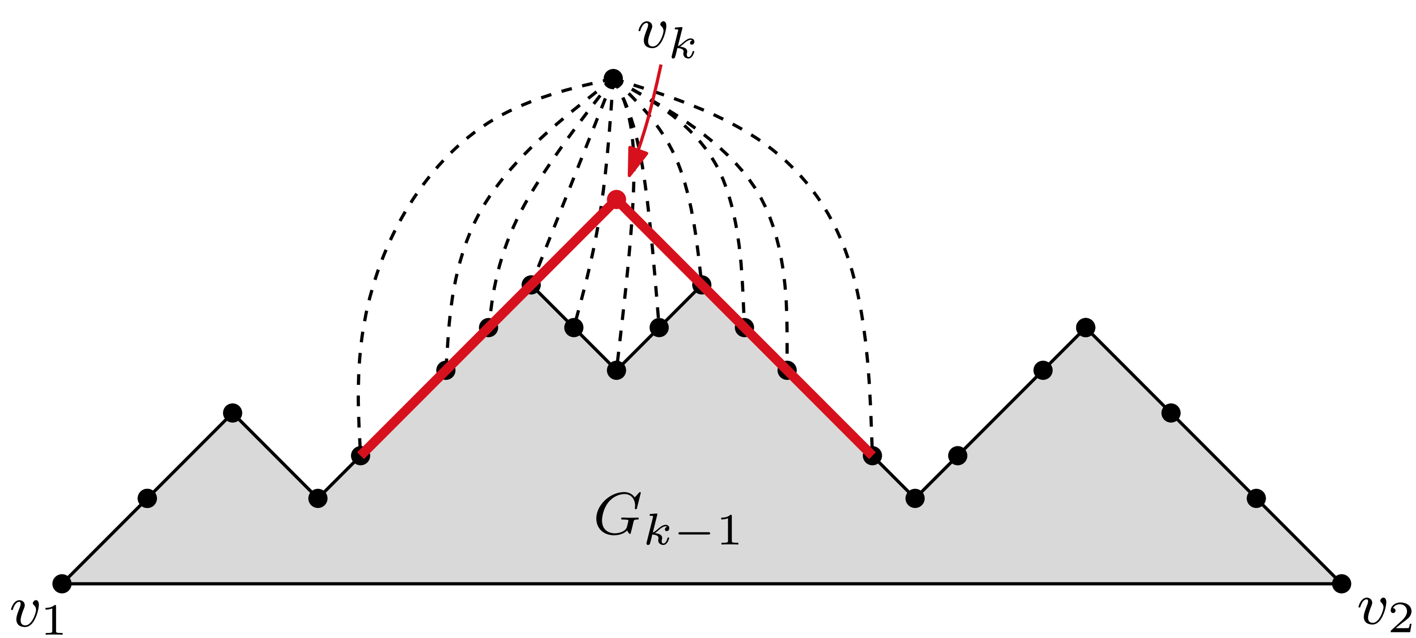

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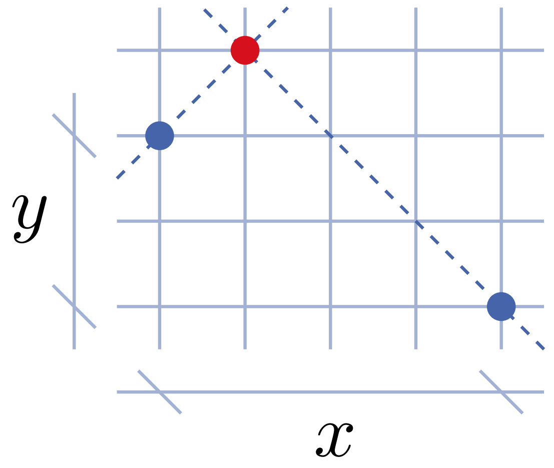

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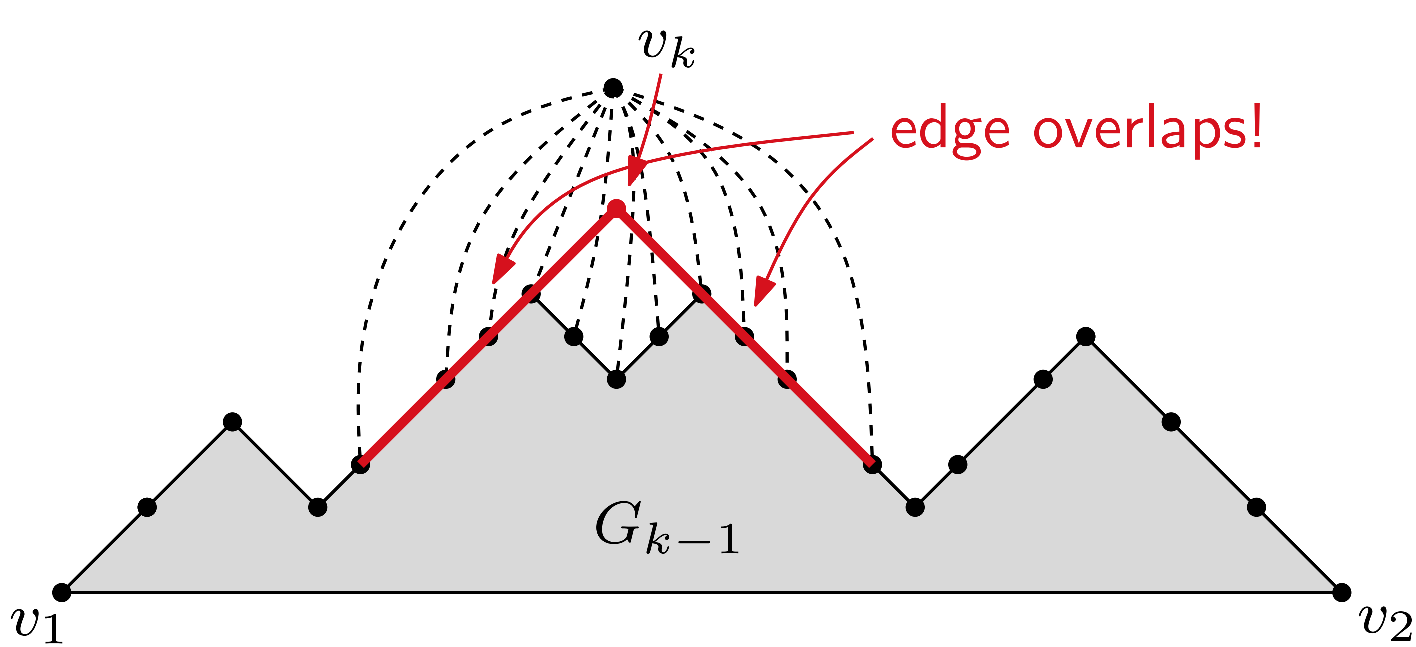

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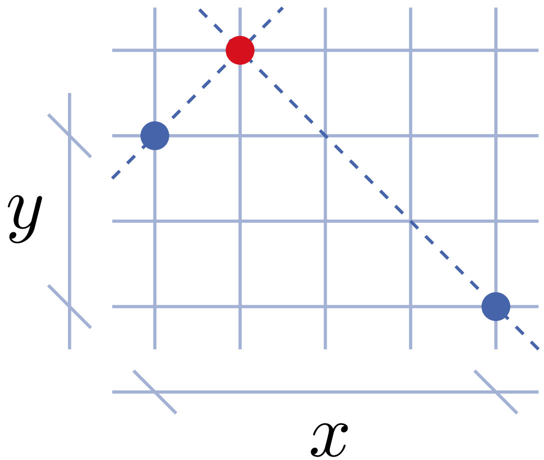

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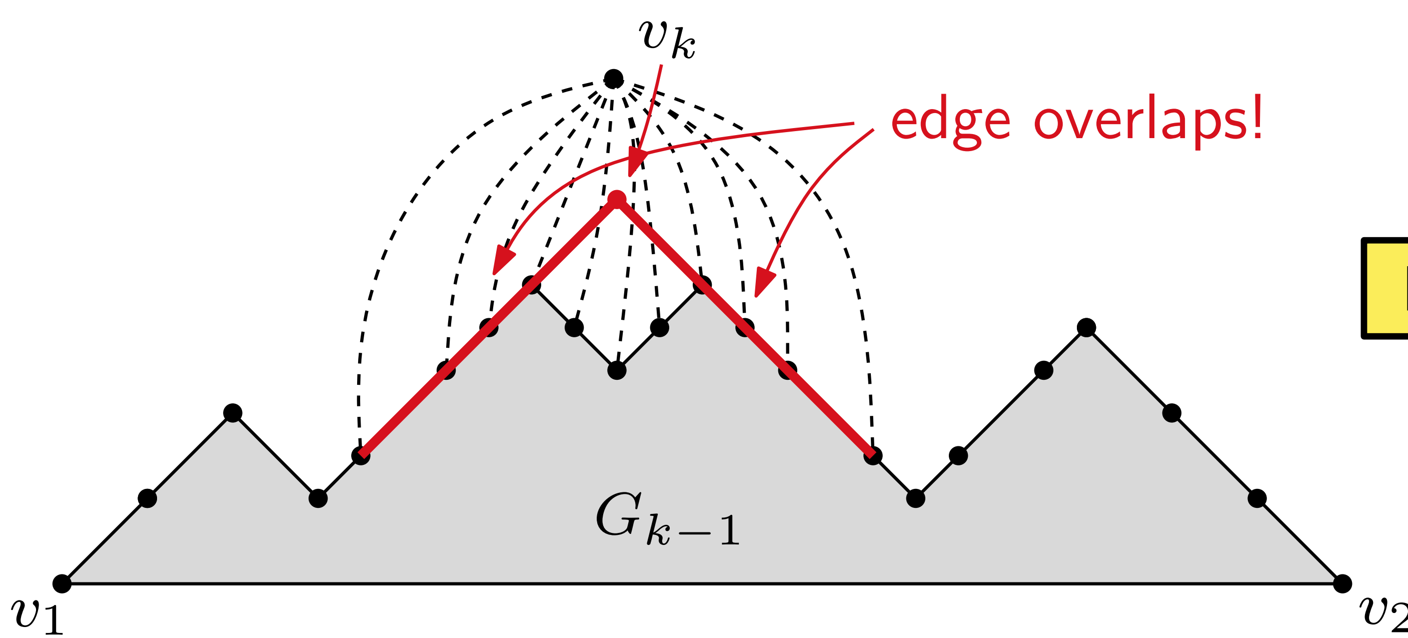

How to fix this?

Shift Algorithm [de Fraysseix, Pach, Pollack 1988]

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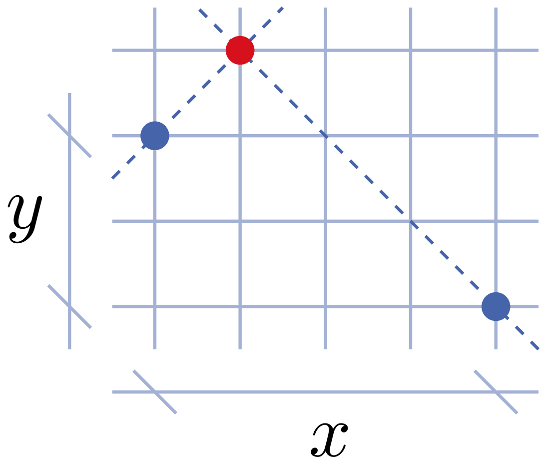

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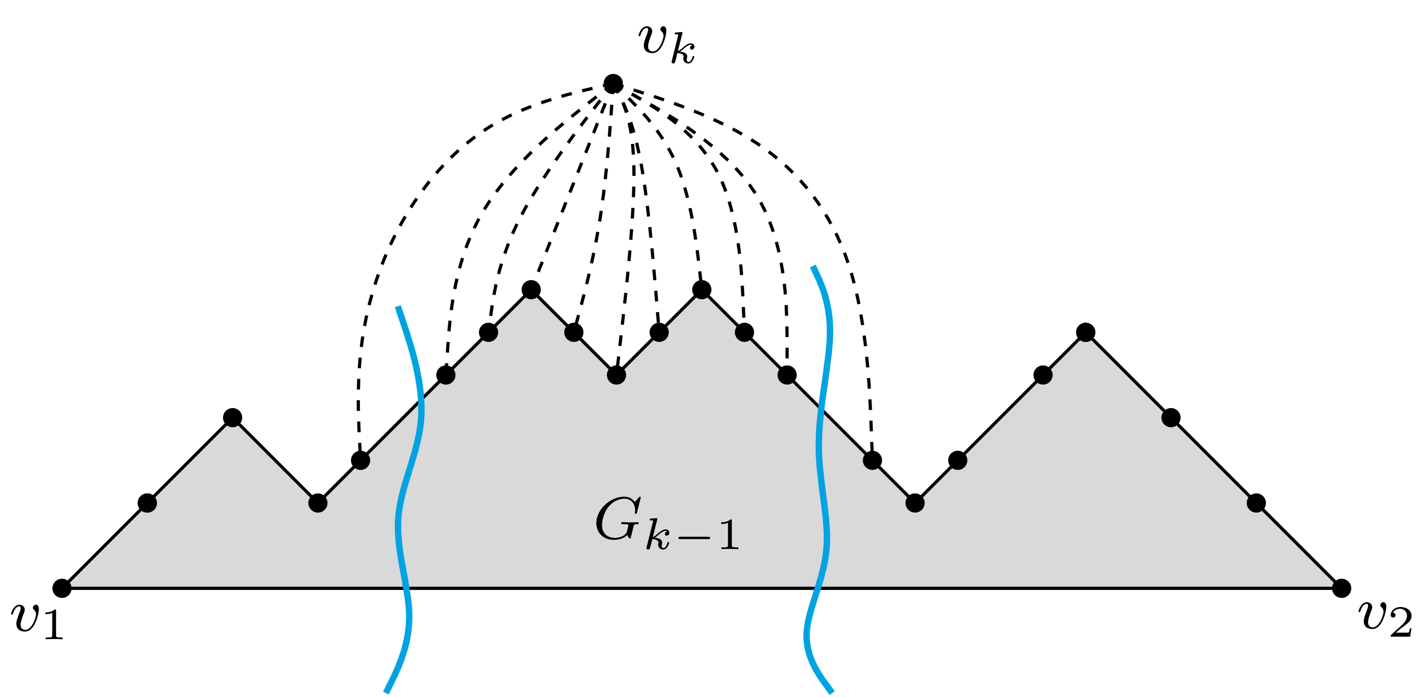

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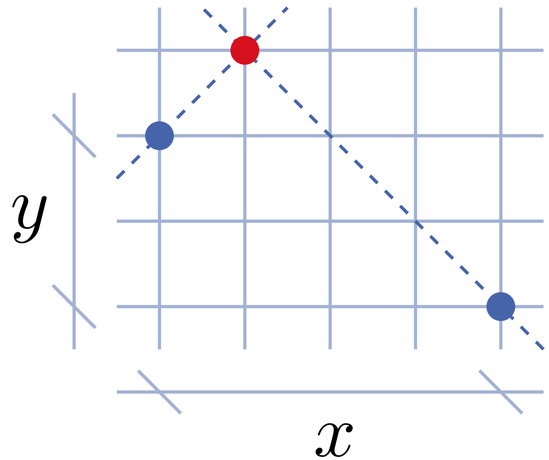

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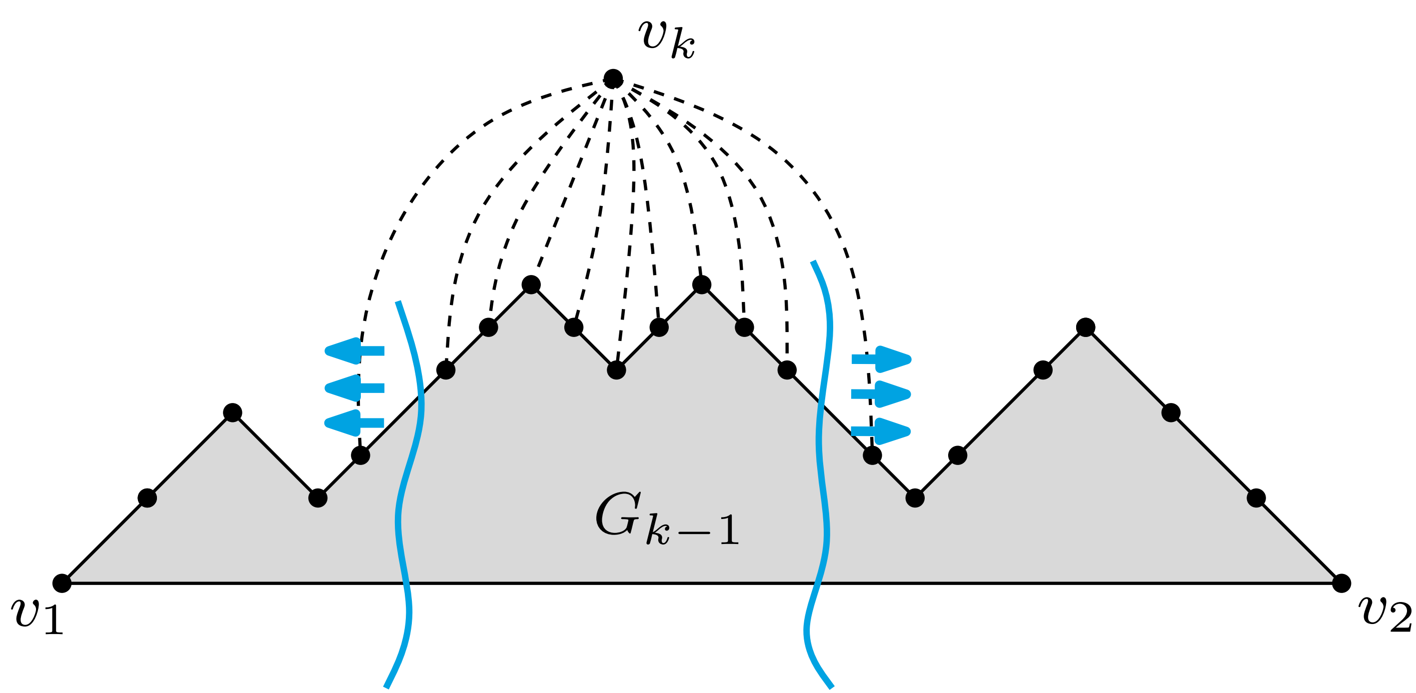

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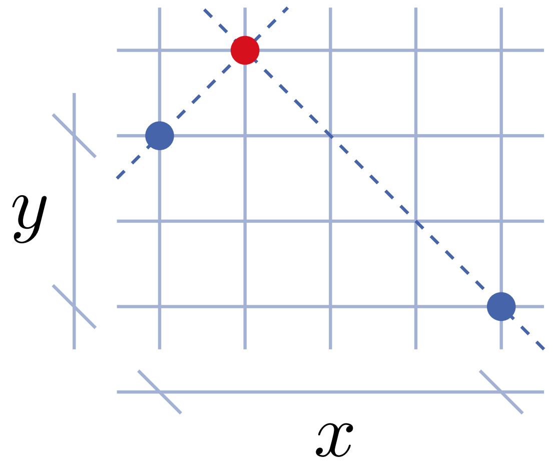

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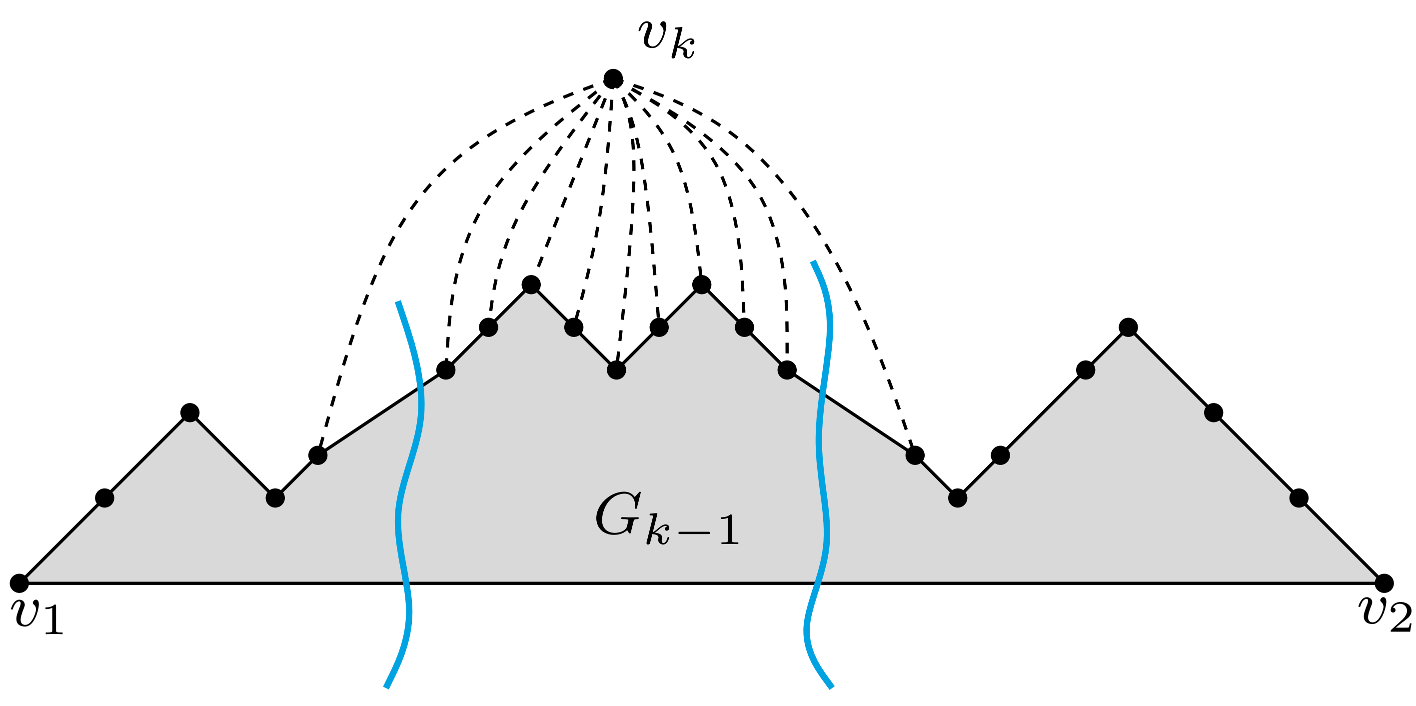

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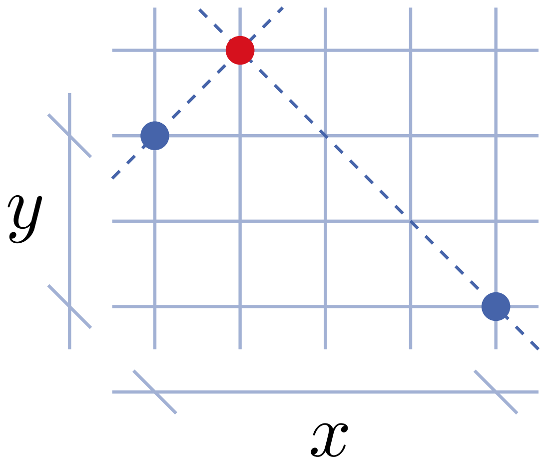

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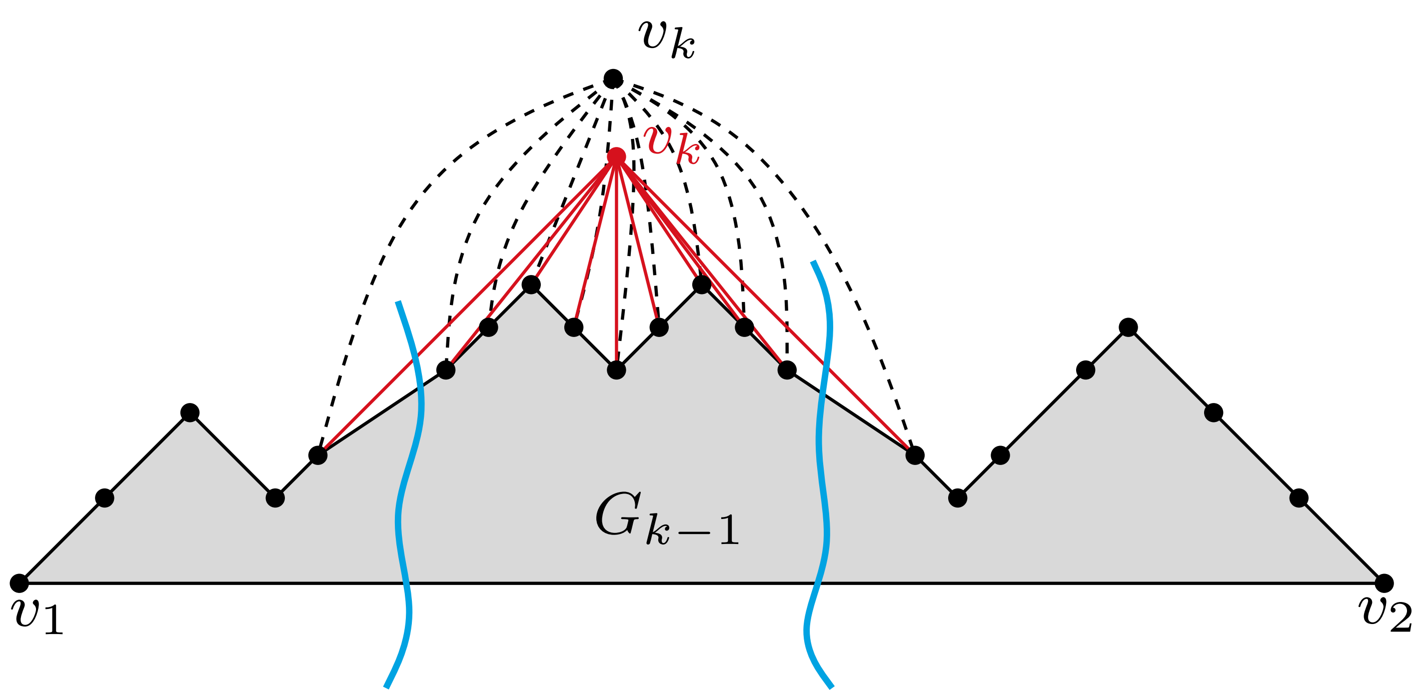

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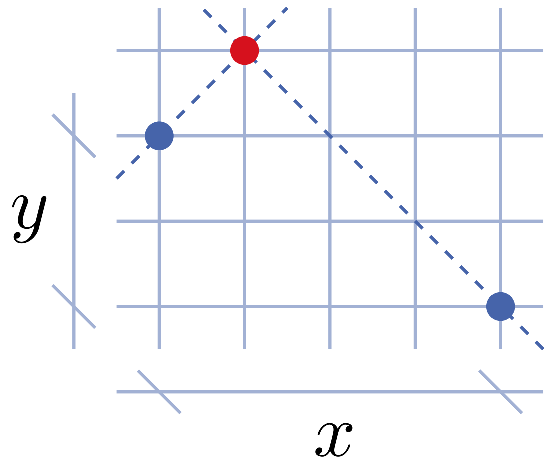

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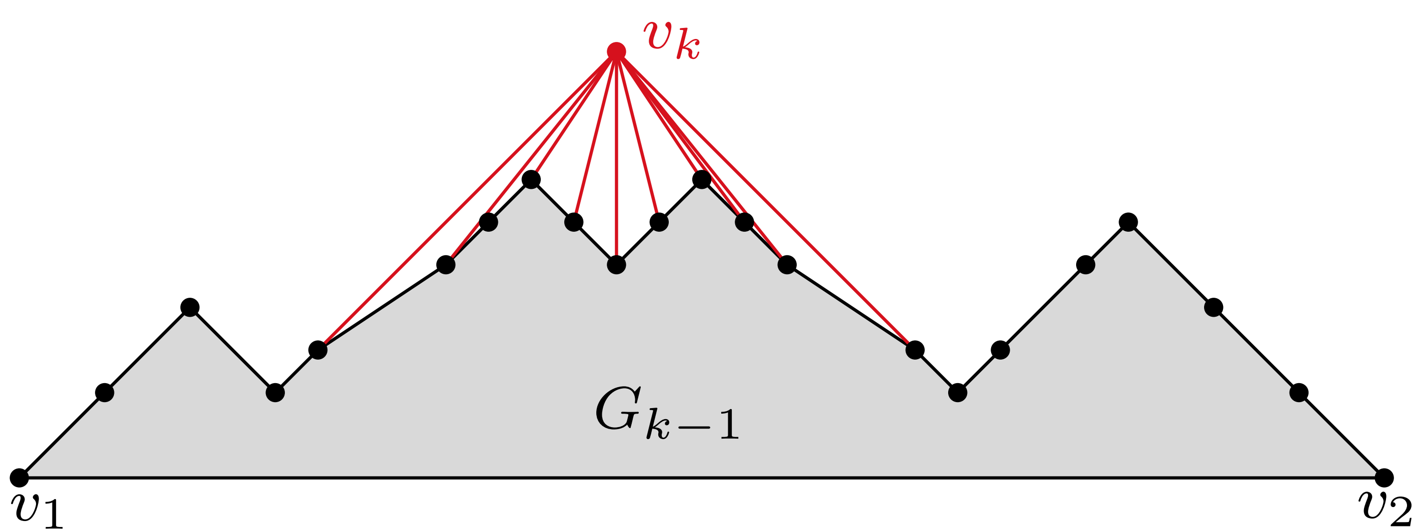

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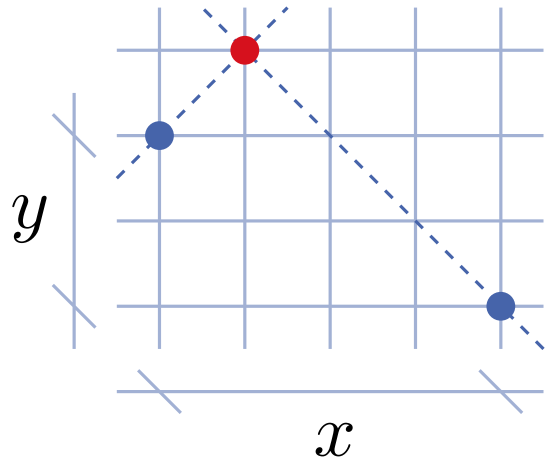

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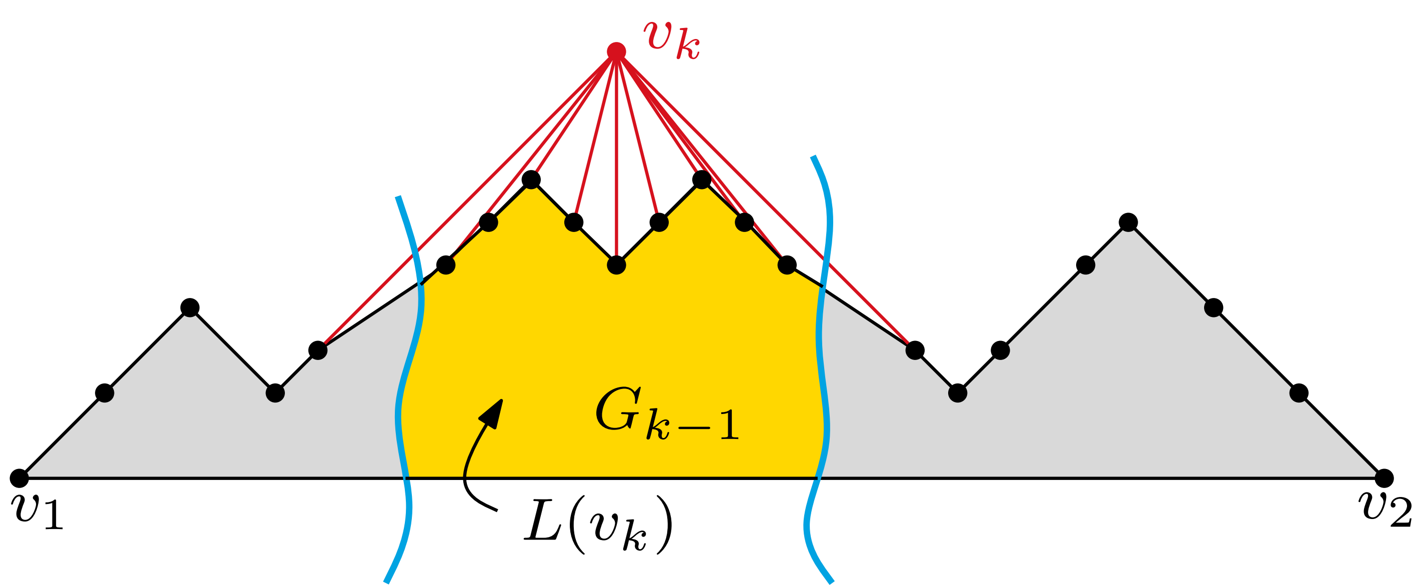
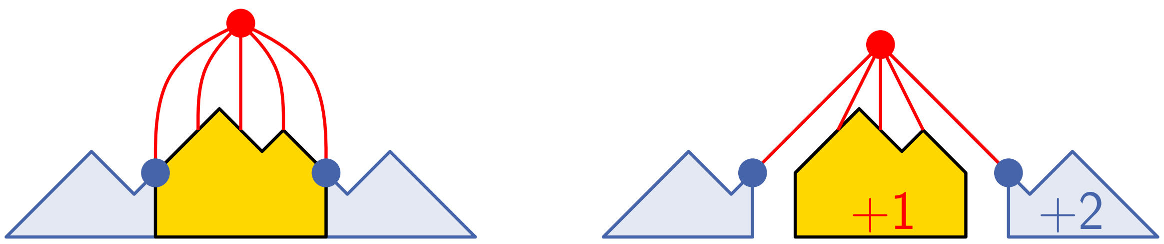
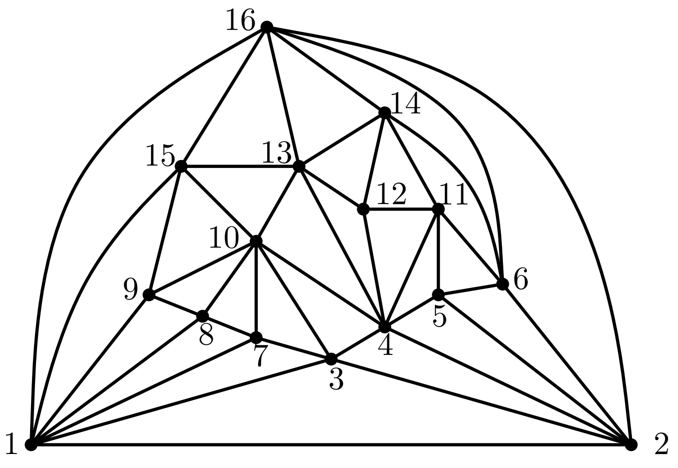
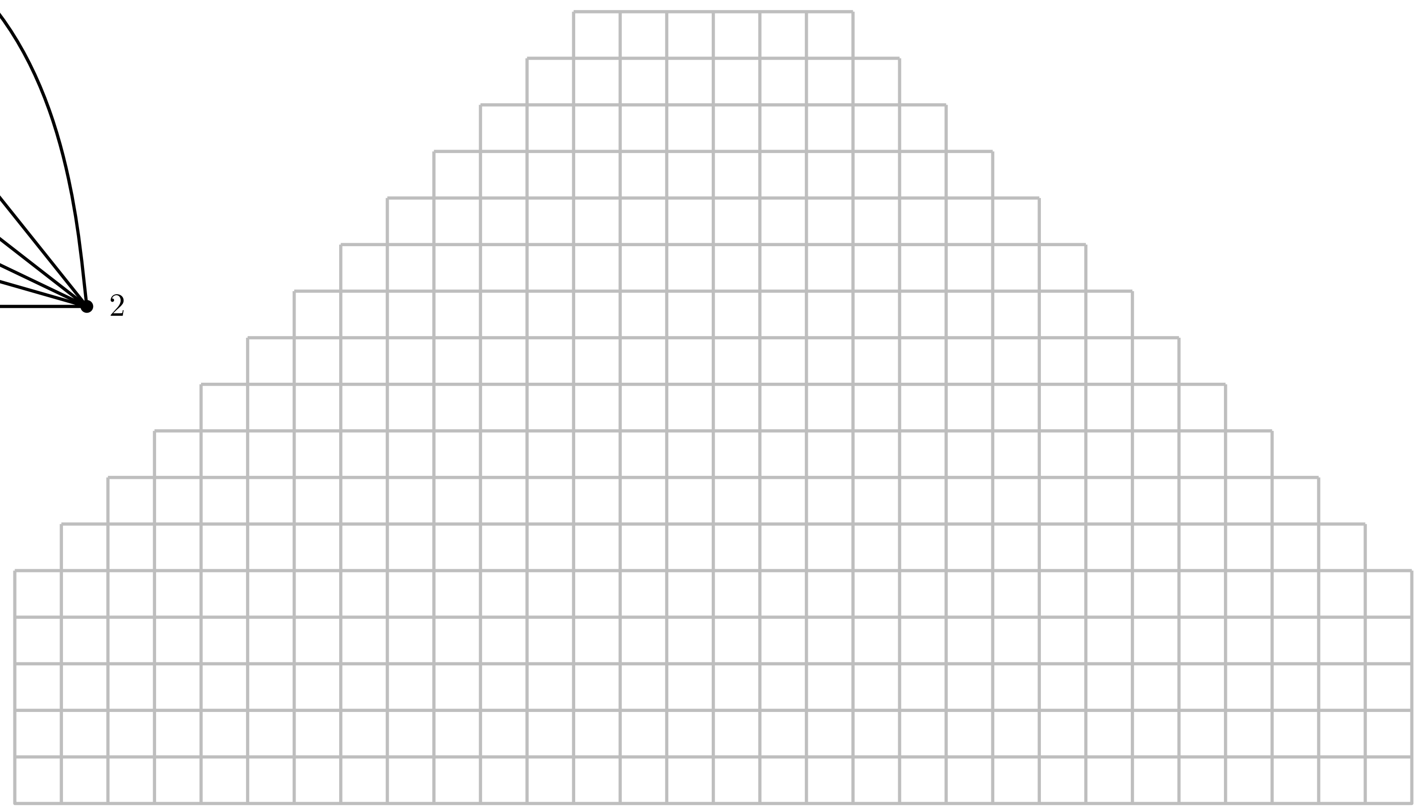


Shift Algorithm: Example







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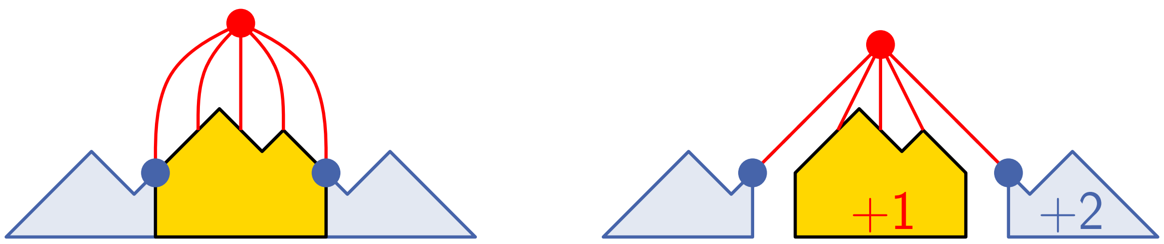

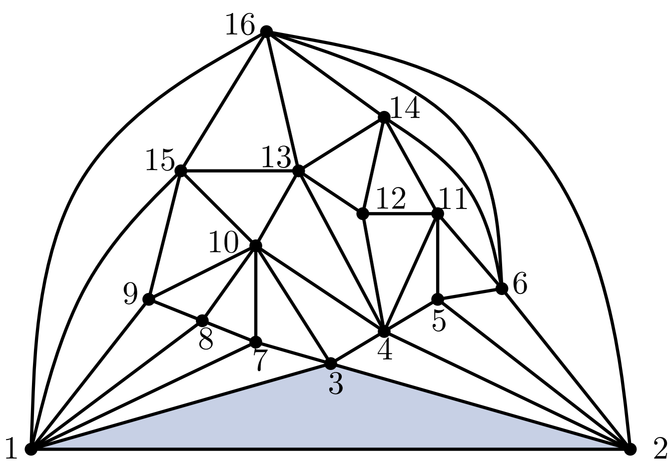

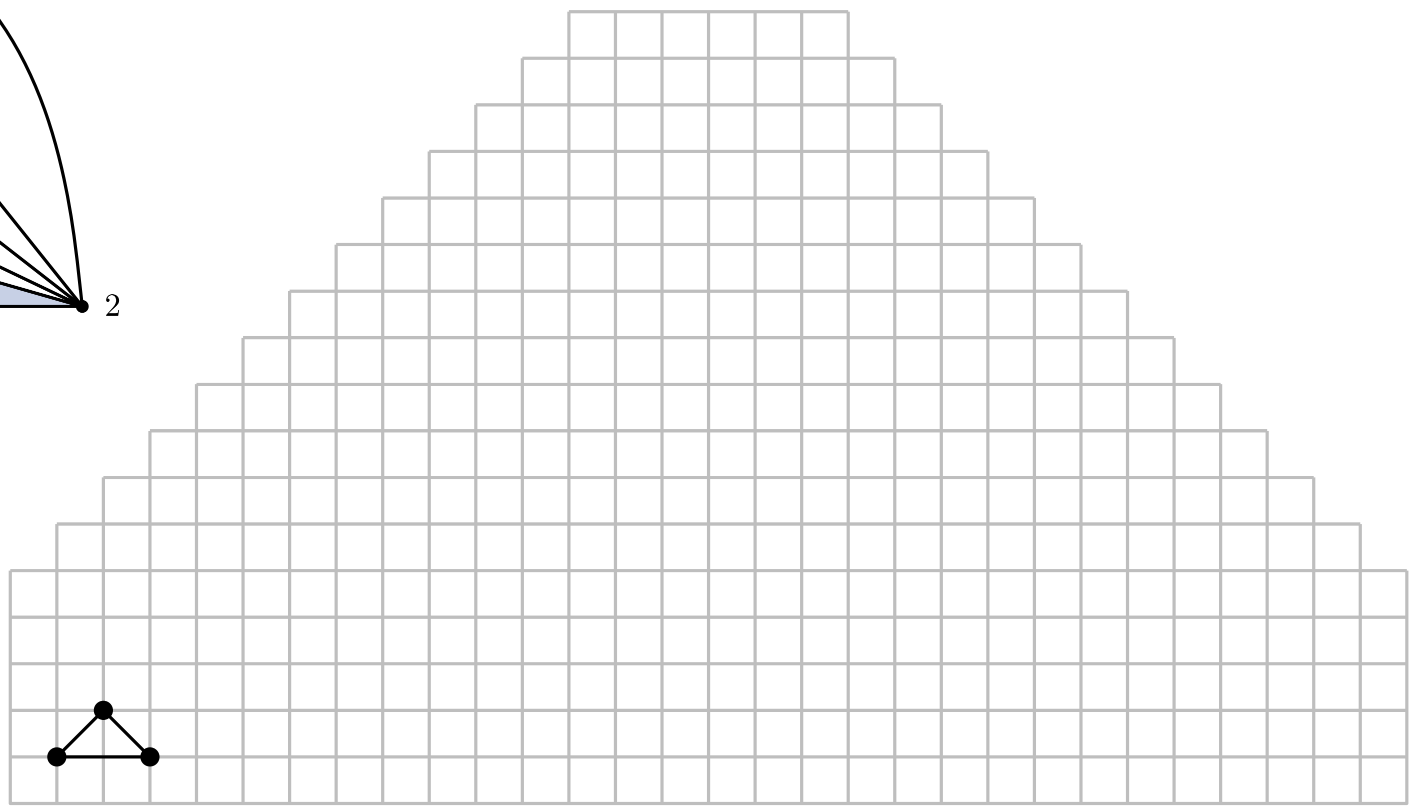

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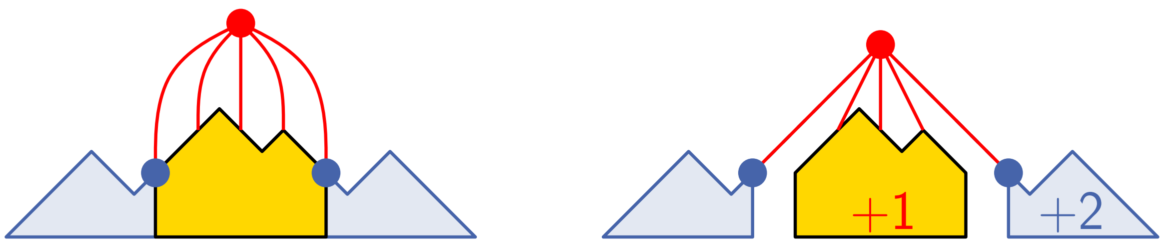

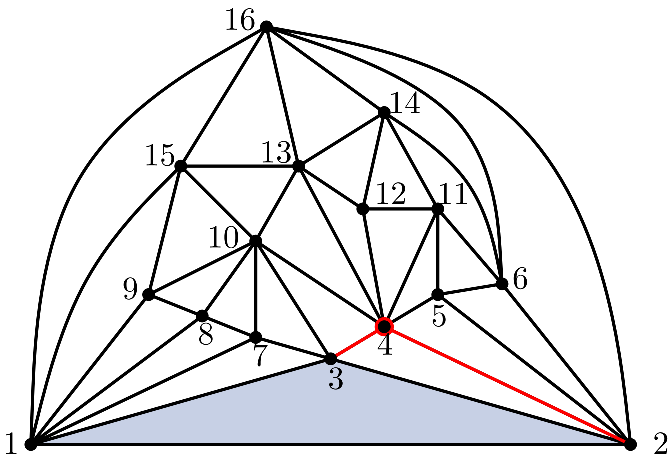

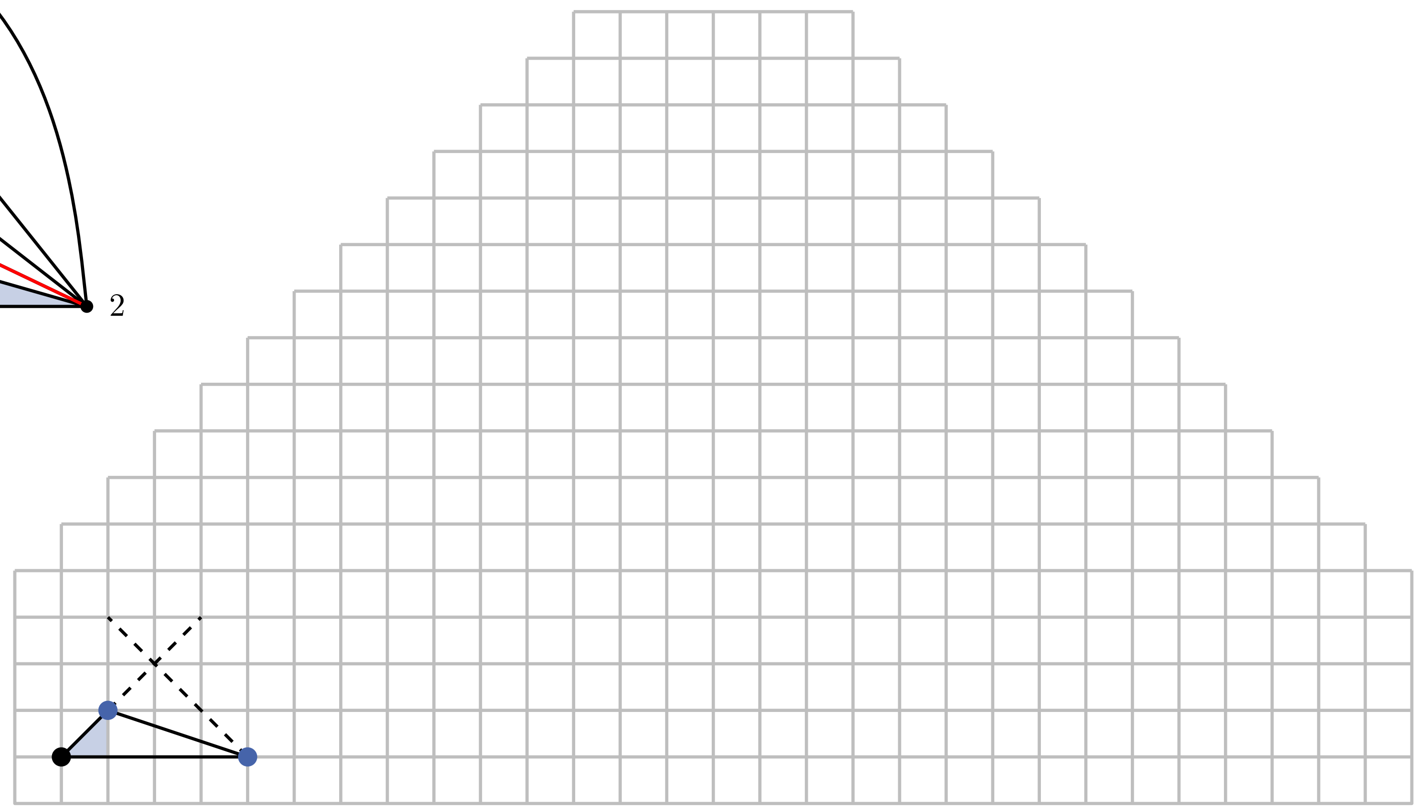

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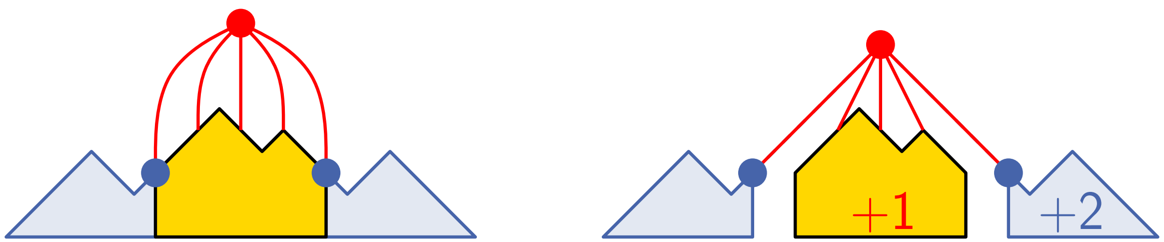

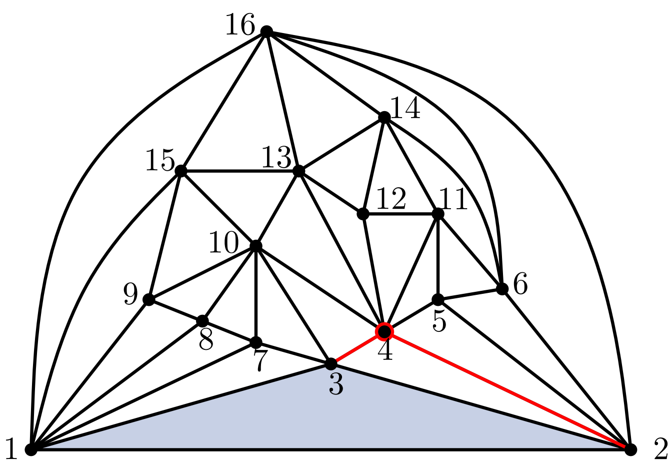

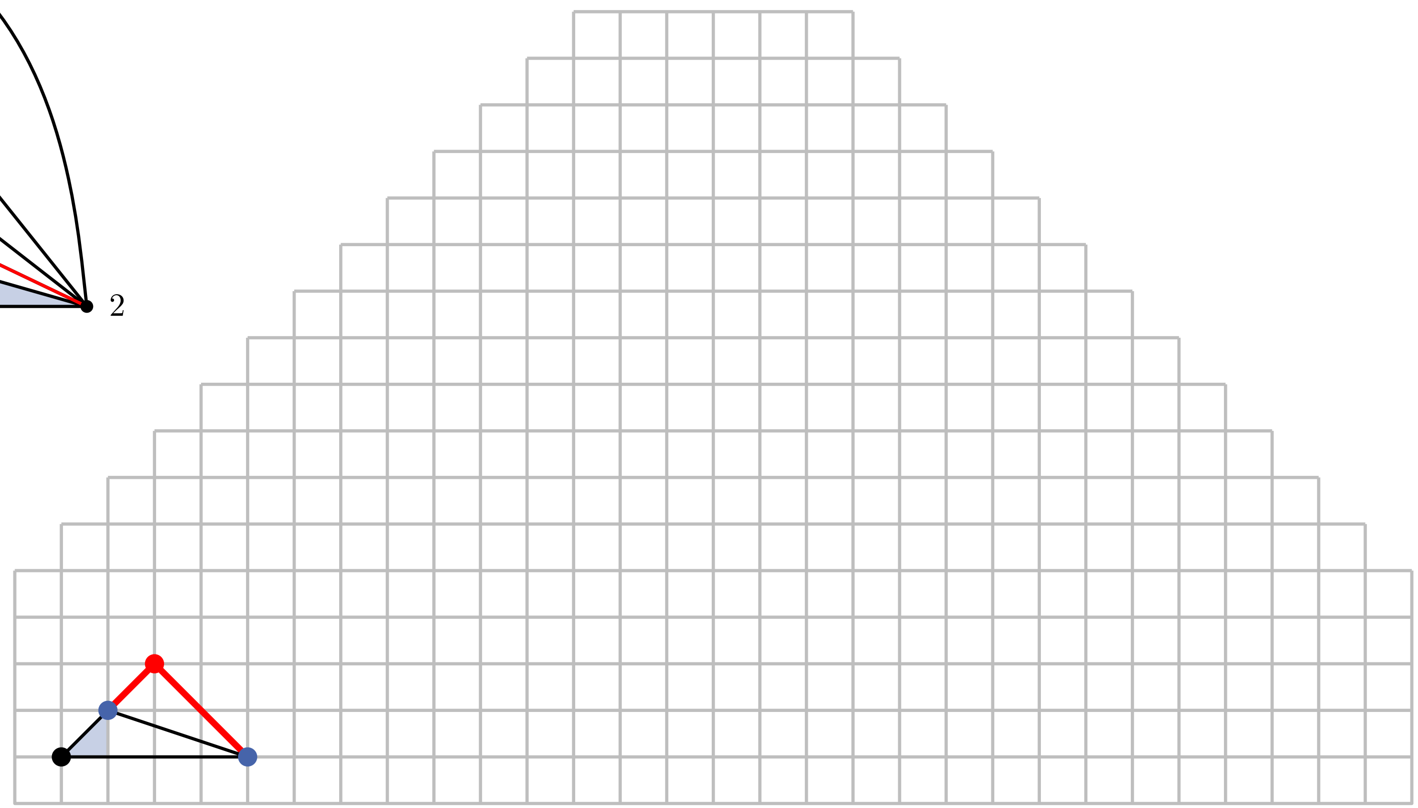

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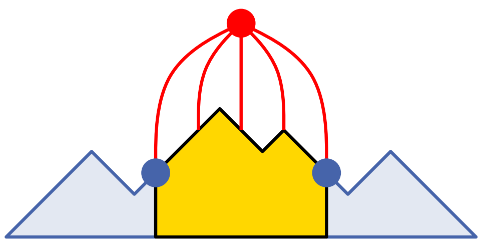

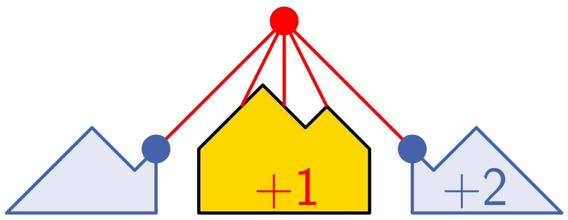

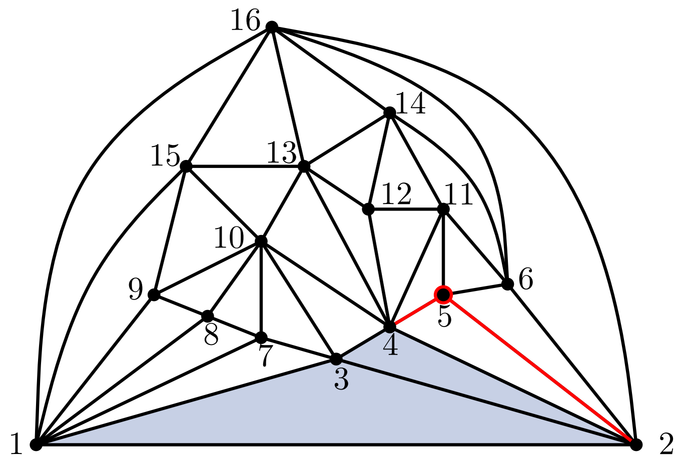

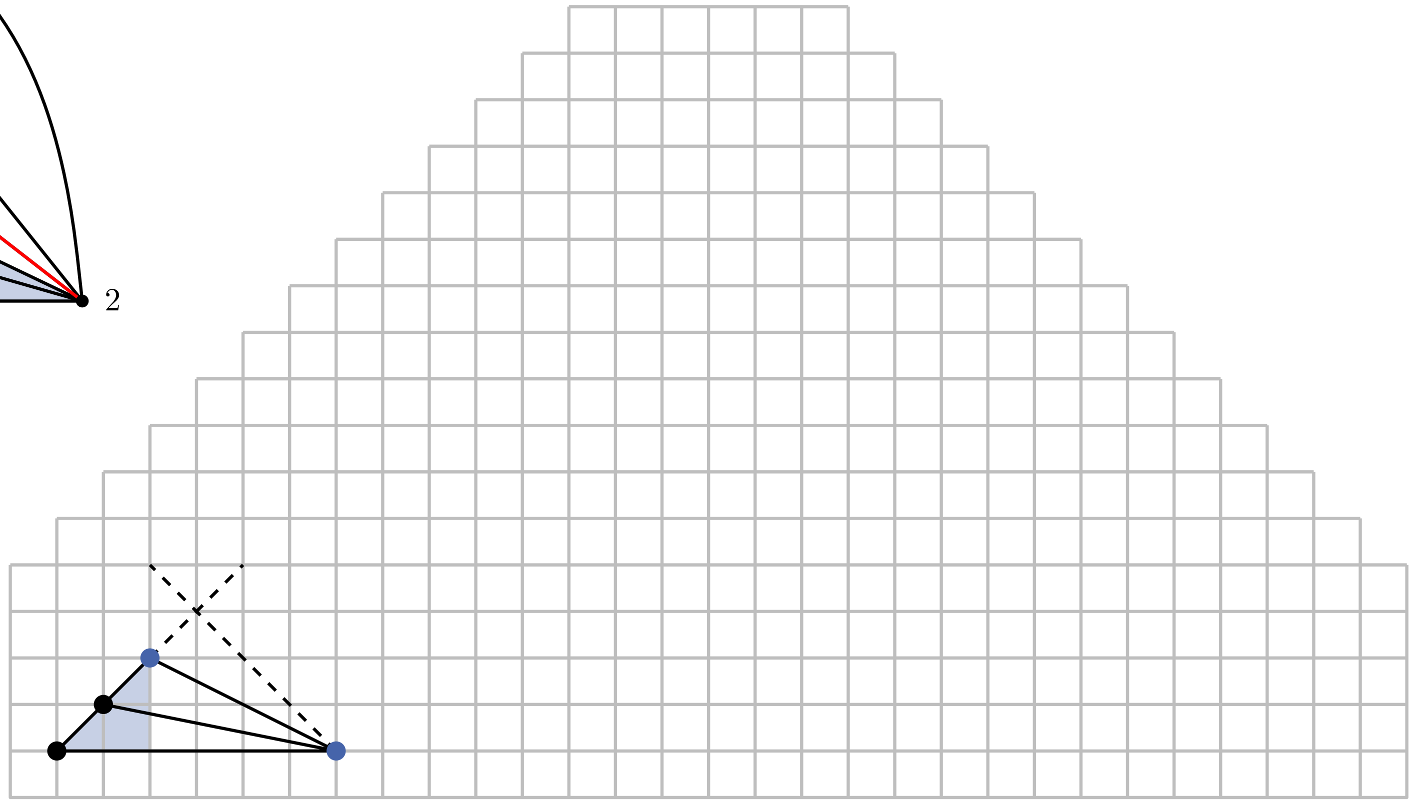

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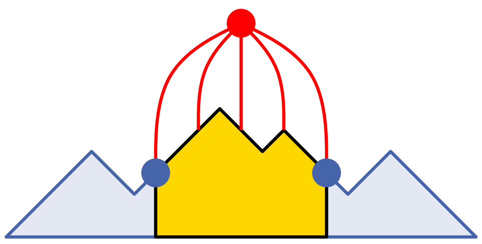

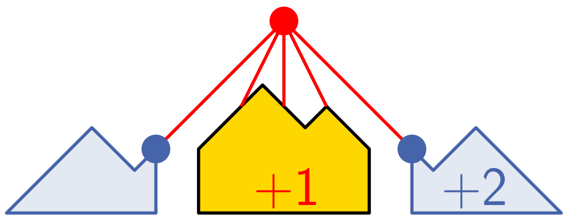

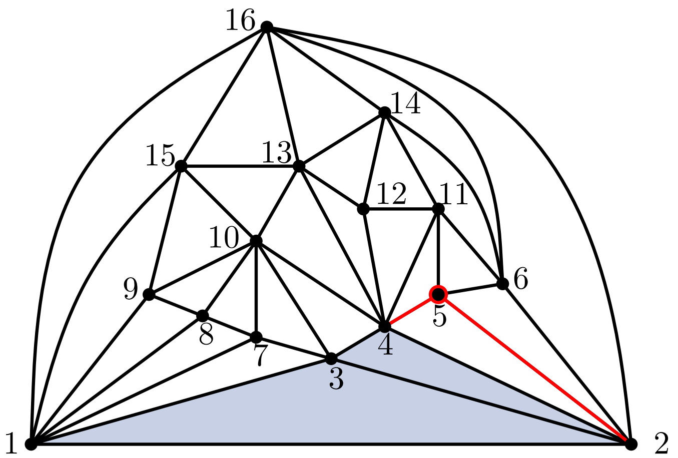

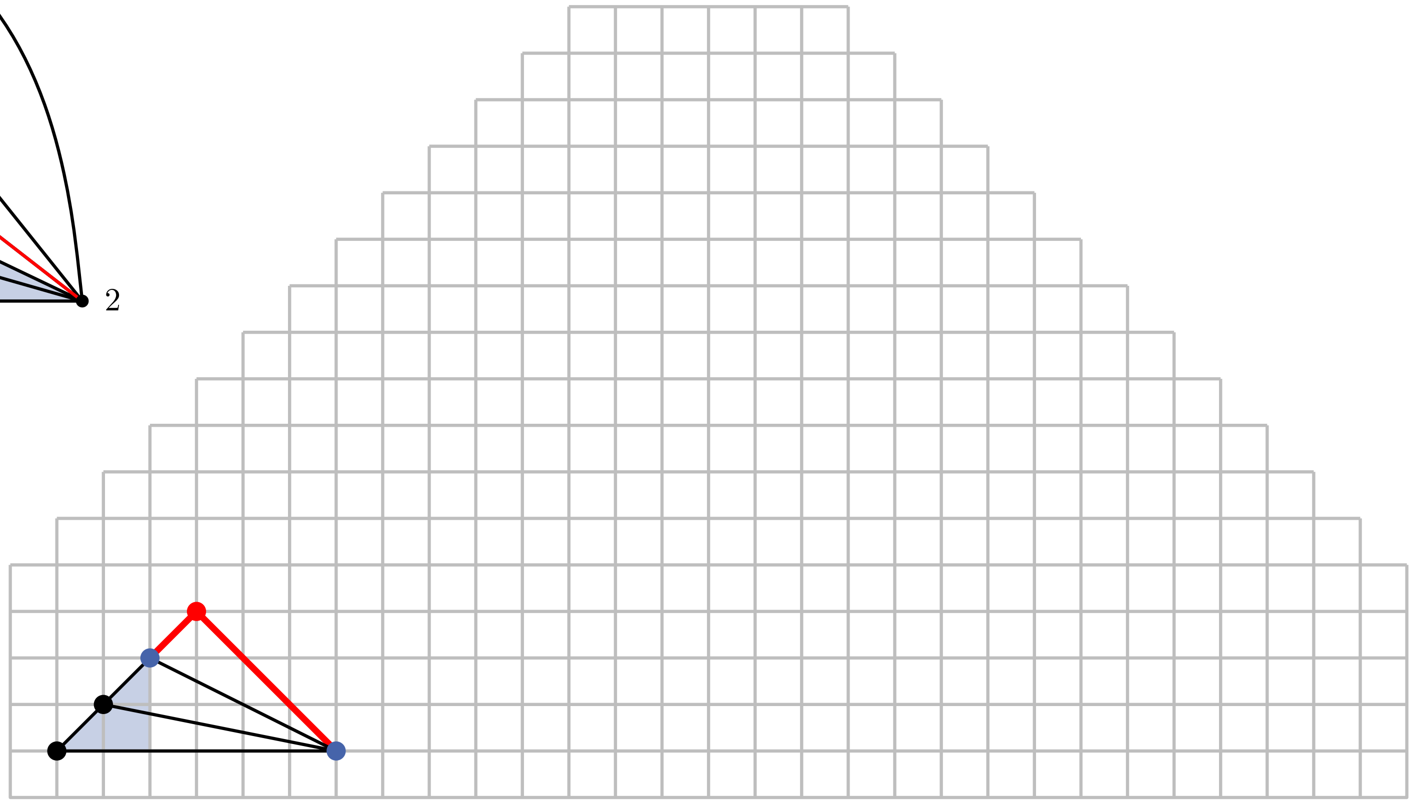

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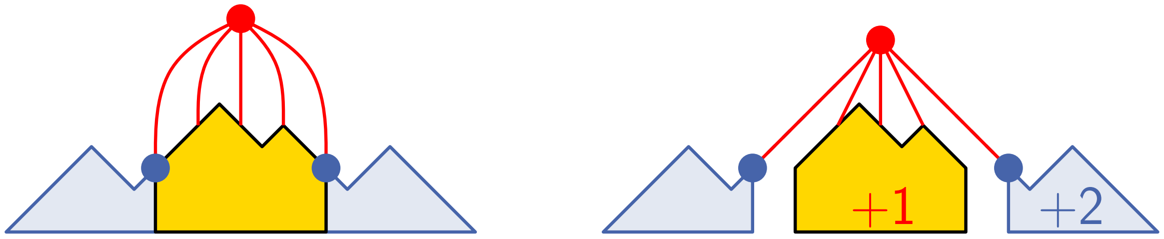
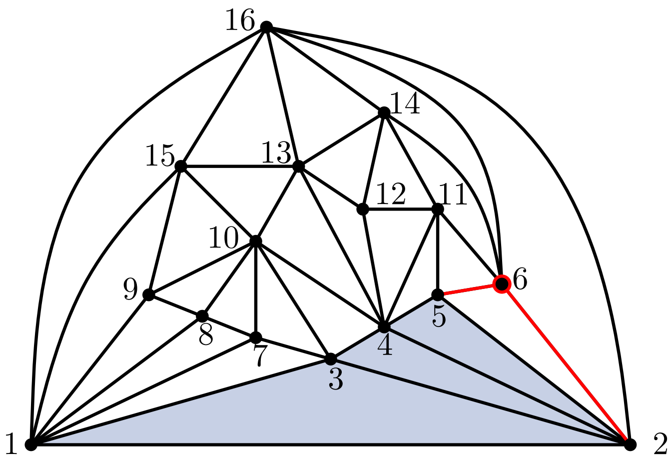
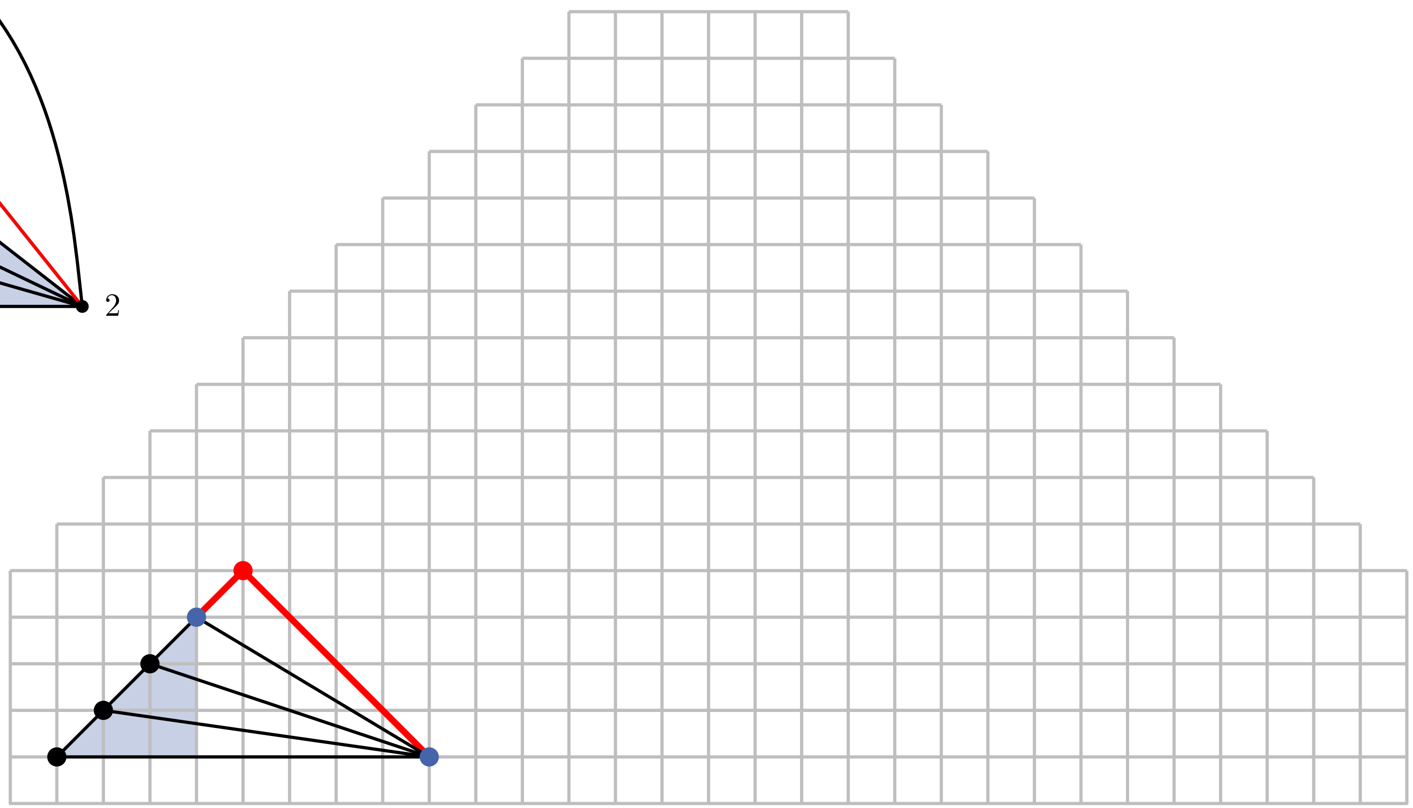
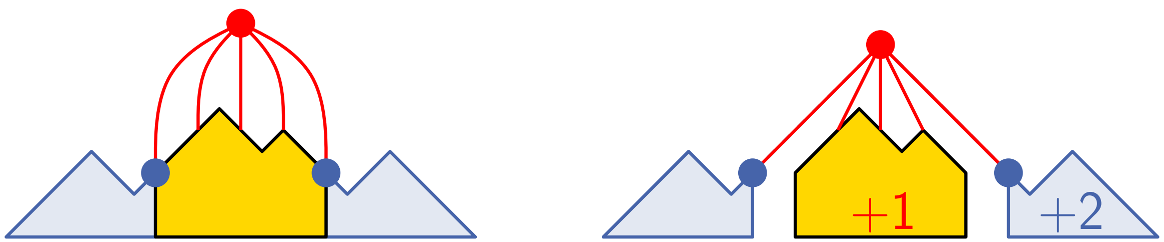
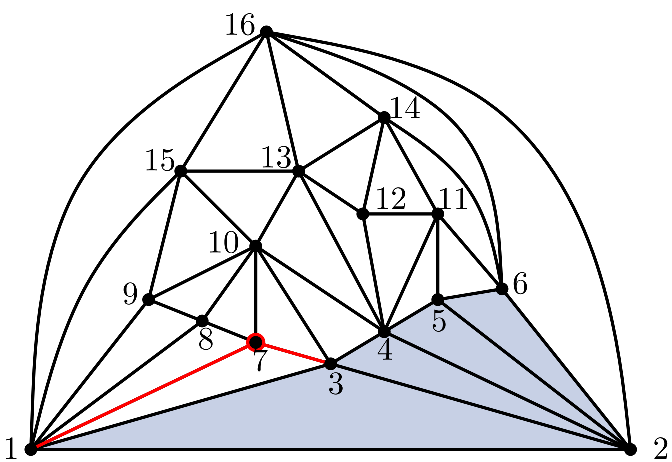


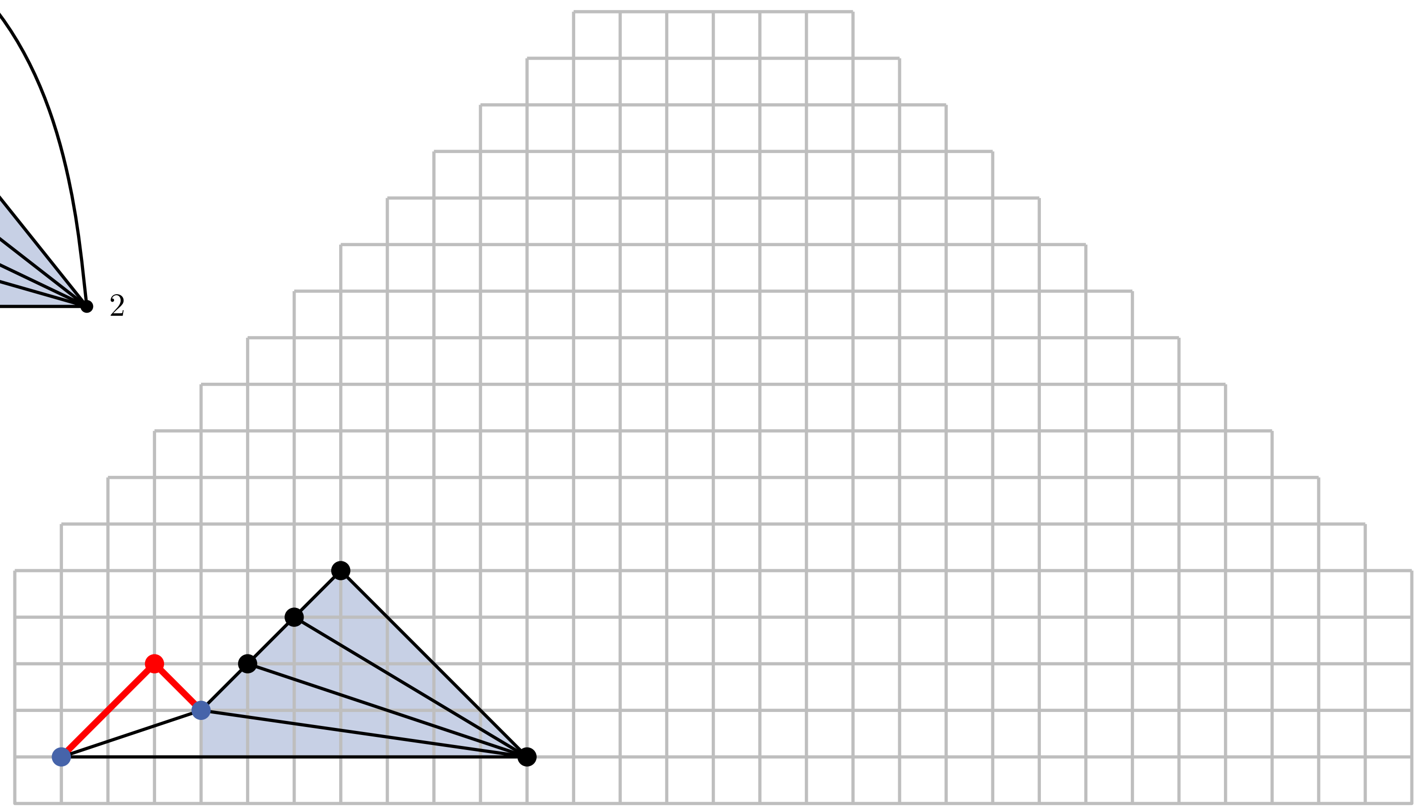




Shift Algorithm: Example







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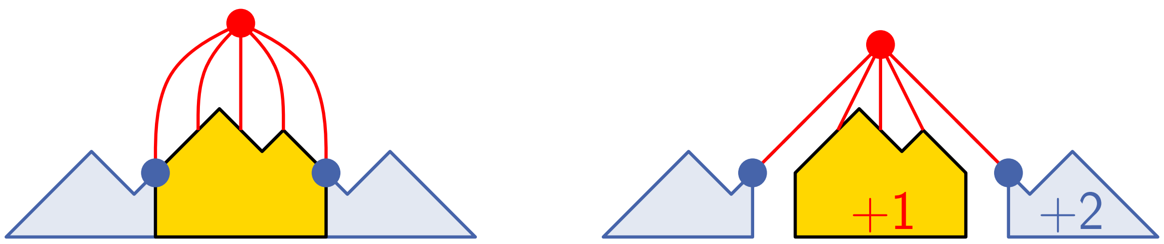

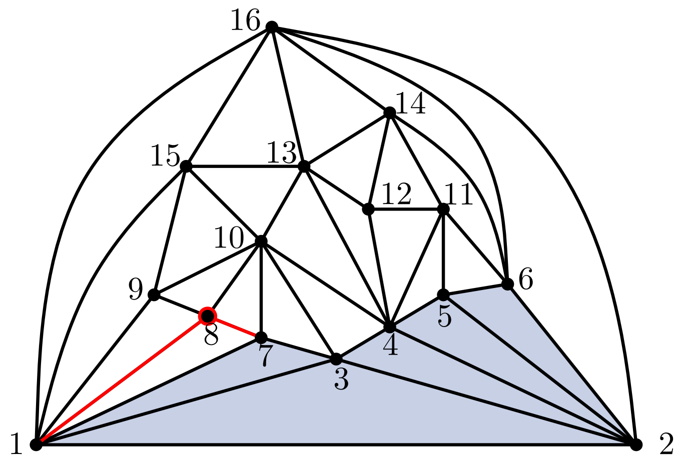

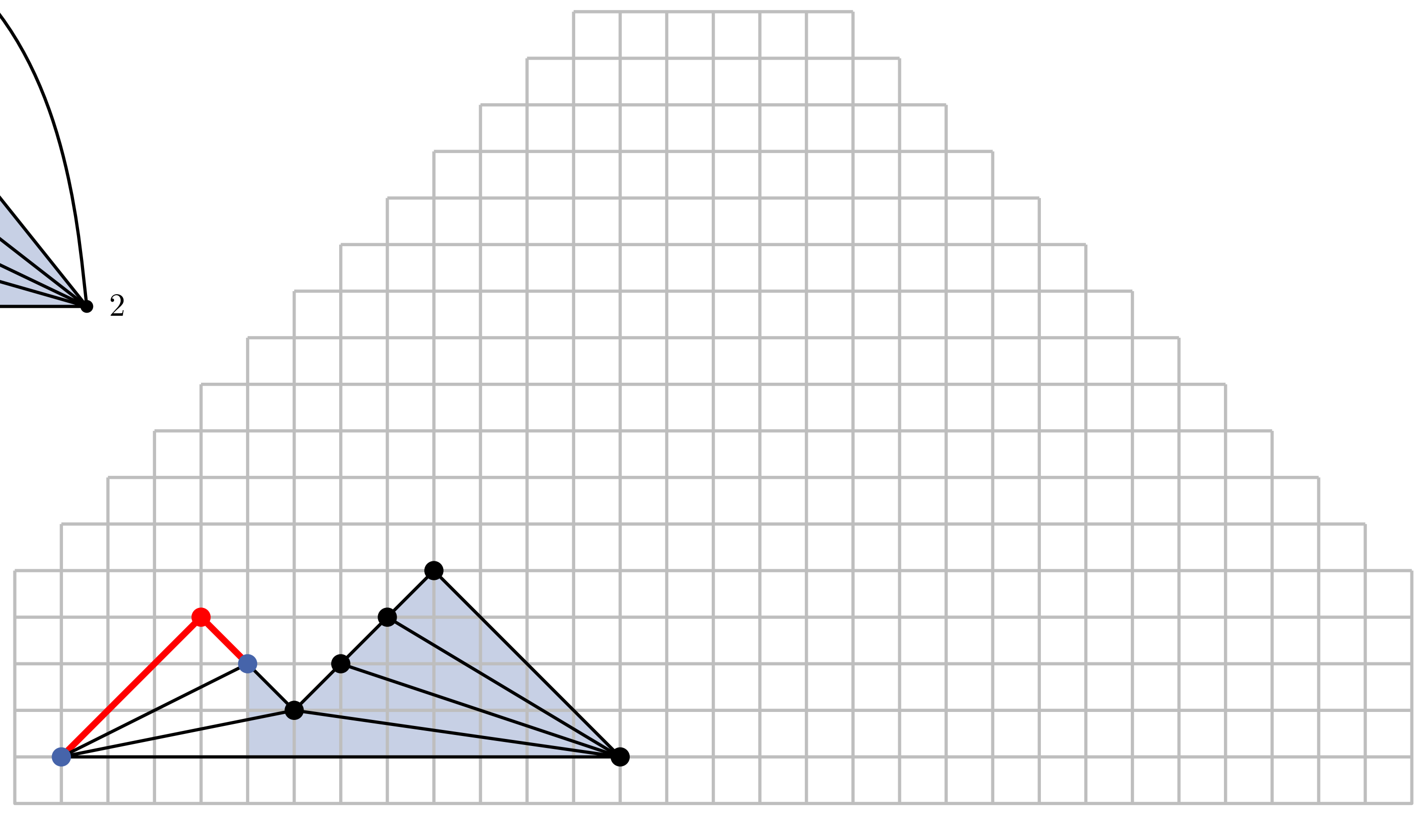

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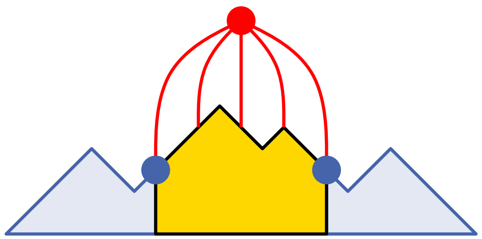

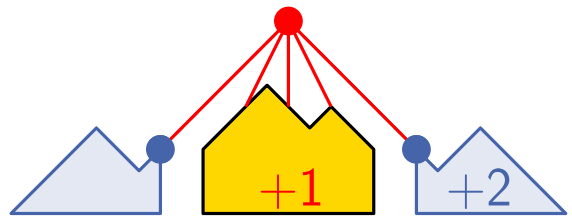

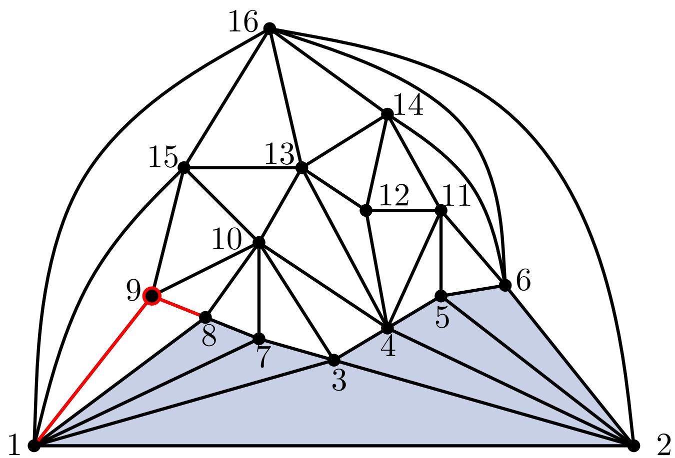

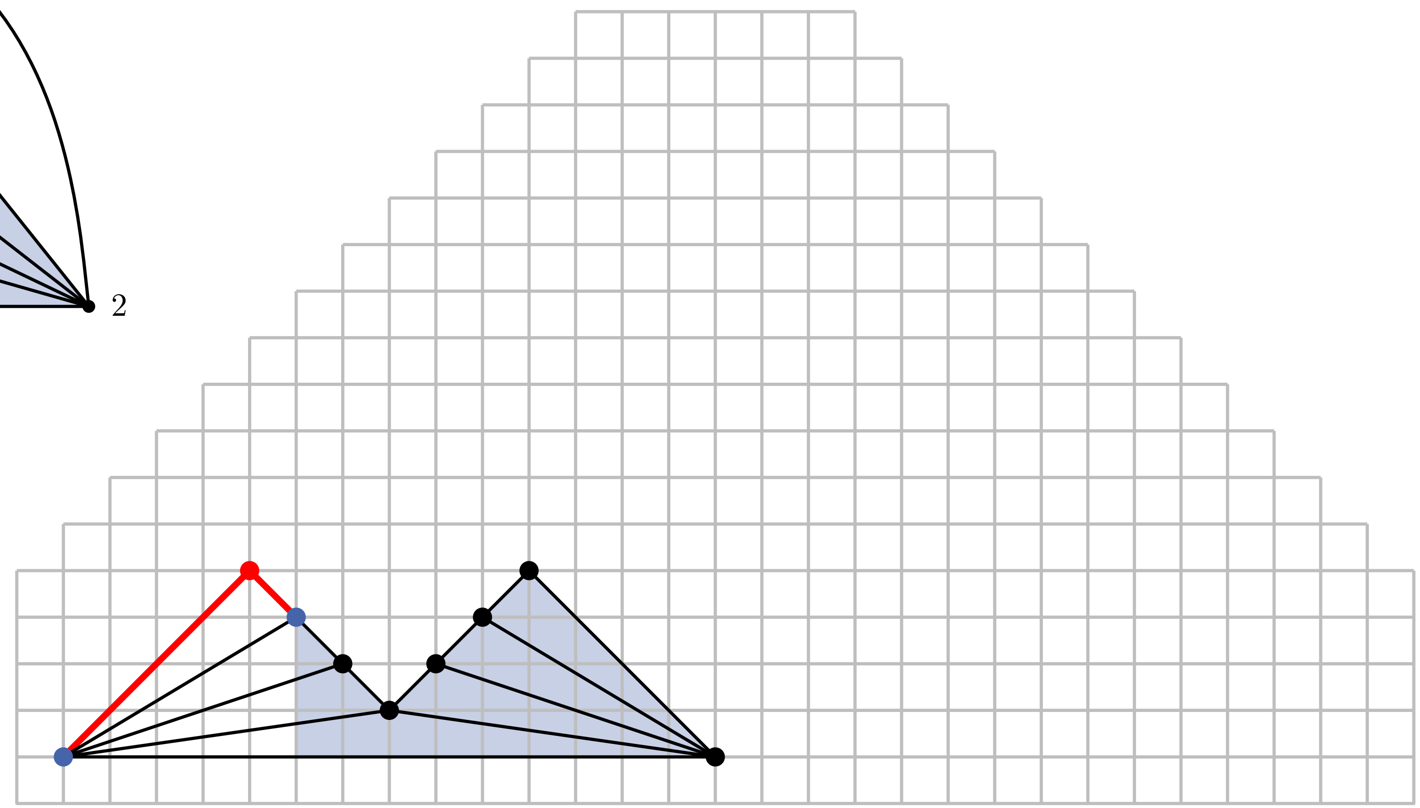

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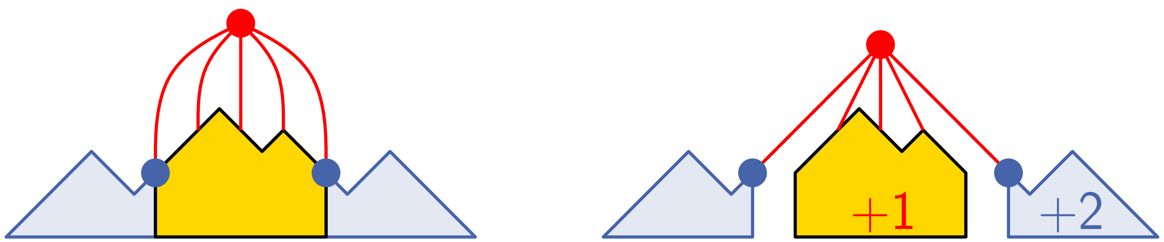

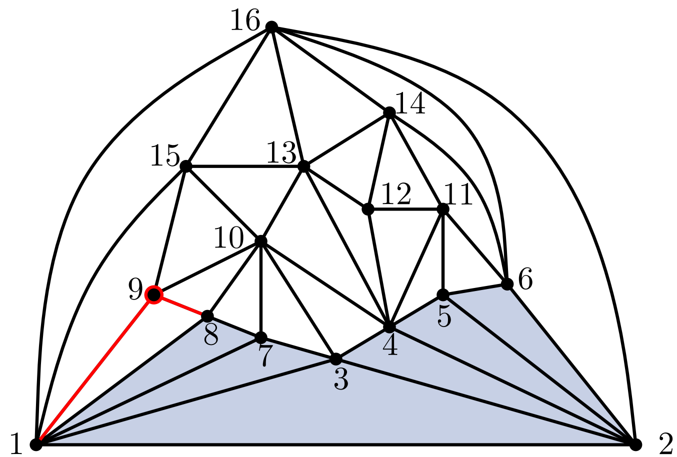

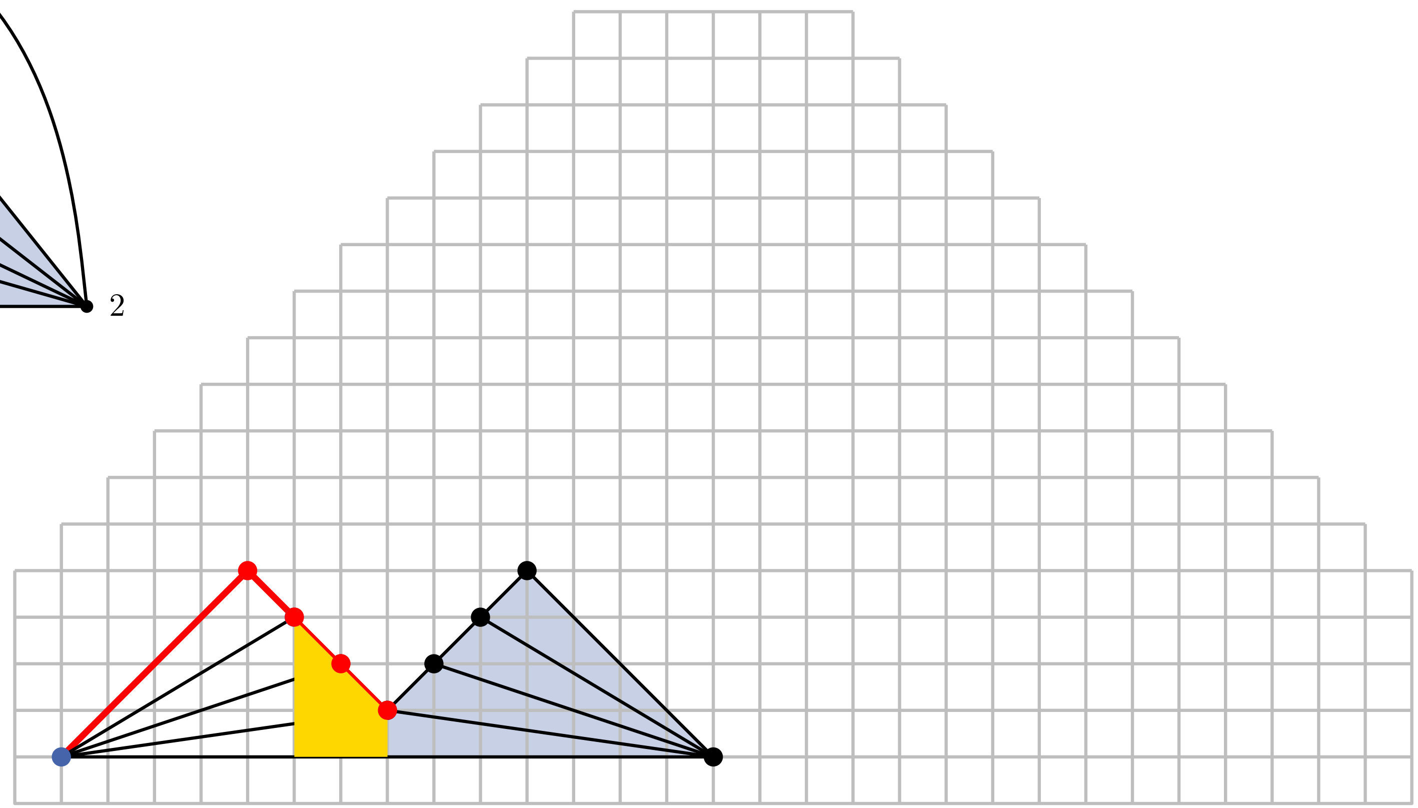

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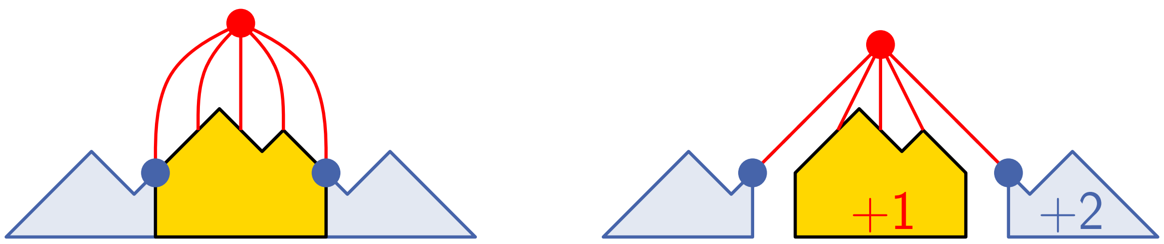

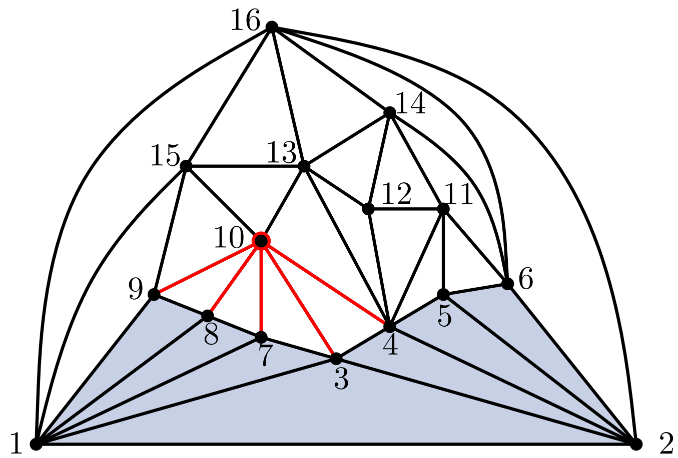

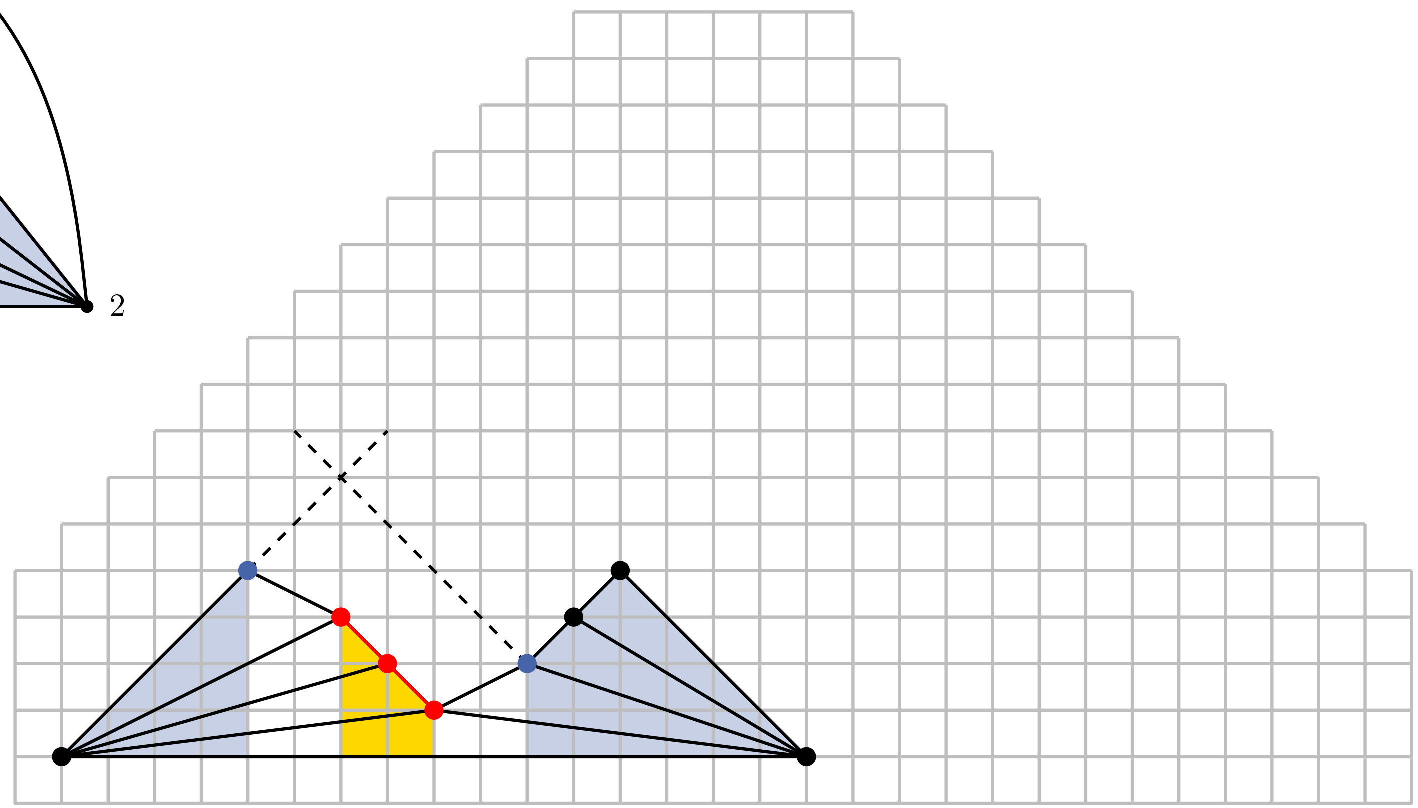

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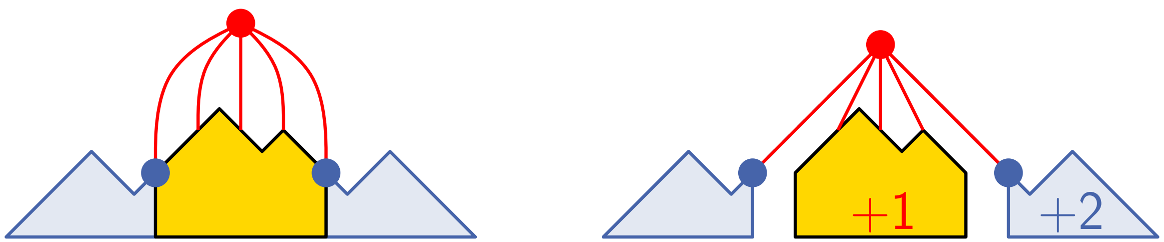

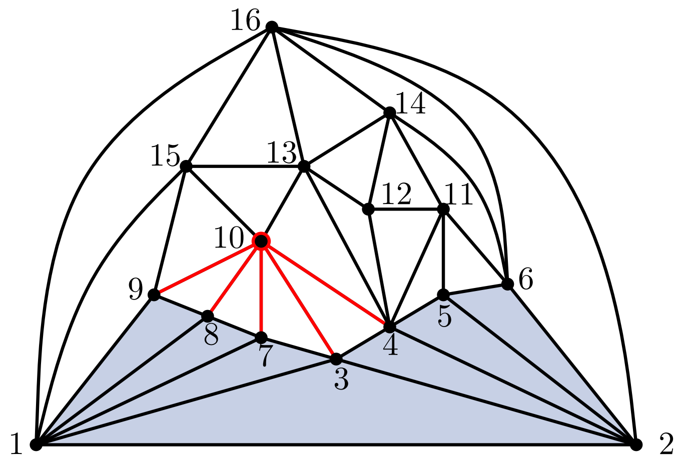

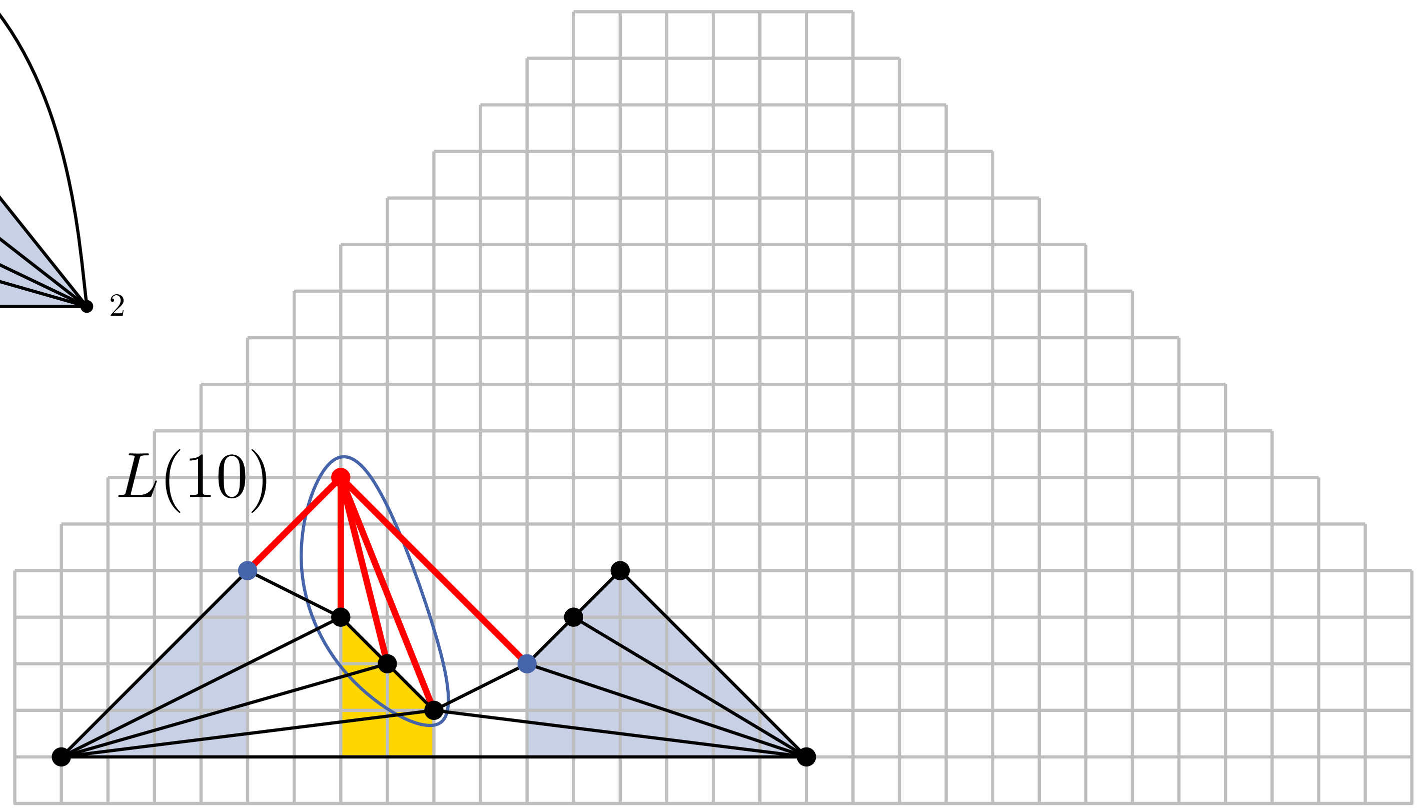

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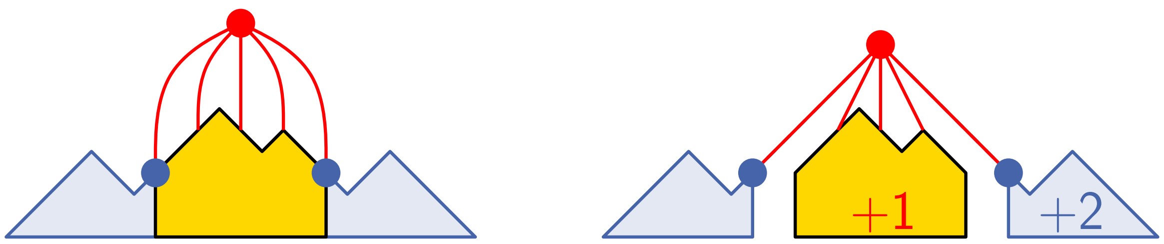

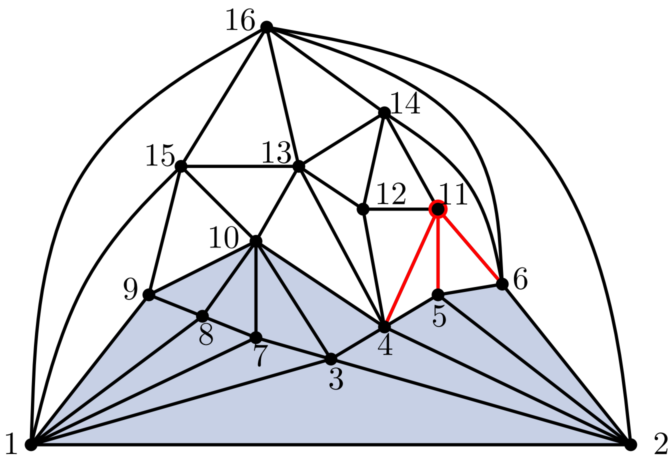

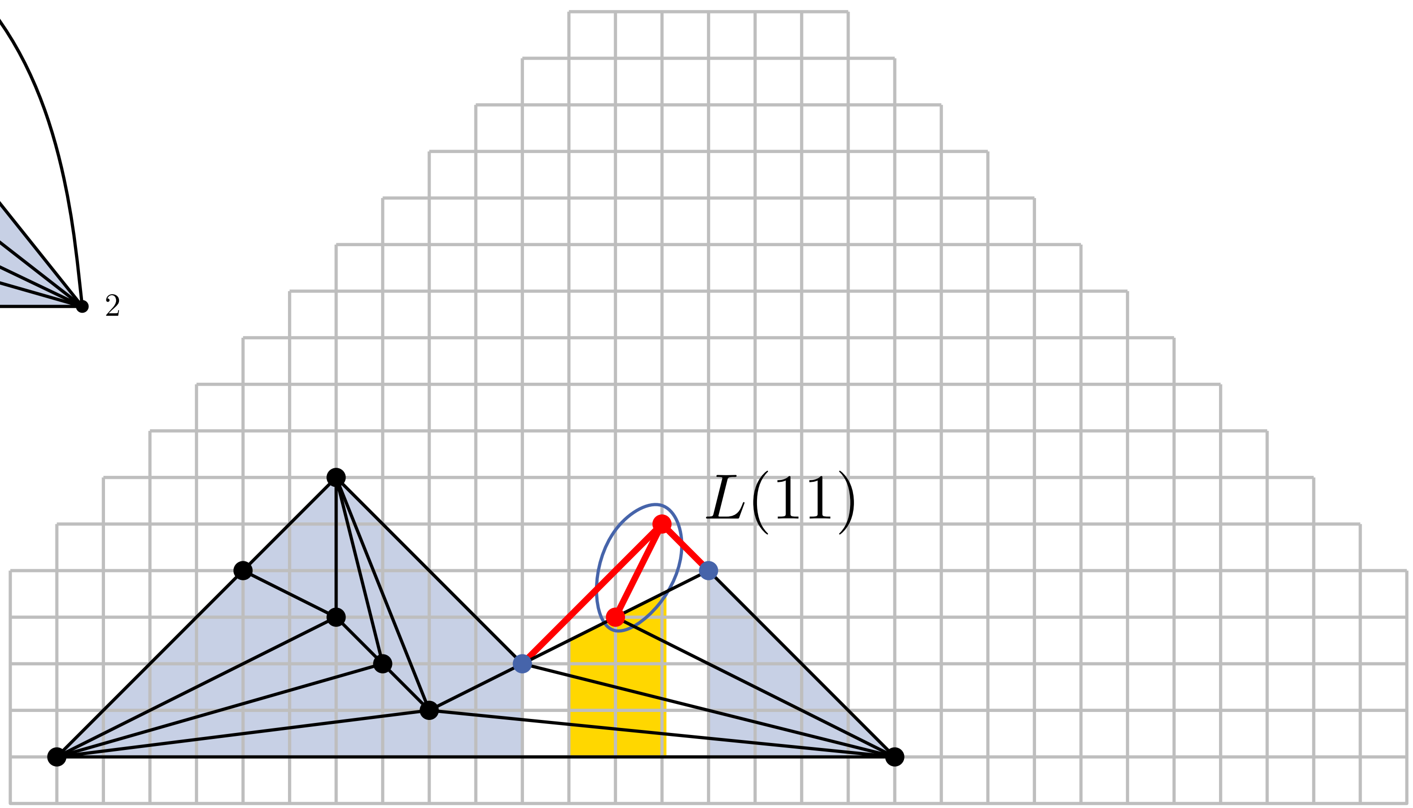

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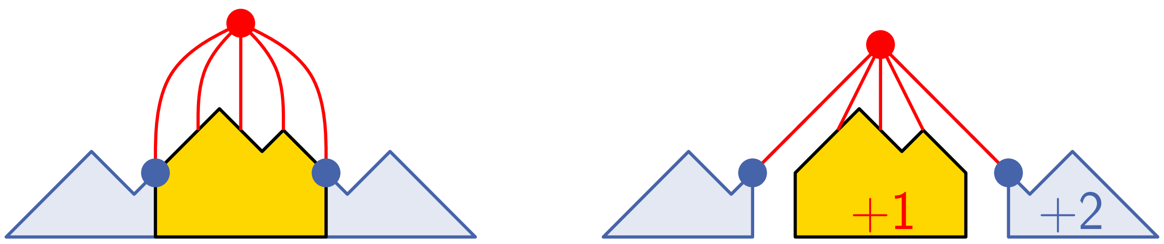

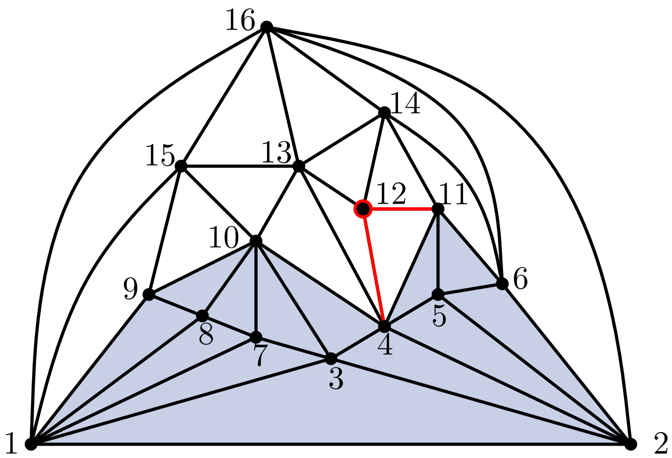

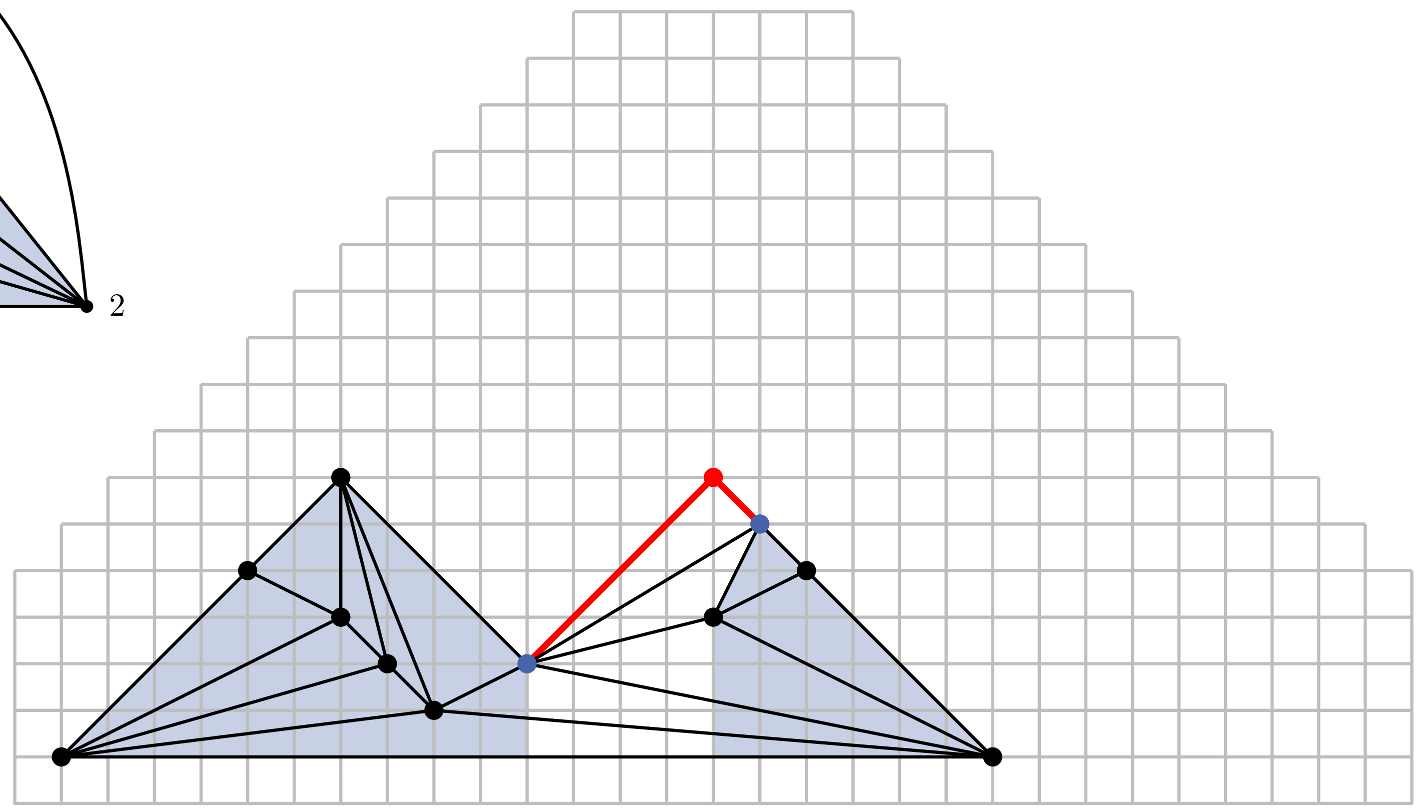

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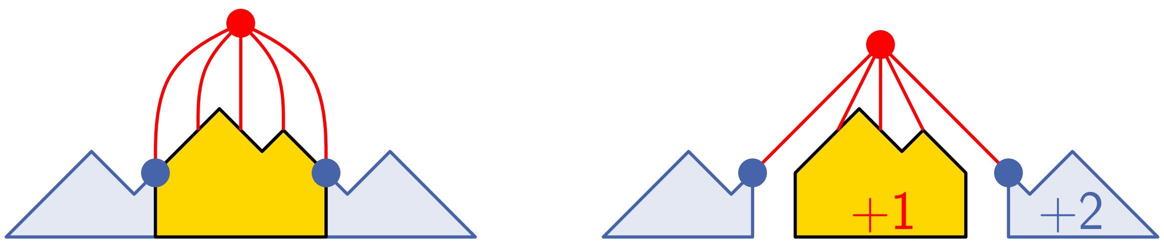

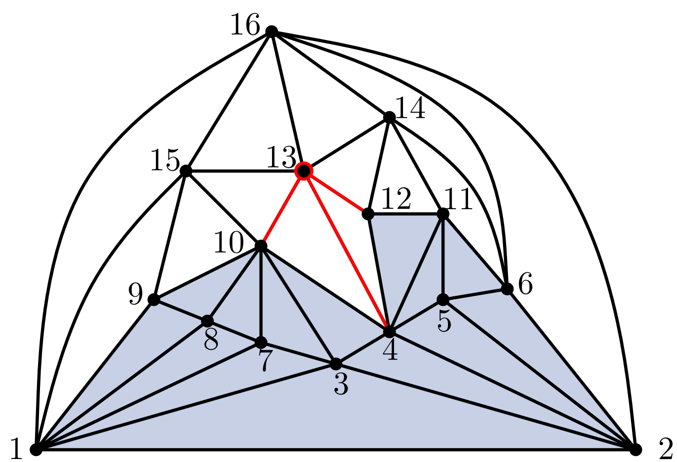

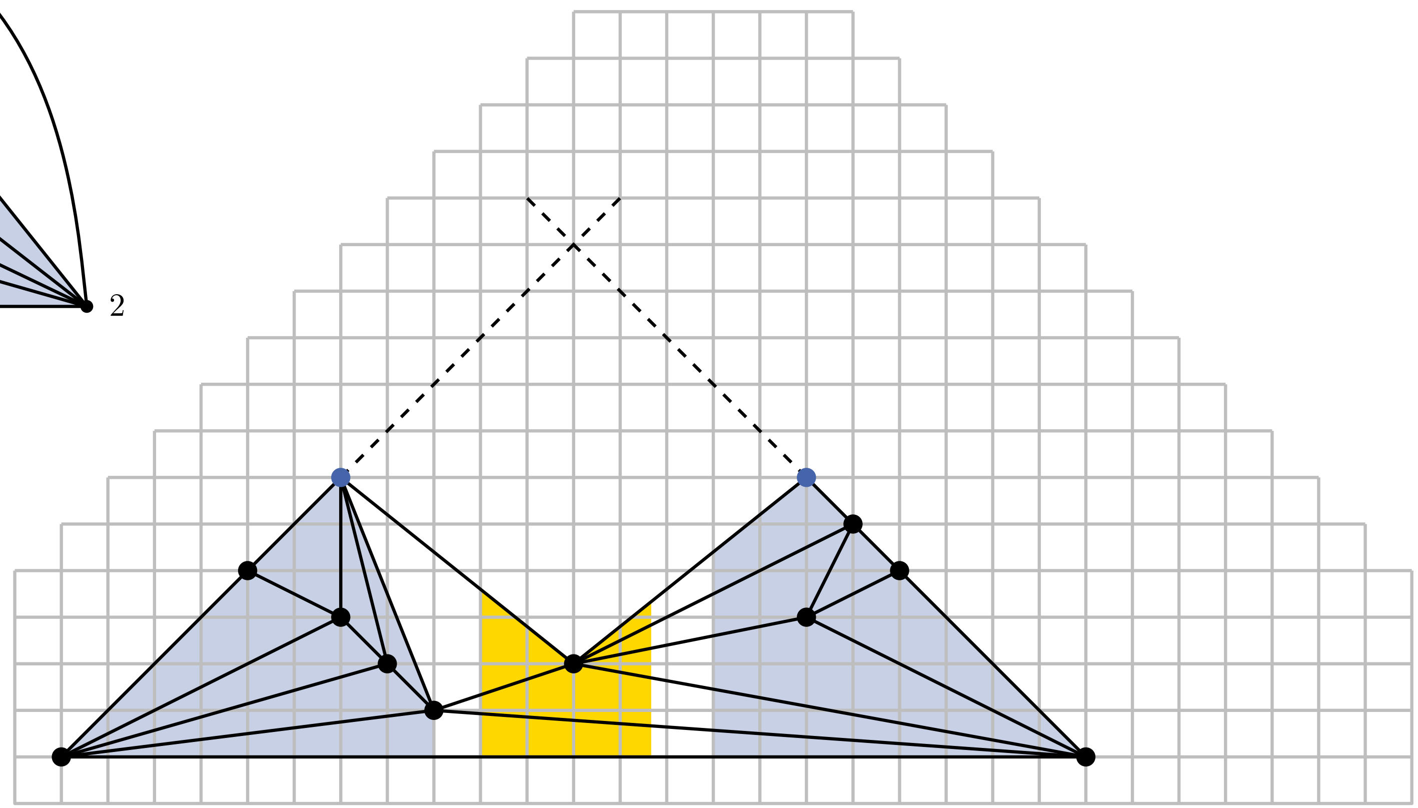

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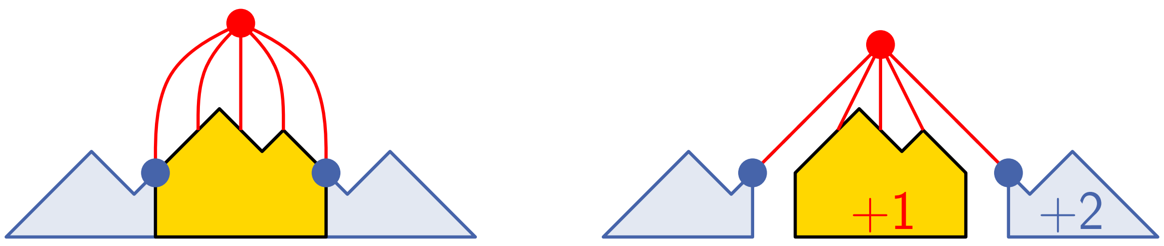

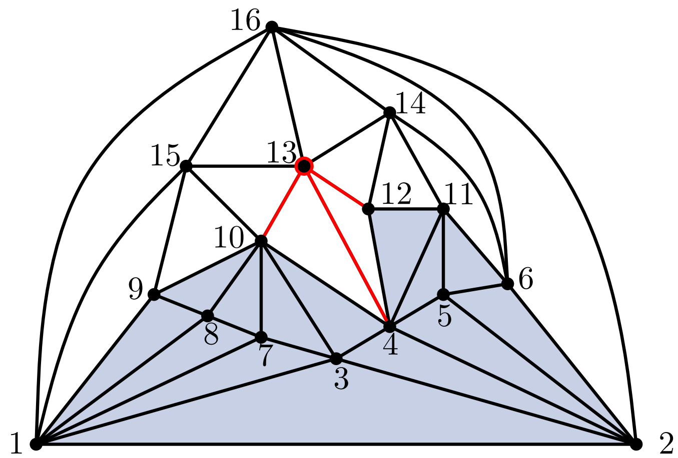

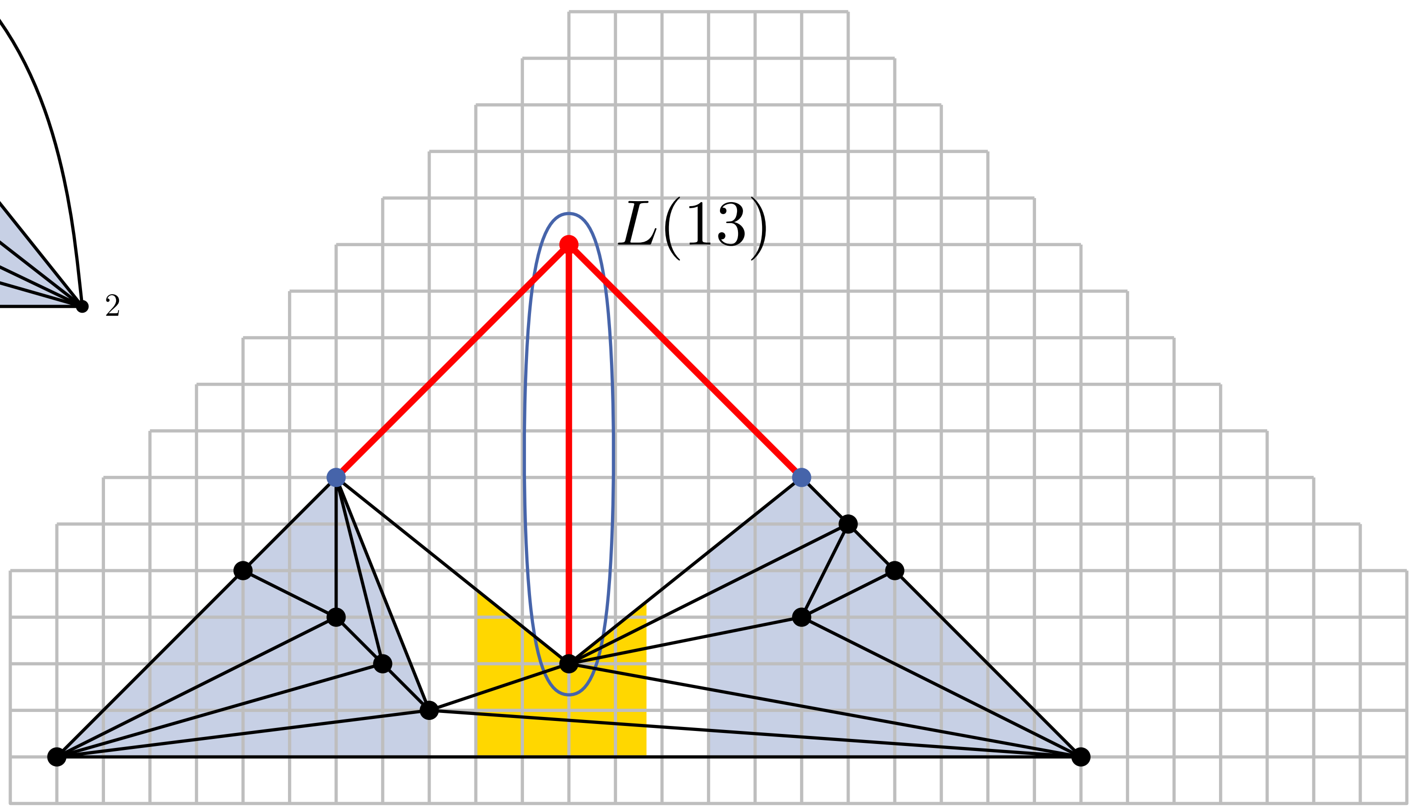

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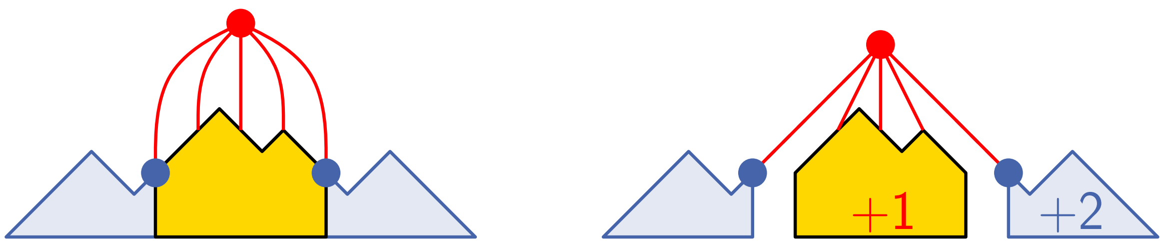

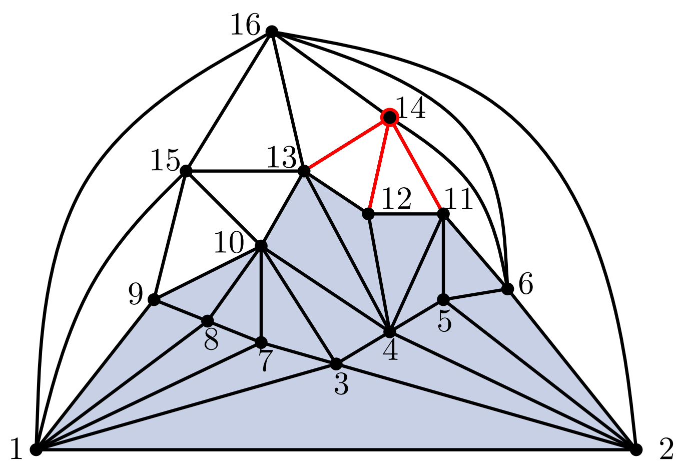

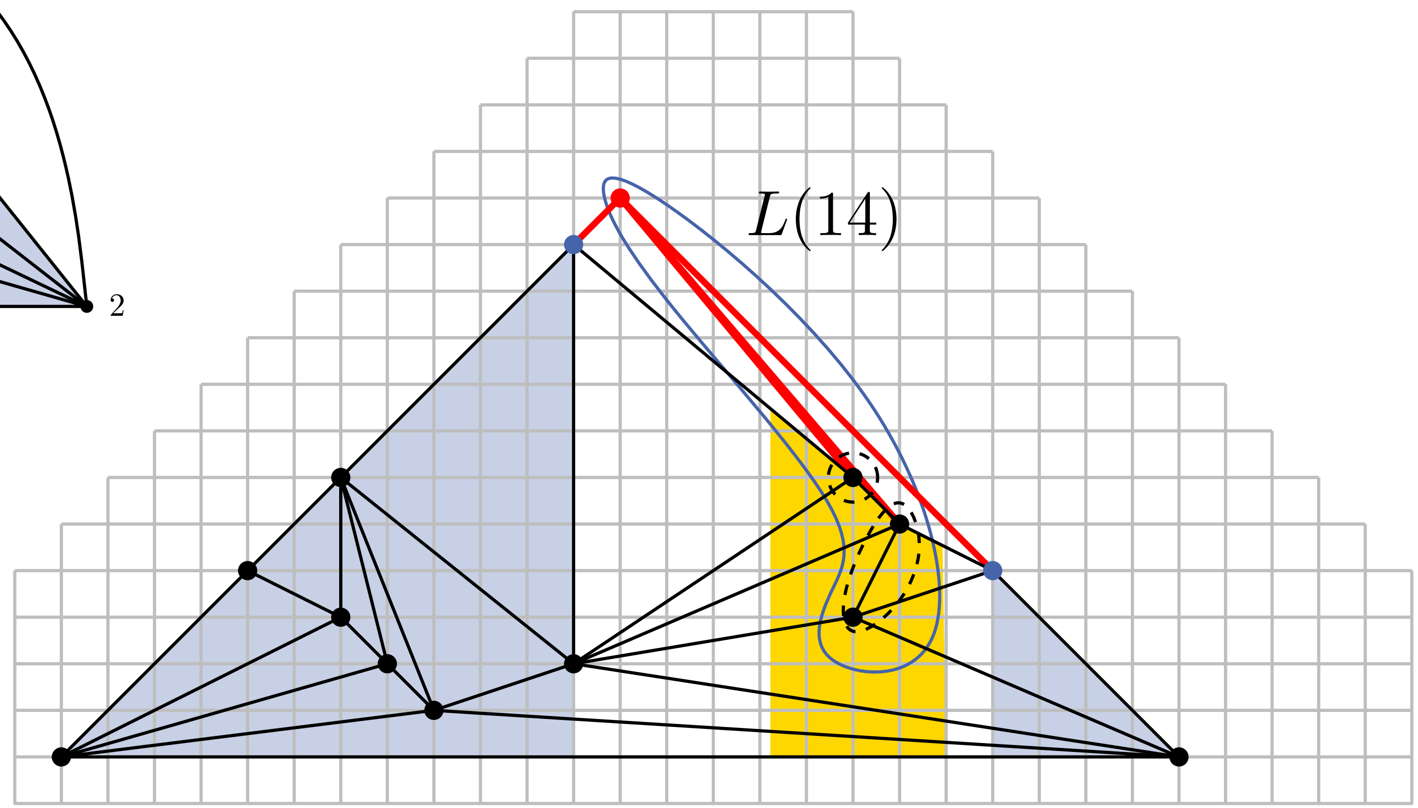

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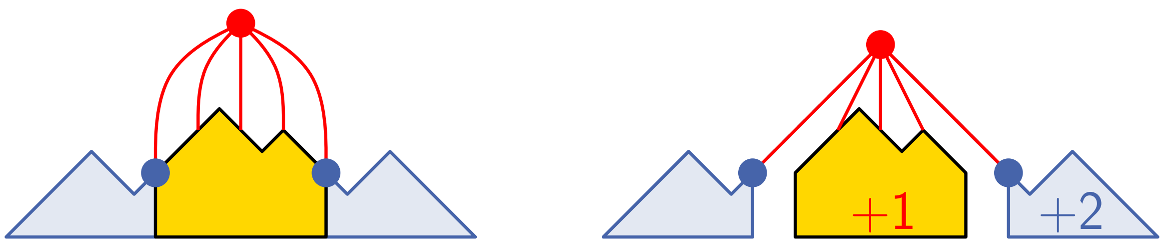

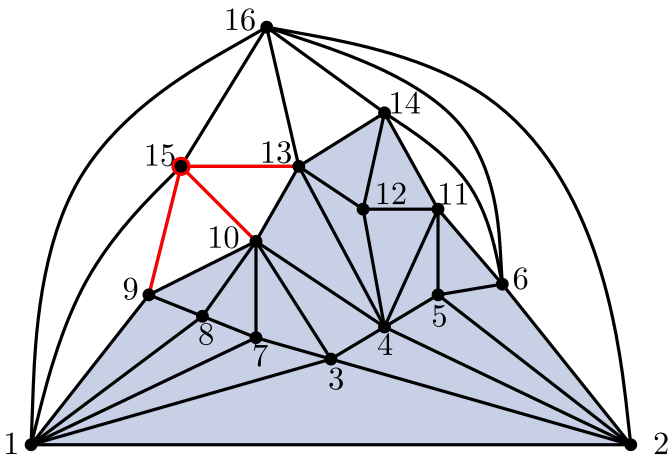

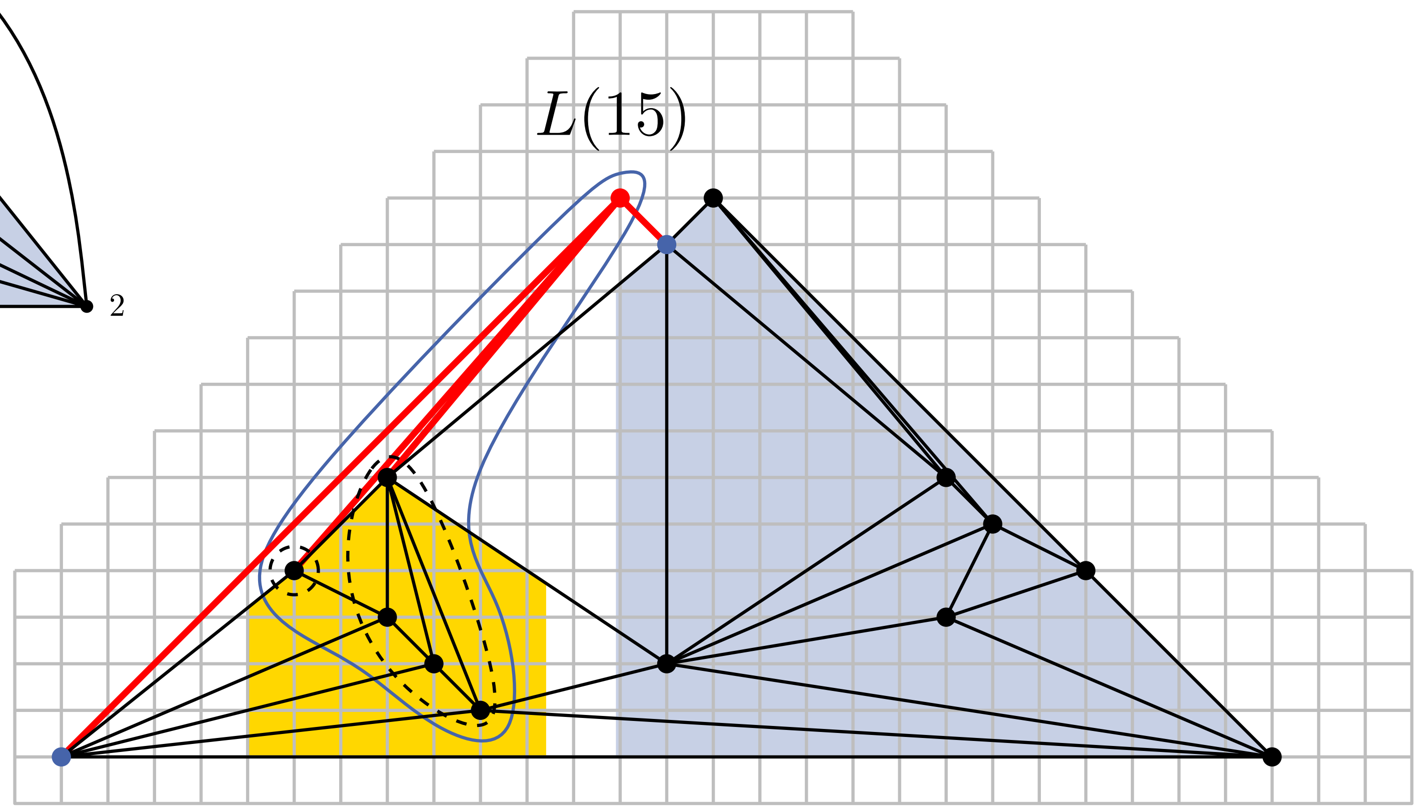

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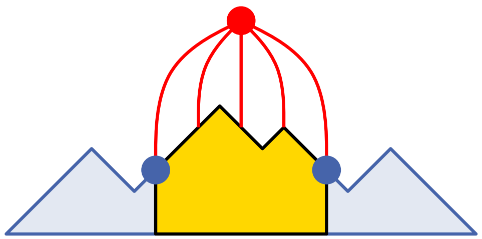

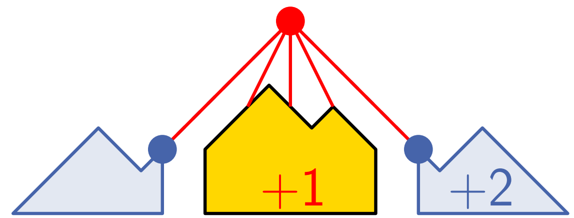

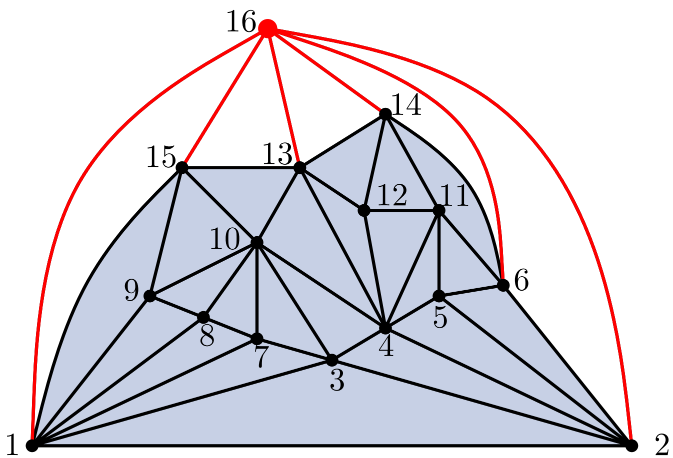

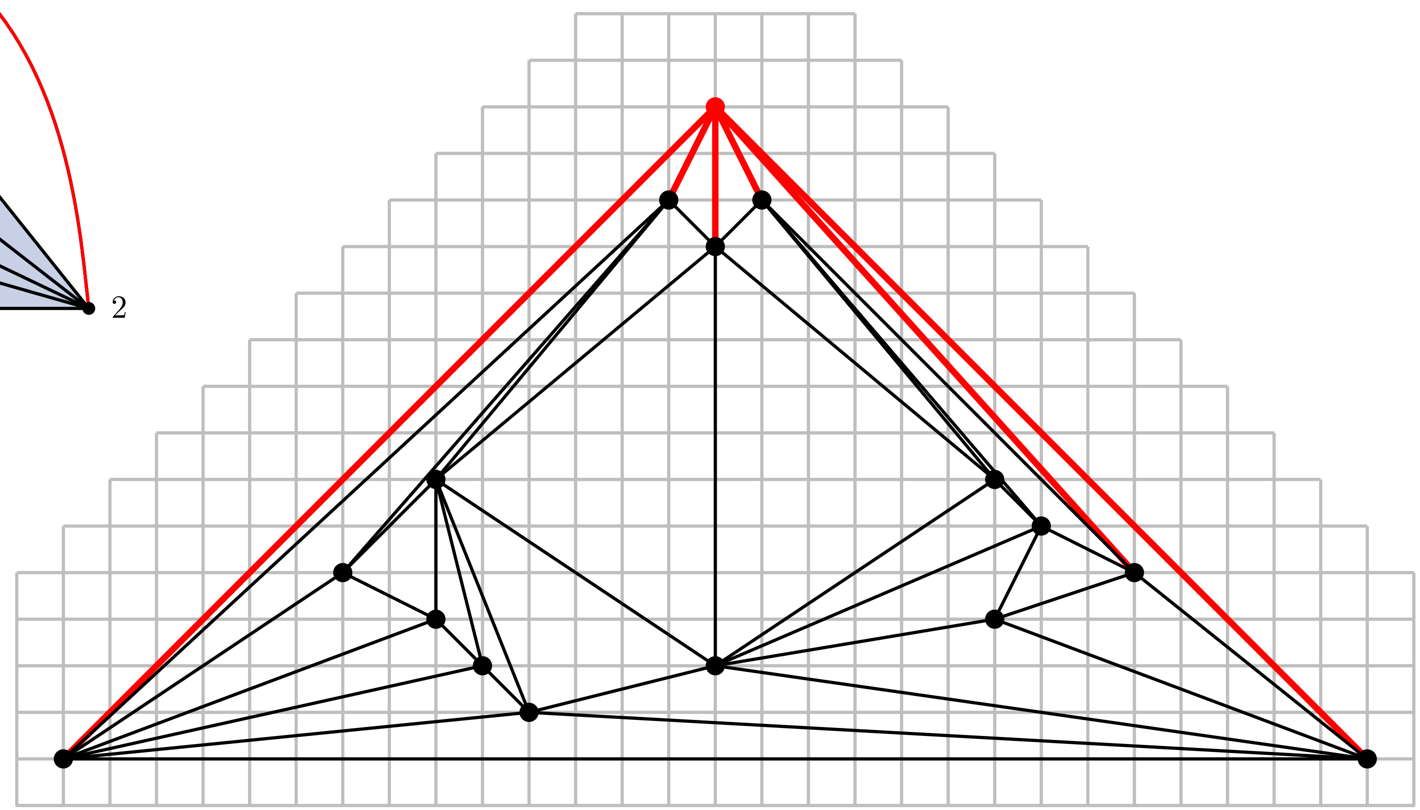

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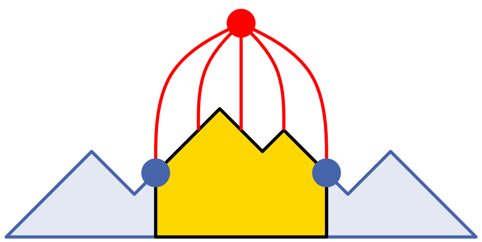
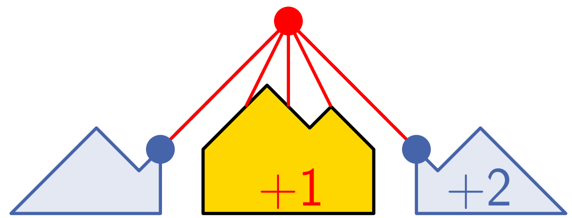
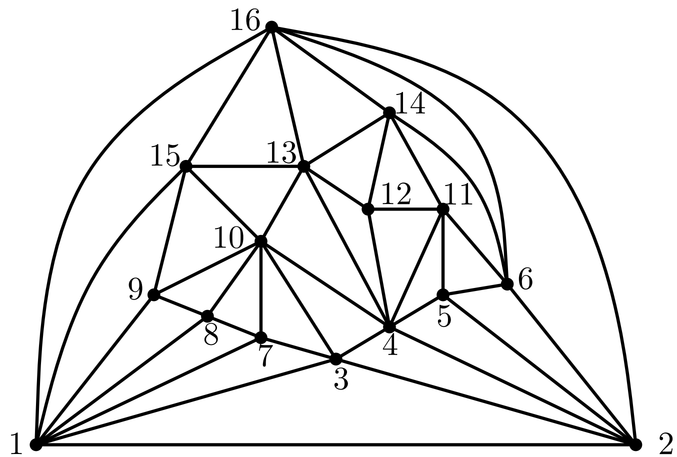
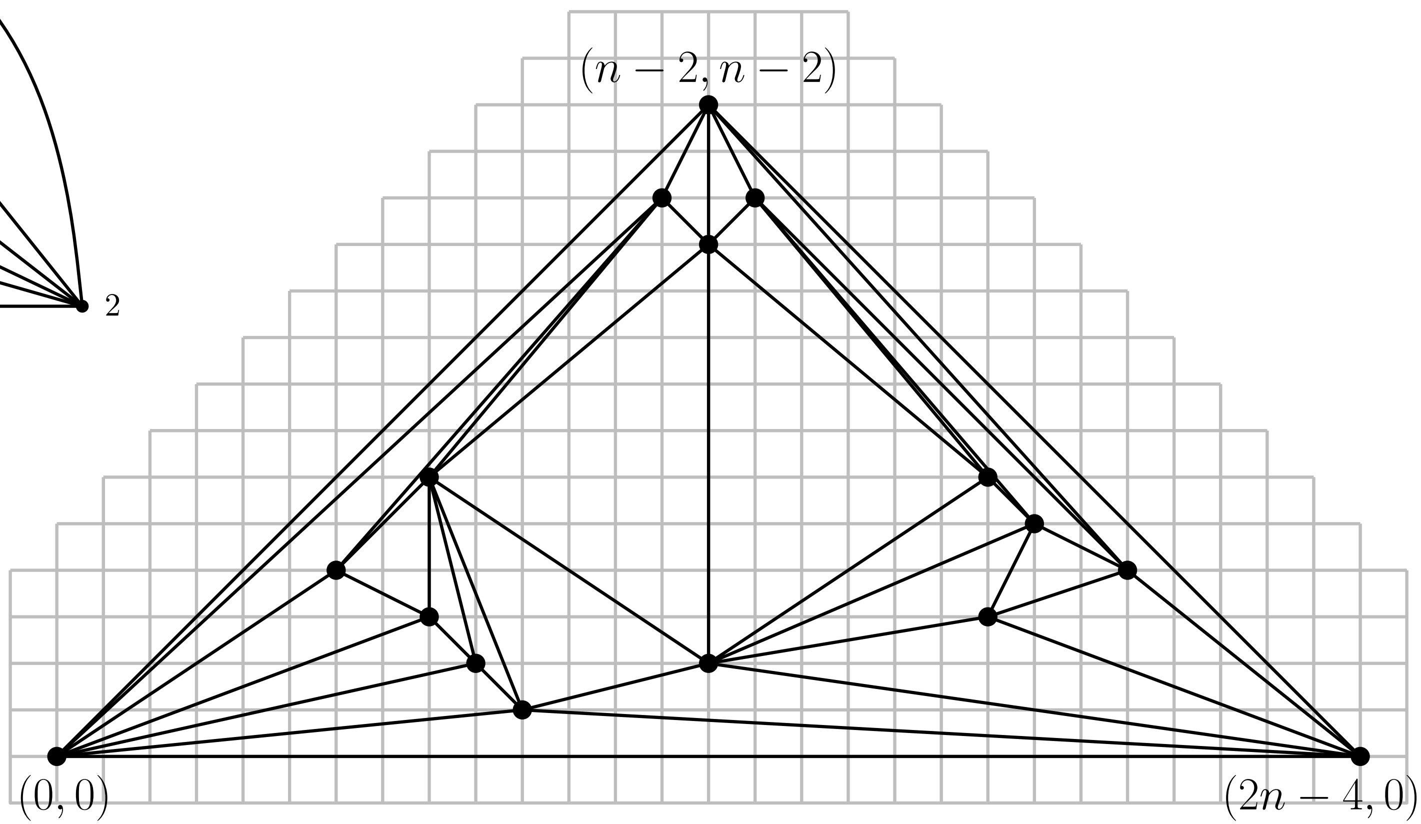
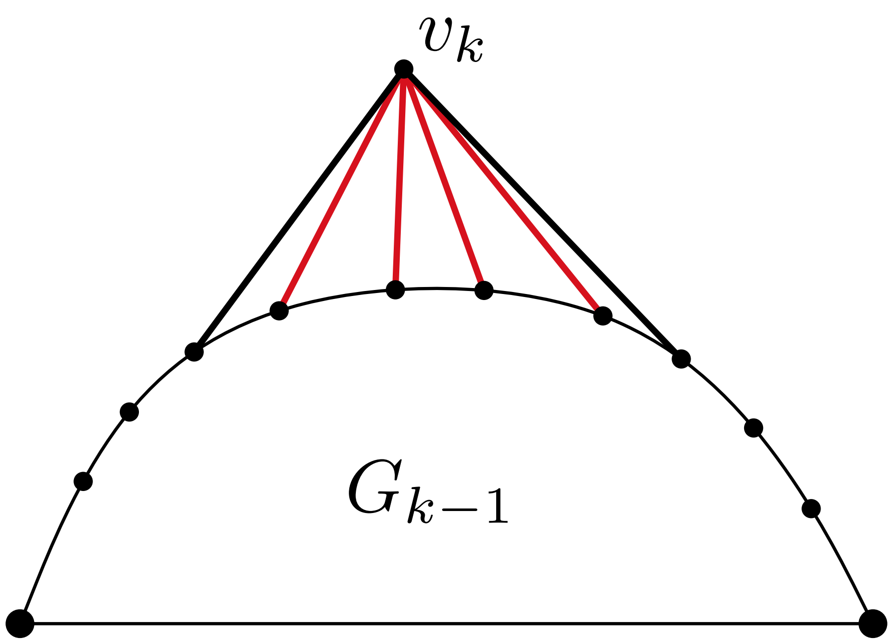








Proof of Planarity



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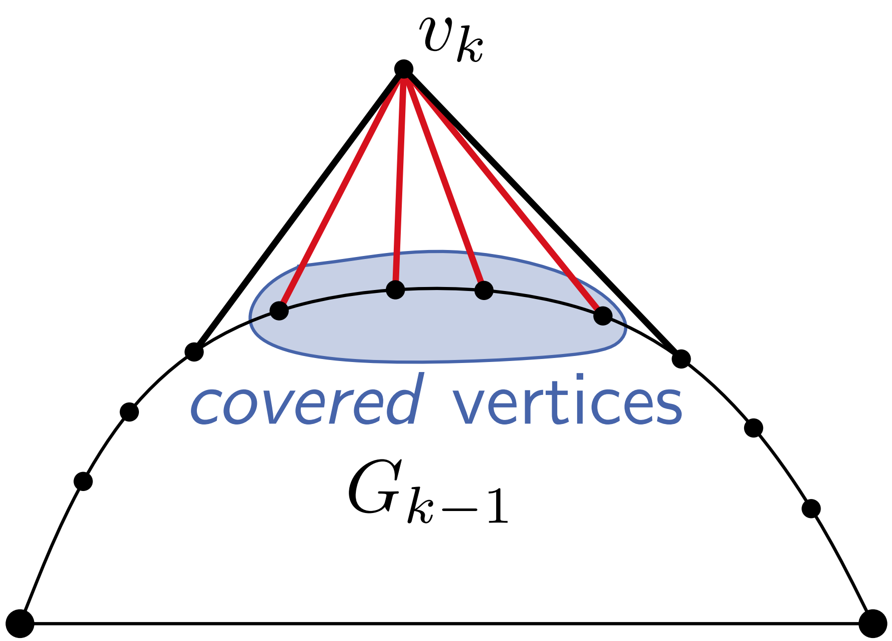

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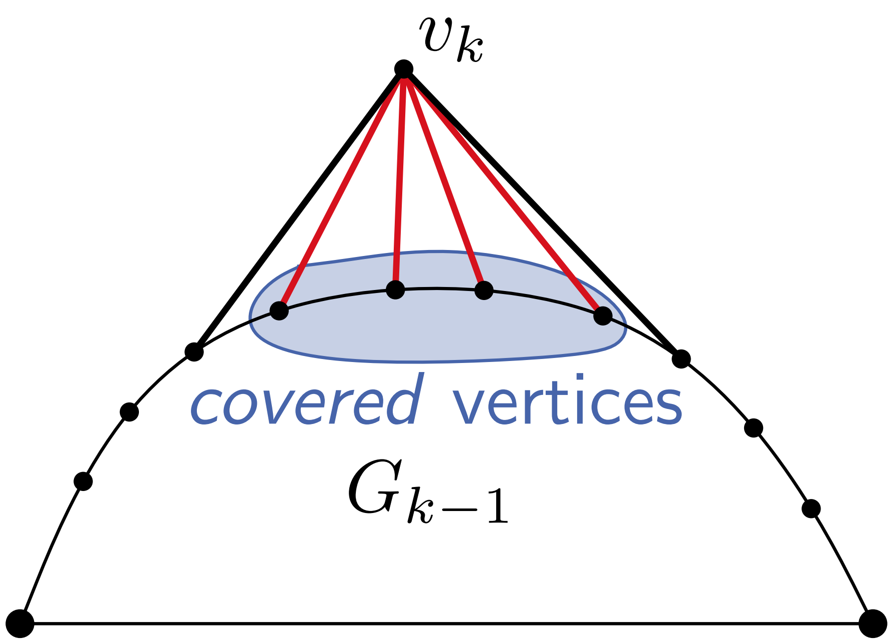

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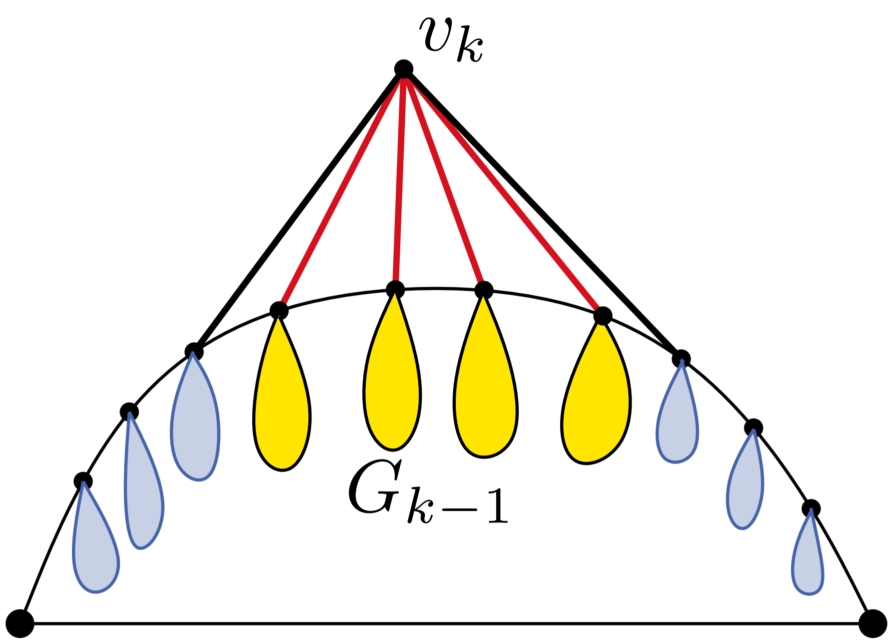

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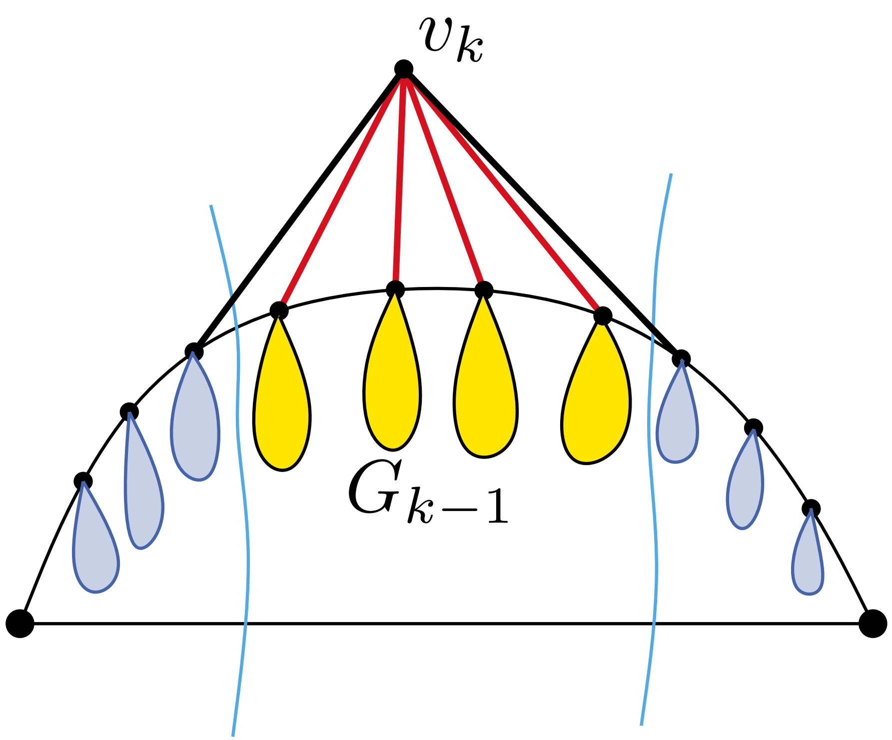

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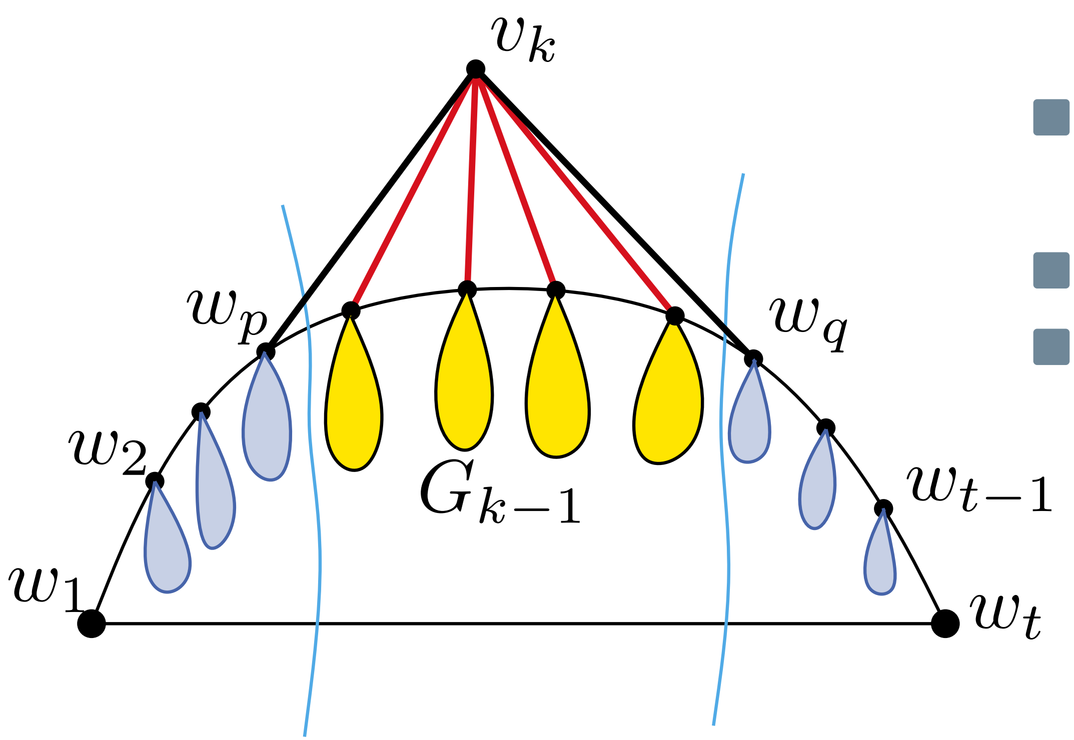

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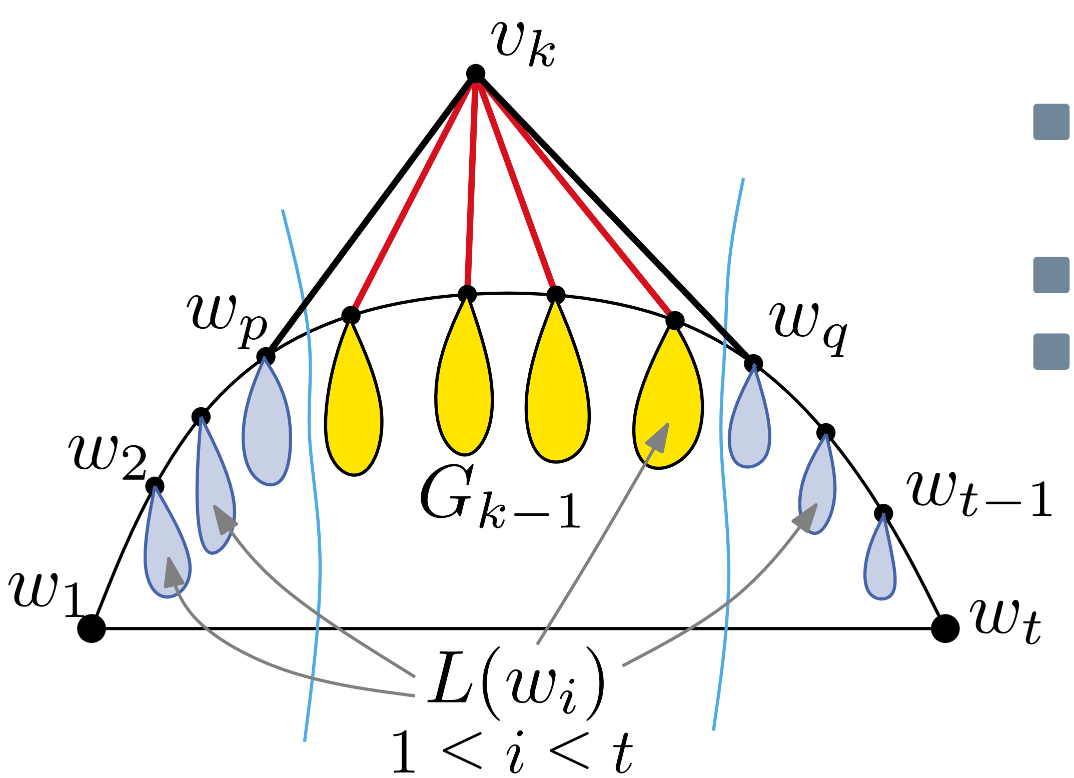

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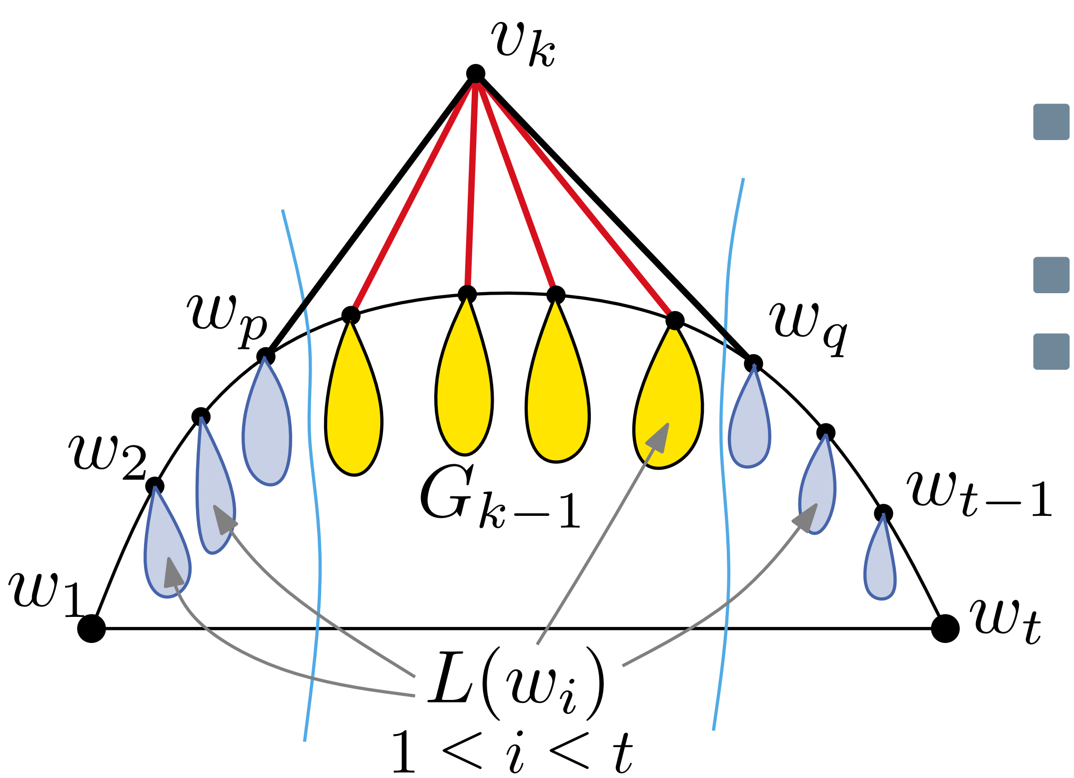

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Lemma: Let G_k be a planar straight-line grid drawing. Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$. If we shift each $L(w_i)$ by δ_i to the right, we get another planar straight-line grid drawing.

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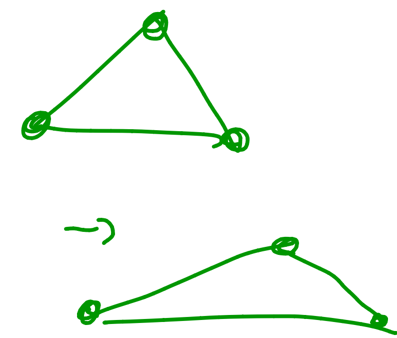

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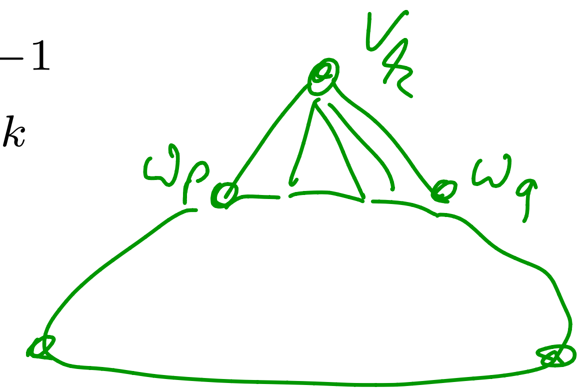

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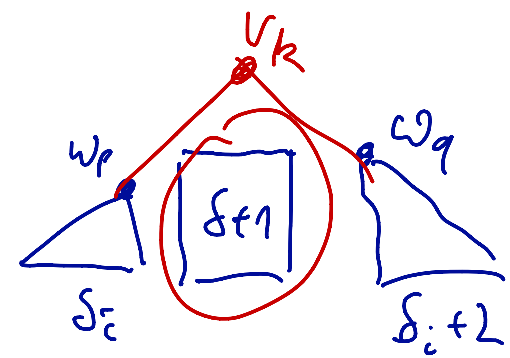

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- complete the drawing of G_k by placing v_k; now v_k is moved rigidly with $L(w_{p+1}), \dots, L(w_{q-1})$ by δ

Overview

Canonical ordering

Shift algorithm

Implementation

Naïve Implementation

$\texttt{Shift}$(plane graph $G = (V, E)$)

let $v_1, \ldots, v_n$ canonical ordering of G
for $i = 1$ **to** n **do** $L(v_i) \leftarrow \{v_i\}$
$P(v_1) \leftarrow (0, 0)$; $P(v_2) \leftarrow (2, 0)$; $P(v_3) \leftarrow (1, 1)$
for $k = 4$ **to** n **do**
- let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ be vertices of C_{k-1}
- let $w_p, \ldots, w_q$ be neighbors of v_k
- **for** $v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do** $x(v) \leftarrow x(v) + 1$
- **for** $v \in \cup_{j=q}^{t} L(w_j)$ **do** $x(v) \leftarrow x(v) + 2$
- $P(v_k) \leftarrow$ intersection point of lines with slope ± 1 from $P(w_p)$ and $P(w_q)$
- $L(v_k) \leftarrow \cup_{j=p+1}^{q-1} L(w_j) \cup \{v_{\cancel{i}\,k}\}$

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$O(n)$ time

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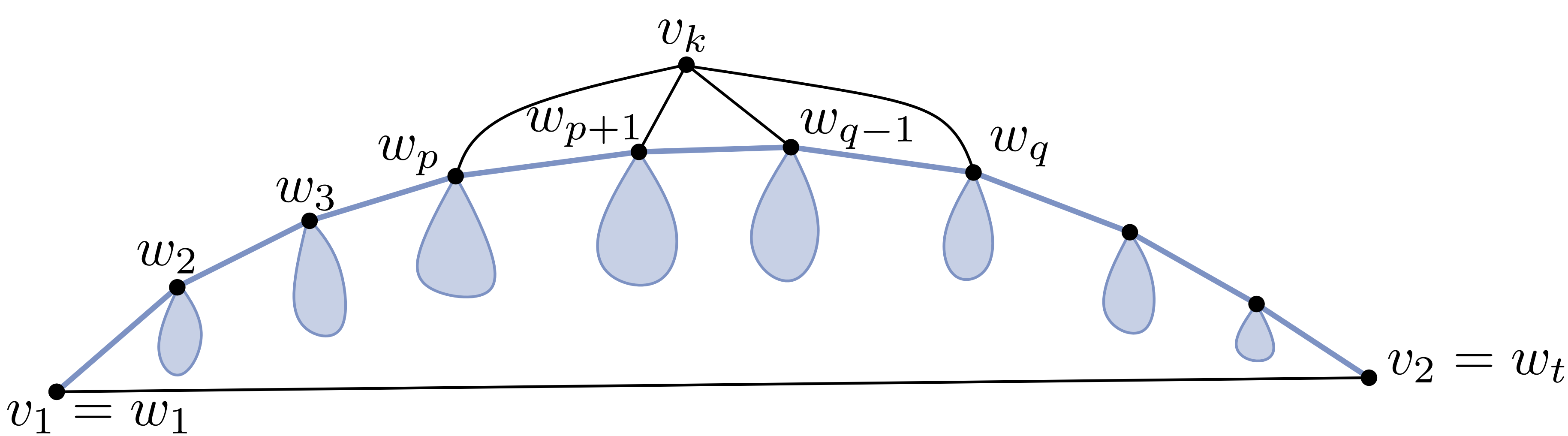
Running time?

$\rightarrow O(n^2)$

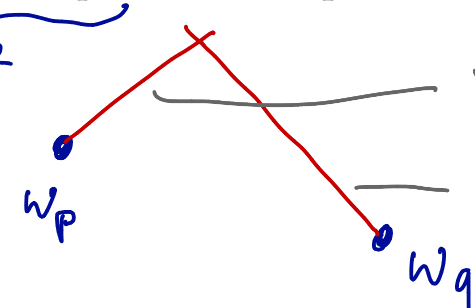
Towards a Faster Implementation

Vertex coordinates



- $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$ (1)
- $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$ (2)
- $x(v_k) - x(w_p) = \underbrace{\frac{1}{2}(\underbrace{x(w_q) - x(w_p)}_{} + y(w_q) - y(w_p))}_{\text{x-distance}}$ (3)

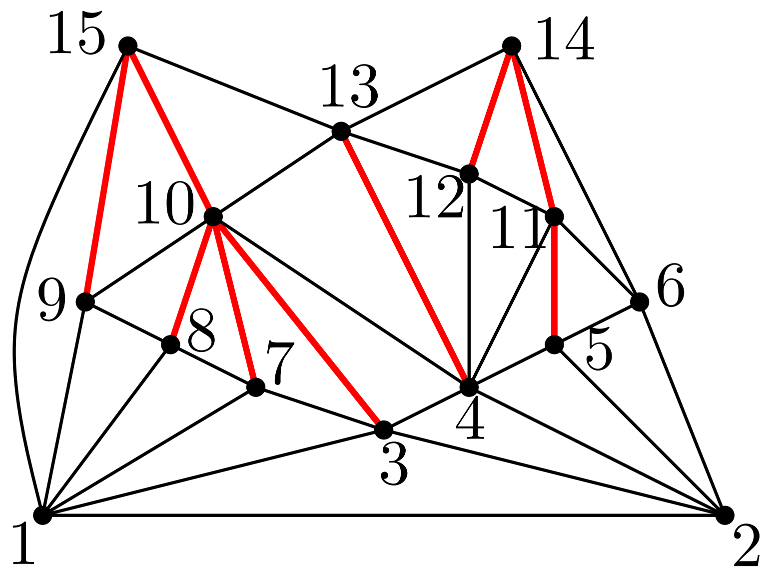
compute intersection point



$y = y(w_p) + 1 \cdot (x - x(w_p))$

$y = y(w_q) - 1 \cdot (x - x(w_q))$

Relative x-Distances



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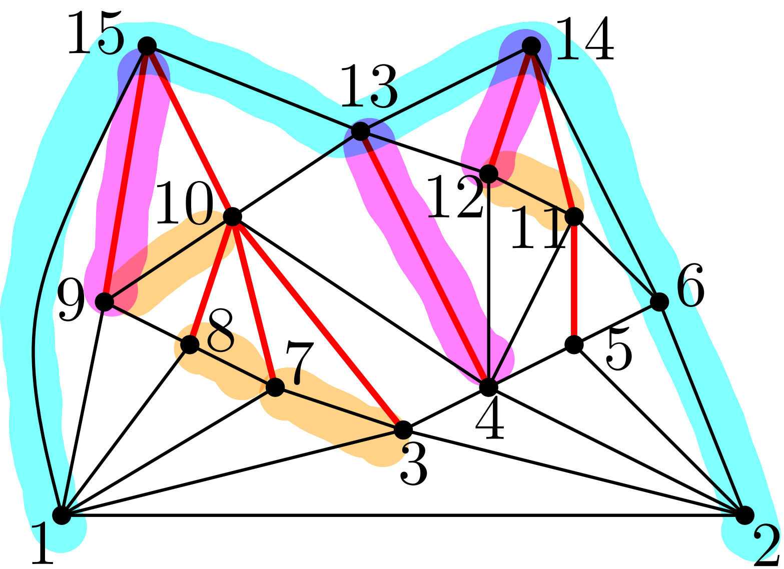
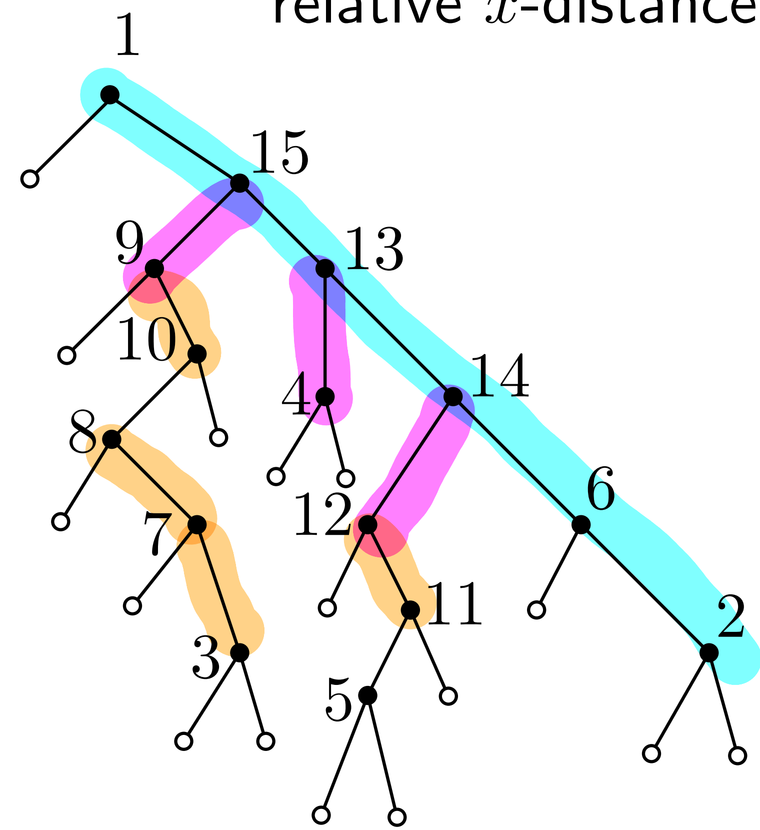


relative x-distance tree T



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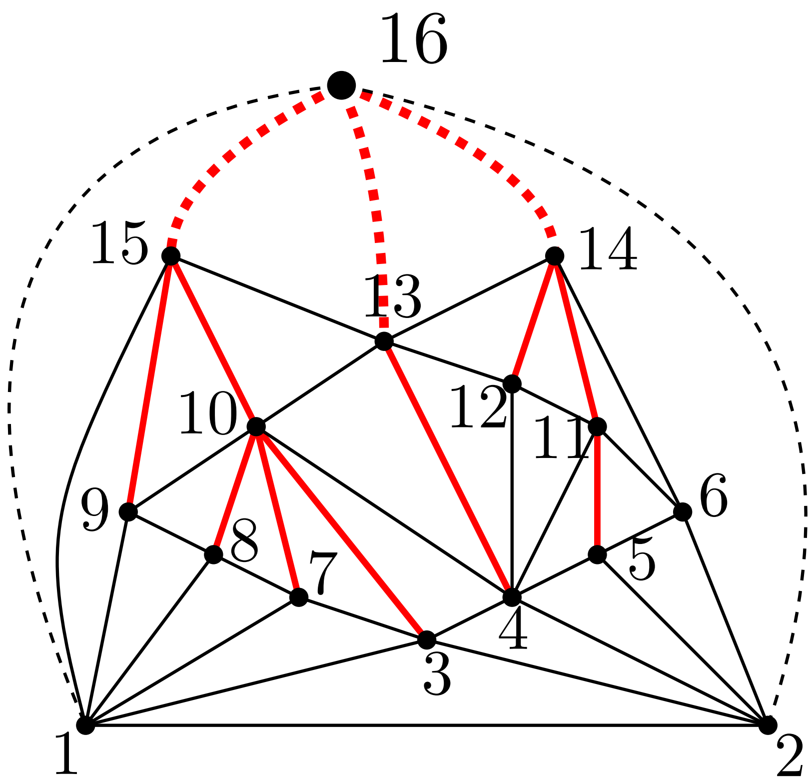

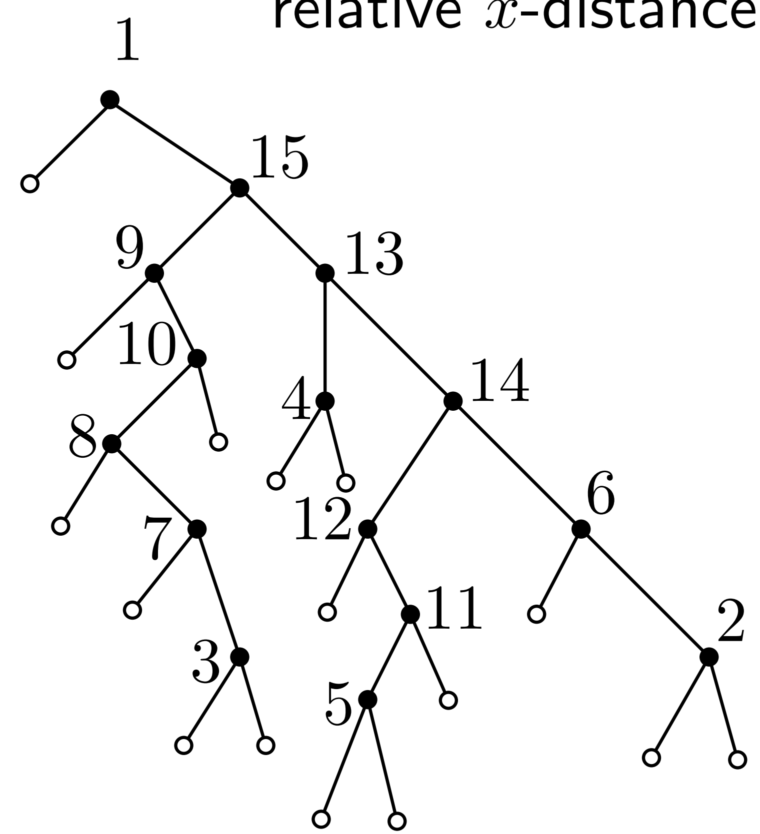

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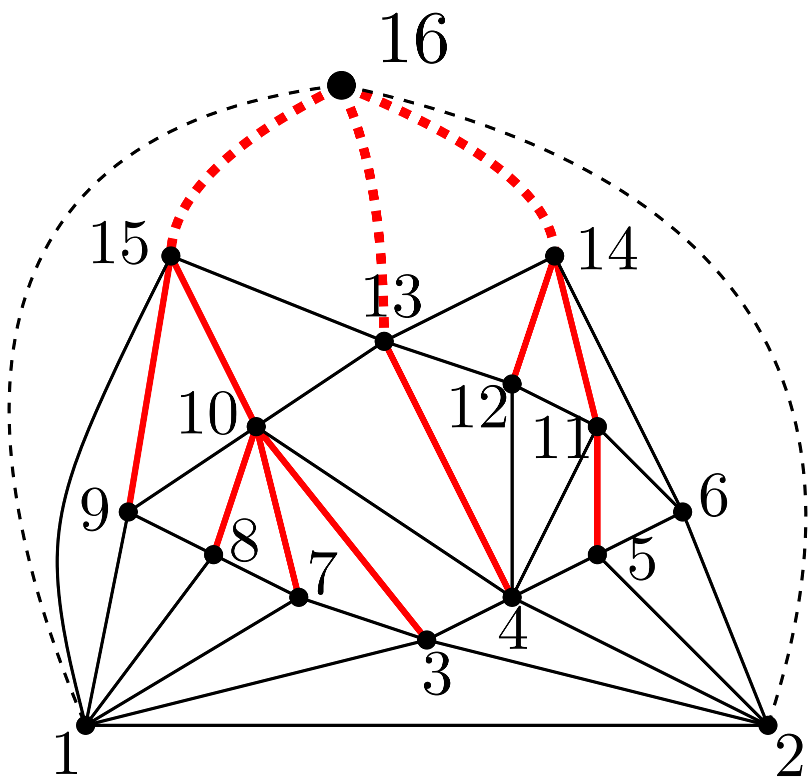

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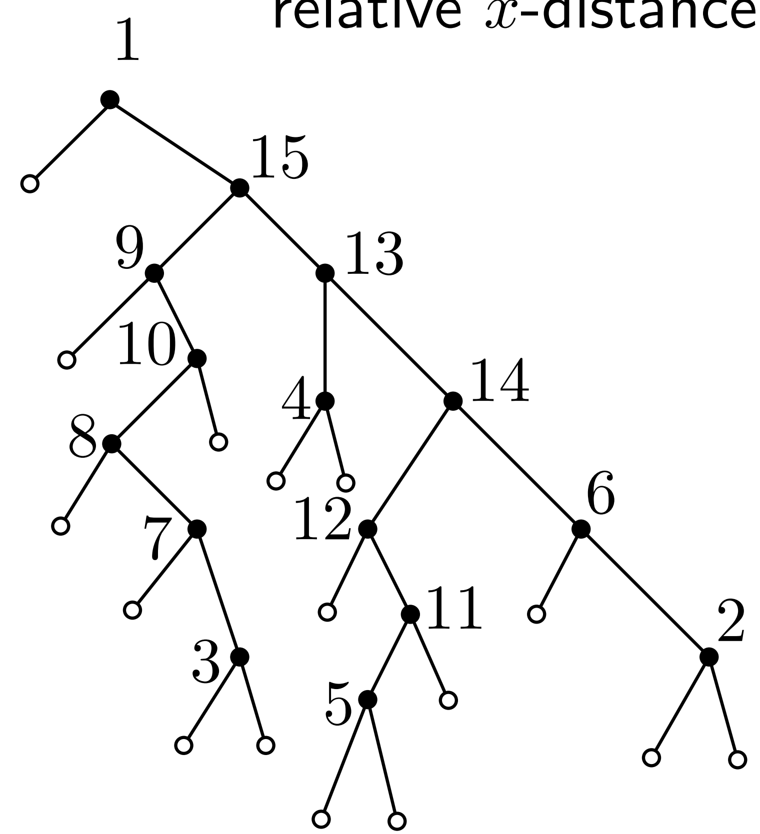

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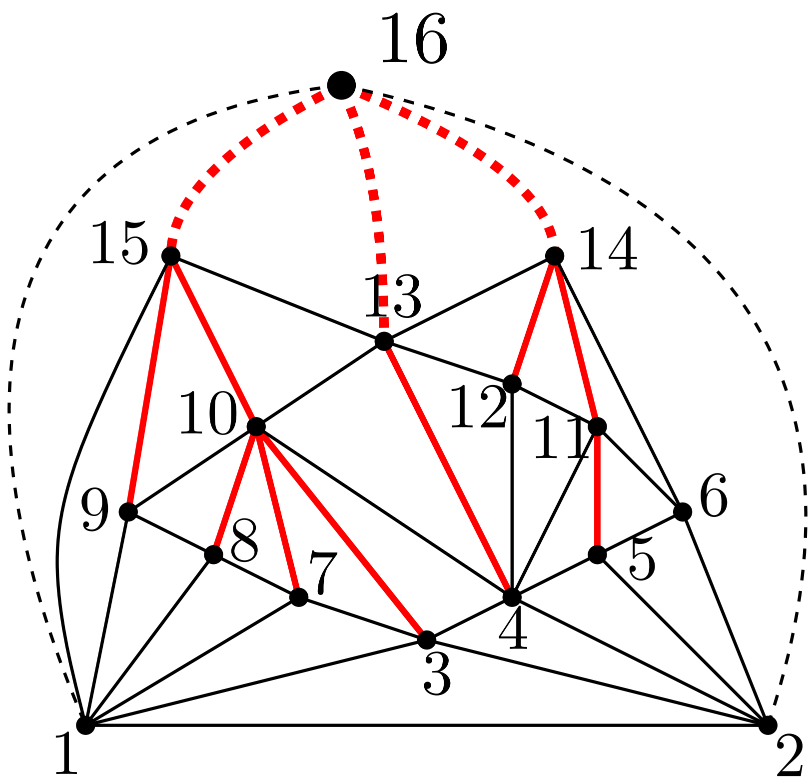

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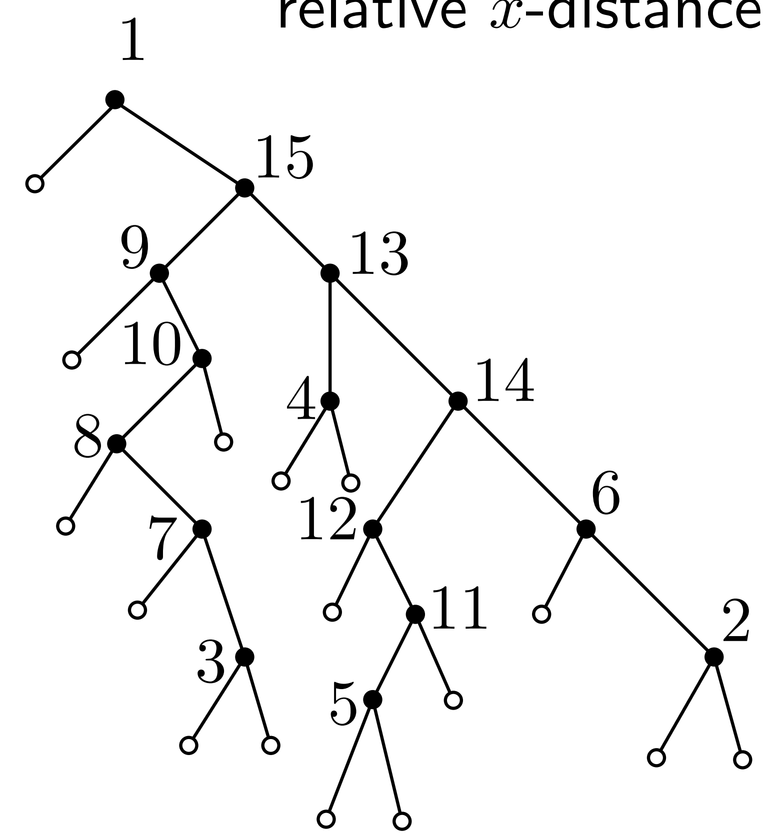

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Linear-Time Implementation

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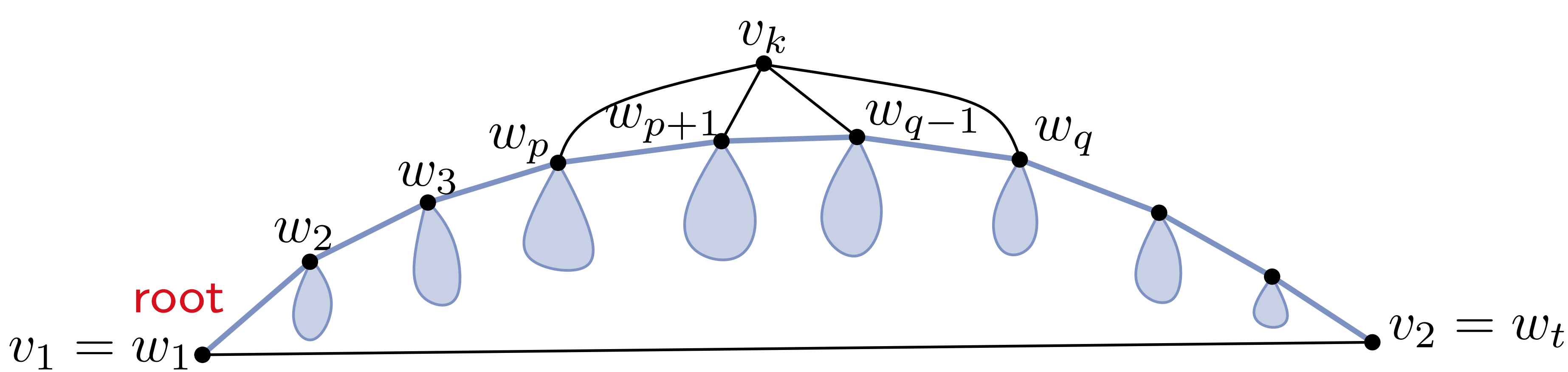

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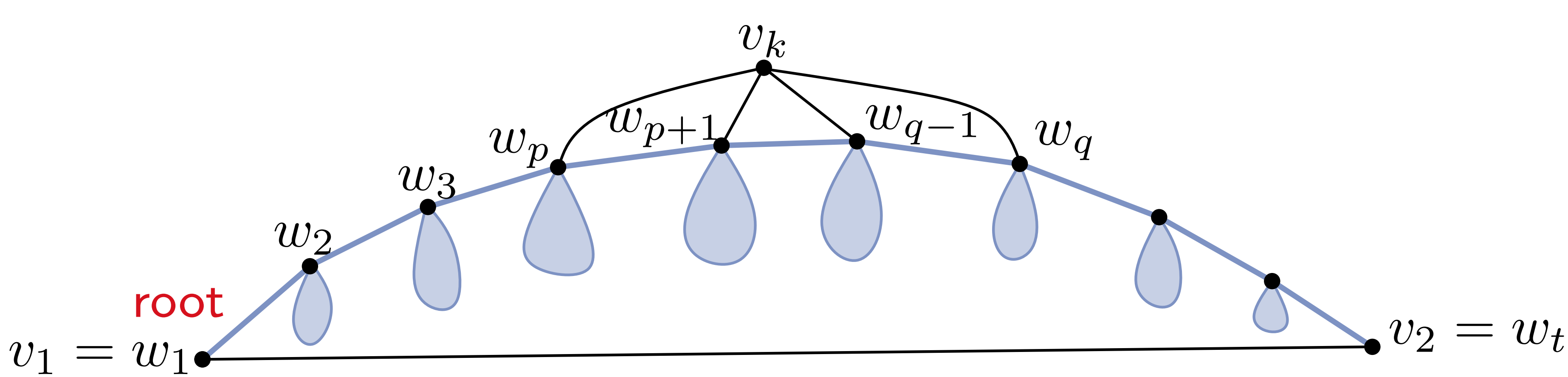

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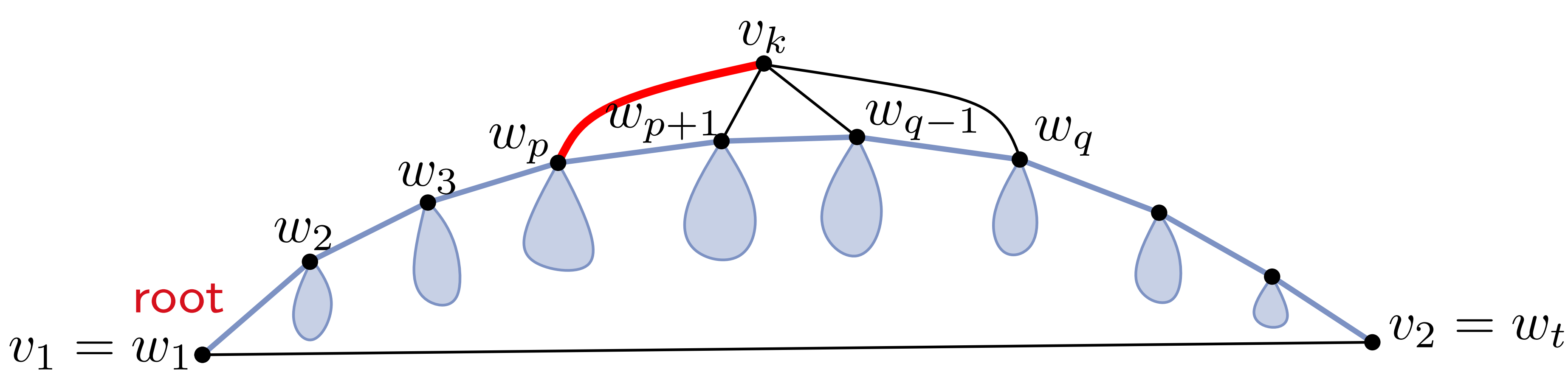

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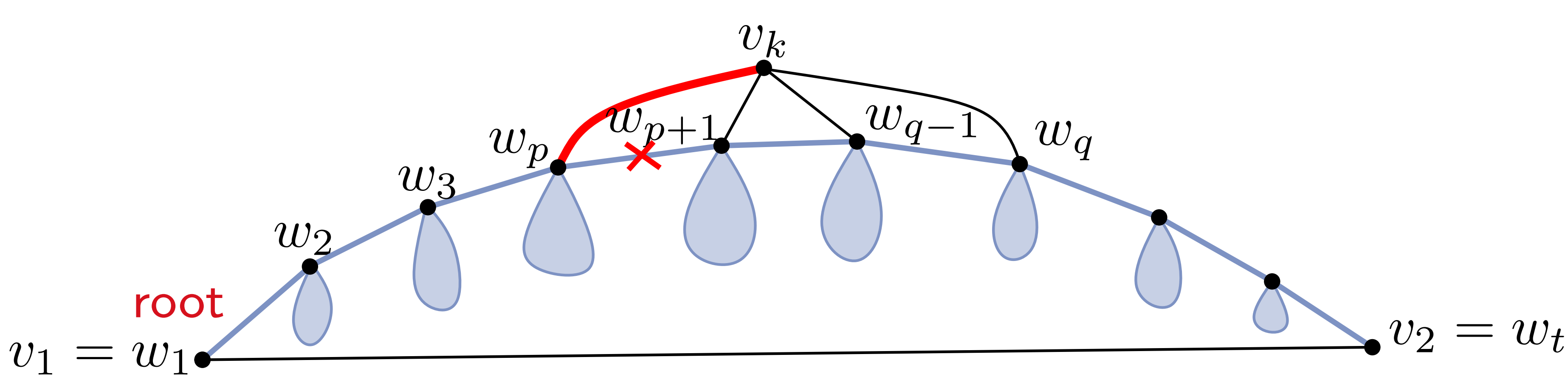

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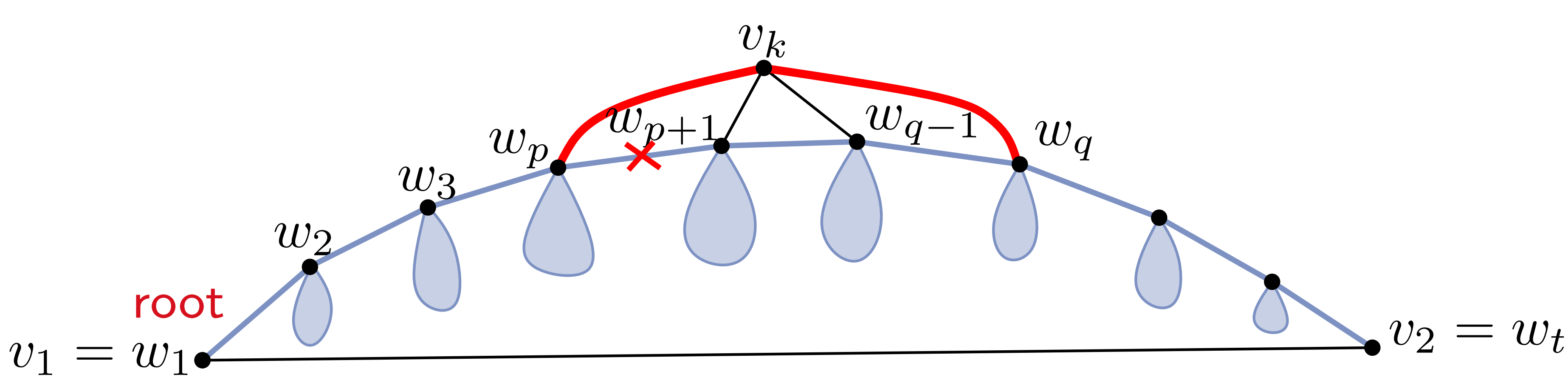

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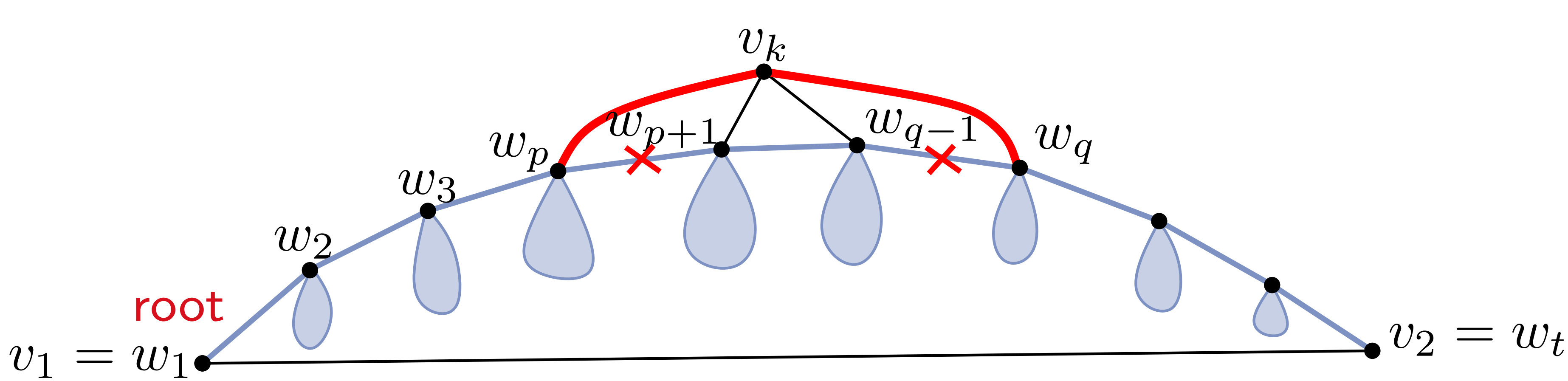

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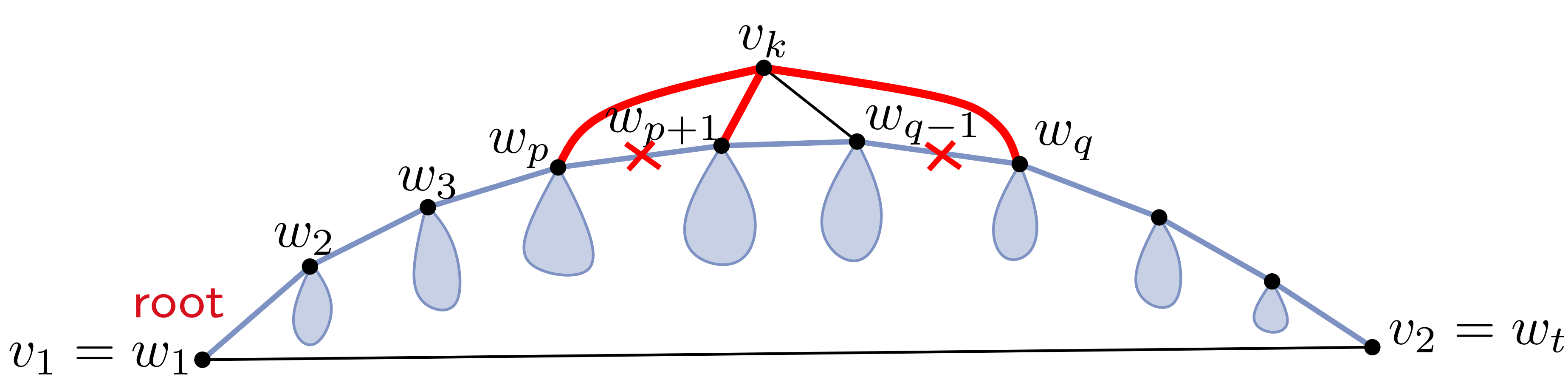

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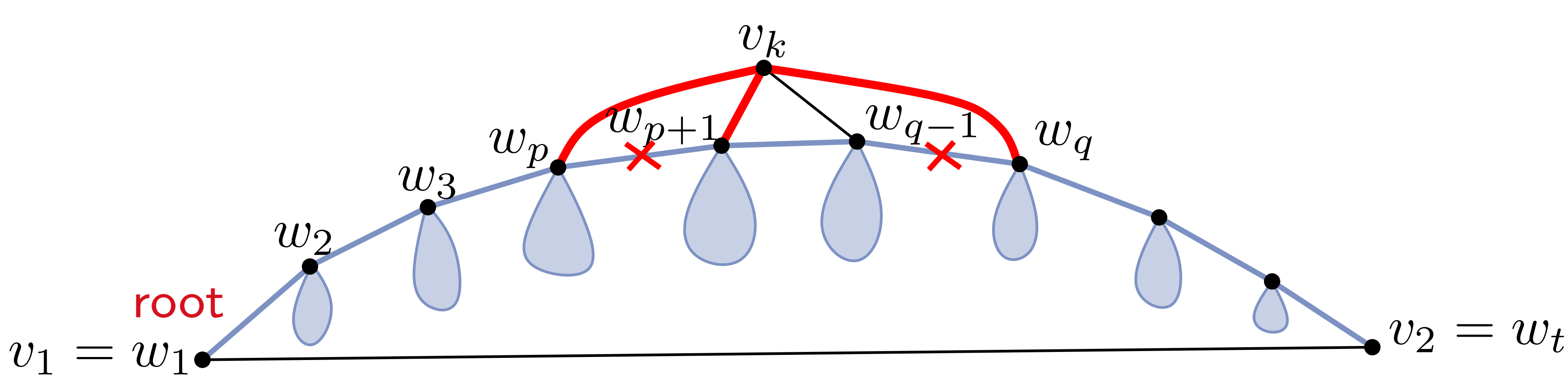

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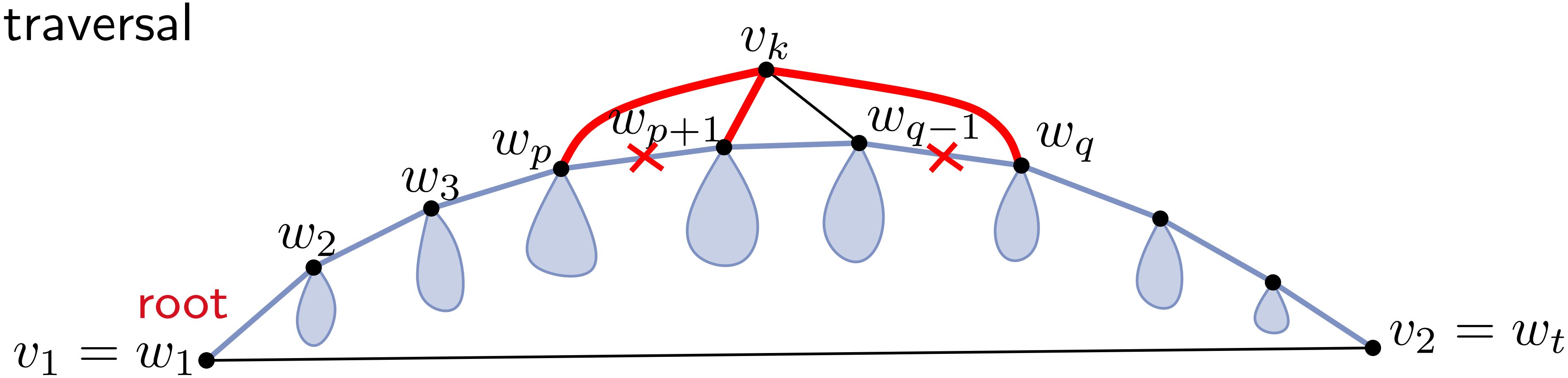
- Once T is completed, compute x-coordinates by a linear-time pre-order traversal



Summary

Theorem: Every n-vertex embedded planar graph $G = (V, E)$ has a straight-line planar drawing on a grid of size $(2n - 4) \times (n - 2)$. [de Fraysseix, Pach, Pollack 1988]

Theorem: The corresponding shift algorithm can be implemented to run in $O(n)$ time. [Chrobak, Payne 1995]