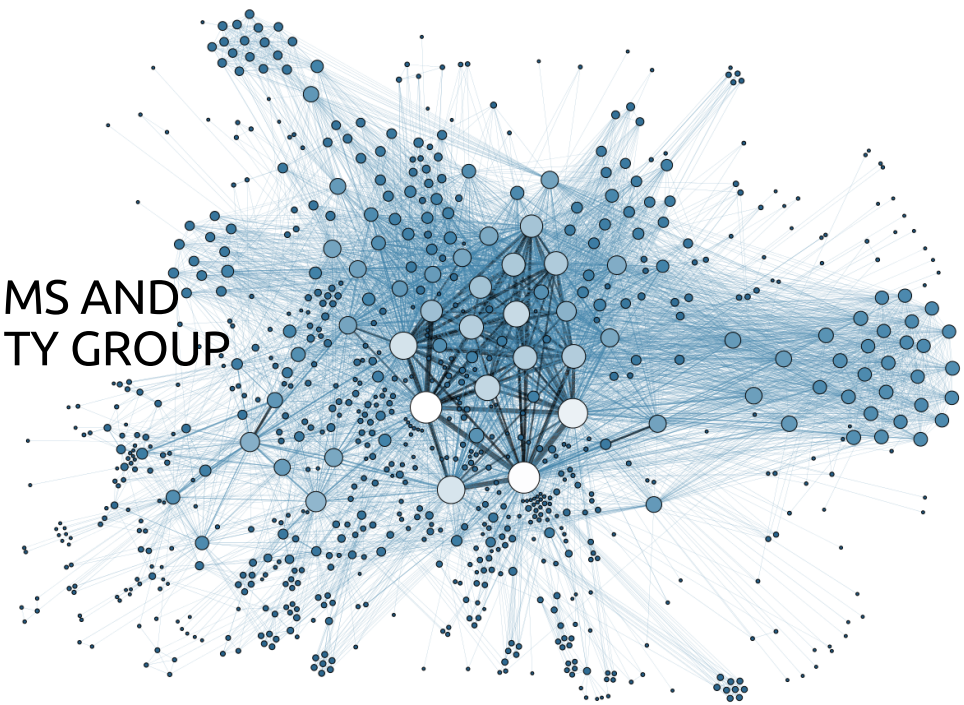
Force- and Stress-Based Graph Drawing

Lecture Graph Drawing Algorithms · 192.053

Martin Nöllenburg

08.05.2018



TU WIEN

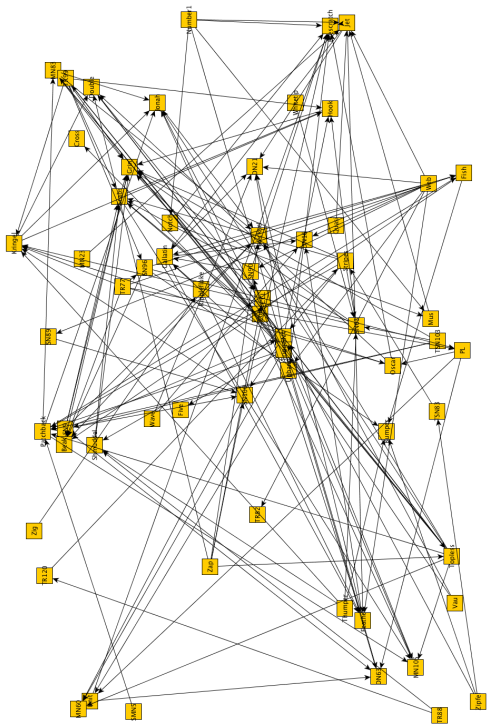
ALGORITHMS AND
COMPLEXITY GROUP

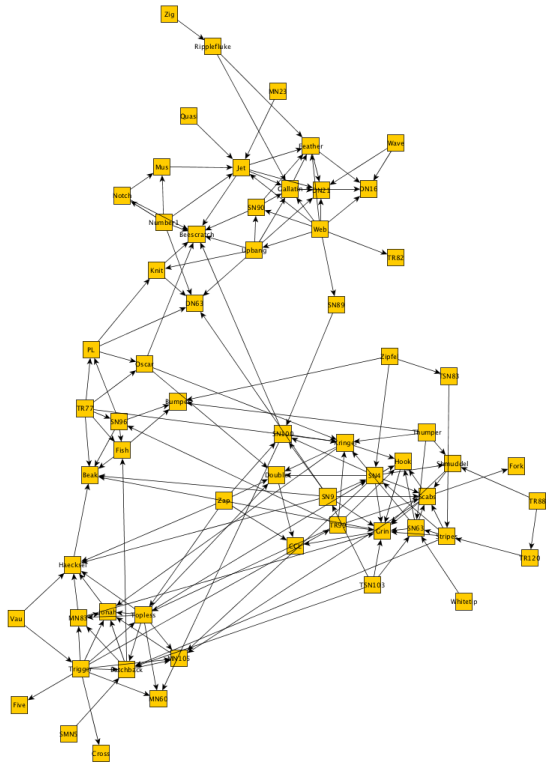
Drawing Arbitrary Graphs

In the previous lectures we studied graphs with structural properties (trees, series-parallel, planar).
What if we don't know the structure of our input graph?

What is a minimal set of constraints and criteria?

Assumptions: simple, undirected graph





- few edge crossings
- maximize minimum angle
- short, uniform edge lengths
- good use of screen space, distribute vertices
- show grouping structure
- adjacent vertices close, non-adj. farther

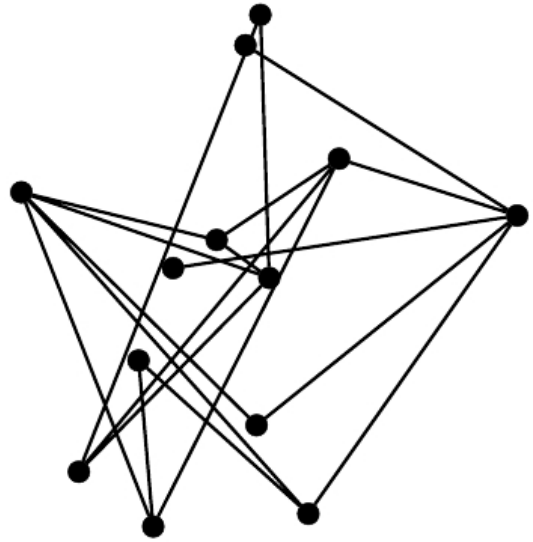
Overview

Force-based graph layout

Scalability and faster force computation

Stress-based graph layout

Physical Model



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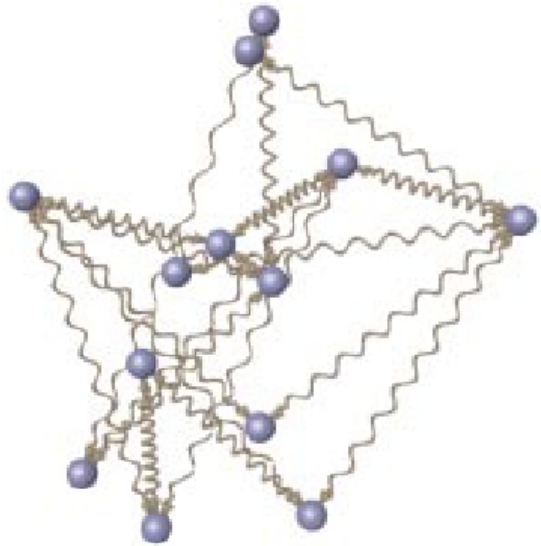

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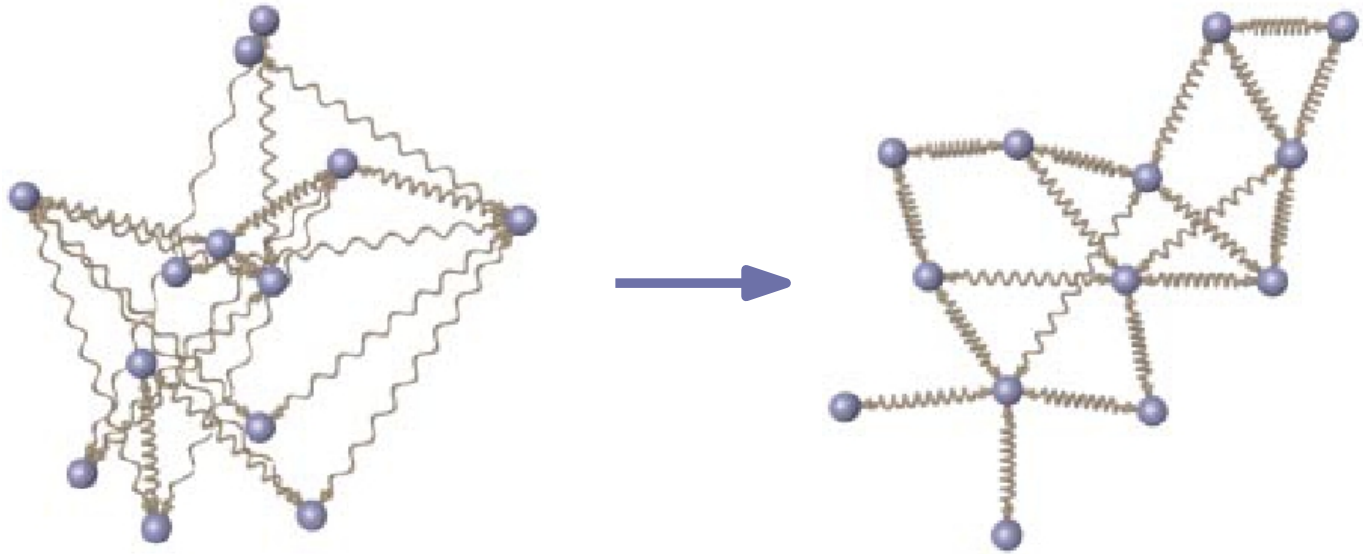

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The vertices are placed in some initial layout and let go so that the spring forces on the rings move the system to a minimal energy state."

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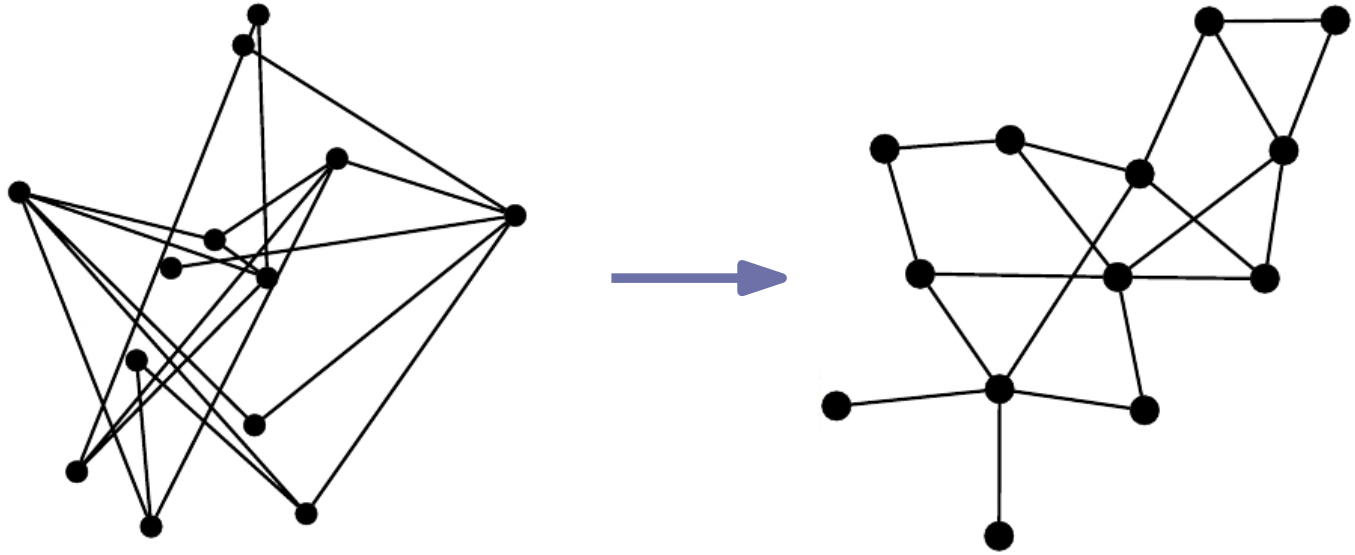

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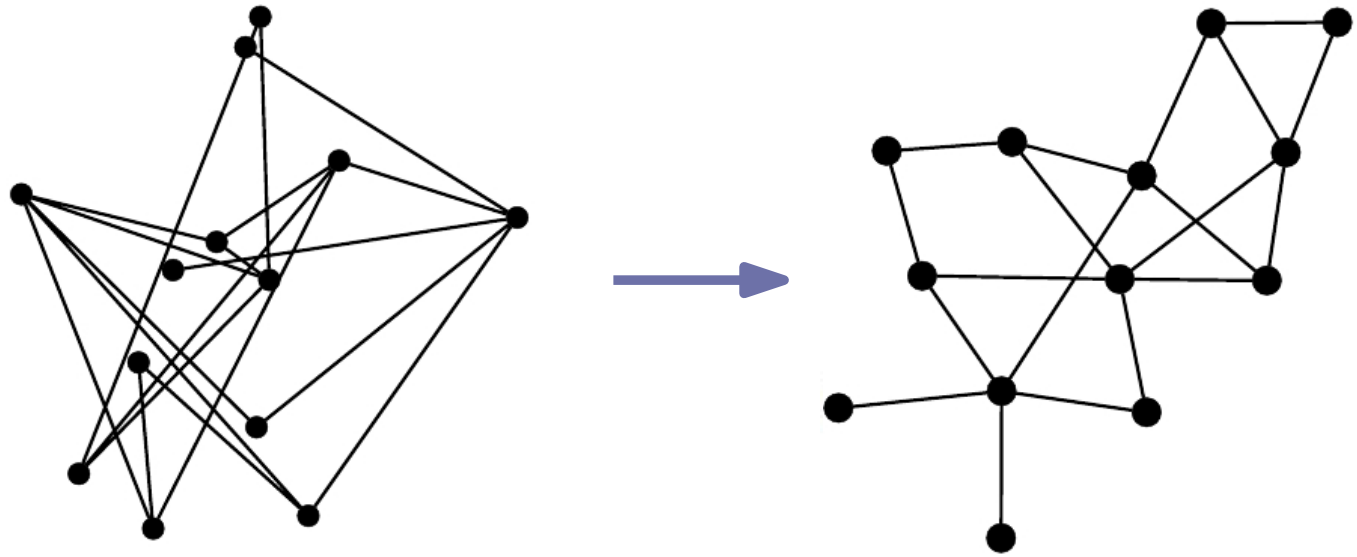

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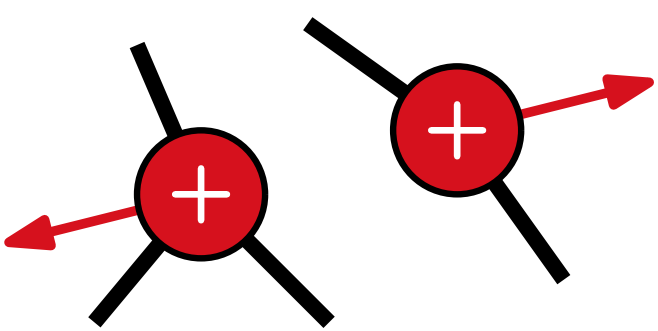

In addition consider (non-adjacent) vertices as electrically charged particles that repel each other.

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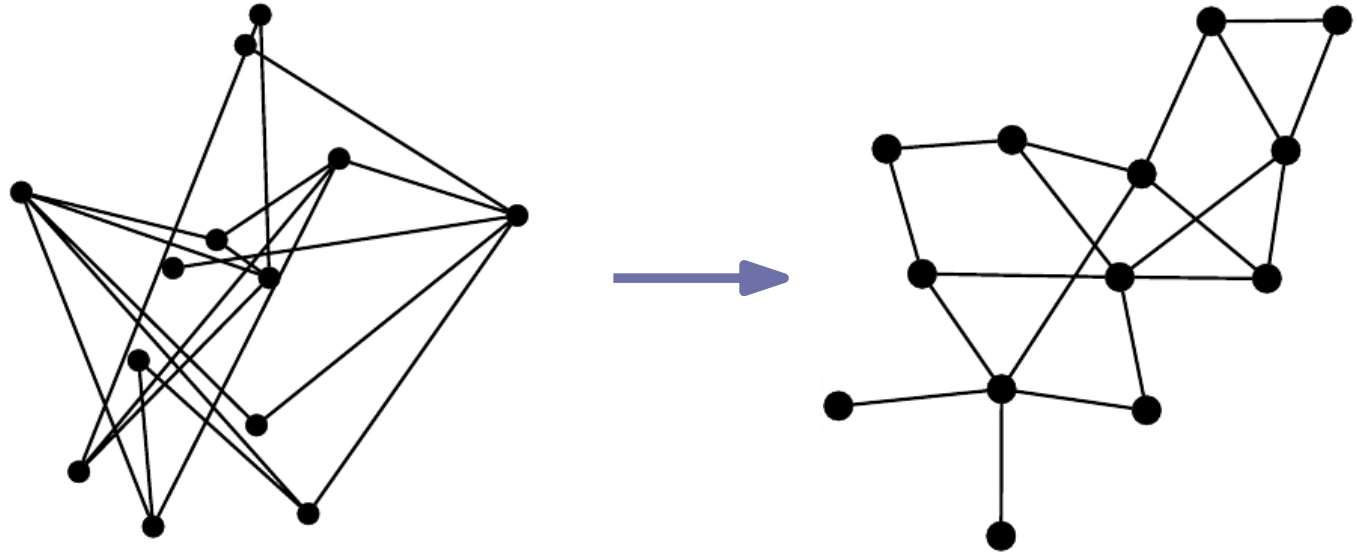
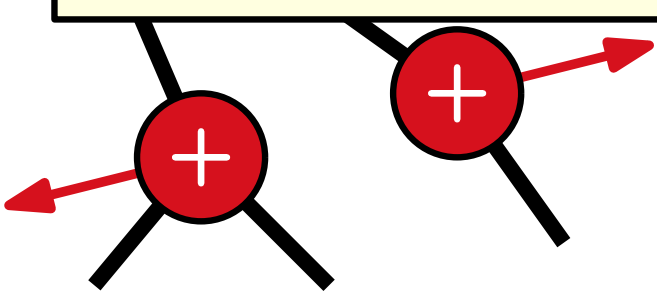


[Eades, 1984] "To embed a graph we replace the vertices by steel rings and r
f

So-called **spring-embedder** or **spring-electrical** algorithms that work according to this or similar principles are among the most frequently used graph-drawing methods in practice.



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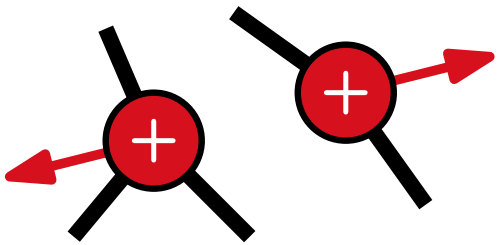
Notation

$\ell = \ell(e)$	ideal spring length for edge e
$p_v = (x_v, y_v)$	position of node v
$\|\|p_u - p_v\|\|$	Euclidean distance between u and v
$\overrightarrow{p_u p_v}$	unit vector pointing from u to v

Spring-Embedder (Eades, 1984)

Model:

- repulsive force between two non-adjacent vertices u and v



$$f_{\text{rep}}(p_u, p_v) = \frac{c_{\text{rep}}}{||p_v - p_u||^2} \cdot \overrightarrow{p_u p_v}$$

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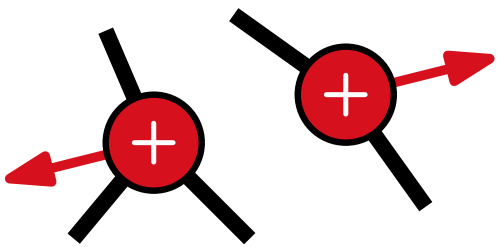

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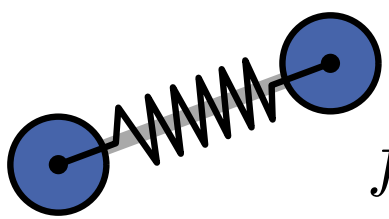
- attractive force between adjacent vertices u and v



$$f_{\text{spring}}(p_u, p_v) = c_{\text{spring}} \cdot \log \frac{||p_u - p_v||}{\ell} \cdot \overrightarrow{p_v p_u}$$

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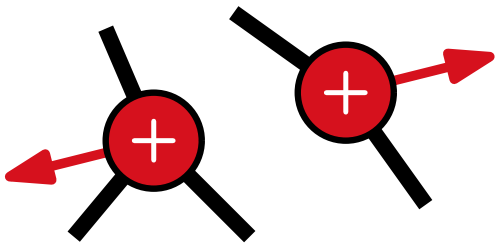

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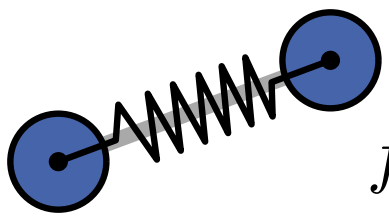

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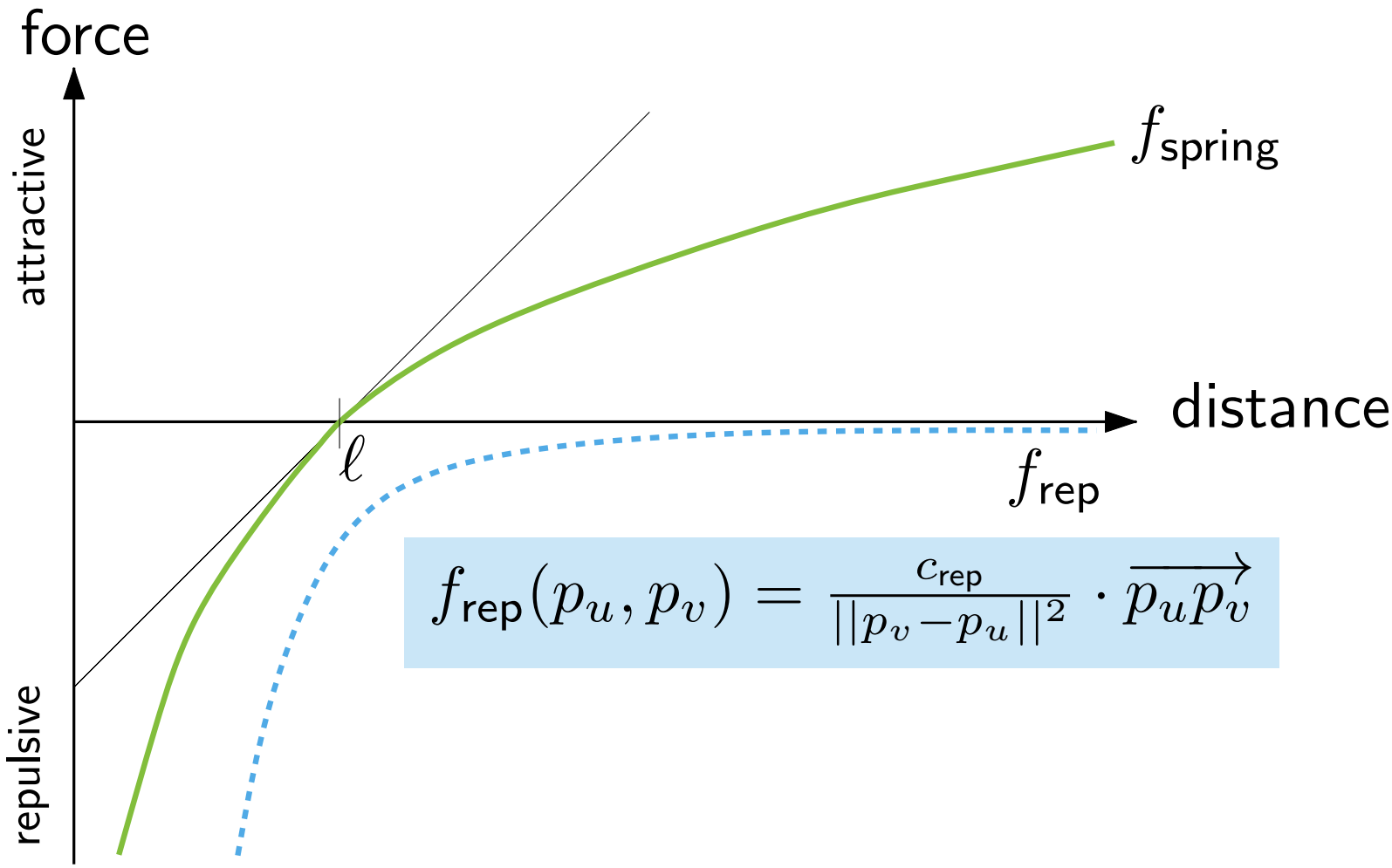
- resulting displacement vector for vertex v

$$F_v = \sum_{uv \notin E} f_{\text{rep}}(p_u, p_v) + \sum_{uv \in E} f_{\text{spring}}(p_u, p_v)$$

Diagram of Spring-Embedder Forces (Eades, 1984)



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Algorithm Spring-Embedder (Eades, 1984)

Input: $G = (V, E)$ connected undirected graph with initial placement $p = (p_v)_{v \in V}$, number of interations $K \in \mathbb{N}$, threshold $\varepsilon > 0$, constant $\delta > 0$

Output: Layout p with low energy

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

  foreach $v \in V$ **do**

    $F_v(t) \leftarrow \sum_{uv \notin E} f_{\text{rep}}(p_u, p_v) + \sum_{uv \in E} f_{\text{spring}}(p_u, p_v)$

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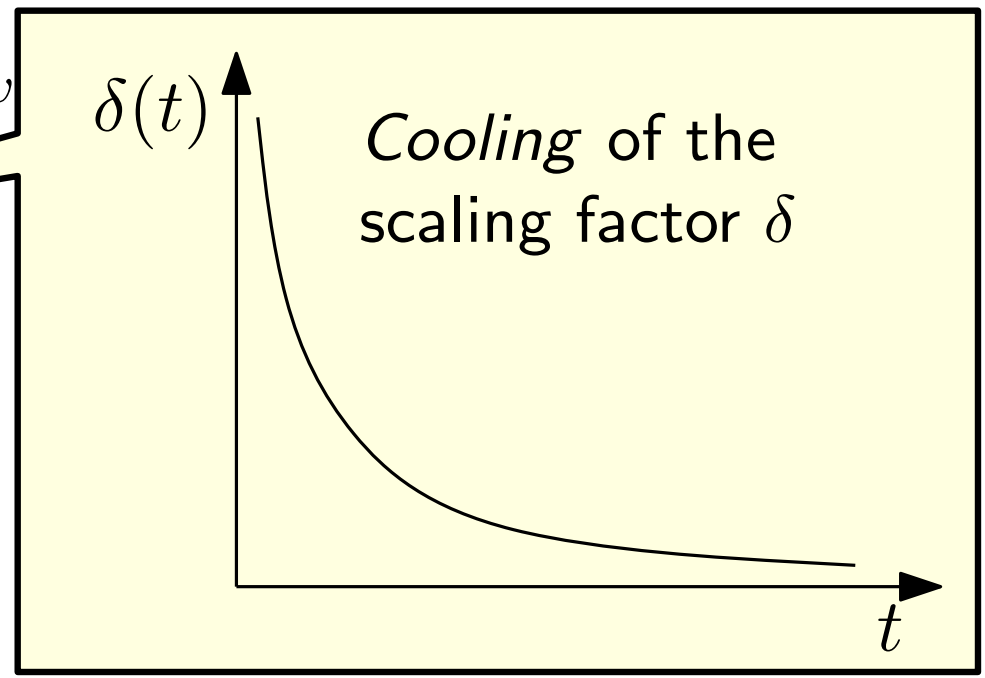

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$O(|V|^2)$ $O(|E|)$

What is the running time of the algorithm?

Discussion

Advantages

- very simple and easy-to-implement algorithm
- good results for small to medium-sized graphs
- empirically good representation of symmetries and structure

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- system is not stable at the end
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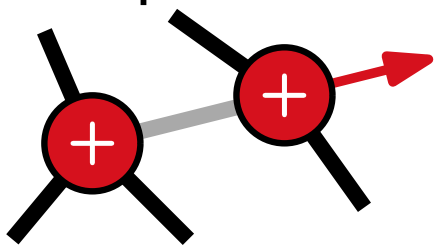
Influence

- original paper by Peter Eades has 1819 citations
- basis for many subsequent ideas and algorithms

Variant: Fruchterman & Reingold (1991)

Model:

- repulsive force between **all** vertex pairs u and v



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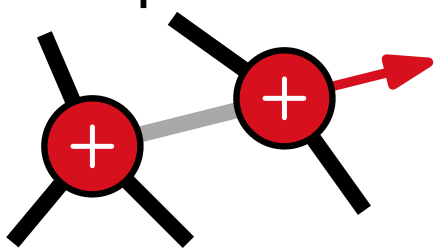

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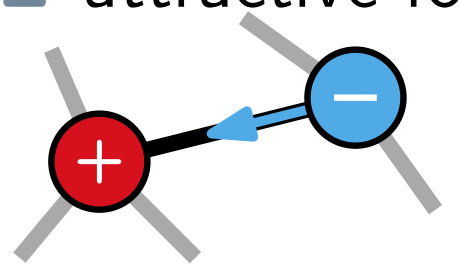

$$f_{\text{attr}}(p_u, p_v) = \frac{||p_u - p_v||^2}{\ell} \cdot \overrightarrow{p_v p_u}$$

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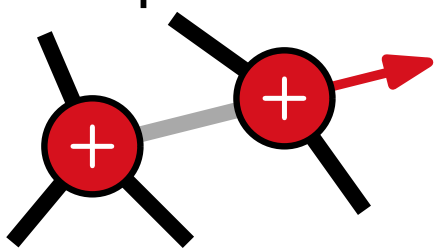

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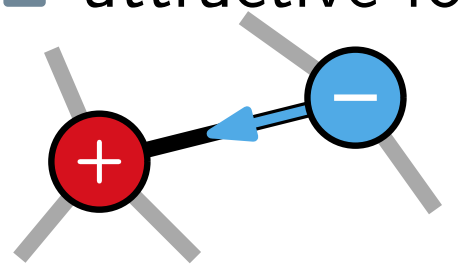

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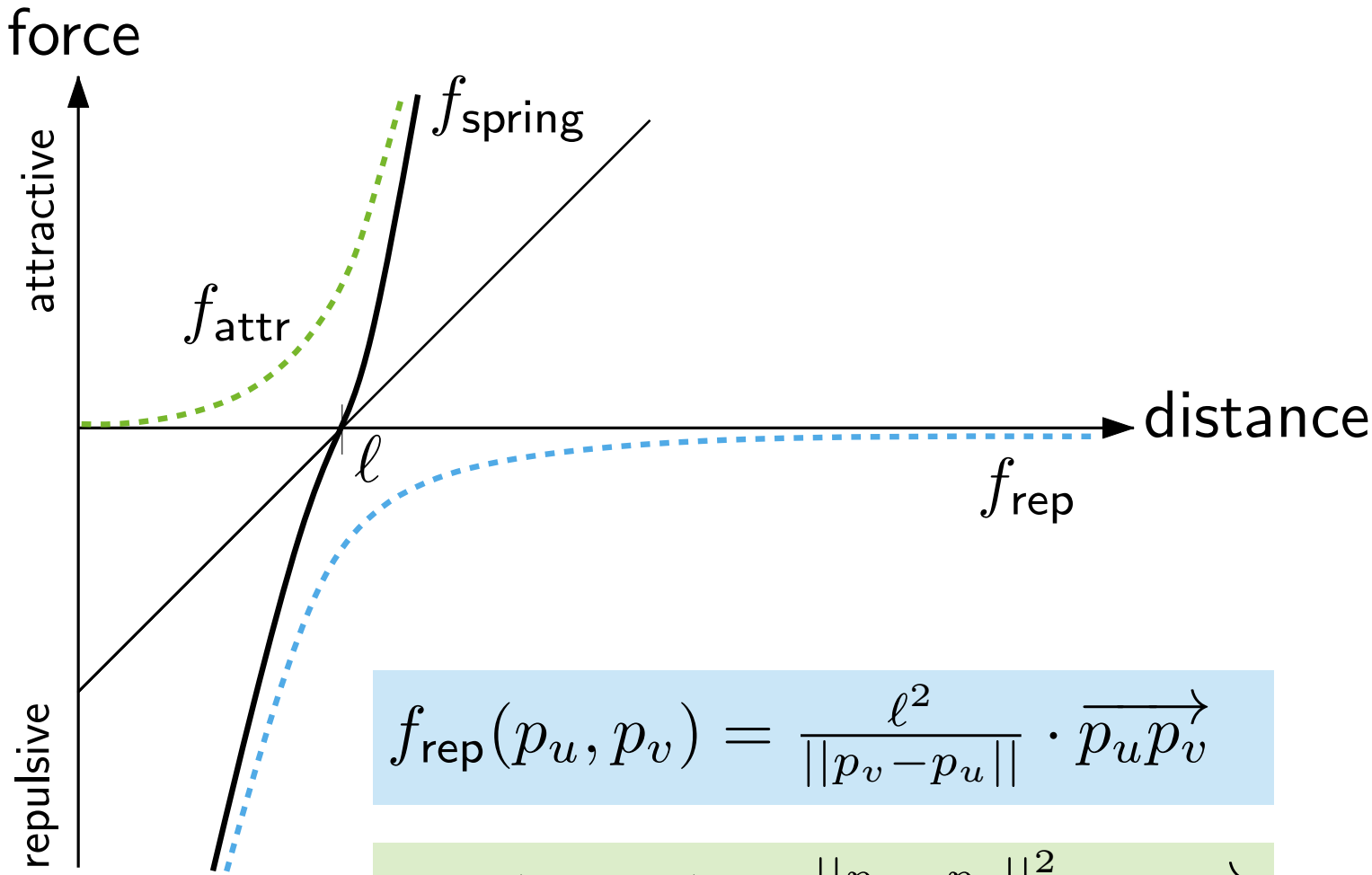
$$f_{\text{spring}}(p_u, p_v) = f_{\text{rep}}(p_u, p_v) + f_{\text{attr}}(p_u, p_v)$$

Diagram of Fruchtermann & Reingold Forces



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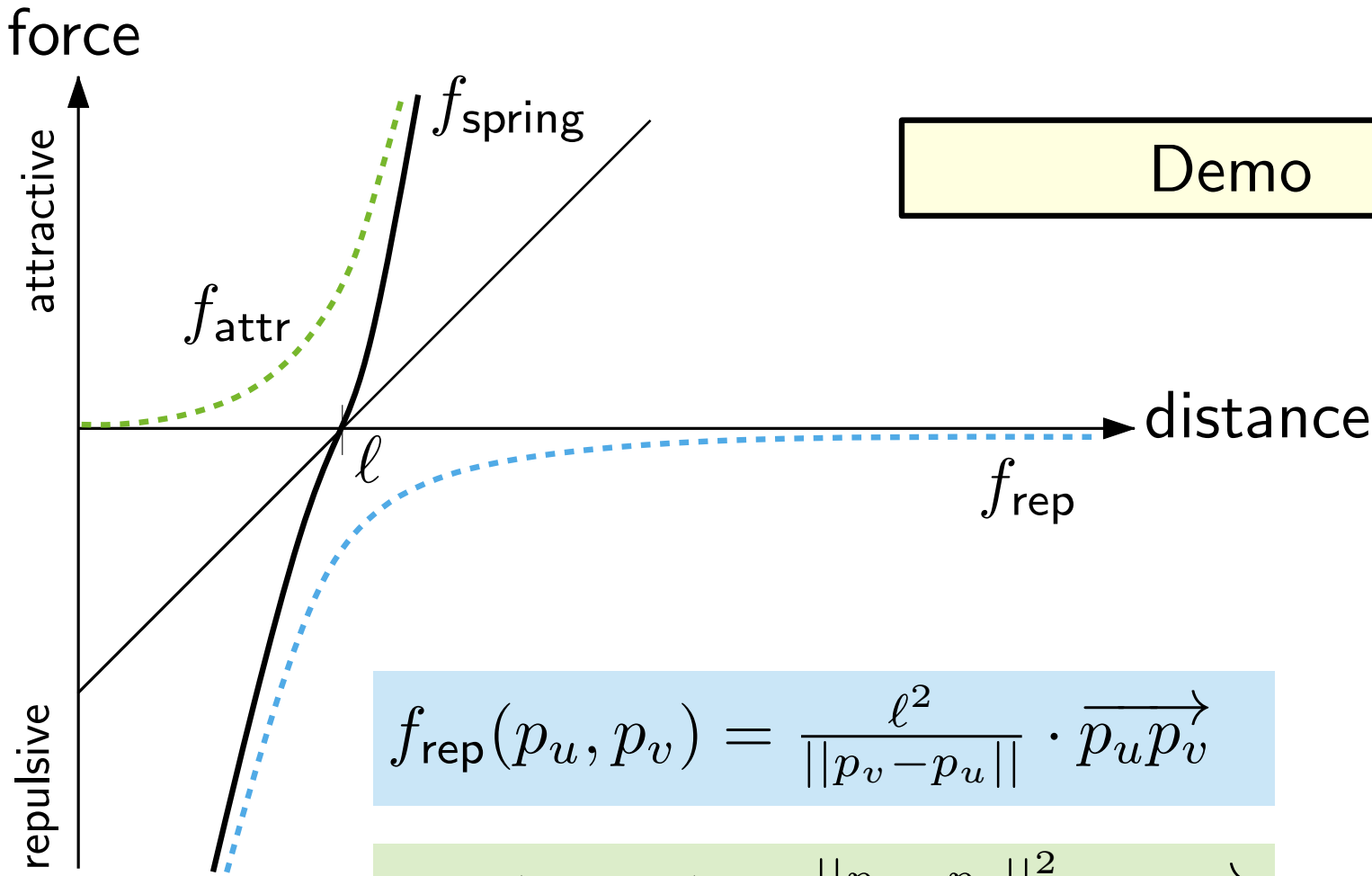

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Other Possible Modifications

- **Inertia**
- **Gravitation**
- **Magnetic forces**

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  define vertex mass as $\Phi(v) = 1 + \deg(v)/2$

  set $f_{\mathsf{attr}}(p_u, p_v) \leftarrow f_{\mathsf{attr}}(p_u, p_v) \cdot 1/\Phi(v)$

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  define barycenter $p_{\text{bary}} = 1/|V| \cdot \sum_{v \in V} p_v$

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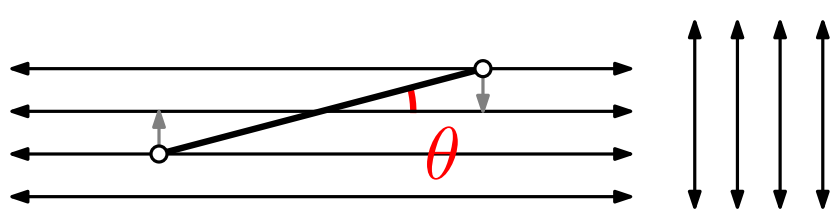
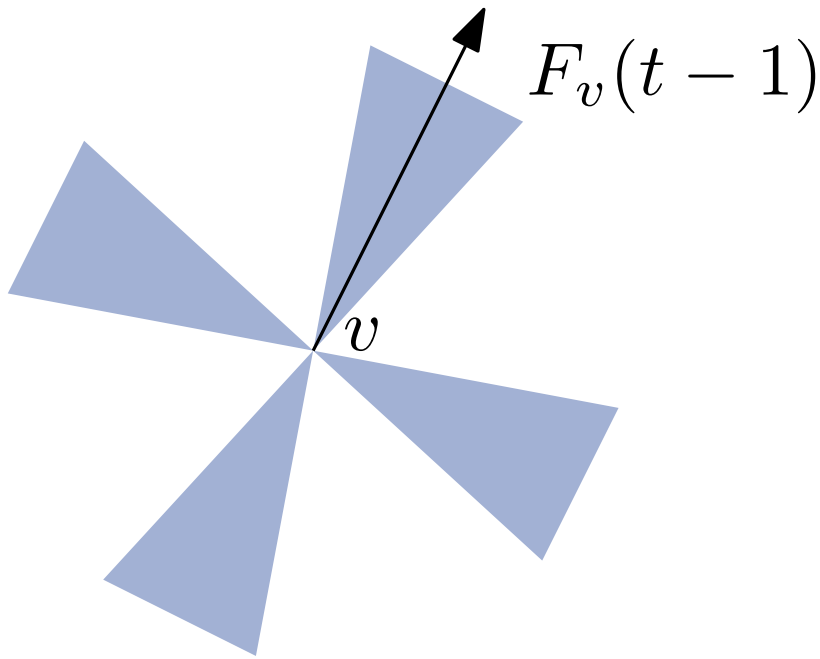
- **Magnetic forces**
  - define magnetic fields (e.g. vertical, horizontal)
  - angle θ between edge and the direction of the field
  - define force that reduces this angle



Adaptive Displacement (Frick, Ludwig, Mehldau 1995)

- store previous displacement vector $F_v(t-1)$



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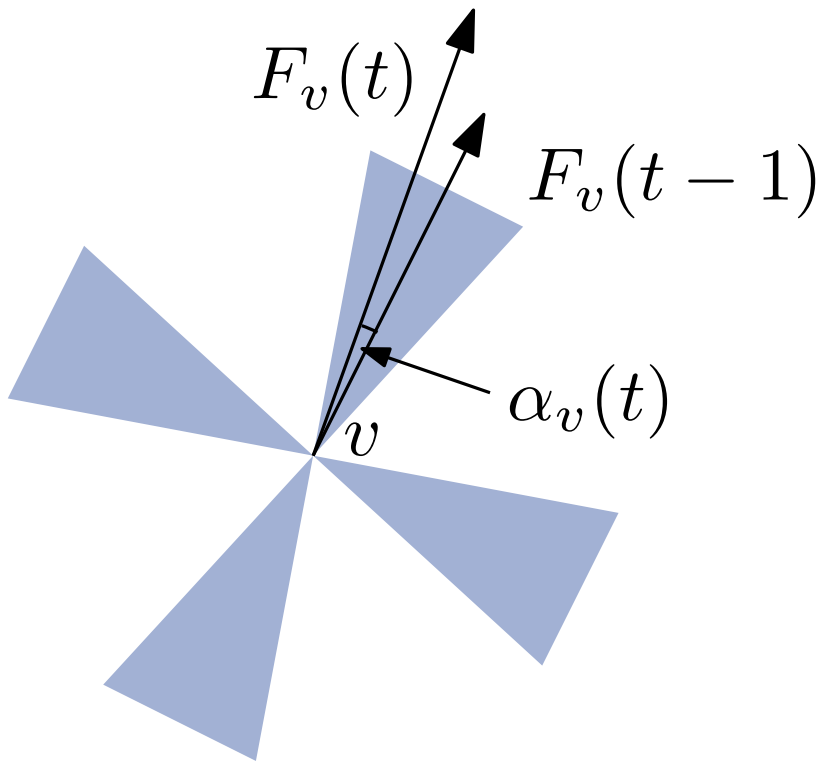

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local temperature

- $\cos(\alpha_v(t)) \approx 1$:
  similar direction
  $\rightarrow$ increase the temperature

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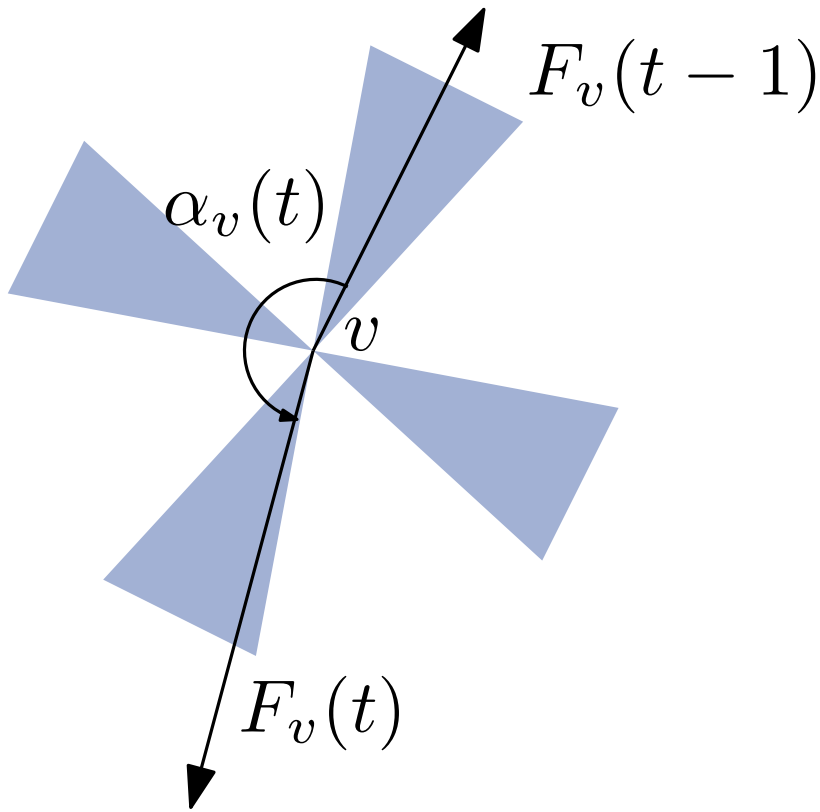

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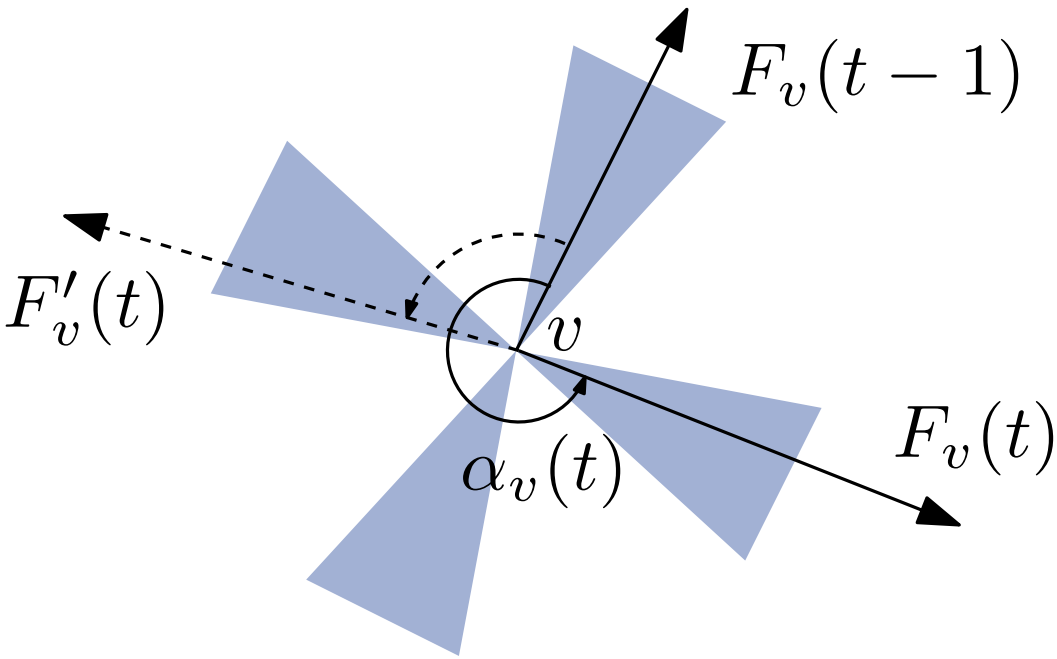
  $\rightarrow$ increase the temperature
- $\cos(\alpha_v(t)) \approx -1$:
  oscillation
  $\rightarrow$ reduce the temperature

Adaptive Displacement (Frick, Ludwig, Mehldau 1995)



- store previous displacement vector $F_v(t-1)$

local temperature

- $\cos(\alpha_v(t)) \approx 1$:
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  $\rightarrow$ increase the temperature
- $\cos(\alpha_v(t)) \approx -1$:
  oscillation
  $\rightarrow$ reduce the temperature
- $\cos(\alpha_v(t)) \approx 0$:
  rotation if it occurs repeatedly
  $\rightarrow$ reduce the temperature

Discussion

Advantages

- algorithm remains simple and implementation friendly
- additional modifications improve layout quality and lead to faster convergence

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- Variants of the Fruchterman & Reingold algorithm are probably the most popular force-based methods (paper cited 4350 times)

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- stability is not guaranteed
- local minima possible
- quadratic time for repulsive forces

Can we reduce this?

Influence

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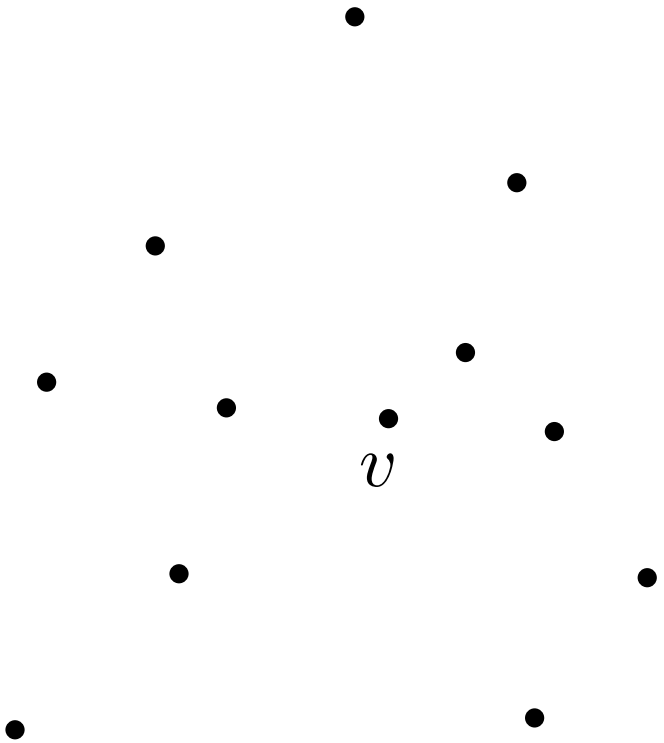
Overview

Force-based graph layout

Scalability and faster force computation

Stress-based graph layout

Local Repulsion: Grid Version (Fruchterman, Reingold, 1990)



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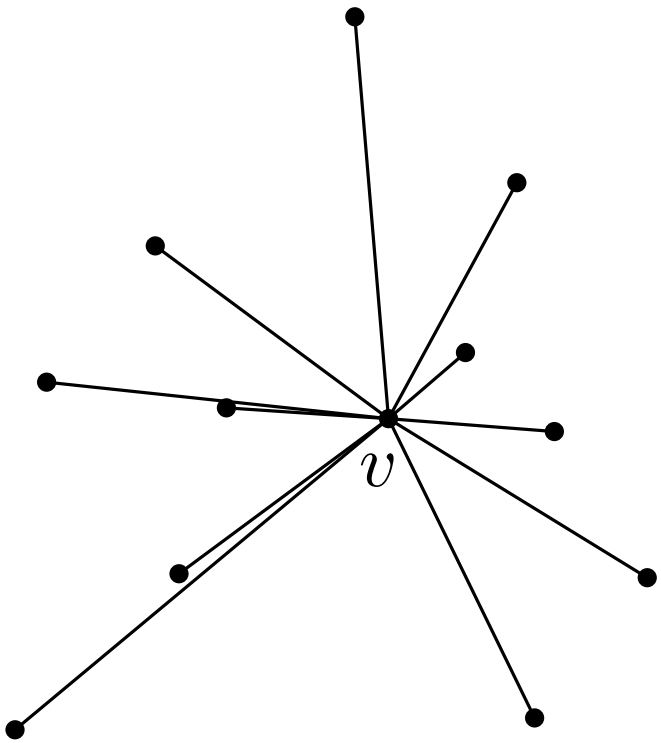

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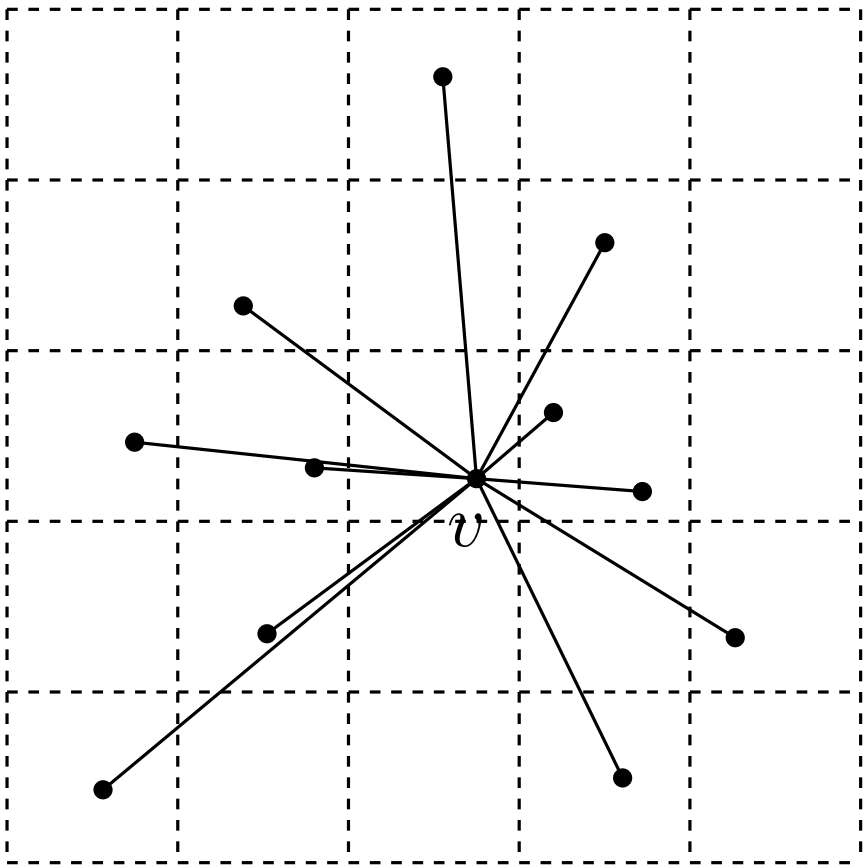

- subdivide plane by a grid

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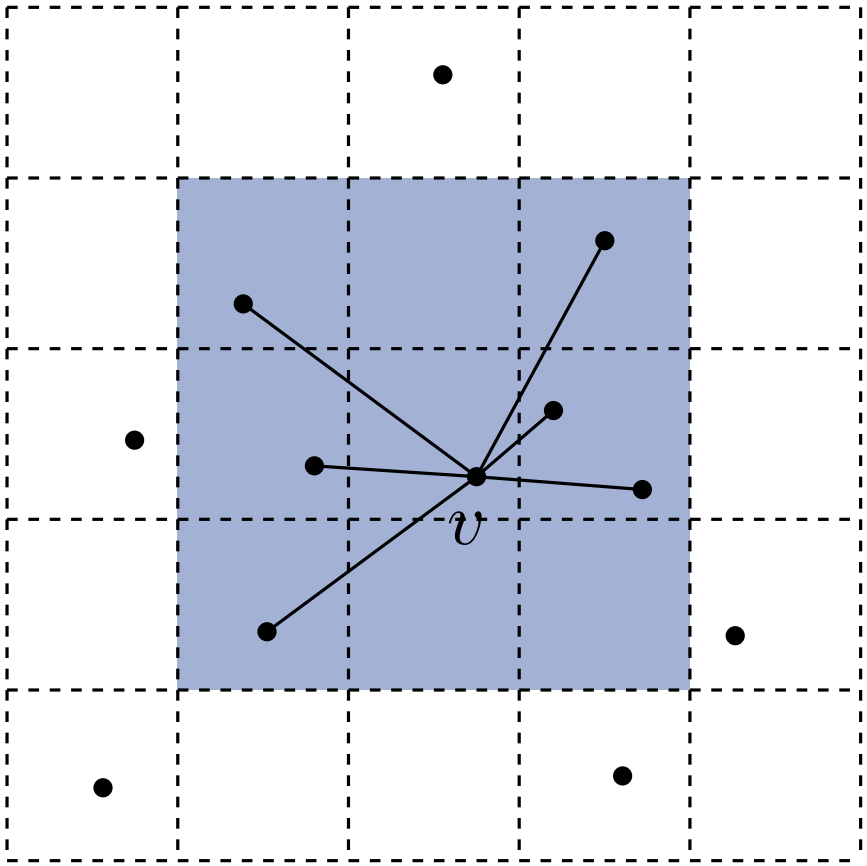

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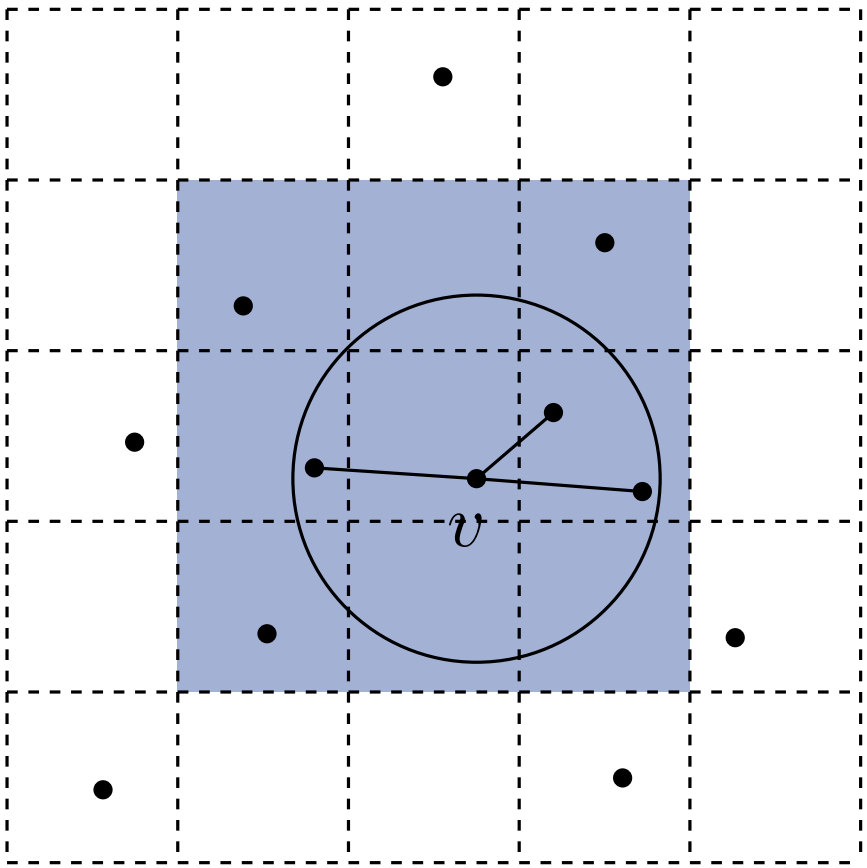

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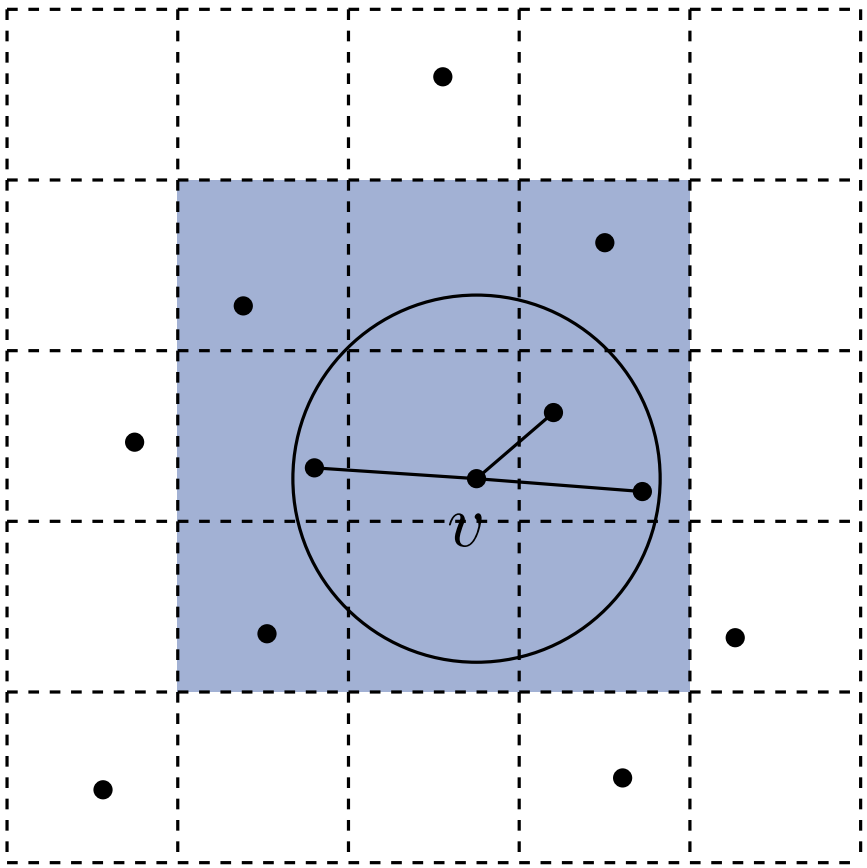

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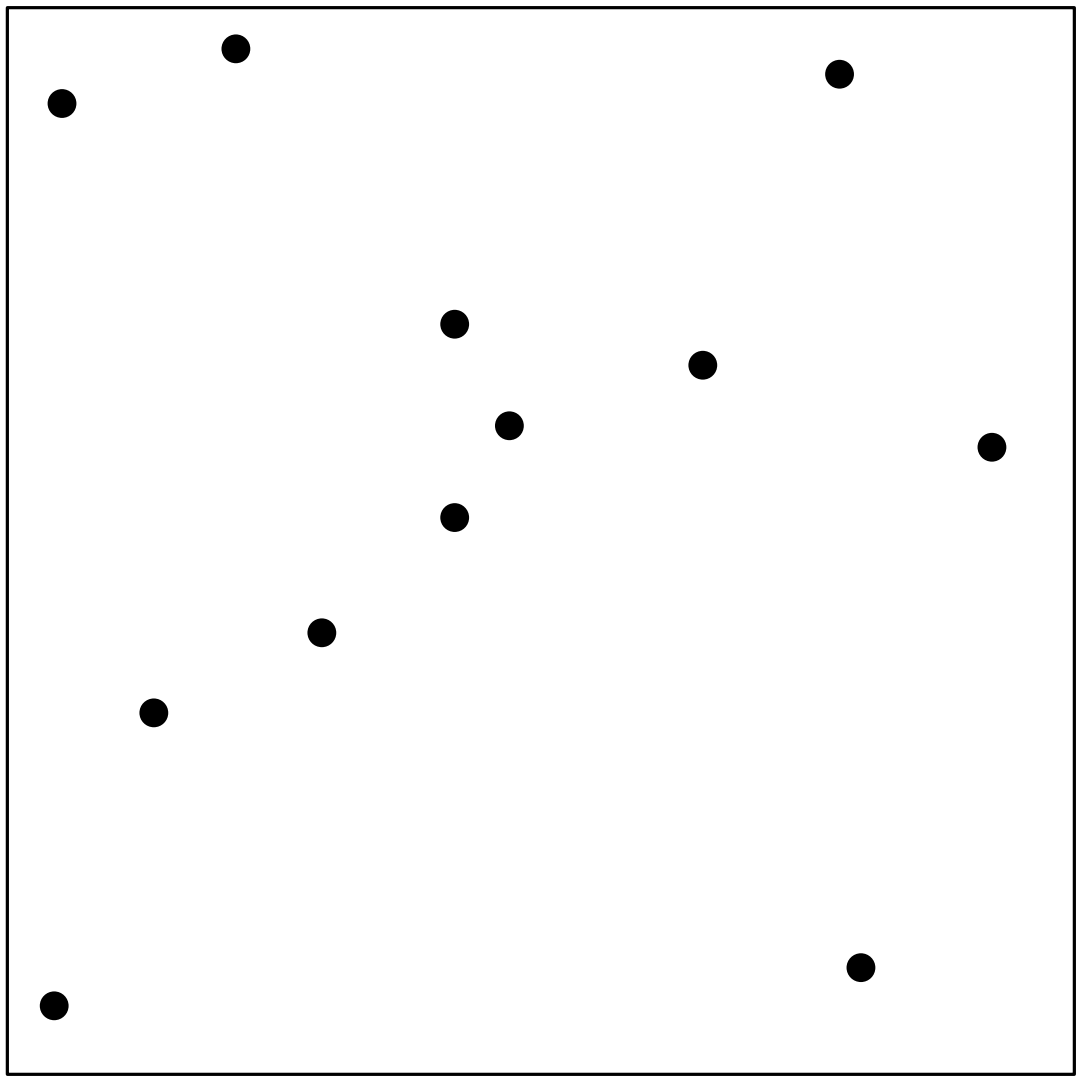
Discussion

- useful idea to improve runtime
- worst-case no improvement
- quality loss (e.g. oscillation around $d_{\max}$)

Quadtree Data Structure

Quadtrees store point sets in $\mathbb{R}^2$ in a hierarchical, geometric data structure

- if square has ≥ 2 points, then split into four subsquares and recurse



R_0

QT

r_0

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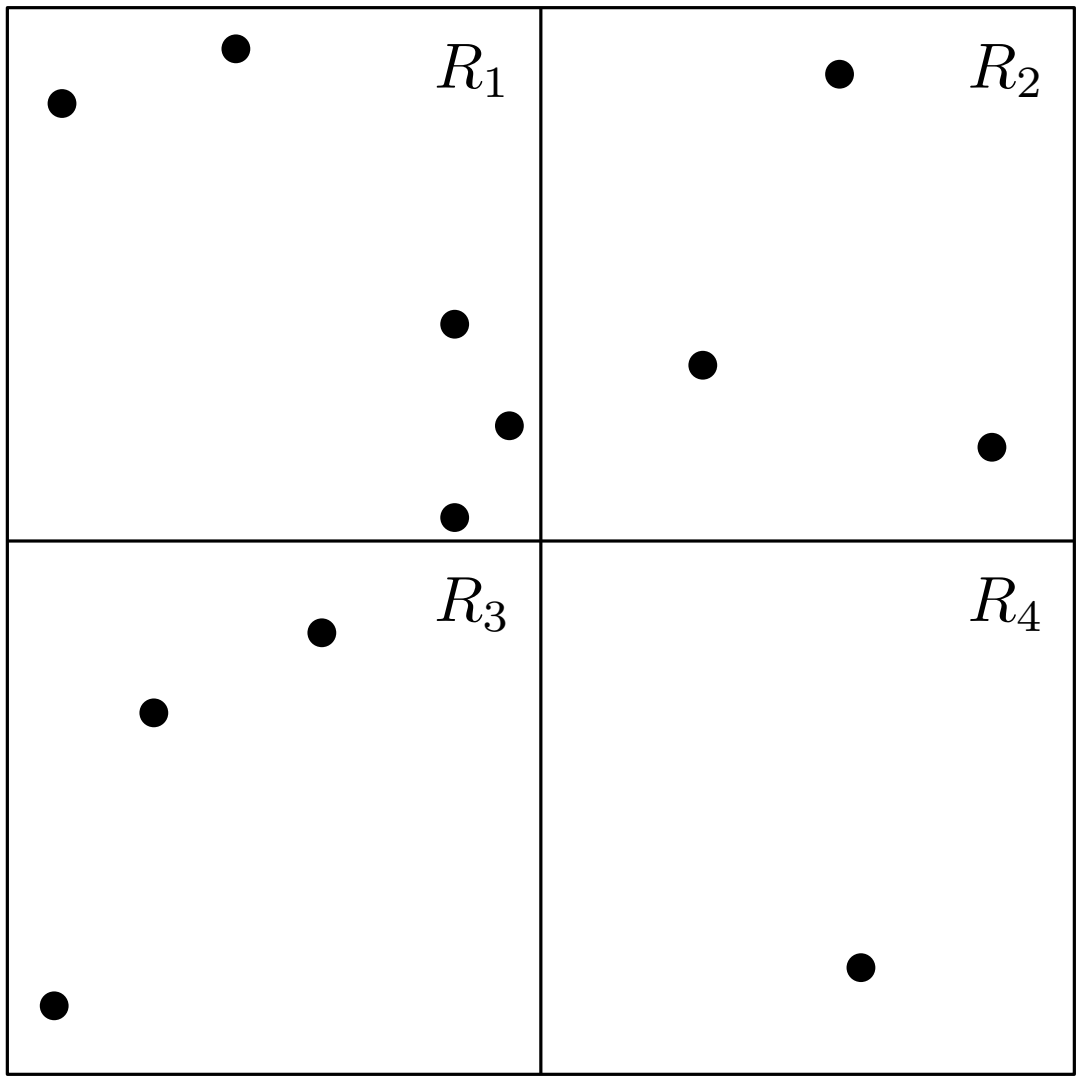

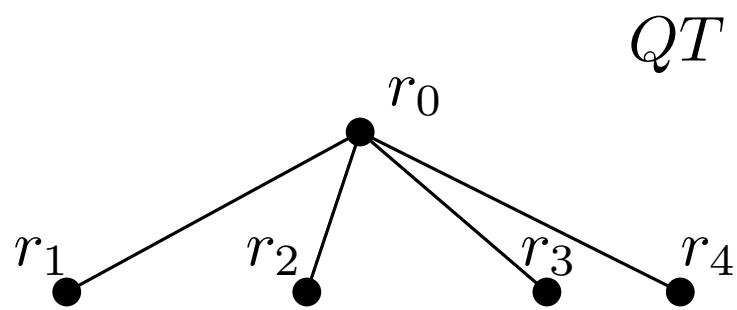

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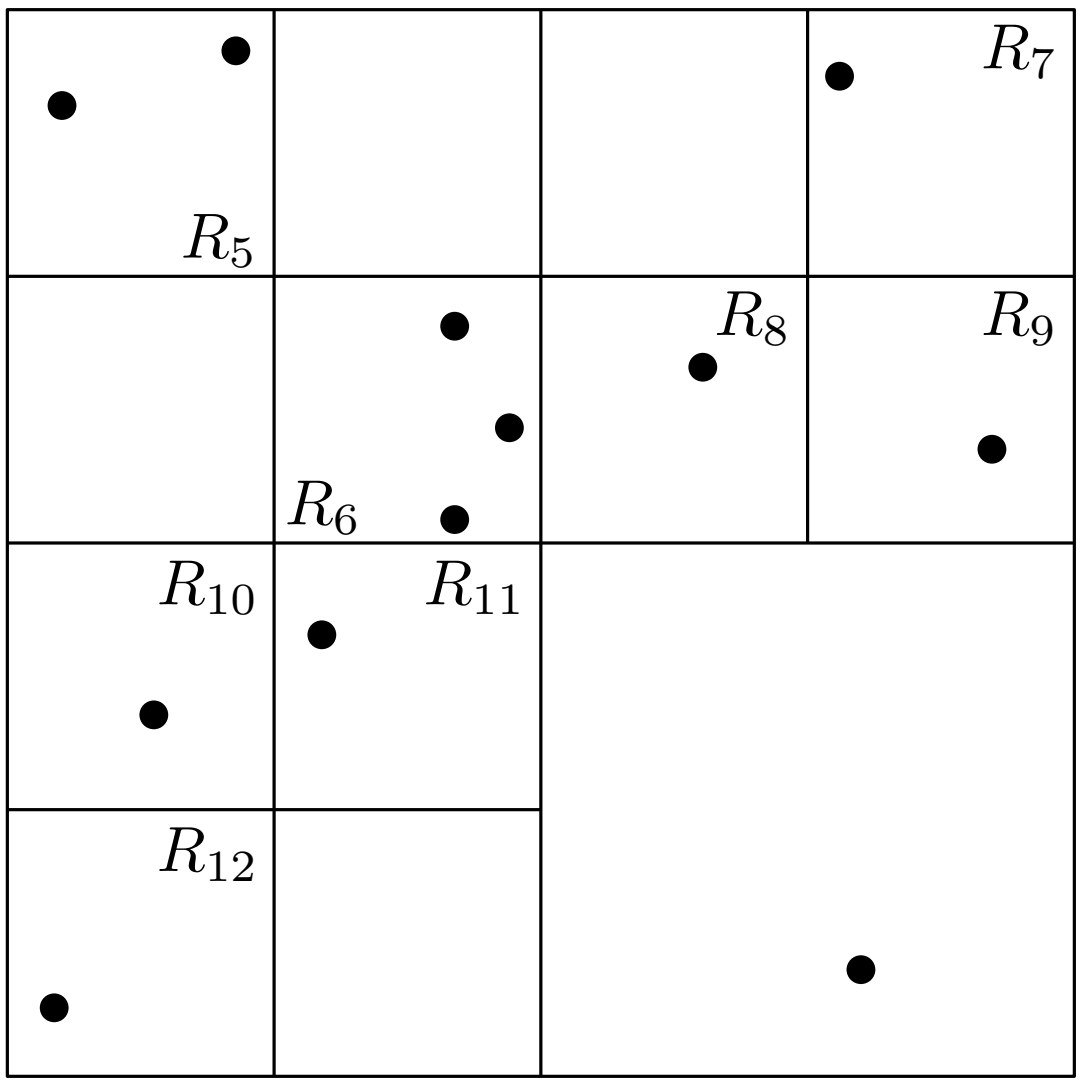

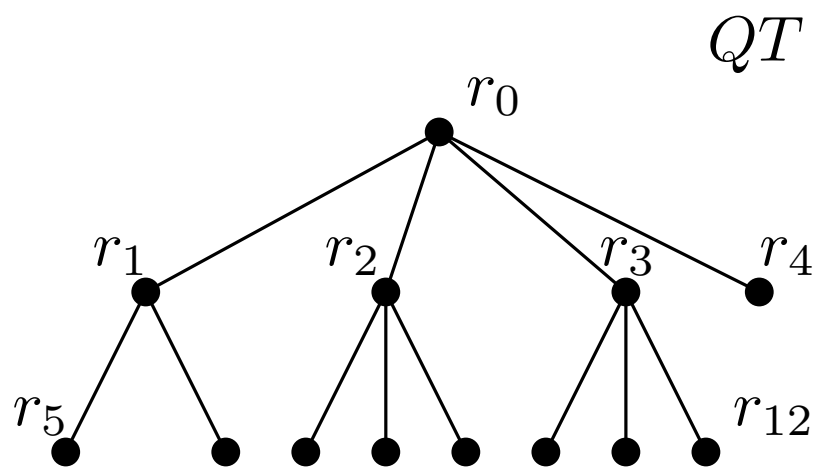

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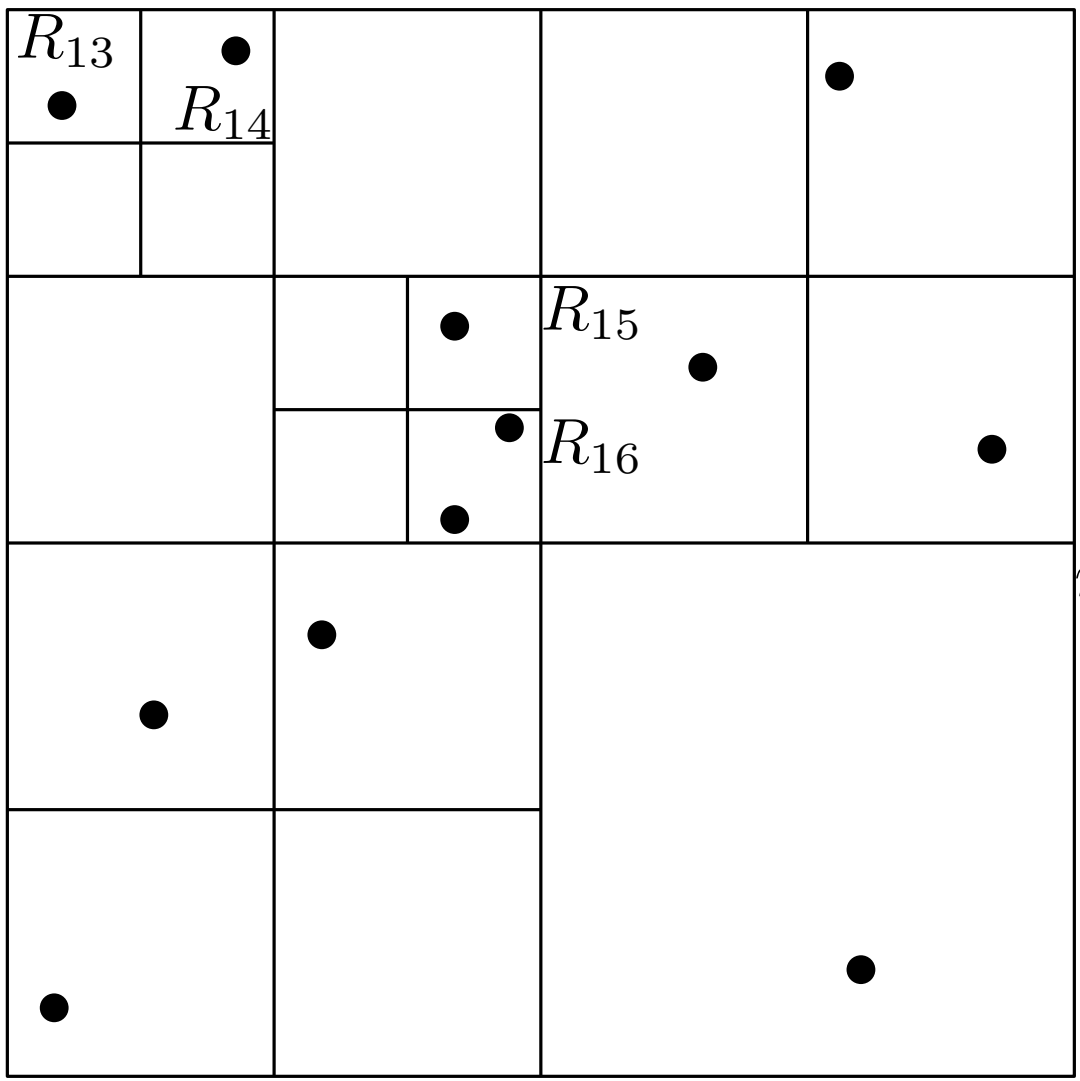

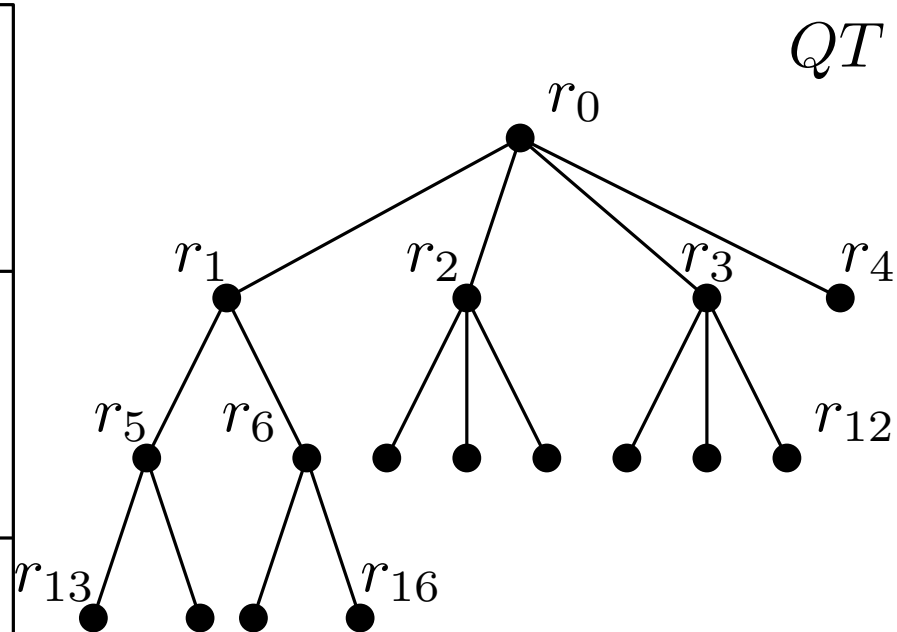

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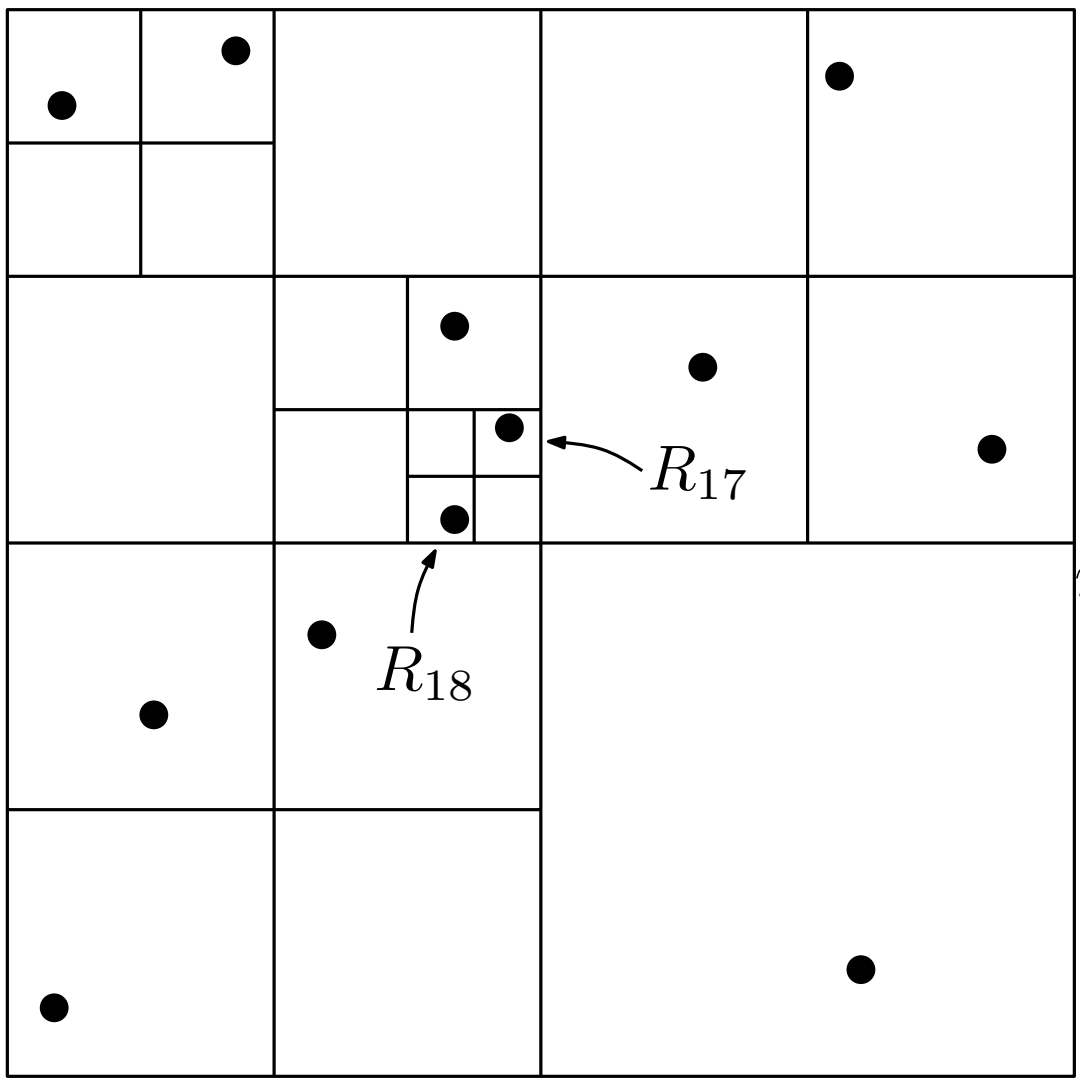
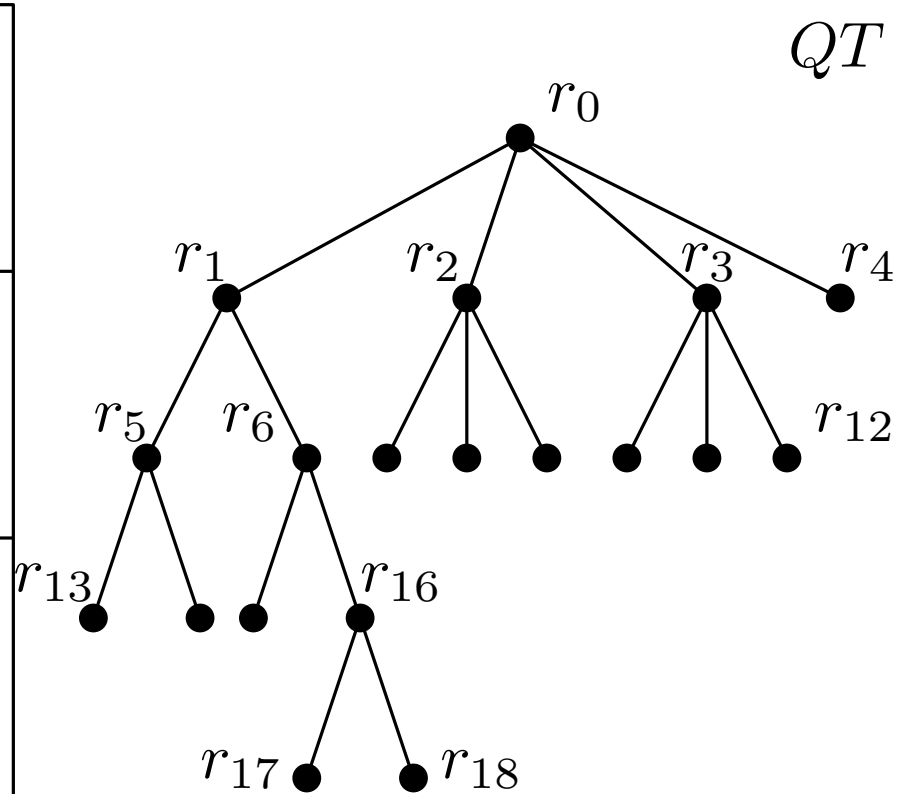
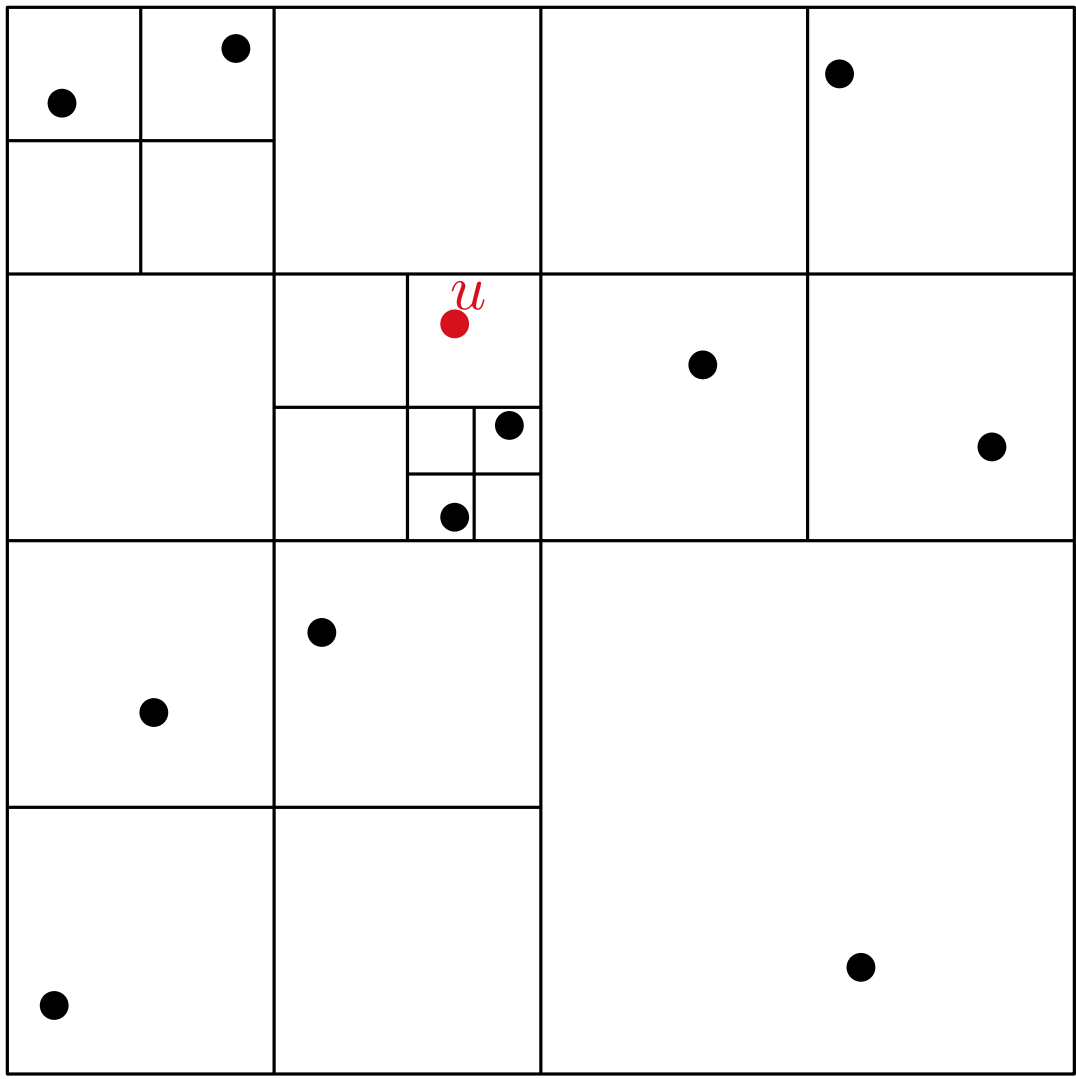
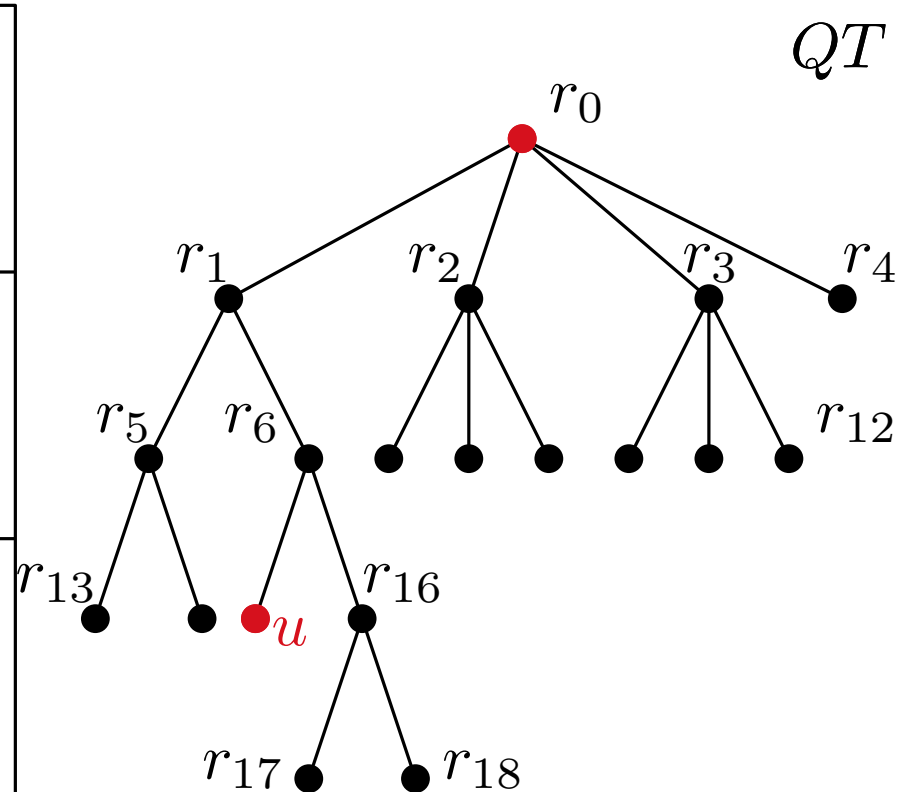




Force Approximation with Quadtrees (Barnes, Hut, 1986)





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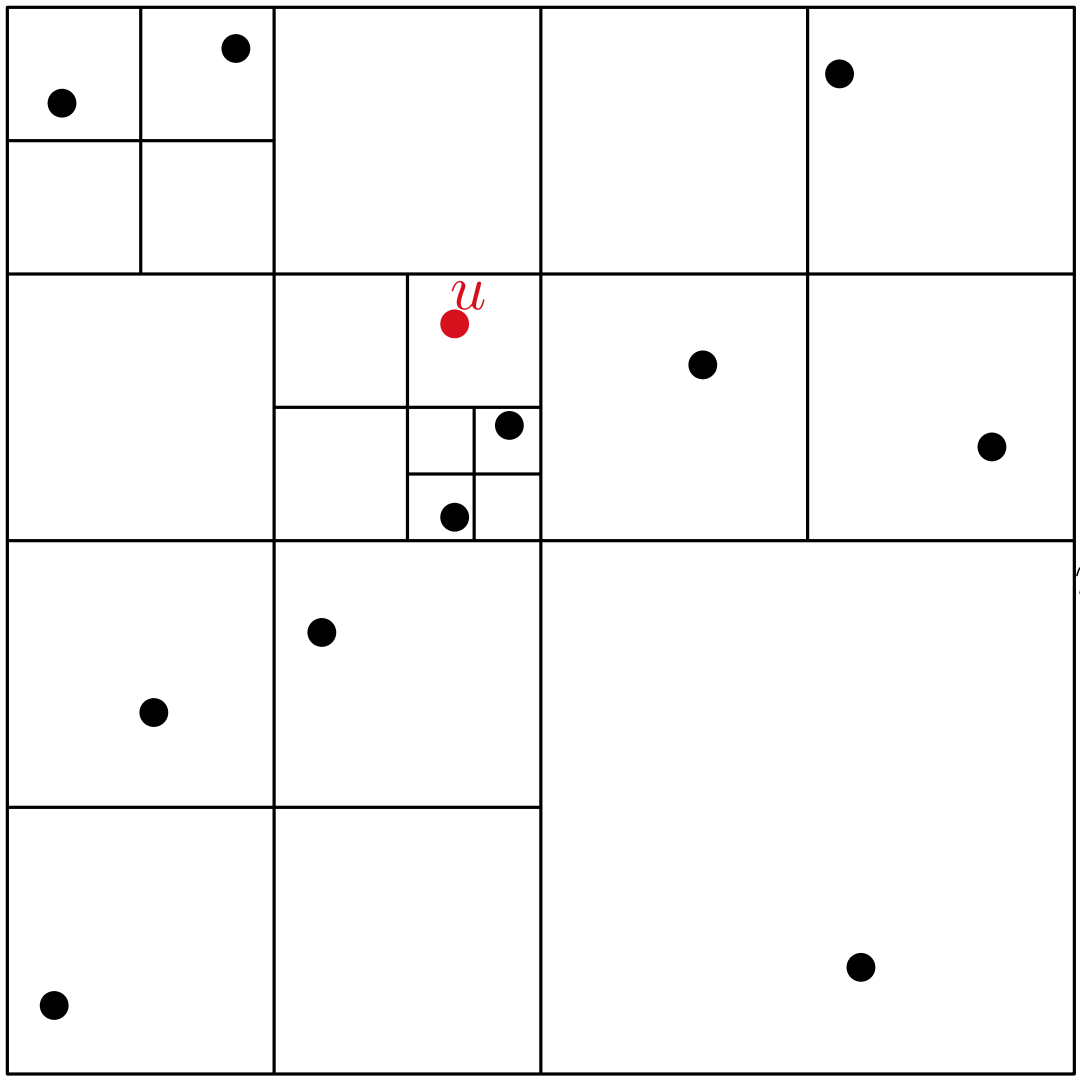

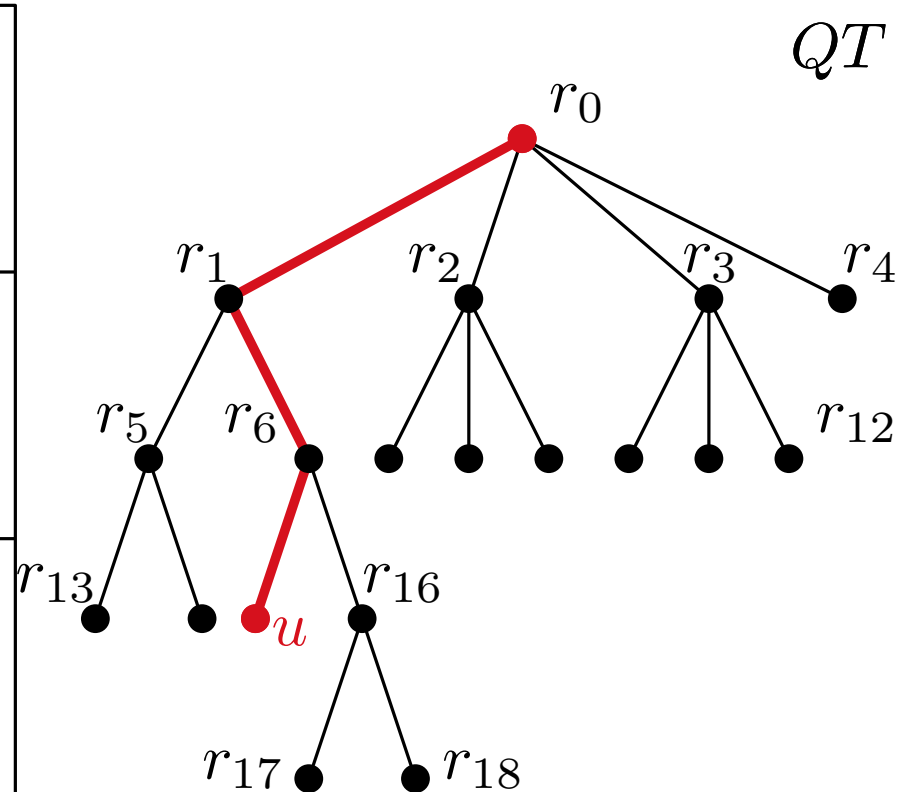

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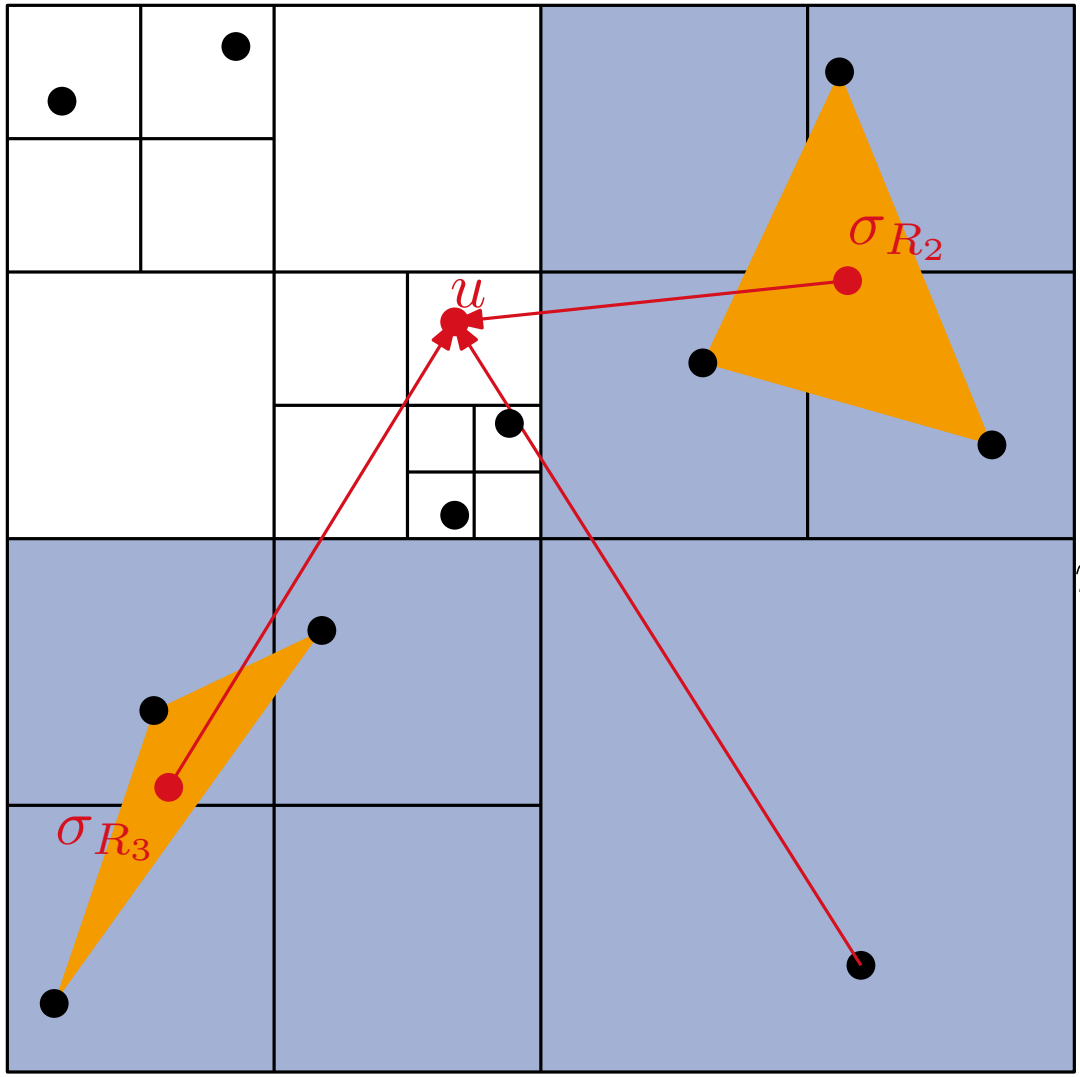

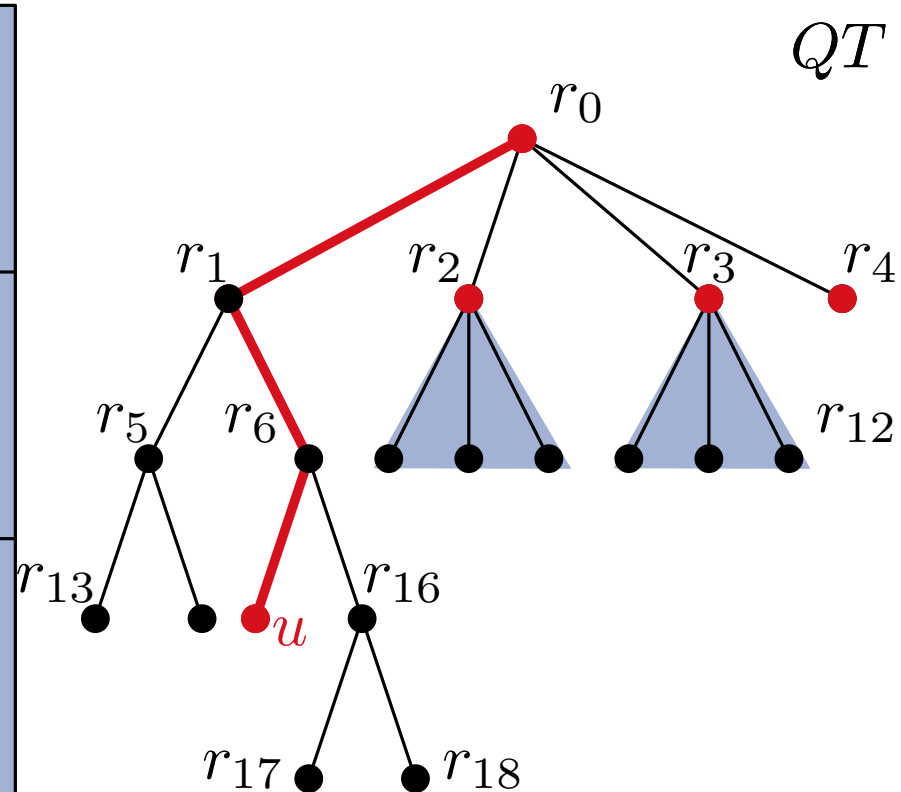

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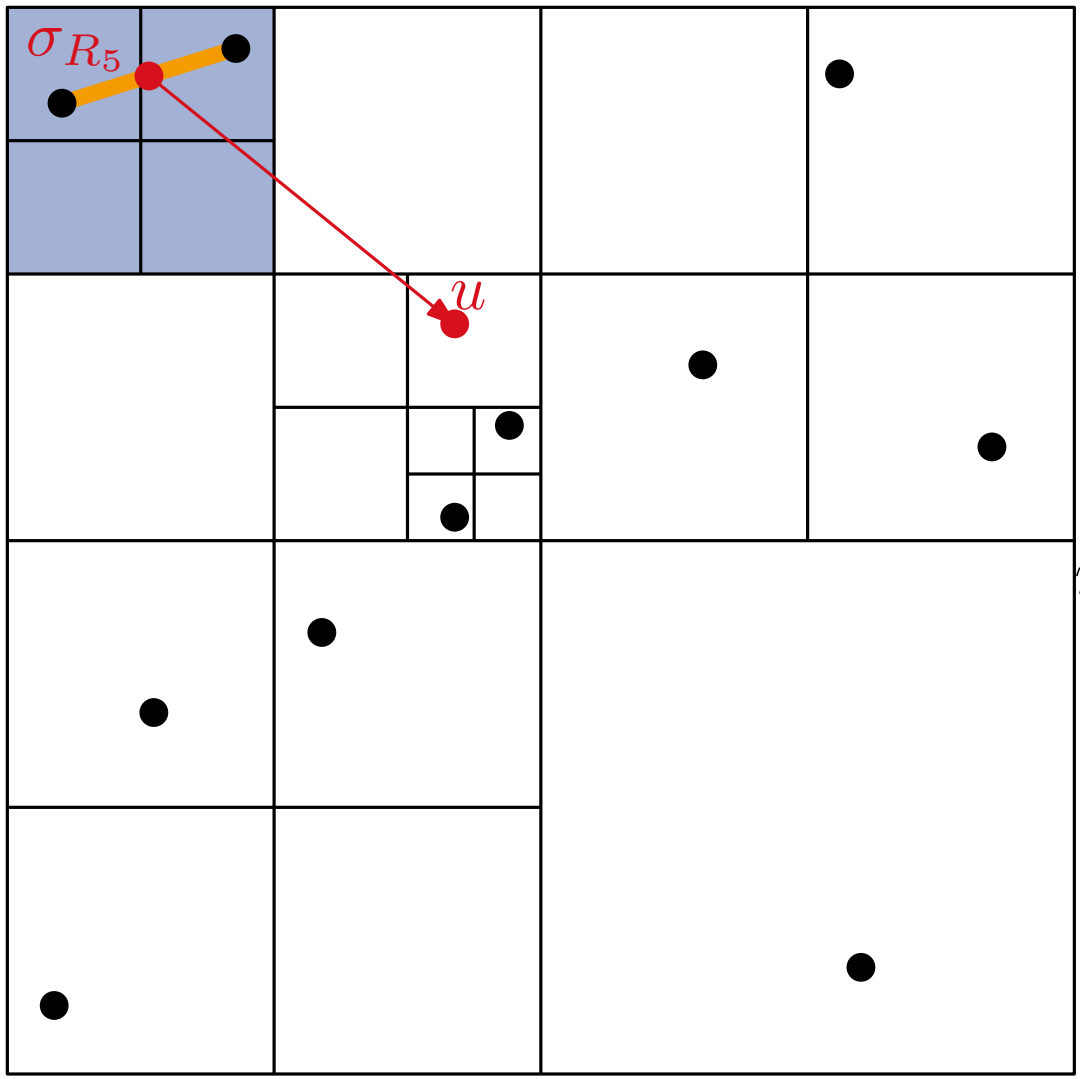

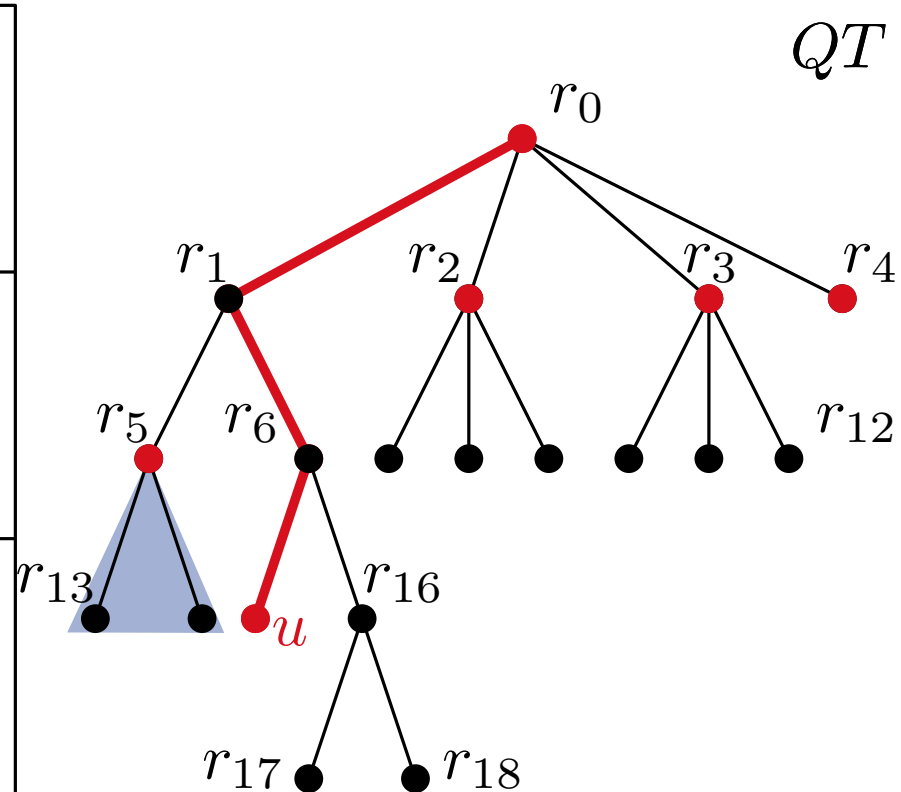

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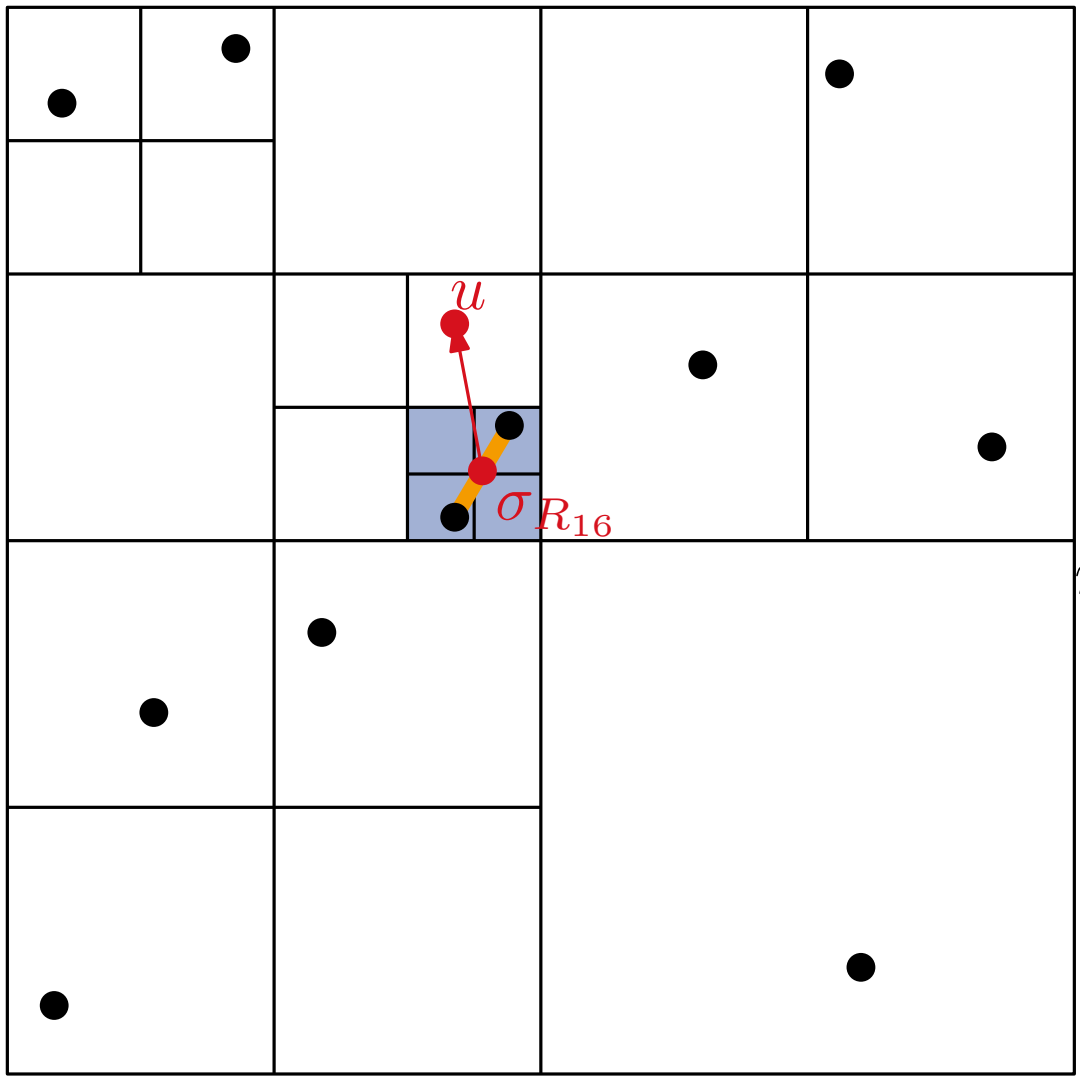

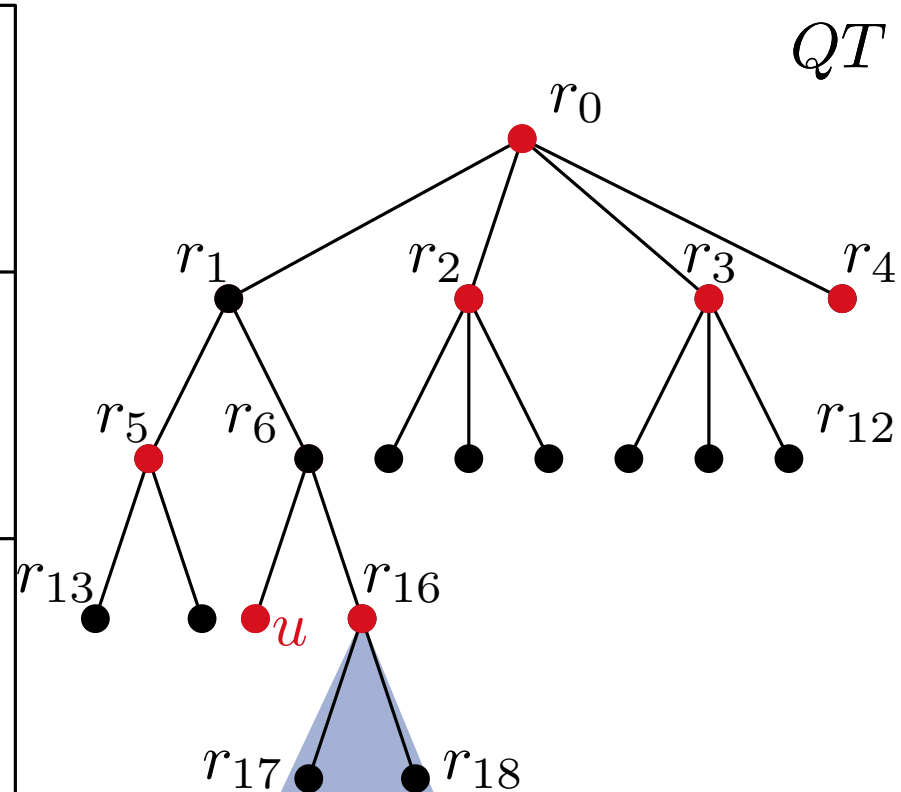

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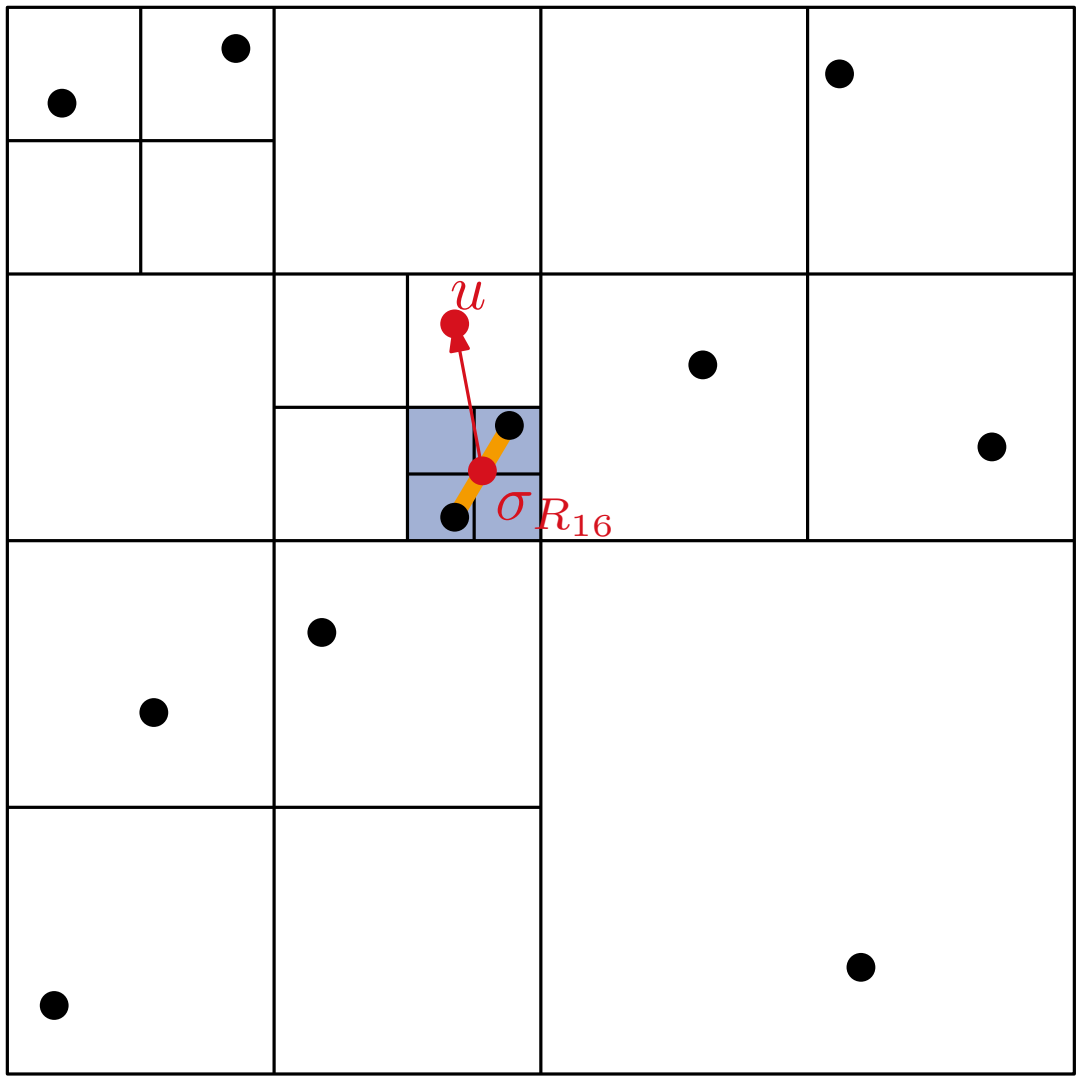
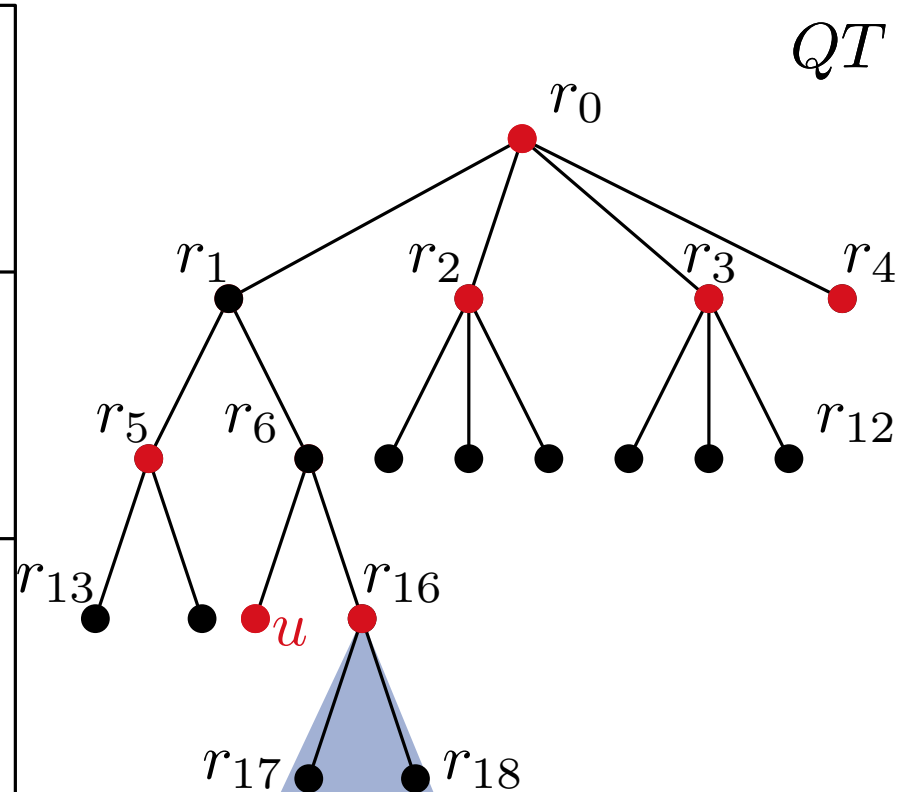




If the point distribution is more or less balanced, the quadtree has logarithmic height

$\Rightarrow$ compute repulsive forces in $O(n \log n)$ time.

Overview

Force-based graph layout

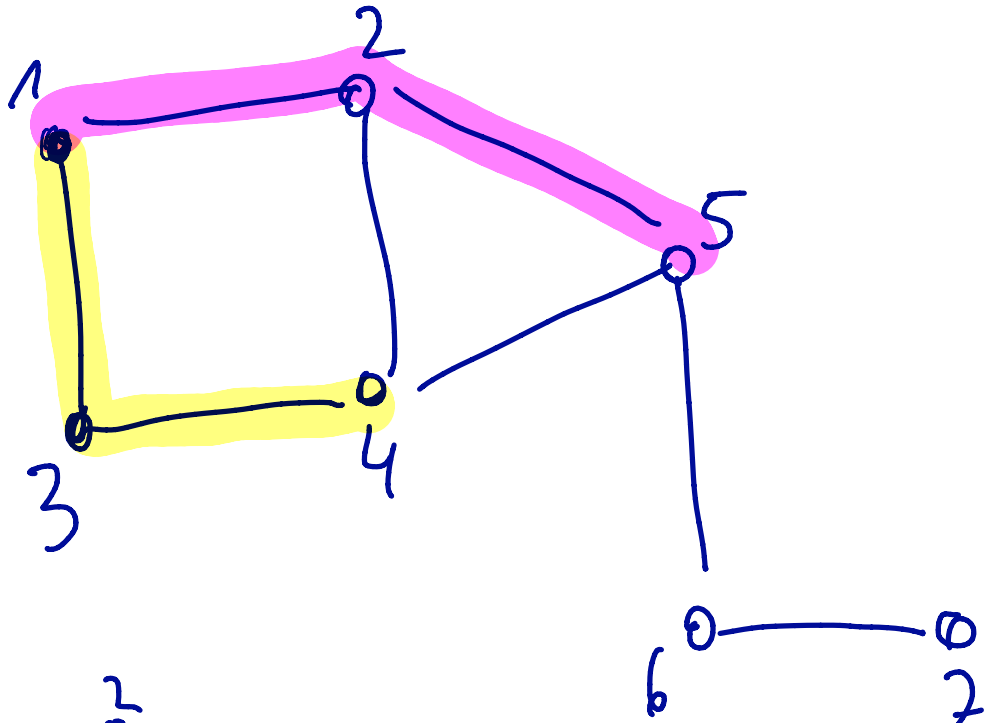
Scalability and faster force computation

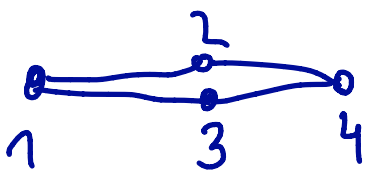
Stress-based graph layout

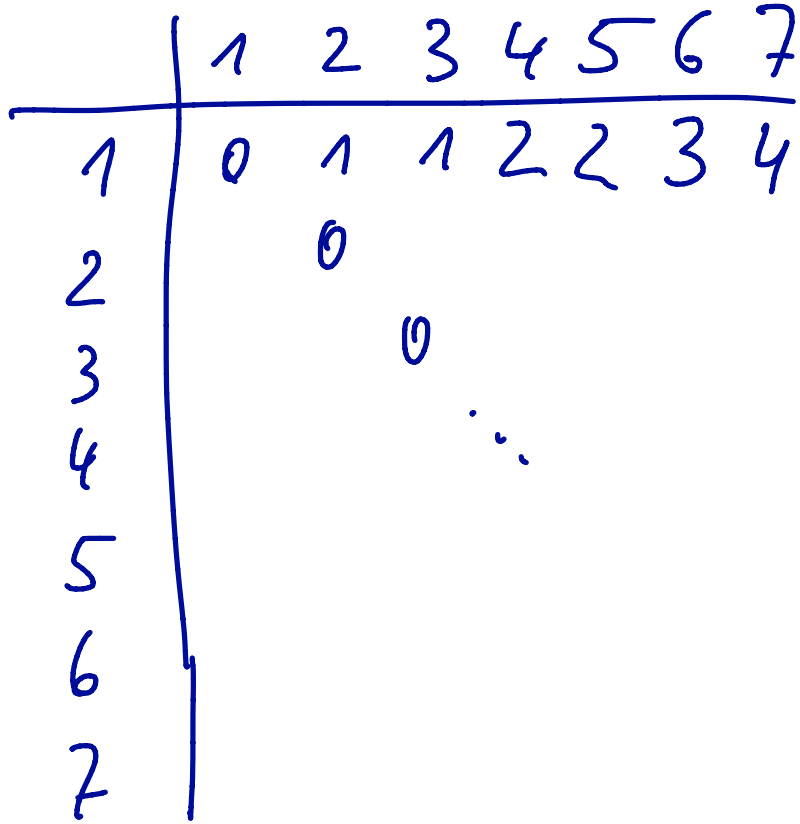
Stress Model

Idea: compute ideal distances between any two vertices and minimize deviation

What would be suitable pairwise distances?







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For a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ define distance matrix $D = (d_{ij}) \in \mathbb{R}^{n \times n}$, where d_{ij} is the shortest-path distance between v_i and v_j in G.

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$$s(p) = \sum_{i<j} w_{ij}(||p_i - p_j|| - d_{ij})^2$$

of the resulting layout, where w_{ij} is a weight term, e.g., $w_{ij} = 1/d_{ij}^2$ or $w_{ij} = d_{ij}^{-\alpha}$ for a parameter α.

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This model is also known as **multidimensional scaling (MDS)**.

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Step 1: compute all pairwise distances in matrix $D = (d_{ij})$
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- Gansner at el. (2004) showed that the following iterative solution process minimizes $s(p)$ using systems of linear equations:

$$L^w \cdot p(t+1) = L^{w,d}(t) \cdot p(t)$$

$$L^w_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \\ \sum_{k \neq i} w_{ik} & \text{if } i = j \end{cases} \qquad L^{w,d}_{ij}(t) = \begin{cases} -\frac{w_{ij} d_{ij}}{||p(t)_i - p(t)_j||} & \text{if } i \neq j \\ -\sum_{k \neq i} L^{w,d}_{ik} & \text{if } i = j \end{cases}$$

until $\frac{s(p(t)) - s(p(t+1))}{s(p(t))} < \varepsilon$.

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Practically efficient solvers are **Cholesky factorization** or the **conjugate gradient** (CG) method with $O(n^2)$ time per iteration.

Scalability of Stress-Based Methods

For large graphs the full stress model is too slow to solve with super-quadratic running times.

Several faster approximations have been developed, e.g.

- **PivotMDS** uses distances to a smaller set of $k \ll n$ pivot elements and saves a linear factor in Step 1 [Brandes, Pich 2006]

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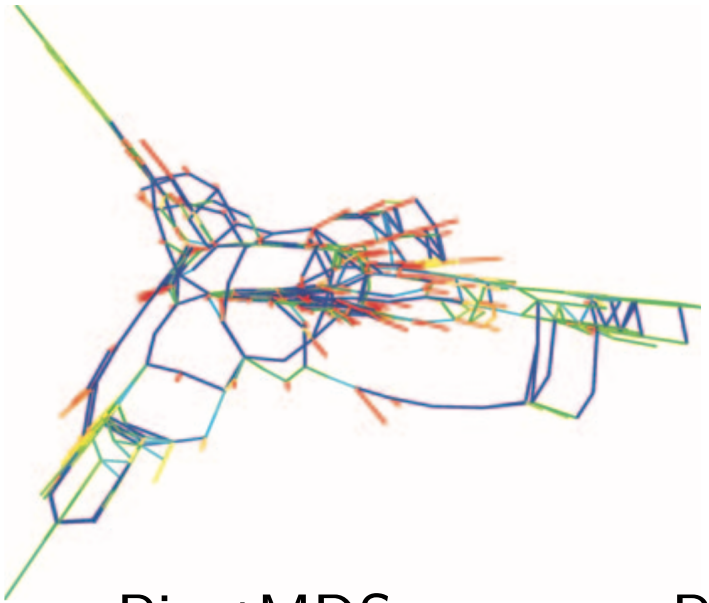
- **Sparse Stress** computes stress to a suitable smaller set $\mathcal{P}$ of $k \ll n$ pivot elements

$$s'(p) = \sum_{ij \in E} w_{ij}(||p_i - p_j|| - d_{ij})^2 + \sum_{i \in V} \sum_{j \in \mathcal{P} \setminus N(i)} w'_{ij}(||p_i - p_j|| - d_{ij})^2,$$

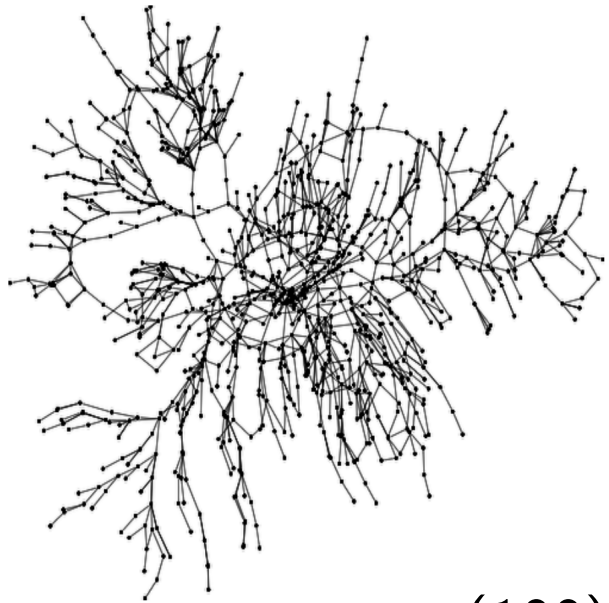
  which allows solving one iteration in $O(kn + m)$ time and precomputing the distances to $\mathcal{P}$ in $O(km + kn \log n)$ time. [Ortmann et al. 2017]

Examples [Gansner et al. 2013] [Ortmann et al. 2017]

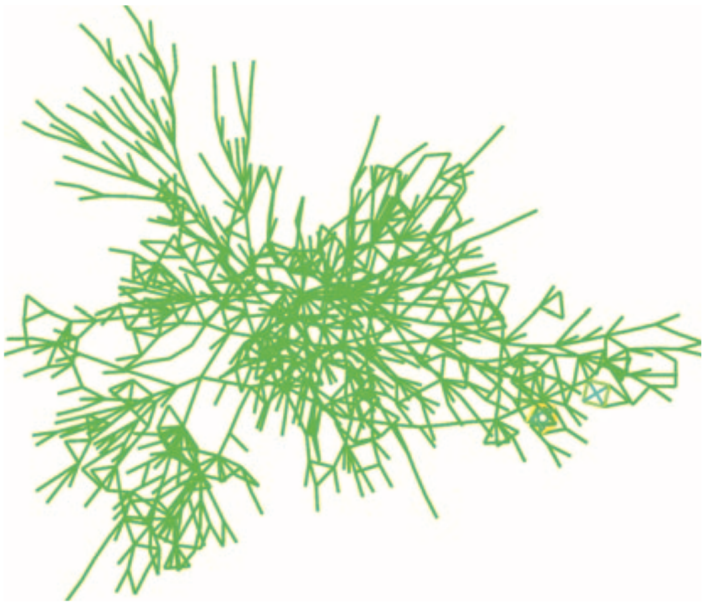


PivotMDS

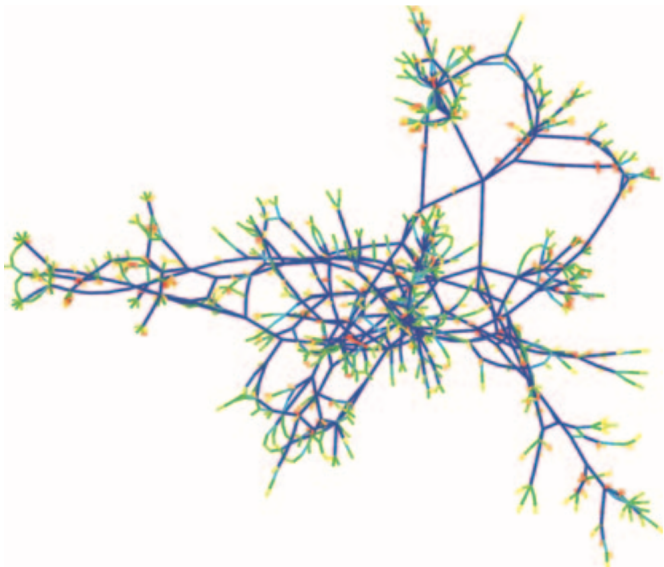
PivotMDS + local sparse stress



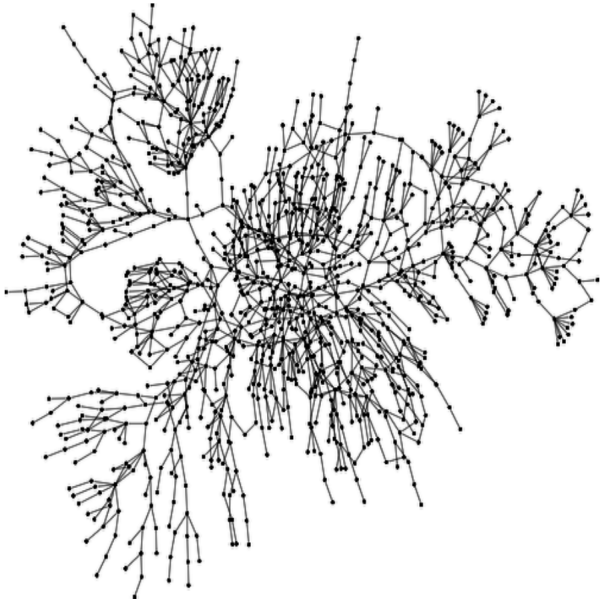
sparse stress (100)



MaxEnt



spring-electrical model



full stress

Some Running Times [Ortmann et al. 2017]

(in seconds)

graph	full stress	sparse 200	sparse 100	sparse 50	maxent*	MARS 200*	MARS 100*	GRIP*	1-stress	Pivot MDS
dwt1005	1.26	0.33	0.15	**0.09**	0.47	1.02	2.36	0.06	0.08	0.06
1138bus	2.20	0.41	0.16	**0.09**	0.91	3.16	1.96	0.20	0.06	0.04
plat1919	9.70	1.00	0.45	**0.24**	1.15	6.80	4.79	0.19	0.31	0.20
3elt	31.82	2.28	0.93	**0.43**	2.26	16.31	8.43	0.71	0.37	0.23
USpowerGrid	36.48	1.85	0.67	**0.37**	2.53	13.54	7.62	1.67	0.28	0.21
commanche	340.10	10.78	3.63	**1.51**	3.60	22.72	12.43	2.29	0.47	0.35
LeHavre	475.05	12.75	4.90	**2.19**	6.31	27.57	19.50	10.18	0.81	0.54
pesa	373.23	9.61	4.14	**1.50**	5.96	50.10	42.68	3.56	0.95	0.60
bodyy5	463.47	12.53	4.31	**2.01**	9.97	46.63	9.27	10.43	1.64	1.04
finance256	1016.92	10.44	4.27	**2.28**	14.76	32.16	24.66	12.12	2.51	1.60
btree	7.79	0.42	0.18	**0.09**	0.63	2.70	1.48	0.06	0.06	0.03
qh882	6.61	0.65	0.28	**0.15**	0.97	8.45	5.79	0.15	0.17	0.14
lpship04l	18.30	0.73	**0.31**	0.18	0.99	7.06	7.63	0.16	0.15	0.10

graph	n	m
dwt1005	1005	3808
1138bus	1138	1458
plat1919	1919	15240
3elt	4740	13722
USpowerGrid	4941	6594
commanche	7920	11880**
LeHavre	11730	15133**
pesa	11738	33914
bodyy5	18589	55346
finance256	20657	71866
btree (binary tree)	1023*	1022
qh882	1764*	3354
lpship04l	2526*	6380

Summary

Force-based approaches

- easily understandable and implementable
- no special requirements on the input graph
- for small and sparse graphs amazingly good layouts (symmetries, clusters, ...)
- easily adaptable and configurable
- robust and scalable (with extra work)

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Stress-based approaches

- distance optimization usually gives high quality
- no special requirements on the input graph
- iterative solving of equation systems or gradient descent

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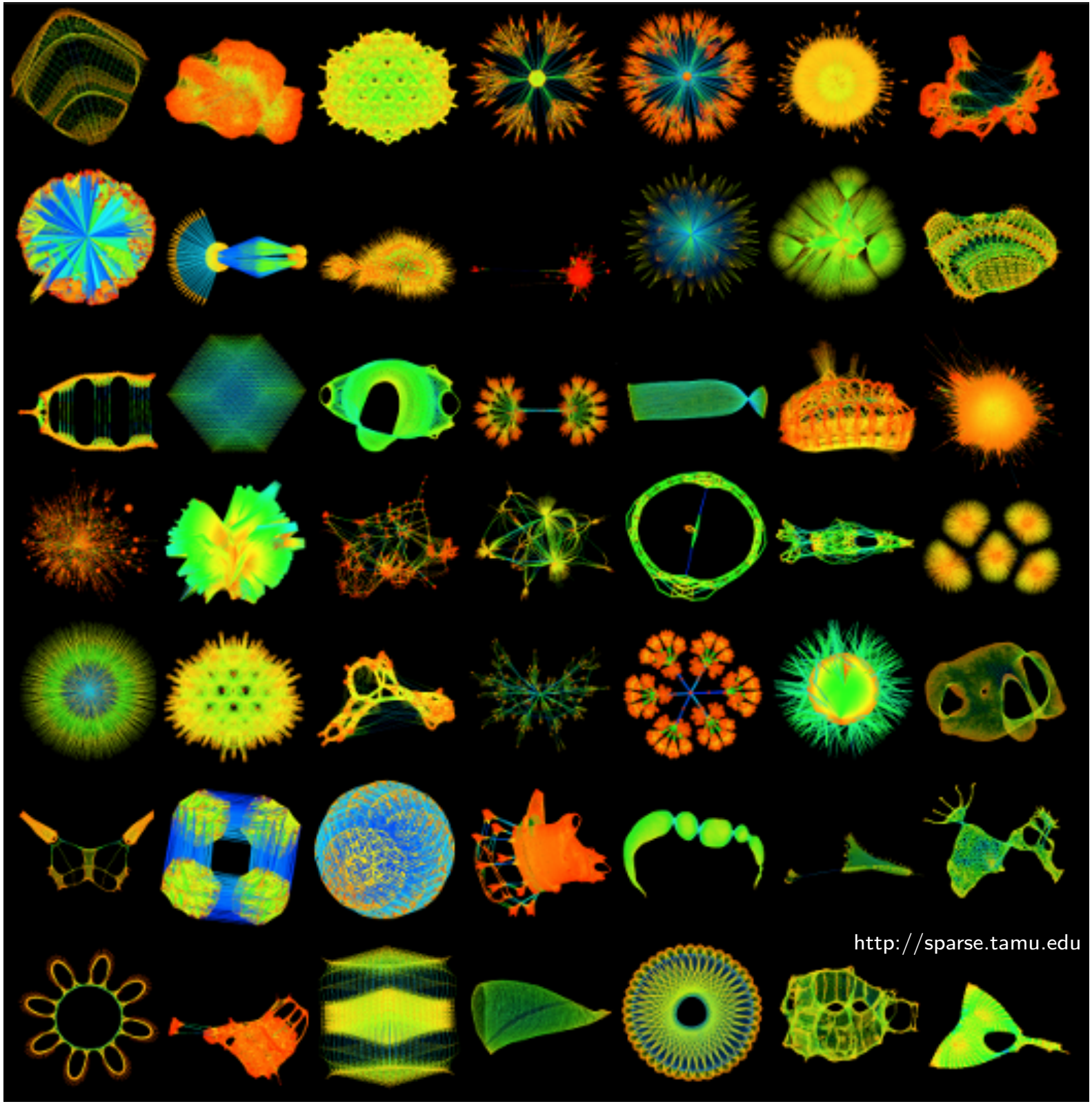
But...

- no guarantees on convergence time
- dependence on initial layout $\rightarrow$ slow convergence
- full stress too slow for large graphs, but fast approximations

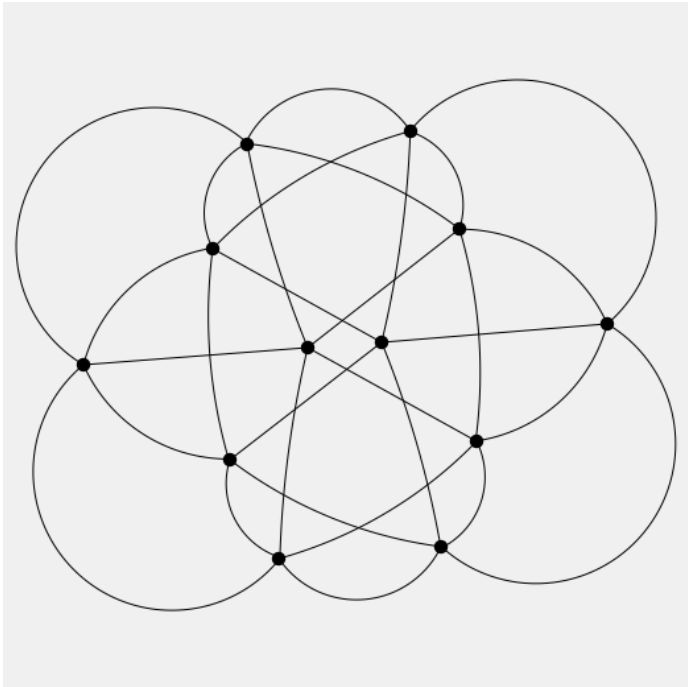
Gallery

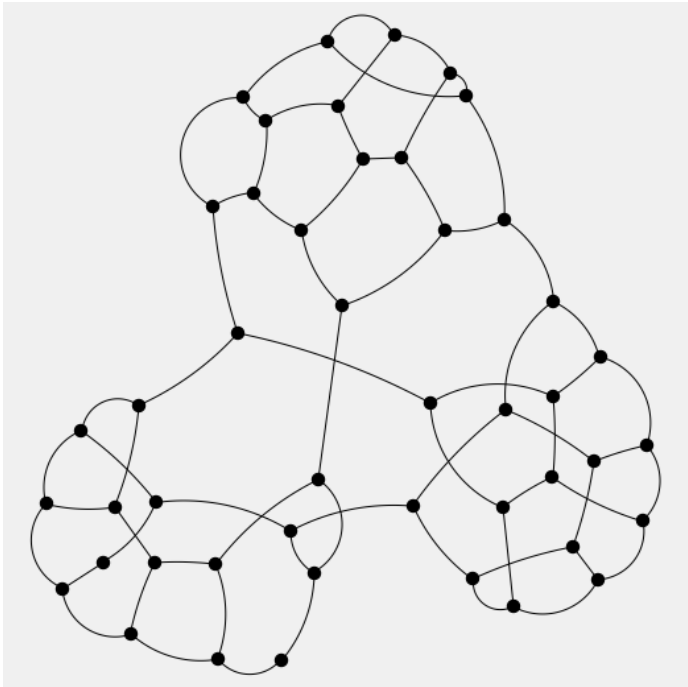


Other Examples

Lombardi-Spring-Embedder (Chernobelskiy et al. 2012)

- edges are circular arcs
- goal: optimal angular resolution $2\pi/\deg(v)$ at each node v
- additional rotational forces





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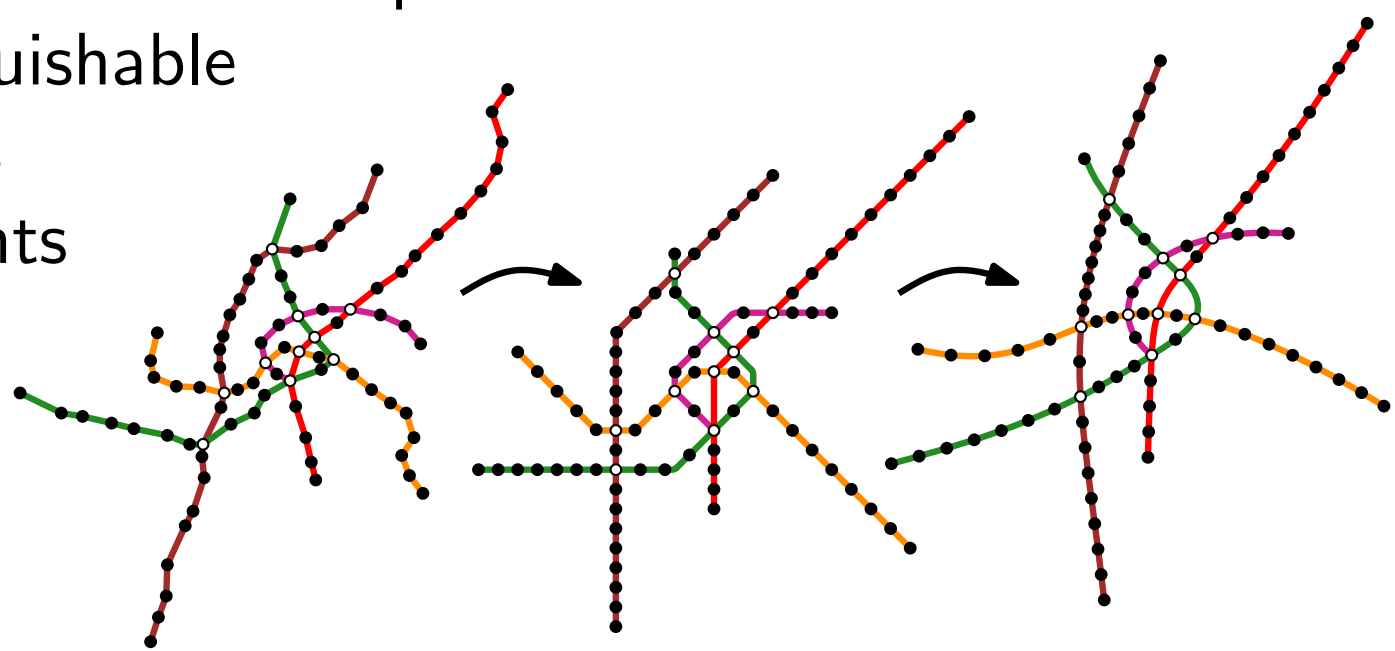
Metro Maps with Bézier curves (Fink et al. 2013)

- model paths as Bézier curves
- forces on nodes and control points:
- lines are distinguishable
- few bend points
- few control points



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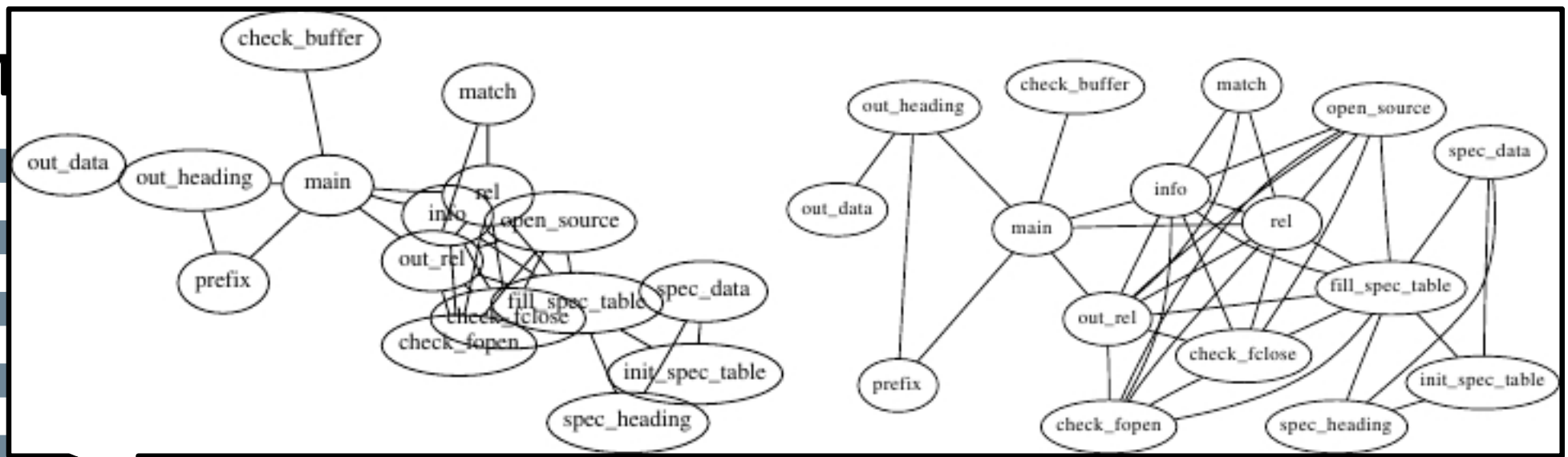
M



- control points

Realistic Node Sizes (Gansner, North 1998)

- node positions are adjusted to avoid overlaps