

# On the Edge-length Ratio of Outerplanar Graphs

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July 2, 2018

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# Introduction

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# Acknowledgement

This presentation is based on the following work:

Sylvain Lazard, William Lenhart, and Giuseppe Liotta. “On the Edge-length Ratio of Outerplanar Graphs”. In: *Graph Drawing and Network Visualization*. Sept. 2017, pp. 17–23.

DOI: [10.1007/978-3-319-73915-1\\_2](https://doi.org/10.1007/978-3-319-73915-1_2)

# Motivation

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- Does a two- or three-connected planar graph admit a unit-length planar straight-line drawing?
- Does a degree four tree have a planar straight-line drawing with vertices at grid positions and edges of the same length?

All these problems are NP-hard.

# Relax problem requirements

- Choose a specific graph family.
- Accept “close enough” quality criteria.



# Preliminary Definitions

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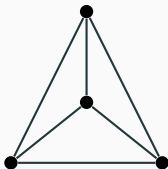
**Definition (outerplanar graph)**

A graph  $G$  is an outerplanar graph if it has a planar drawing for which all vertices belong to the outer edge.

**Definition (outerplanar graph: forbidden minors)**

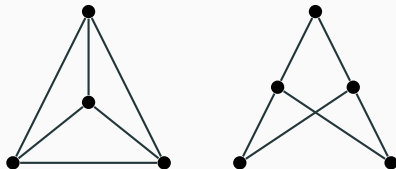
A finite graph  $G$  is outerplanar if and only if its minors include neither  $K_4$  nor  $K_{2,3}$ .

## Outerplanar graphs: negative examples



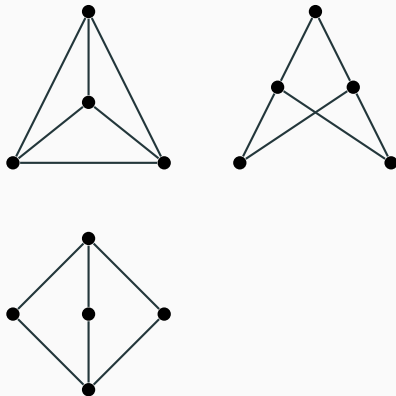
**Figure 1:**  $K_4$  (top left) and embeddings of  $K_{2,3}$  (all others)

# Outerplanar graphs: negative examples



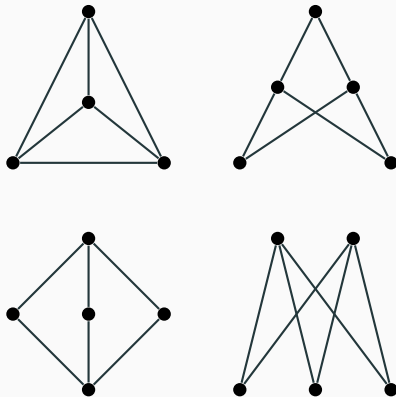
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## Outerplanar graphs: negative examples



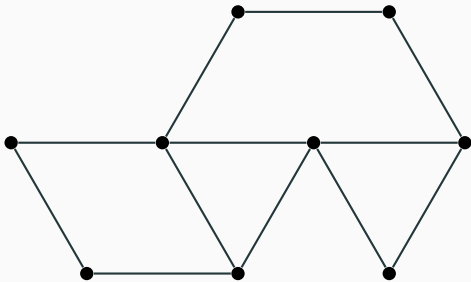
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# Outerplanar graphs: negative examples



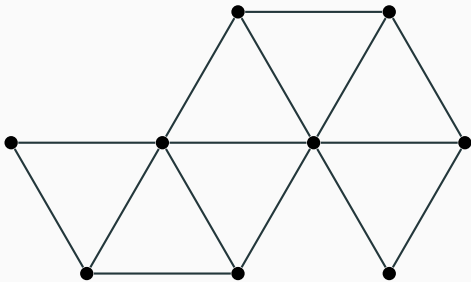
**Figure 1:**  $K_4$  (top left) and embeddings of  $K_{2,3}$  (all others)

## Outerplanar graphs: positive example



**Figure 2:** A non-trivial outerplanar graph

## Outerplanar graphs: maximal example



**Figure 3:** Maximal outerplanar graph



# Main Result

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**Theorem**

*The planar edge-length ratio of an outerplanar graph is strictly less than 2. Also, for any given real positive number  $\varepsilon$ , there exists an outerplanar graph whose planar edge-length ratio is greater than  $2 - \varepsilon$ .*

## Chain definition

A sequence  $T_s, T_{s+1}, \dots, T_t$  of triangles from  $G$ , s.t.

1.  $s \leq 0 \leq t$
2.  $T_0$  has an outer edge from  $G$  (its vertices labeled 0) and a third vertex (labeled 1).
3.  $\forall i \in [1, t]$ : the vertices of  $T_i$  are labeled  $\{i-1, i, i+1\}$ .  
Triangles  $T_i$  and  $T_{i-1}$  share an edge  $(i, i-1)$ .
4.  $\forall i \in [s, -1]$ : the vertices of  $T_i$  are labeled  $\{i, i+1, i+2\}$ .  
Triangles  $T_i$  and  $T_{i+1}$  share an edge  $(i+1, i+2)$ .

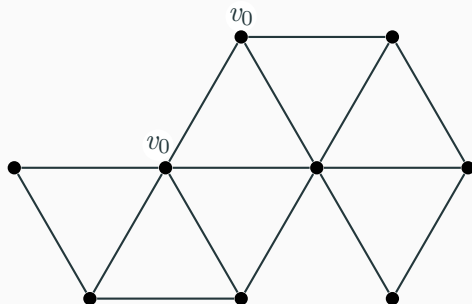
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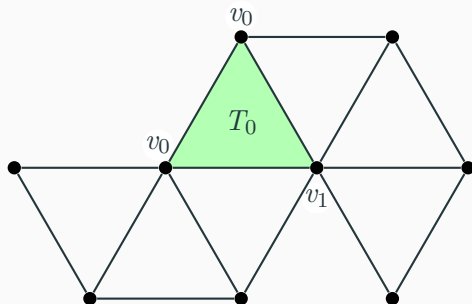
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Triangles  $T_i$  and  $T_{i+1}$  share an edge  $(i+1, i+2)$ .

*Property:* this prohibits fans with more than 3 triangles for all vertices, except  $v_1$ .

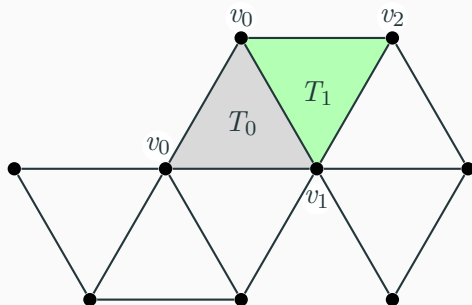
# Chain building example



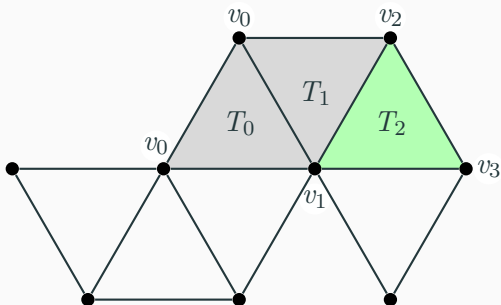
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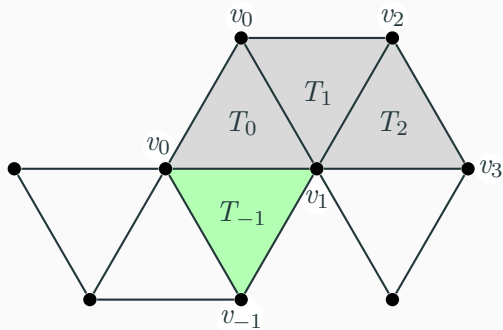


# Chain building example

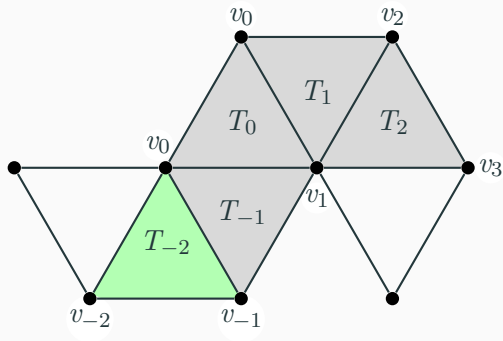




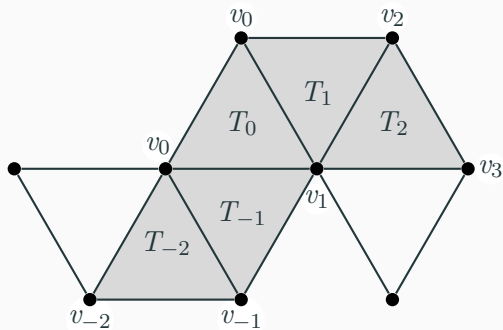
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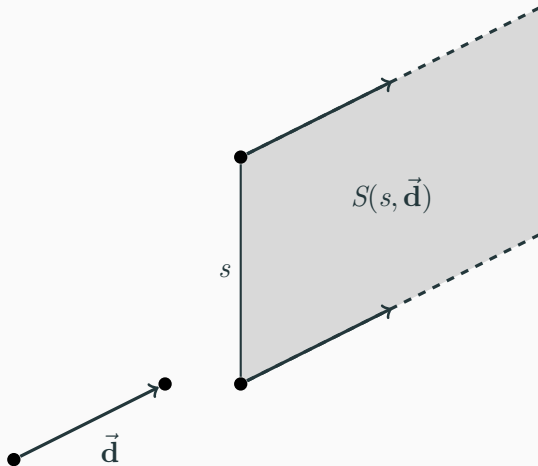


# Chain building example



# Chain properties

# Half-inifinite strip

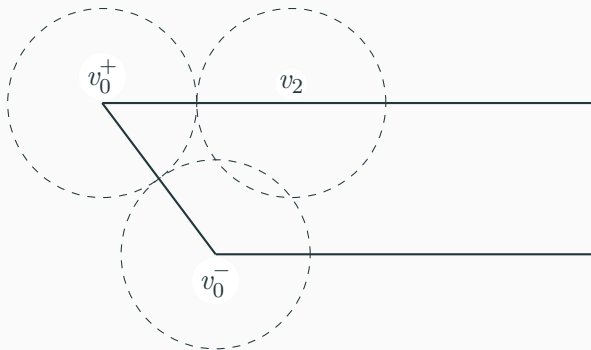


# Chain drawing lemma

## Lemma

*Given a chain  $C$  with  $n$  vertices, an external edge  $e$  of  $C$ , a segment  $s$  of length 1 in the plane and a direction  $d$  s.t. the (smaller) angle between  $s$  and  $d$  is  $\theta < \theta_0 = \arccos(1/4) \approx 75.5^\circ$ , there exists a planar straight-line drawing of  $C$  such that:*

1. *The drawing is completely contained within the strip  $S(s, d)$ .*
2. *For each external edge  $e'$  of  $C$ :*
  - 2.1 *It has length 1.*
  - 2.2 *It is not parallel to  $d$  and forms an angle less than  $\theta_0$  with it.*
  - 2.3 *Strip  $S(e', d)$  is empty.*
3. *Each non-external edge of  $C$  has length greater than  $1/2$ .*



Thank you!



## References

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Sylvain Lazard, William Lenhart, and Giuseppe Liotta. “On the Edge-length Ratio of Outerplanar Graphs”. In: *Graph Drawing and Network Visualization*. Sept. 2017, pp. 17–23. DOI: [10.1007/978-3-319-73915-1\\_2](https://doi.org/10.1007/978-3-319-73915-1_2).

# Backup Slides

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## Backup slide: connected graphs

### **Definition (k-vertex-connected graph)**

A connected graph  $G$  is a  $k$ -vertex-connected ( $k$ -connected) if it has more than  $k$  vertices and remains connected if fewer than  $k$  vertices are removed.

## Backup slide: planar and outerplanar graphs

### **Definition (planar graph)**

A graph  $G$  is a planar graph if it can be embedded in the plane, i.e. it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

### **Definition (planar graph: forbidden minors)**

A finite graph  $G$  is planar if and only if its minors include neither  $K_{5,5}$  nor  $K_{3,3}$ .

### **Definition (outerplanar graph)**

A graph  $G$  is an outerplanar graph if it has a planar drawing for which all vertices belong to the outer edge.

### **Definition (outerplanar graph: forbidden minors)**

A finite graph  $G$  is outerplanar if and only if its minors include neither  $K_4$  nor  $K_{2,3}$ .

## Backup slide: outerplanar graphs

### **Definition (maximal outerplanar graph)**

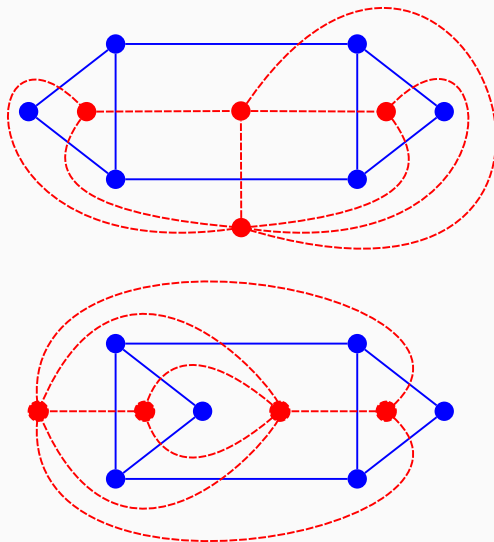
An outerplanar graph  $G$  is a maximal outerplanar graph if no more edges can be added while preserving outerplanarity.

## Backup slide: dual graphs

### **Definition (dual graph)**

Given a plane graph  $G$ , its dual graph  $G^*$  is a plane graph that has a vertex for each face of  $G$ . The dual graph  $G^*$  has an edge whenever two faces of  $G$  are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge.

## Backup slide: dual graphs (example)



**Figure 4:** Two non-isomorphic dual graphs, image source.

## Backup slide: series-parallel graphs

See lecture 3