On the Edge-length Ratio of Outerplanar Graphs

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Introduction

Acknowledgement

This presentation is based on the following work:

Sylvain Lazard, William Lenhart, and Giuseppe Liotta. "On the Edge-length Ratio of Outerplanar Graphs". In: *Graph Drawing and Network Visualization*. Sept. 2017, pp. 17–23. DOI: 10.1007/978-3-319-73915-1_2

Motivation

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- Does a two- or three-connected planar graph admit a unit-length planar straight-line drawing?
- Does a degree four tree have a planar straight-line drawing with vertices at grid positions and edges of the same length?

All these problems are NP-hard.

Relax problem requirements

- Choose a specific graph family.
- Accept “close enough” quality criteria.

Preliminary Definitions

Outerplanar graphs

Definition (outerplanar graph)
A graph G is an outerplanar graph if it has a planar drawing for which all vertices belong to the outer edge.

Definition (outerplanar graph: forbidden minors)
A finite graph G is outerplanar if and only if its minors include neither K_4 nor $K_{2,3}$.

Outerplanar graphs: negative examples

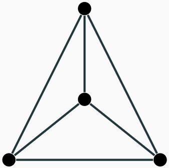
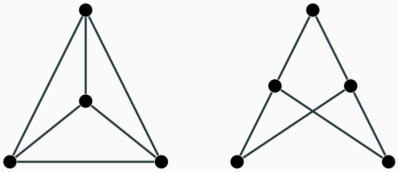


Figure 1: K_4 (top left) and embeddings of $K_{2,3}$ (all others)

Outerplanar graphs: negative examples



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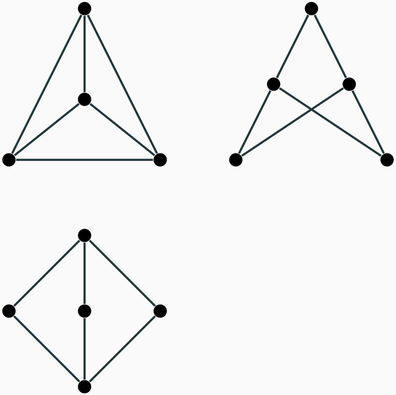
Outerplanar graphs: negative examples



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Outerplanar graphs: negative examples

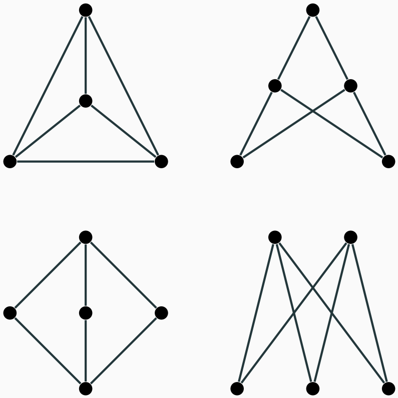

Figure 1: K_4 (top left) and embeddings of $K_{2,3}$ (all others)

Outerplanar graphs: positive example

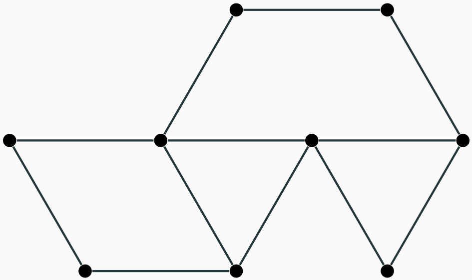

Figure 2: A non-trivial outerplanar graph

Outerplanar graphs: maximal example

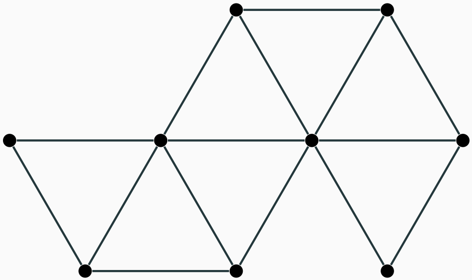

Figure 3: Maximal outerplanar graph

Main Result

Main result

Theorem
The planar edge-length ratio of an outerplanar graph is strictly less than 2. Also, for any given real positive number ε, there exists an outerplanar graph whose planar edge-length ratio is greater than $2 - \varepsilon$.

Chain definition

A sequence $T_s, T_{s+1}, \ldots, T_t$ of triangles from G, s.t.

1. $s \leq 0 \leq t$
2. T_0 has an outer edge from G (its vertices labeled 0) and a third vertex (labeled 1).
3. $\forall i \in [1, t]$: the vertices of T_i are labeled $\{i-1, i, i+1\}$. Triangles T_i and T_{i-1} share an edge $(i, i-1)$.
4. $\forall i \in [s, -1]$: the vertices of T_i are labeled $\{i, i+1, i+2\}$. Triangles T_i and T_{i+1} share an edge $(i+1, i+2)$.

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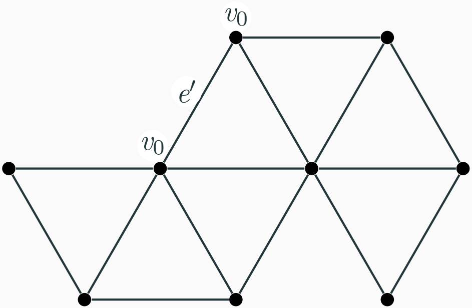
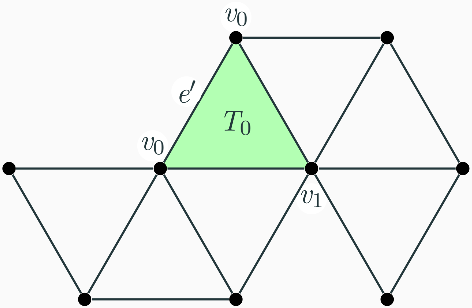
4. $\forall i \in [s, -1]$: the vertices of T_i are labeled $\{i, i+1, i+2\}$. Triangles T_i and T_{i+1} share an edge $(i+1, i+2)$.

Property: this prohibits fans with more than 3 triangles for all vertices, except v_1.

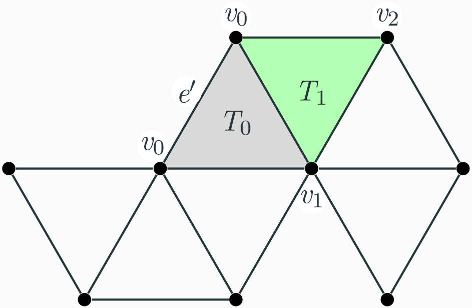
Chain building example



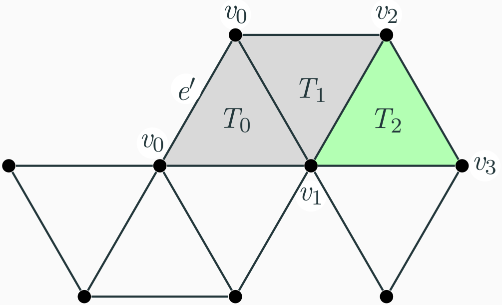
Chain building example



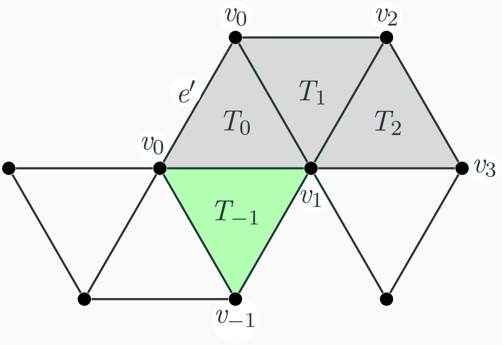
Chain building example



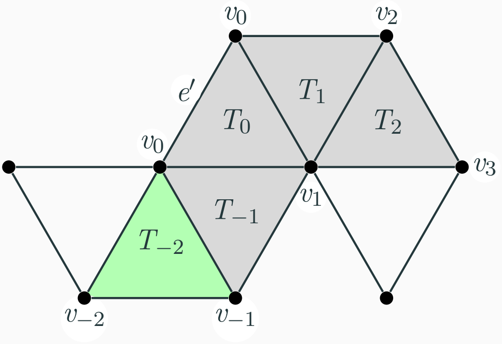
Chain building example



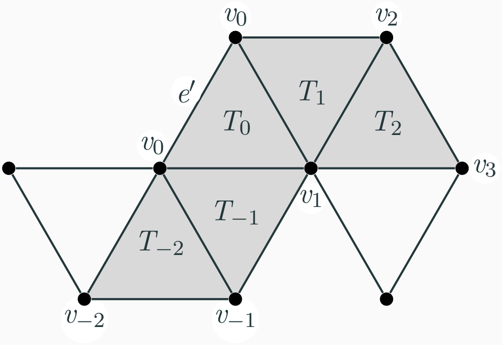
Chain building example



Chain building example



Chain building example



Chain properties

- Outer edge e' determines the maximal chain $C_{e'}$.

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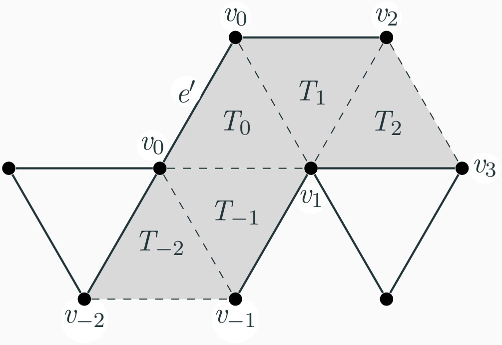
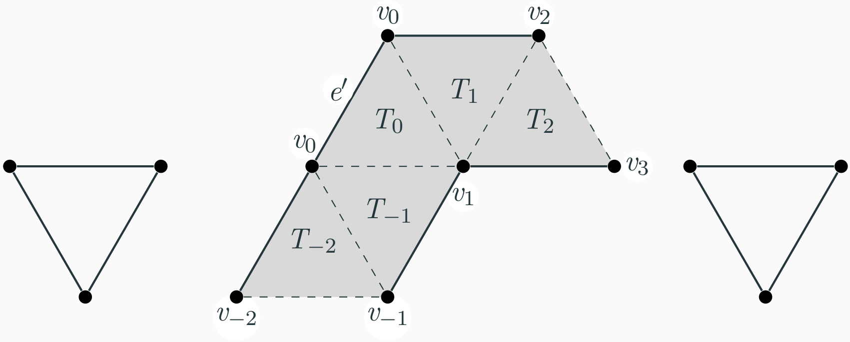
- Edges of $C_{e'}$ can be partitioned into two sets: $S_{e'}$ and $L_{e'}$.
- Graph G can be decomposed into a tree of chains.

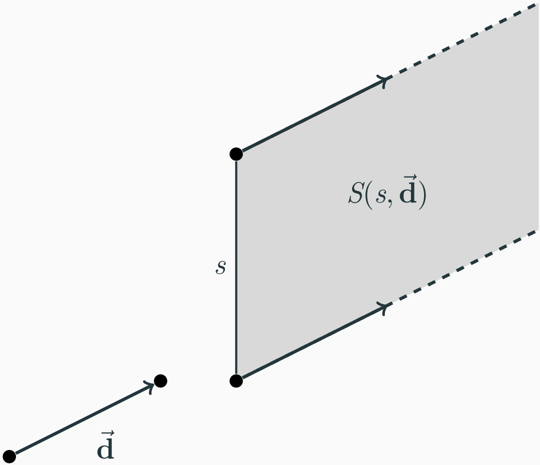
Chain properties: $S_{e'}$ and $L_{e'}$



Chain properties: decomposition



Half-inifinite strip



Lemma

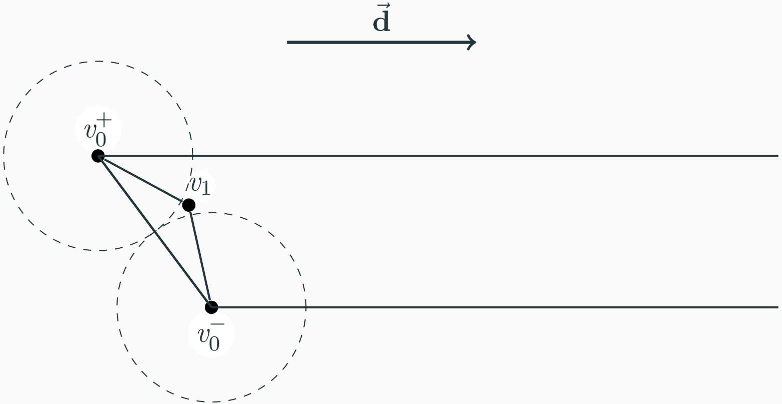
Given a chain C with n vertices, an external edge e of C, a segment s of length 1 in the plane and a direction d s.t. the (smaller) angle between s and d is $\theta < \theta_0 = \arccos(1/4) \approx 75.5°$, there exists a planar straight-line drawing of C such that:

1. *The drawing is completely contained within the strip $S(s, d)$.*
2. *For each external edge e' of C:*
   - 2.1 *It has length 1.*
   - 2.2 *It is not parallel to d and forms an angle less than θ_0 with it.*
   - 2.3 *Strip $S(e', d)$ is empty.*
3. *Each non-external edge of C has length greater than 1/2.*

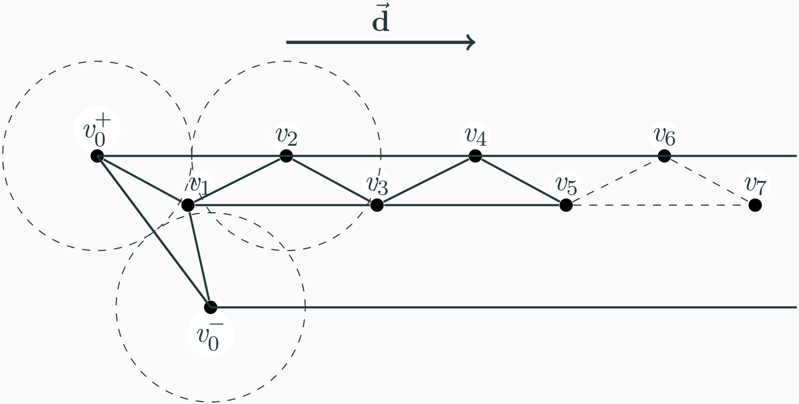
Possible chains

- Single triangle.
- Positive sequence.
- Positive and negative sequence.

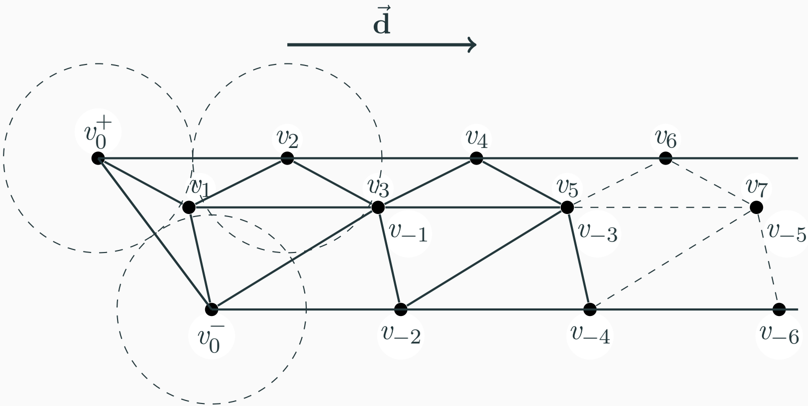
Possible chains: single triangle



Possible chains: positive sequence



Possible chains: positive and negative sequences



Drawing algorithm

- Let $C_{e'}$ be the root of the chain decomposition tree.
- Select a line segment s of length 1 in the plane and an initial direction $\vec{\mathbf{d}}$ not parallel to s s.t. it forms a degree less than $\theta_0 = \arccos(1/4)$ with s.
- Apply Lemma to compute U-strip drawing of $C_{e'}$ in $S(e', \vec{\mathbf{s}})$.
- Each long edge $e \in L_{e'}$ is drawn with length 1 and not parallel to $\vec{\mathbf{d}}$. It forms an angle less than θ_0, so the conditions of the Lemma still apply. Draw C_e in $S(e, \vec{\mathbf{s}})$.

Thank you!

References

Sylvain Lazard, William Lenhart, and Giuseppe Liotta. “On the Edge-length Ratio of Outerplanar Graphs”. In: *Graph Drawing and Network Visualization*. Sept. 2017, pp. 17–23. DOI: 10.1007/978-3-319-73915-1_2.

Backup Slides

Backup slide: connected graphs

Definition (k-vertex-connected graph)
A connected graph G is a k-vertex-connected (k-connected) if it has more than k vertices and remains connected if fewer than k vertices are removed.

Backup slide: planar and outerplanar graphs

Definition (planar graph)
A graph G is a planar graph if it can embedded in the plane, i.e. it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

Definition (planar graph: forbidden minors)
A finite graph G is planar if and only if its minors include neither $K_{5,5}$ nor $K_{3,3}$.

Definition (outerplanar graph)
A graph G is an outerplanar graph if it has a planar drawing for which all vertices belong to the outer edge.

Definition (outerplanar graph: forbidden minors)
A finite graph G is outerplanar if and only if its minors include neither K_4 nor $K_{2,3}$.

Backup slide: outerplanar graphs

Definition (maximal outerplanar graph)
An outerplanar graph G is a maximal outerplanar graph if no more edges can be added while preserving outerplanarity.

Backup slide: dual graphs

Definition (dual graph)
Given a plane graph G, its dual graph G^* is a plane graph that has a vertex for each face of G. The dual graph G^* has an edge whenever two faces of G are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge.

Backup slide: dual graphs (example)

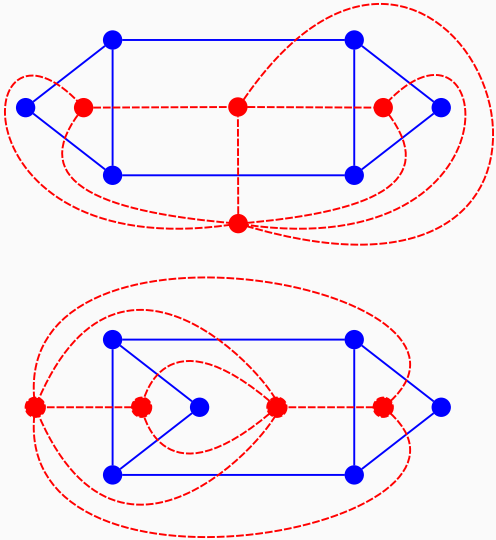


Figure 4: Two non-isomorphic dual graphs, image source.

Backup slide: series-parallel graphs

See lecture 3