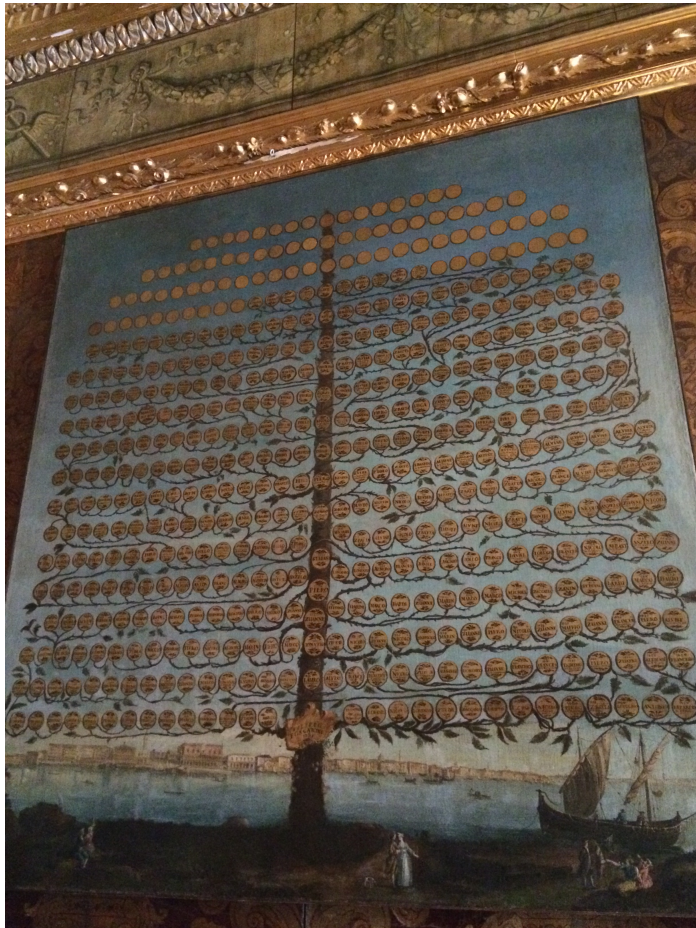
Tree Layout

Lecture Graph Drawing Algorithms · 192.053

Martin Nöllenburg

20.03.2018

ALGORITHMS AND
COMPLEXITY GROUP



Before we start...

- Registration for (mandatory) exercise groups starts today until Friday, March 23, 23:55.
- TUWEL page $\rightarrow$ Exercises has more details
- If no more groups with your favorite topic available, send us a message and we will open more.

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Any further questions?

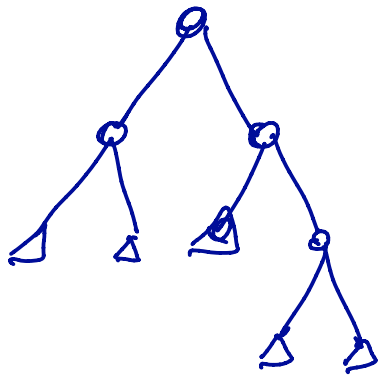
Motivation

What types of tree drawings do we know?

- what are typical drawing conventions?
- which aesthetics shall be optimized?

Let's collect some properties!

- symmetry btw. branches
- uniform edge length
- no overlaps / crossings
- uniform node distances
- upward / downward orientation
- different symbols for inner / leaves



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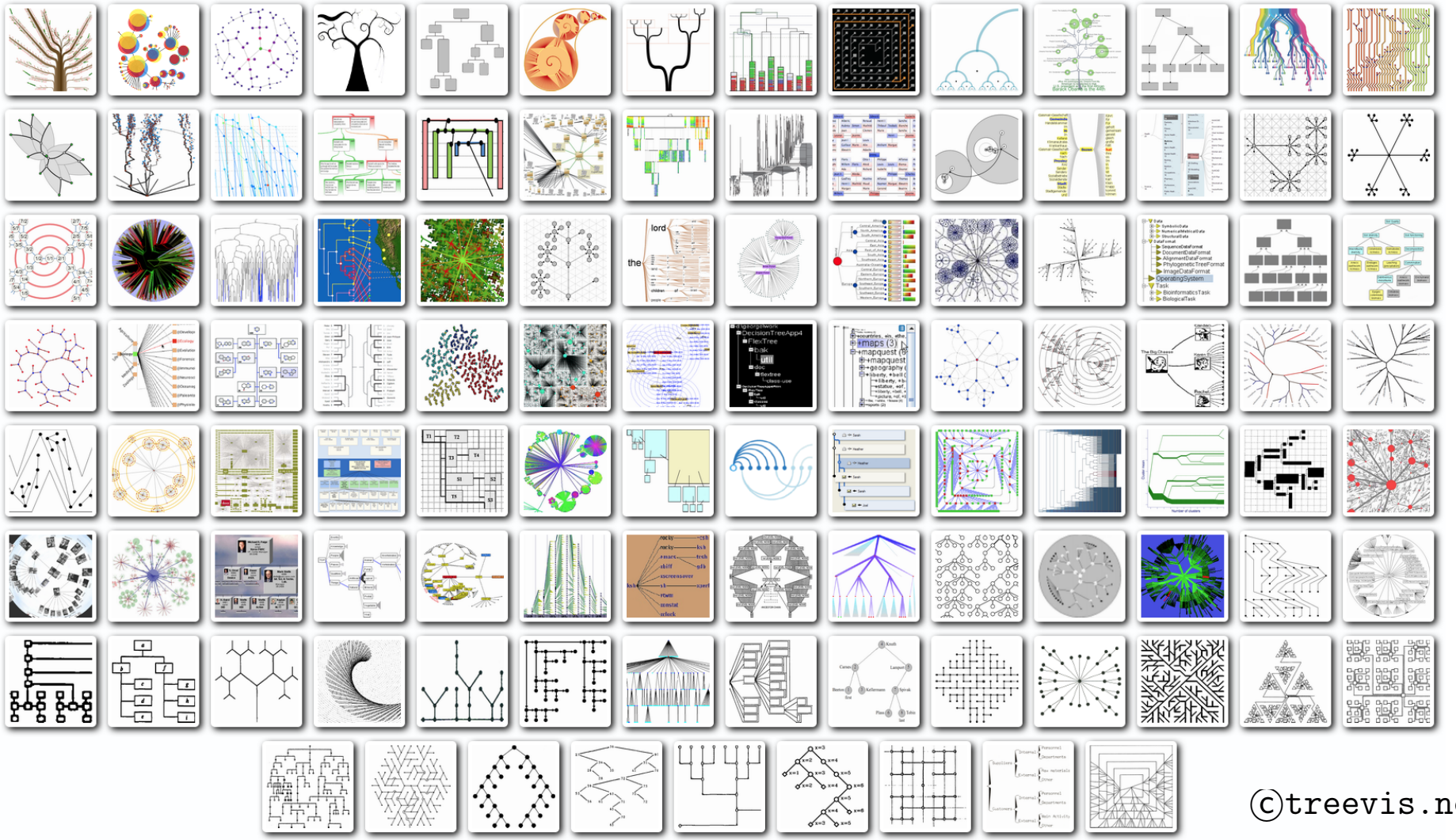


©treevis.net

Overview

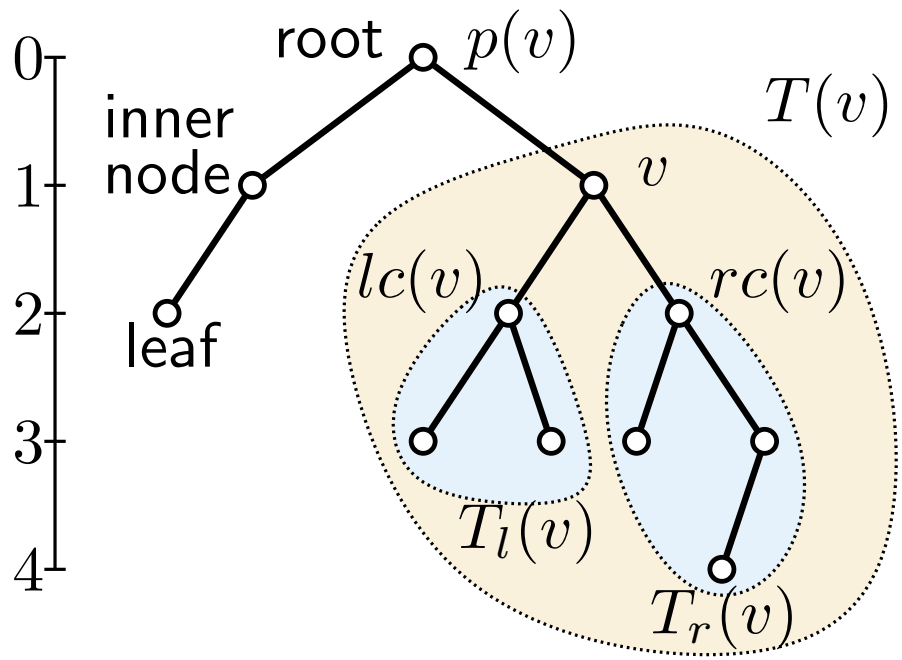
Layered drawings of rooted trees

Radial tree drawings

Compact orthogonal drawings of rooted trees

Layered Binary Tree Layout – Terminology

Let's start with textbook-style tree drawings in computer science:



- root r
- parent $p(v)$
- left child $lc(v)$, right child $rc(v)$
- subtree $T(v)$ rooted at v
- left and right subtrees $T_l(v), T_r(v)$
- leaf vs. inner node
- depth(v): distance to root
- height(T) $= \max_v$ depth(v)
- ordered/unordered: left/right given?

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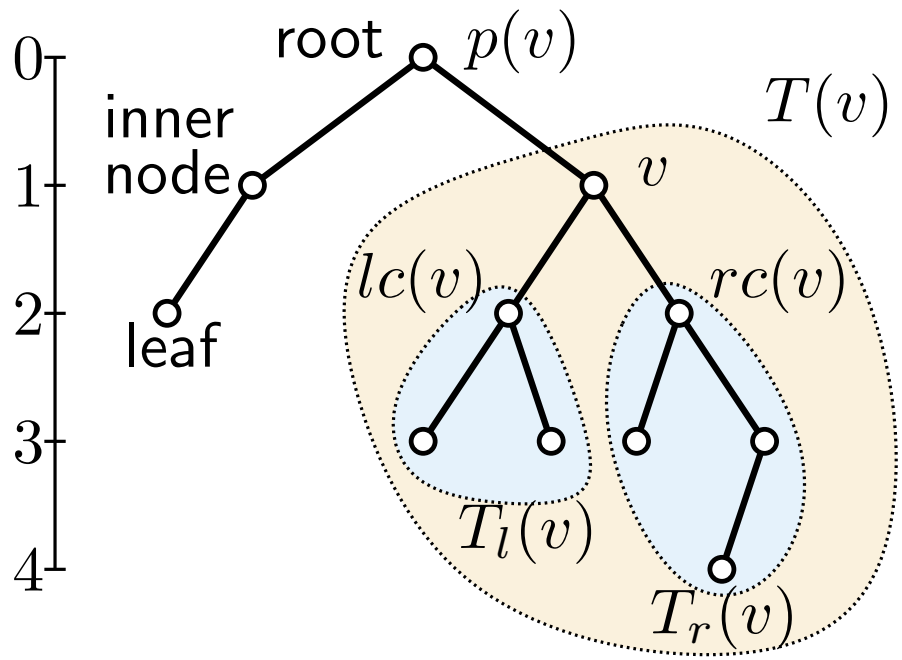

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Input: rooted (ordered) binary tree T with n nodes

Output: layered tree drawing of T with

- y-coordinates given by depth
- parent centered above children
- straight-line edges
- no crossings
- isomorphic subtrees drawn the same
- integer grid positions (?)
- compact

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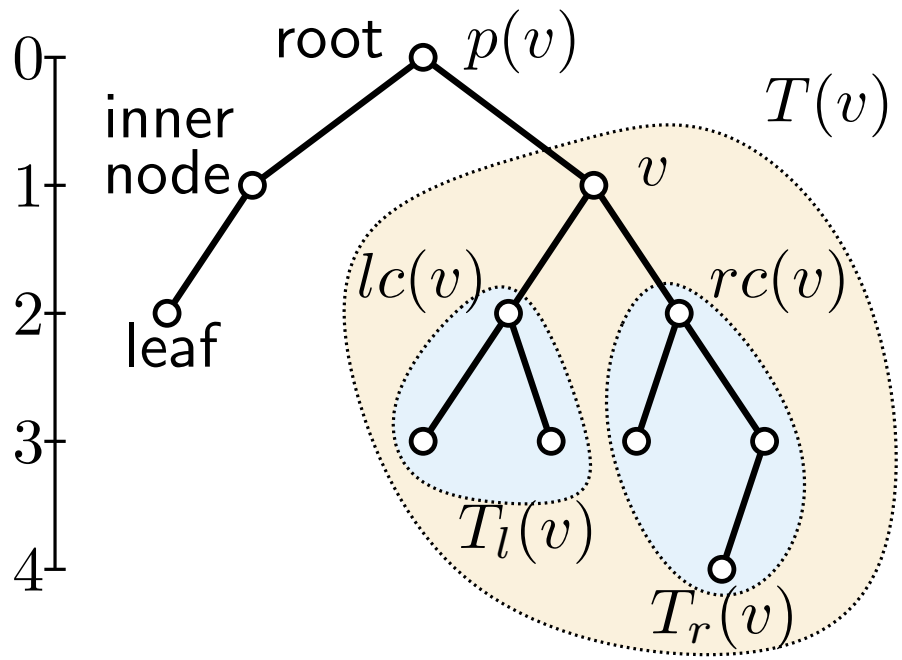

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Simple Algorithms: Tree Traversal

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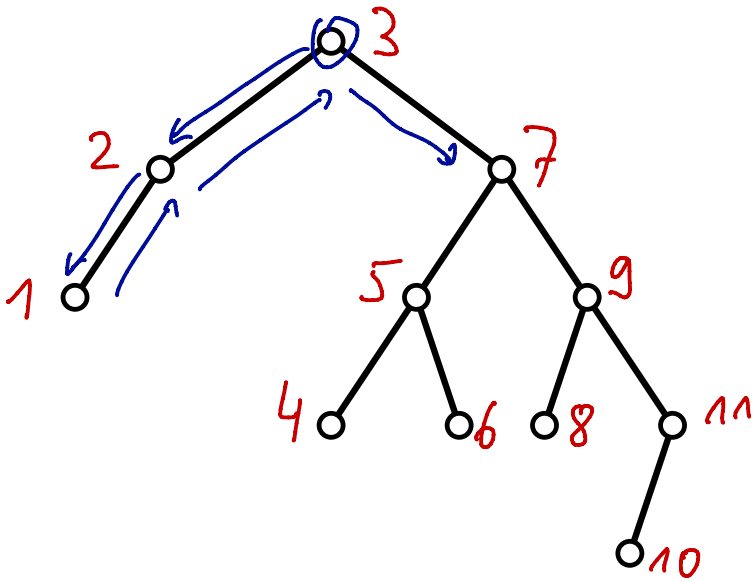

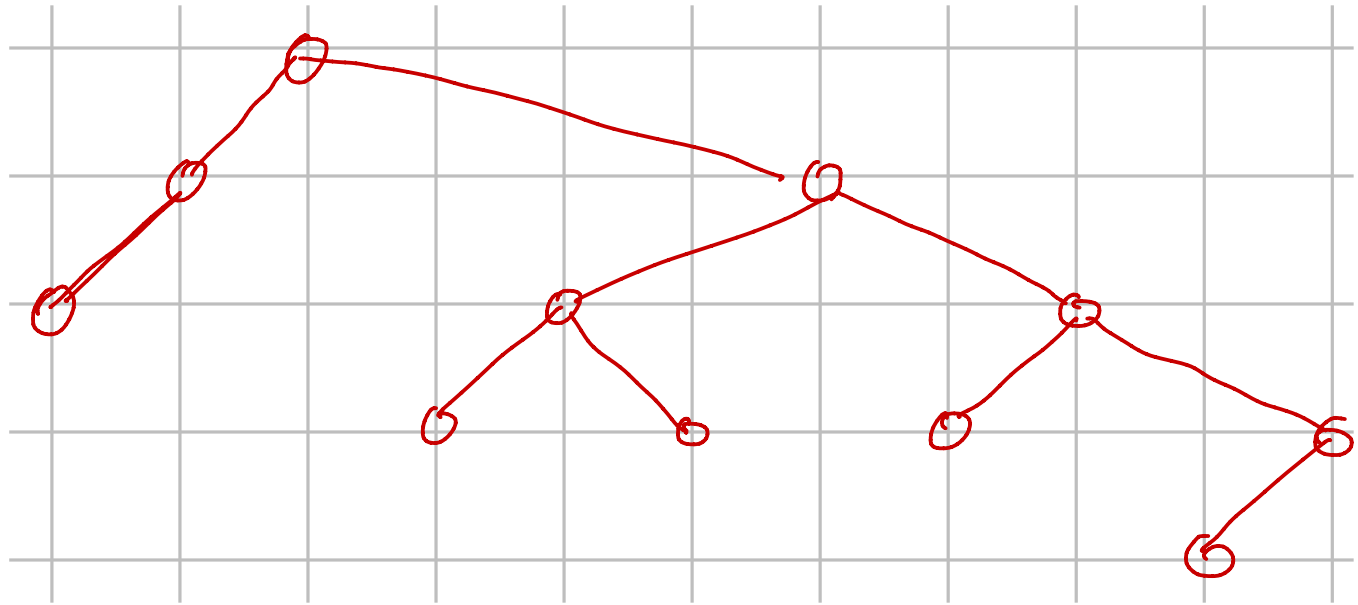

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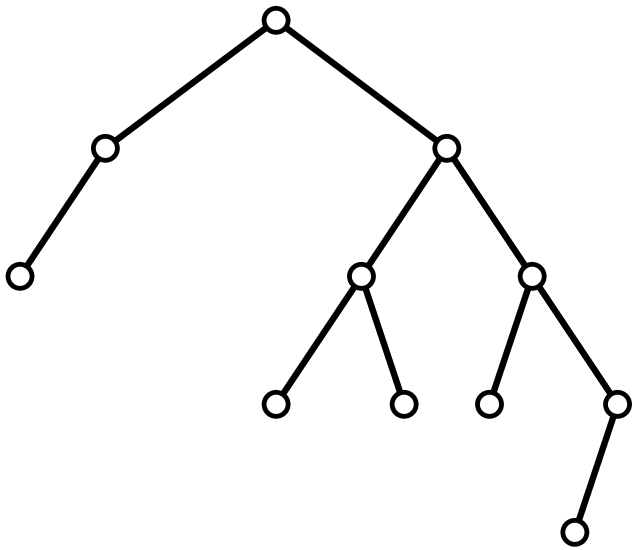
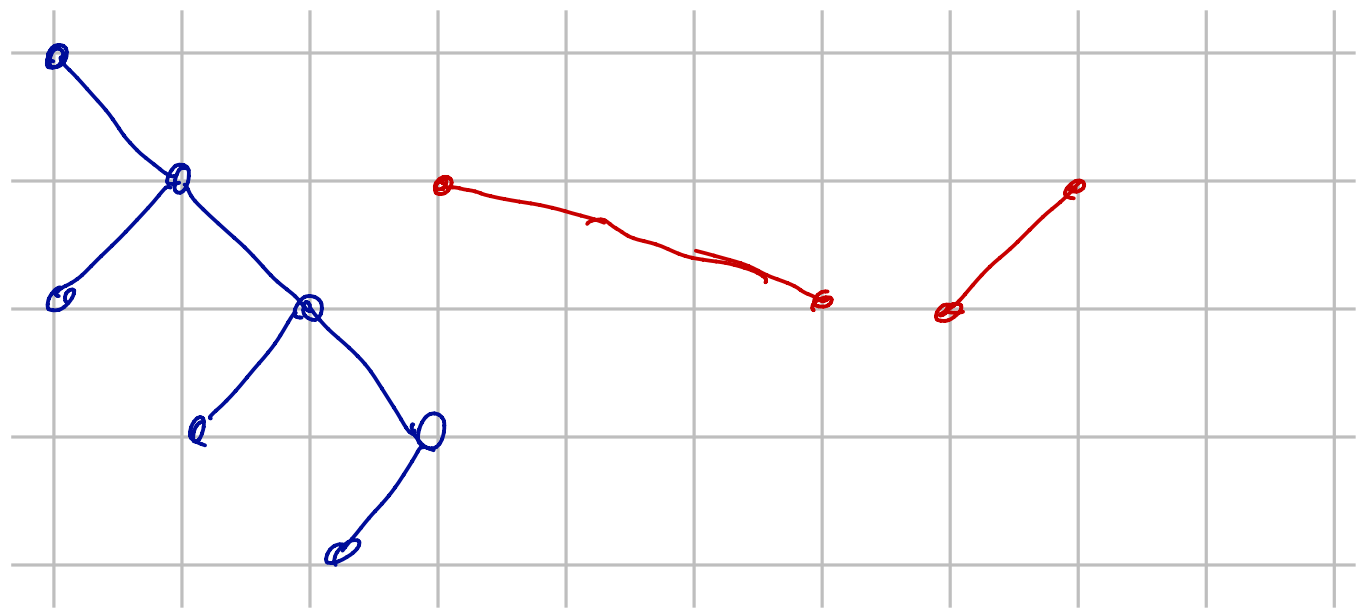


Properties

- running time: $O(n)$
- drawing area: $n \cdot \text{height}(T) = O(n^2)$
- conventions:
  - upward, straight line, no crossings, isomorphic subtrees

issues:

- area
- parent position



Improving Area and Parent Position

Recursive algorithm of Reingold/Tilford (1981):

Base case: leaf v, place at layer $-\text{depth}(v)$ ⊡

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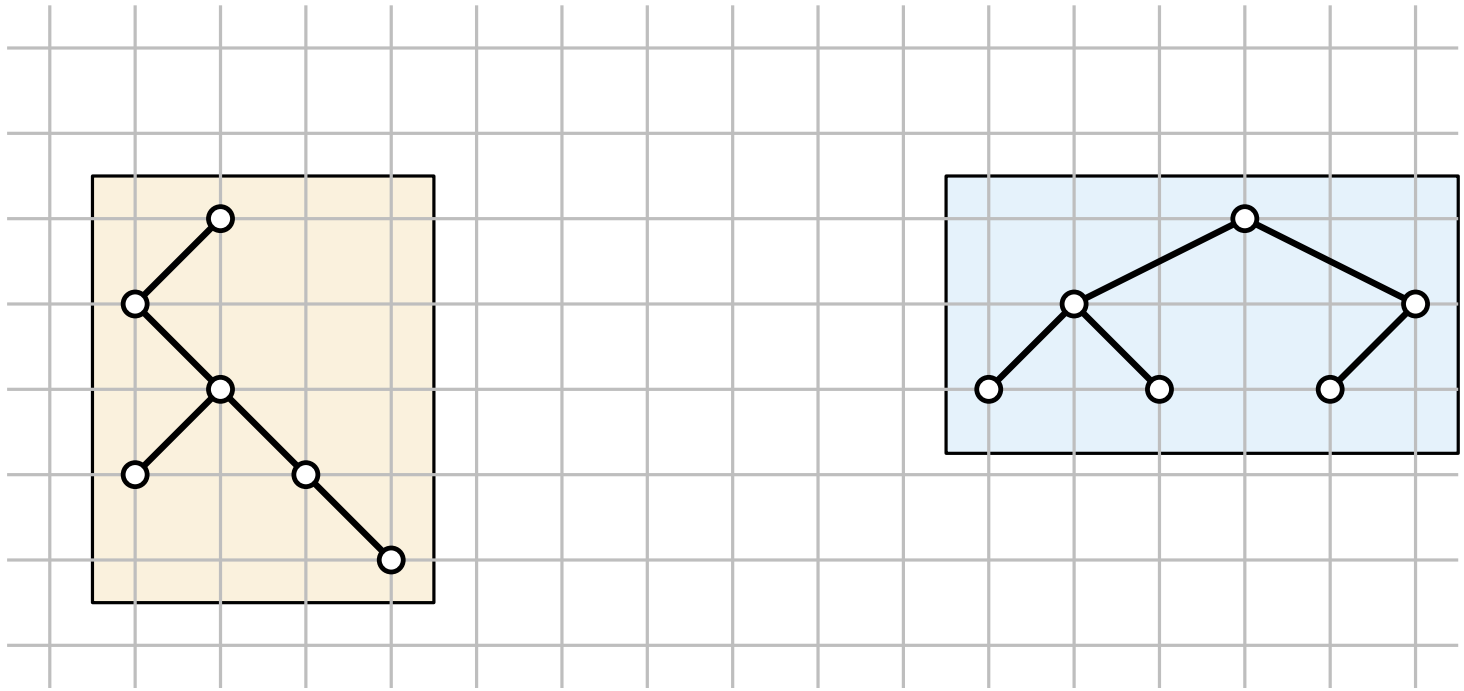

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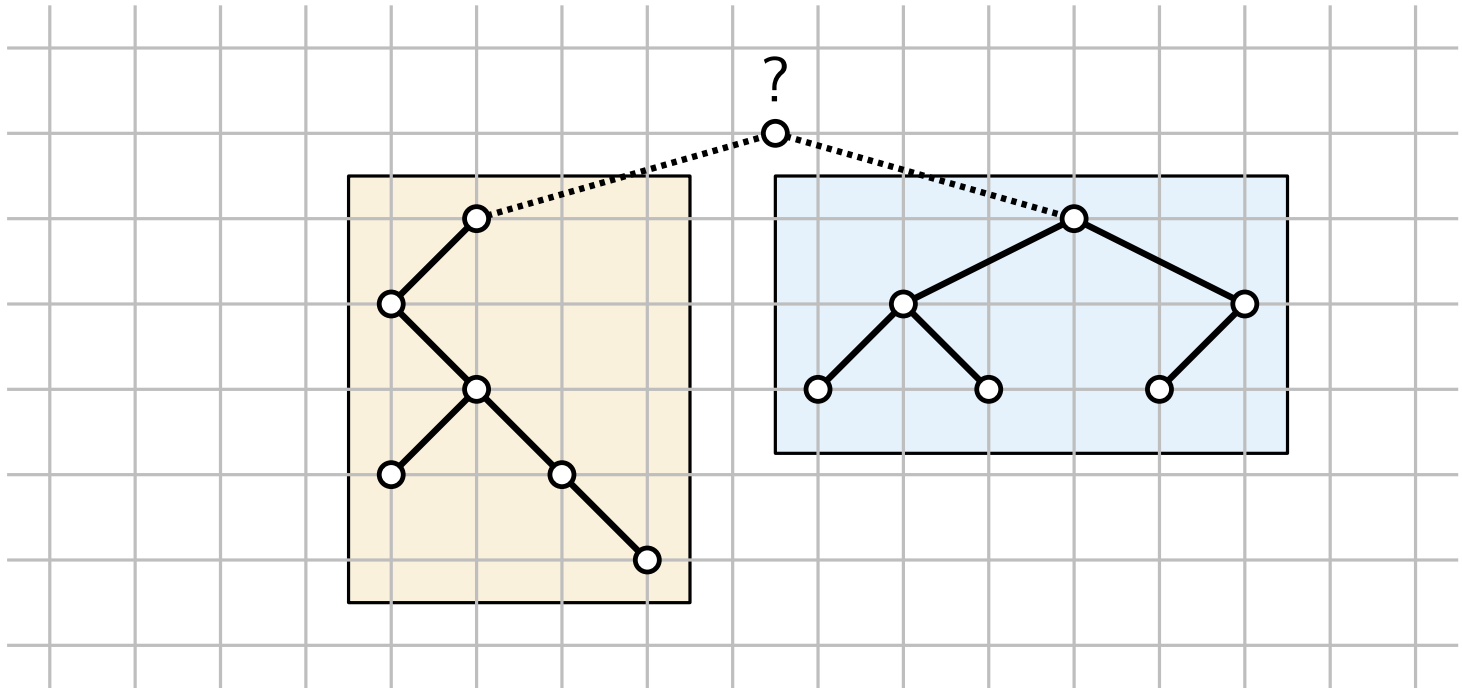
Conquer: place subtree drawings next to each other with distance 2 or 3 and center v above $lc(v)$ and $rc(v)$



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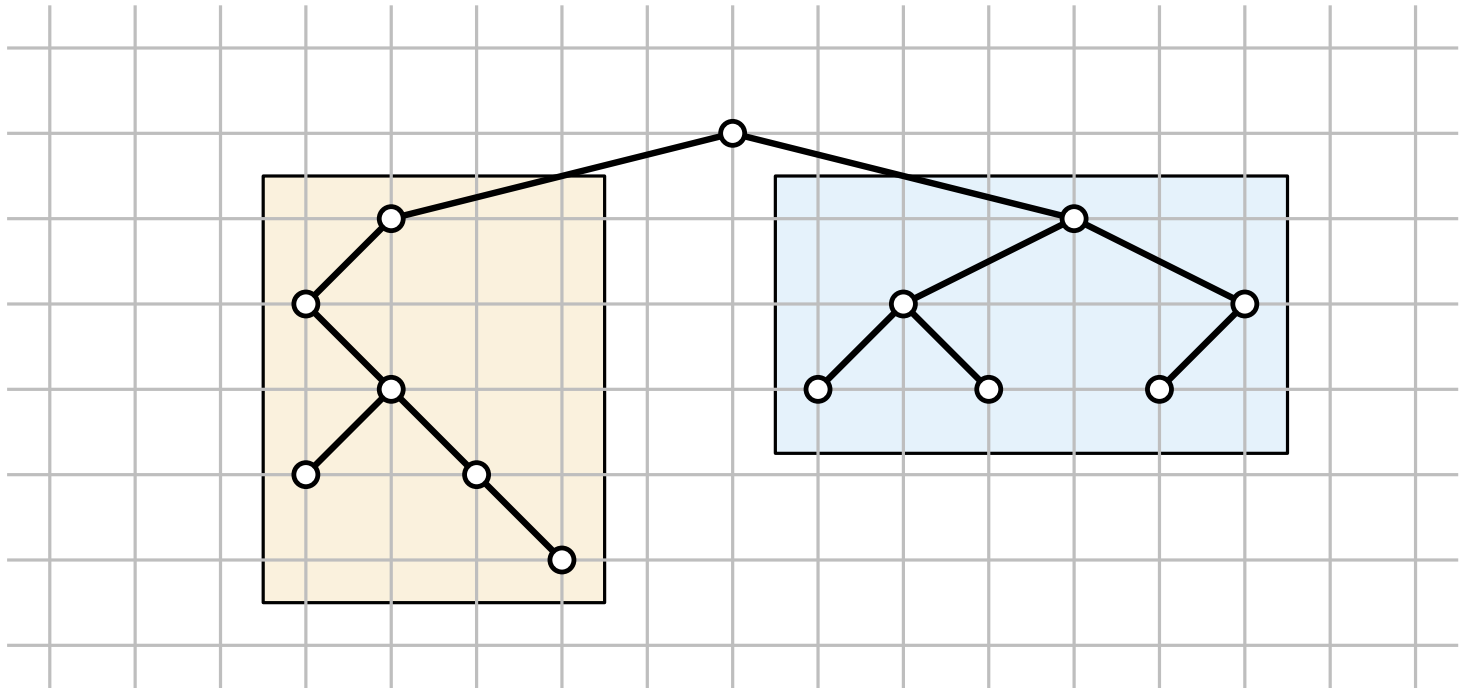

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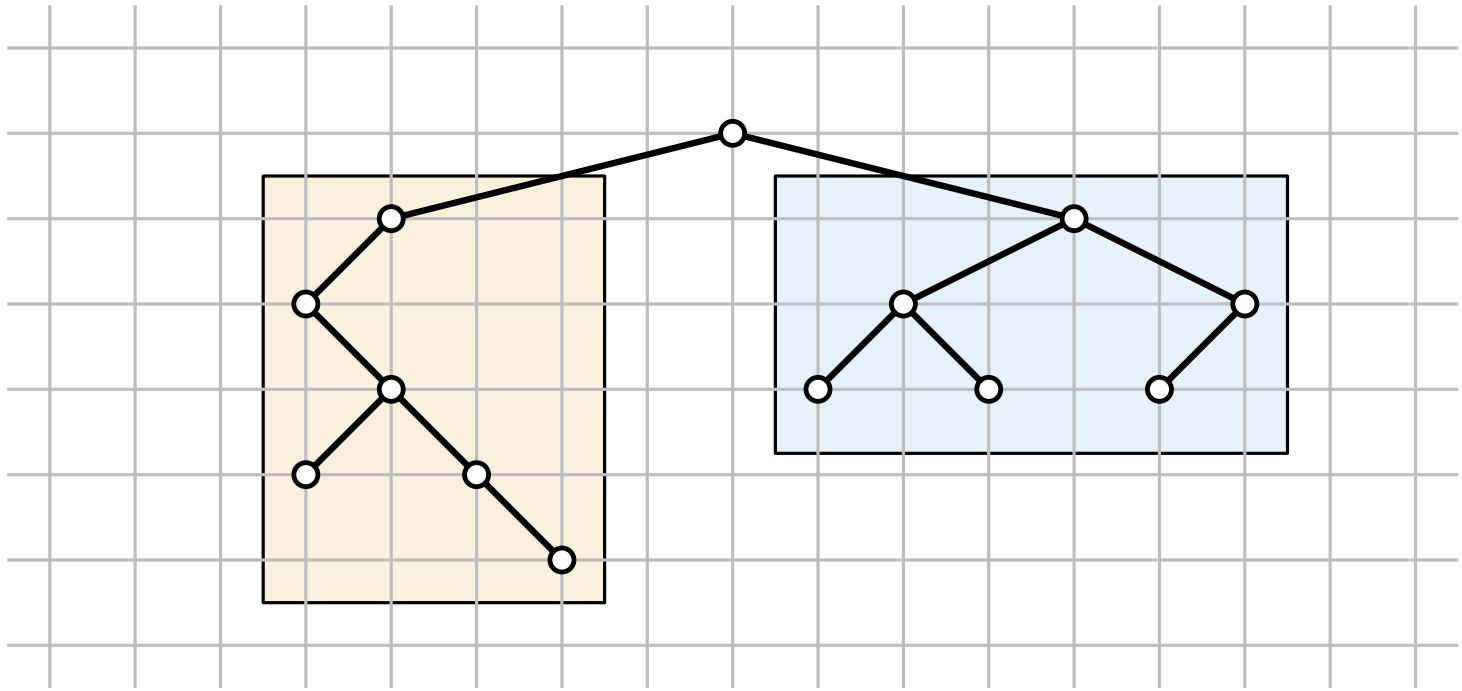

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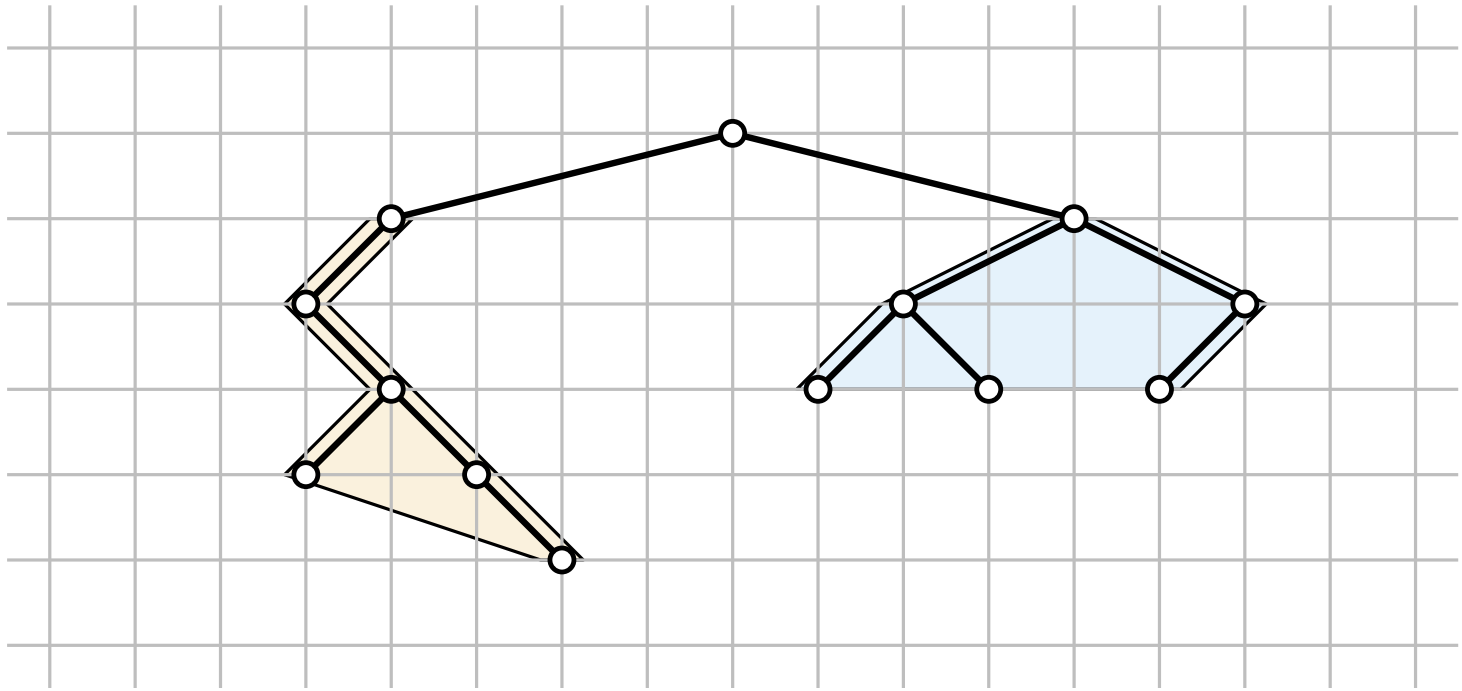

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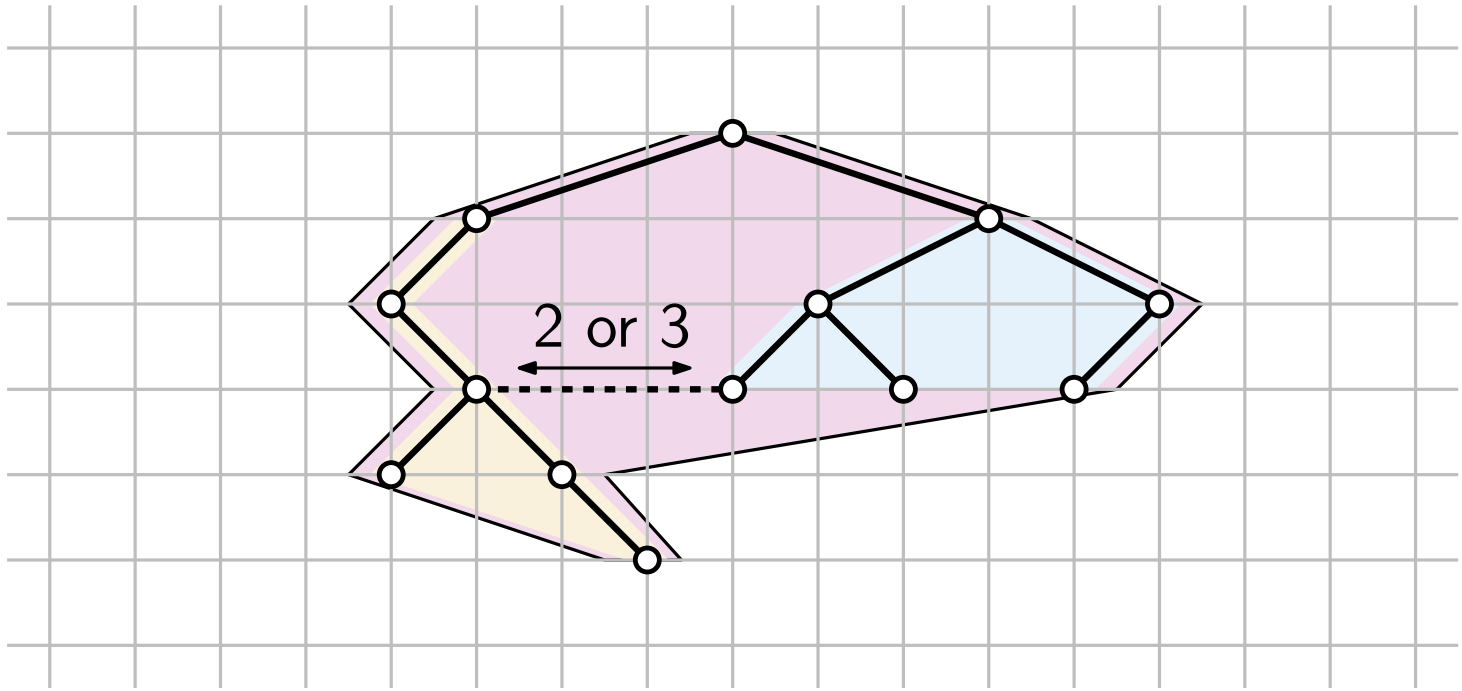

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Improvement: use tight **contour** for more compact placement

Reingold/Tilford: 2-Phase Algorithm

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   - compute contours and x-offsets to parent
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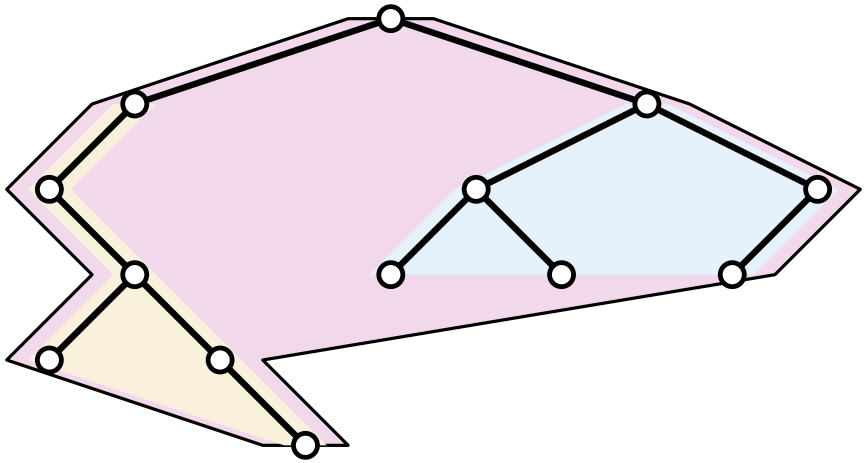

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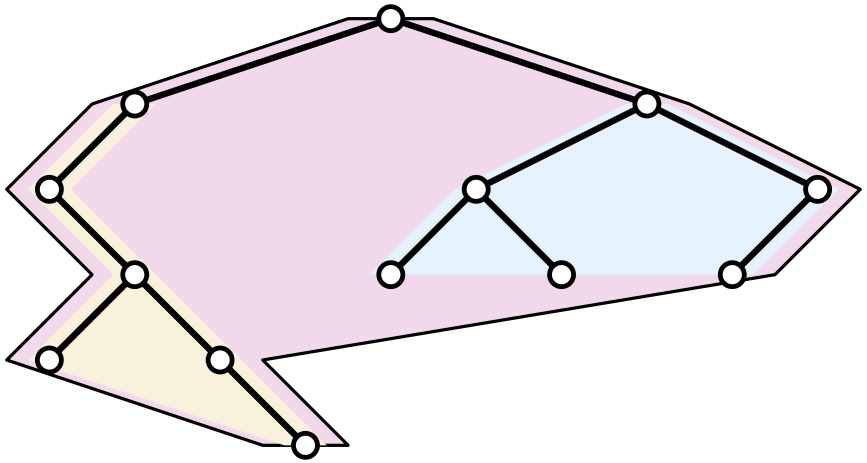

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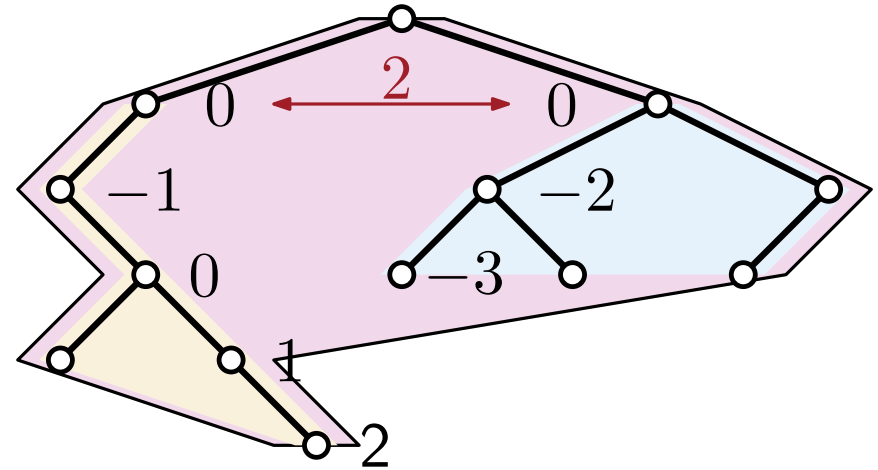

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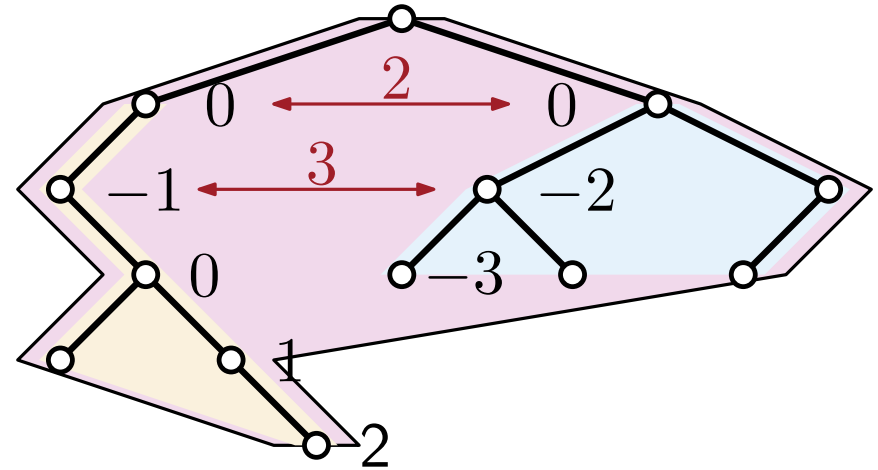

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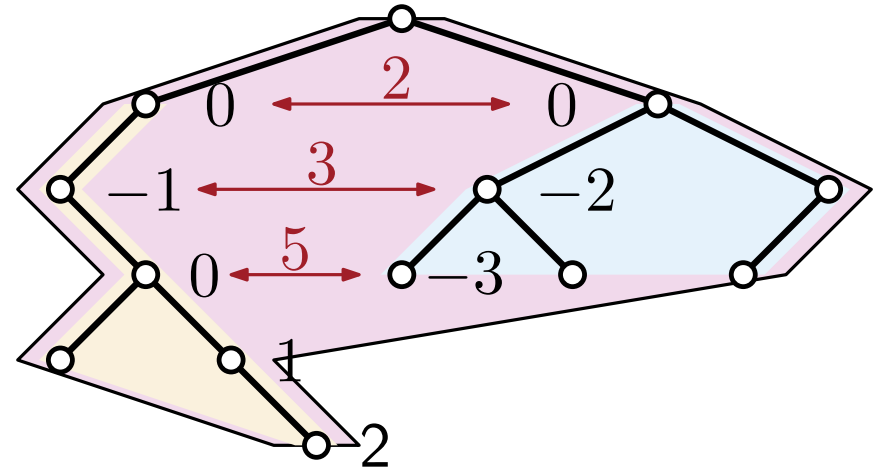

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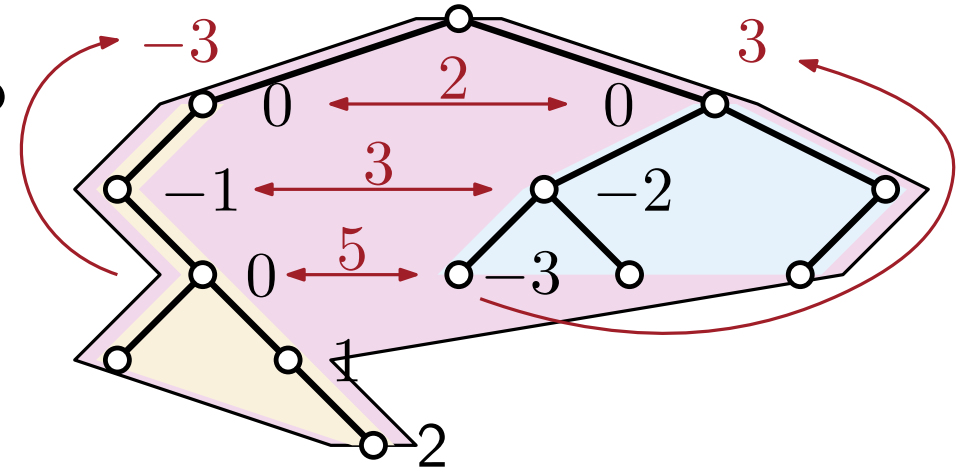

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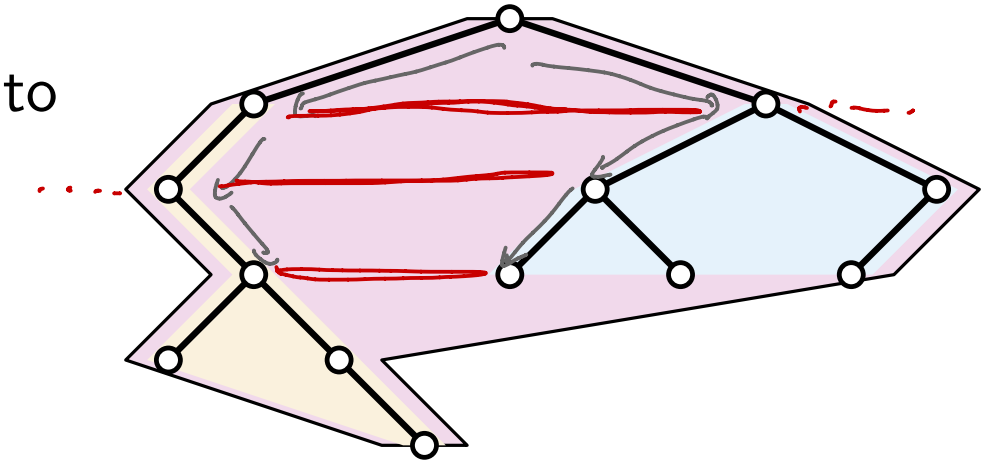

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Running time: postorder $+ \sum_v \min\{\text{height}(T_l(v), \text{height}(T_r(v)\}$

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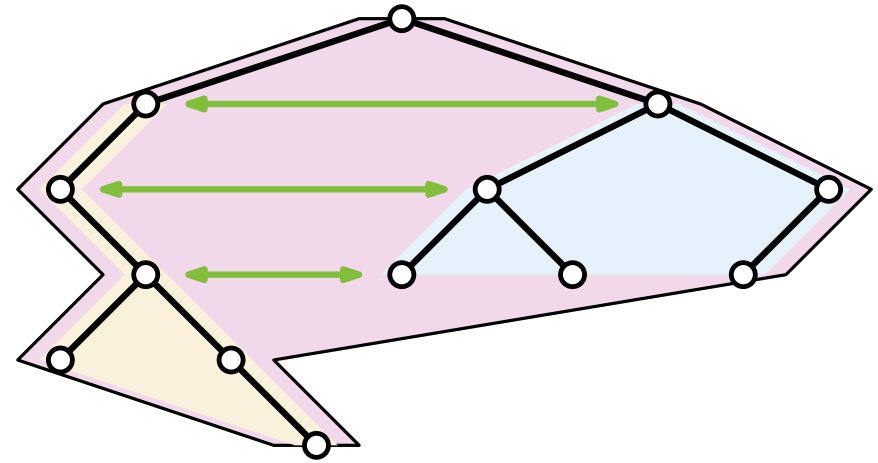

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– each node compared at most twice (once left, once right)
– sums up to $2n$

Running time: postorder $+ \sum_v \min\{\text{height}(T_l(v), \text{height}(T_r(v)\}$ $= O(n)$

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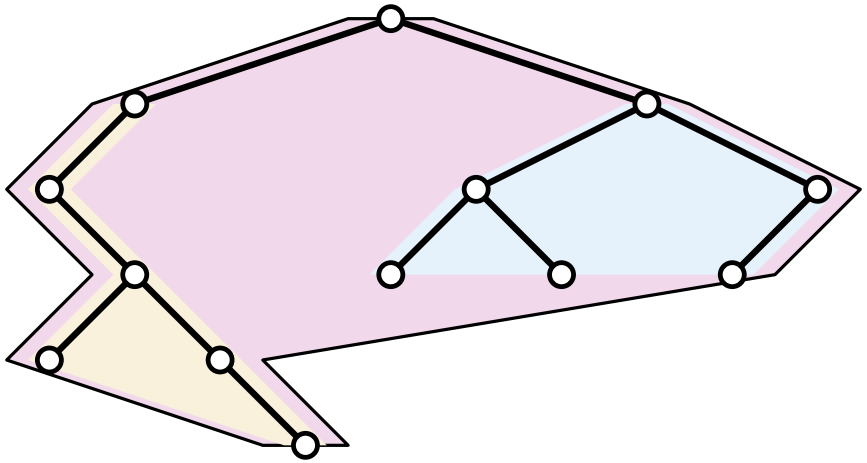

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Phase 2: set $x(\text{root}) = 0$
for all $v \in T$ in preorder
1. set $y(v) = -\text{depth}(v)$
2. set $x(lc(v)) = x(v) + \text{x-offset}(lc(v))$
3. set $x(rc(v)) = x(v) + \text{x-offset}(rc(v))$

Running time: preorder $= O(n)$

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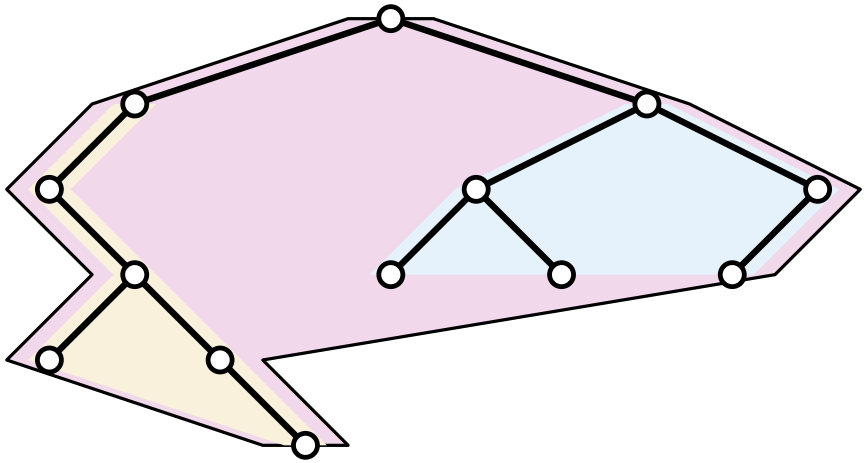

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Theorem: Given a rooted binary tree a layered, straight-line tree drawing with the following properties can be computed in $O(n)$ time:

- parents centered above children
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- isomorphic subtrees are drawn the same
- integer grid coordinates
- area is height$(T) \cdot O(n) = O(n^2)$

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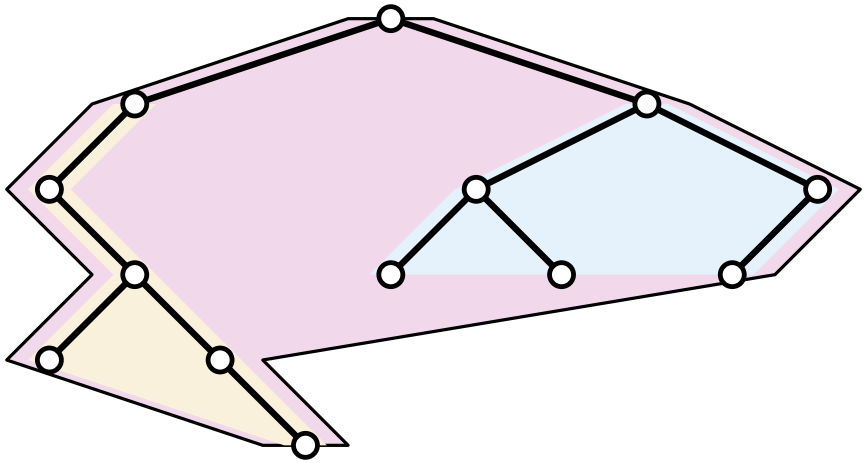

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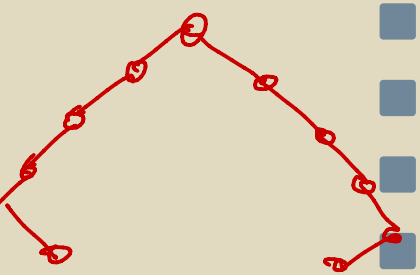
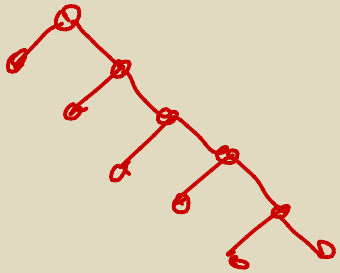
linked list of nodes on contour (left and right) with offsets to parent

Theorem: Given a rooted binary tree a layered, straight-line tree drawing with the following properties can be computed in $O(n)$ time:



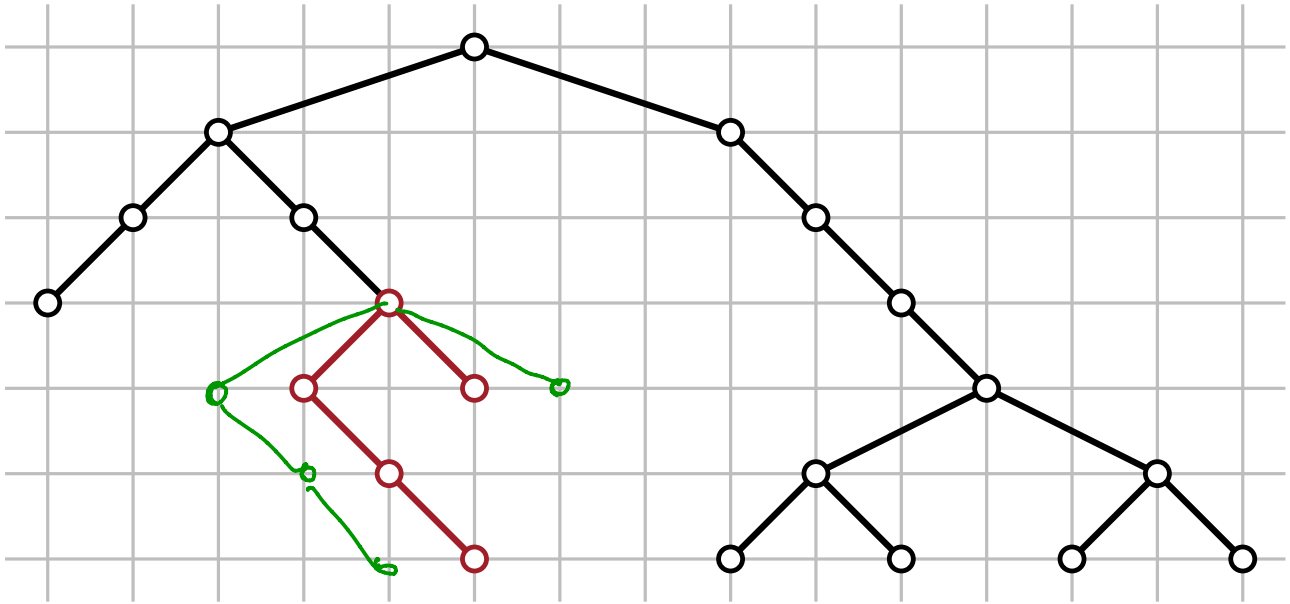
- parents centered above children
- child order preserved
- isomorphic subtrees are drawn the same
- integer grid coordinates
- area is height$(T) \cdot O(n) = O(n^2)$



What is a worst-case example for the area?

Reingold/Tilford: Problems and Extensions

Width minimization:



width 14

Reingold/Tilford: Problems and Extensions

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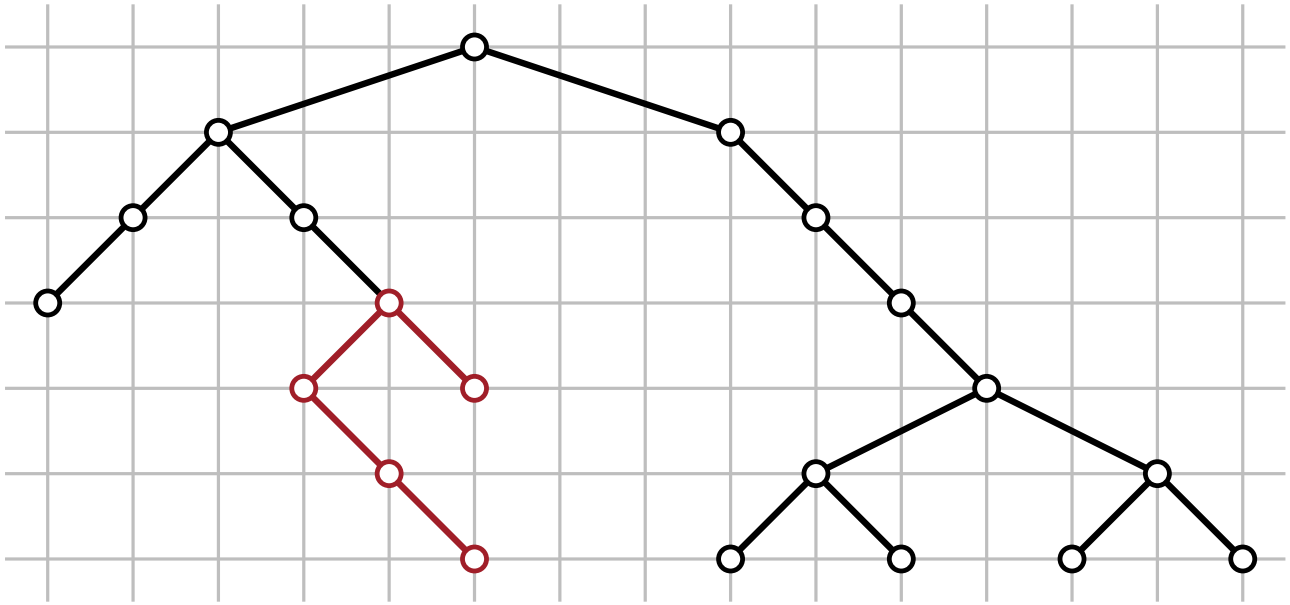

width 14

Could this be done better?

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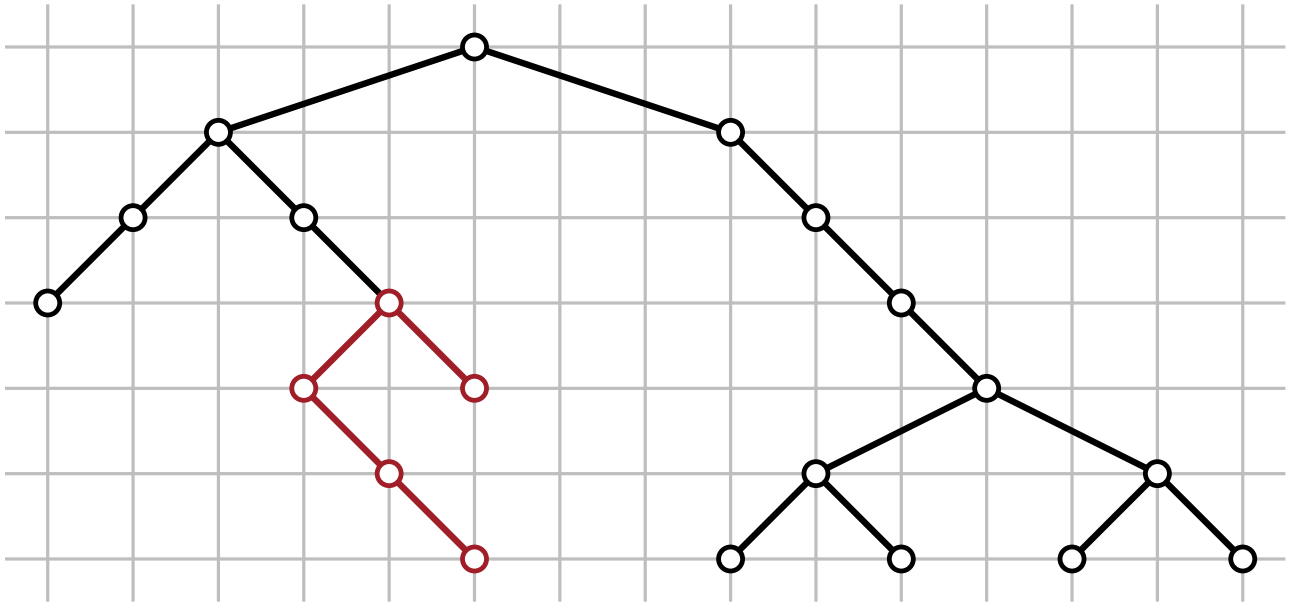

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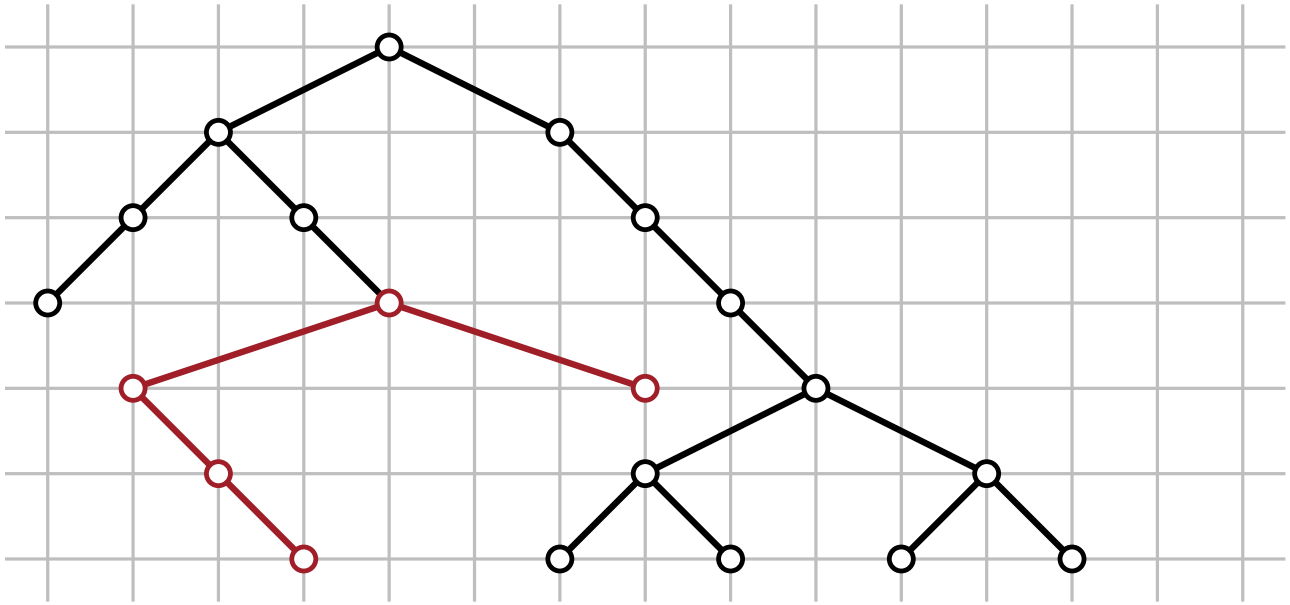

Yes, width 12

- dependencies among subtrees
- isomorphic trees drawn differently
- width minimization on grid is NP-hard!

[Supowit, Reingold '83]

Reingold/Tilford: Problems and Extensions

Width minimization: [Supowit, Reingold '83]

- recursive divide & conquer procedure unable to minimize width
- problem is NP-hard on grid

Reingold/Tilford: Problems and Extensions

Width minimization: [Supowit, Reingold '83]

- recursive divide & conquer procedure unable to minimize width
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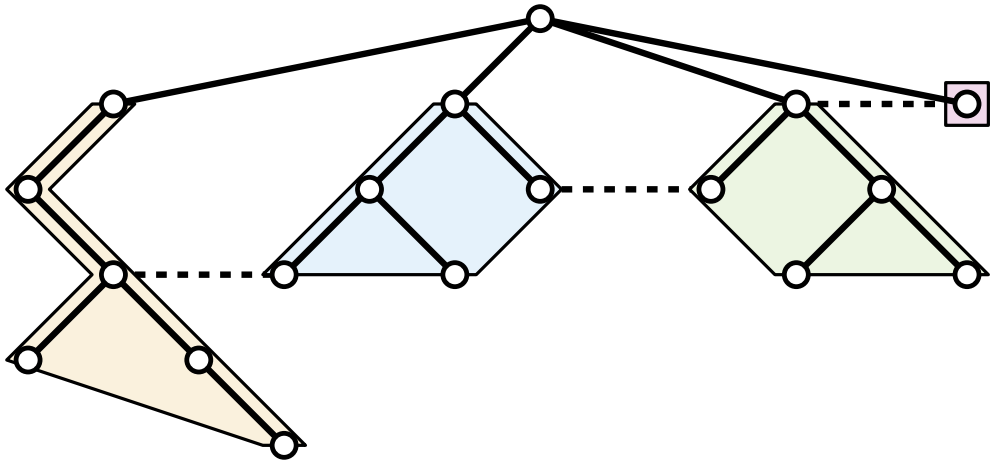
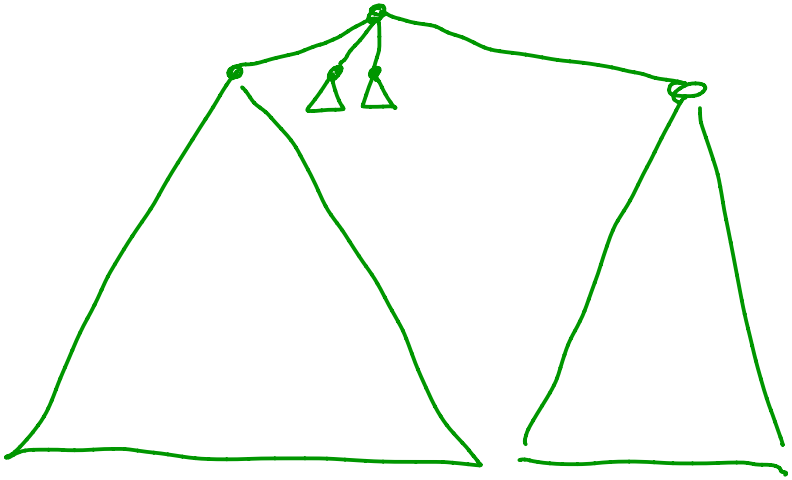
Non-binary Trees: [Walker '90], [Buchheim et al. '02]

- recursive algorithm easily extends
- additional post-processing to improve horizontal balancing
- remains $O(n)$ time





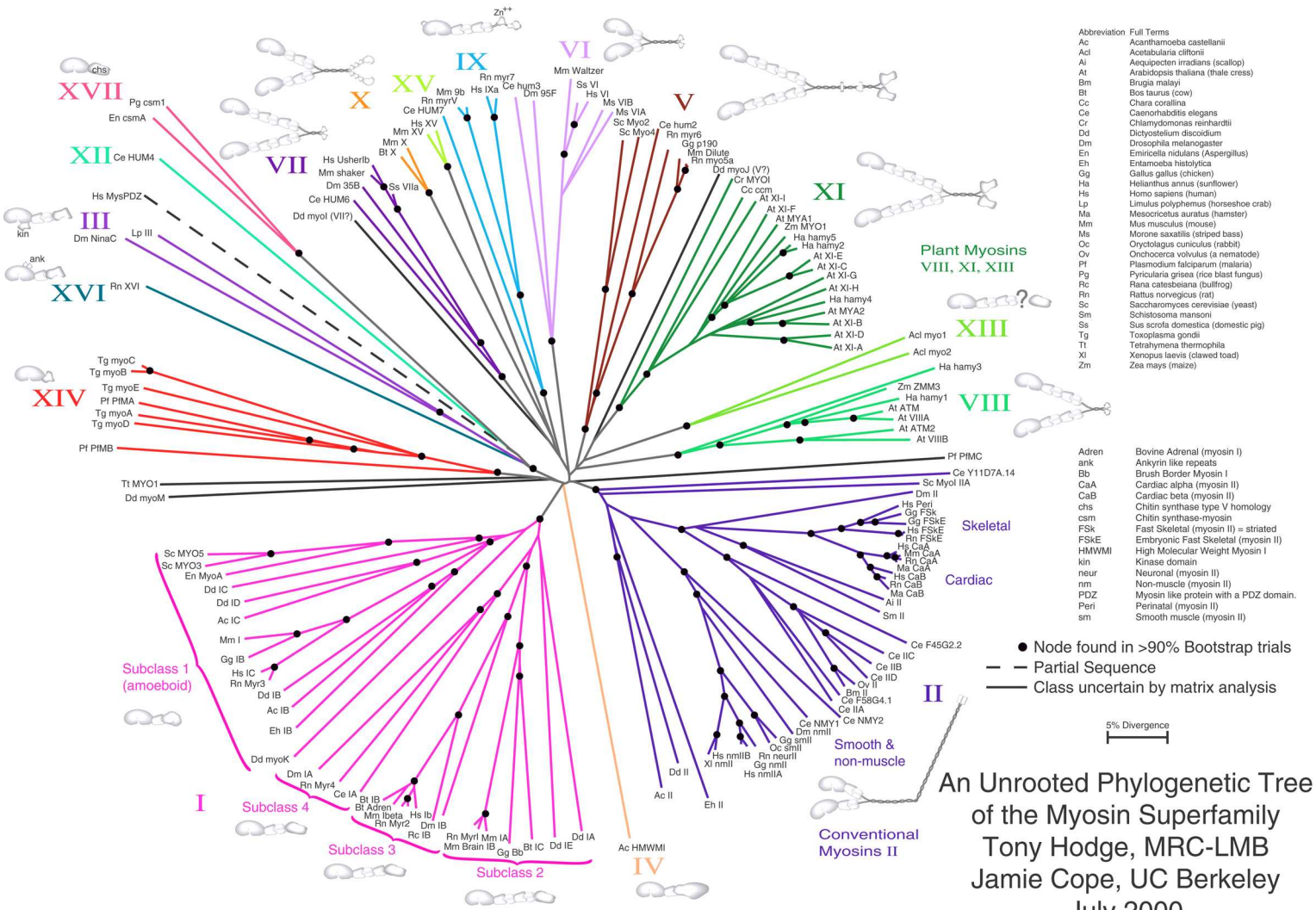
Overview

Layered drawings of rooted trees

Radial tree drawings

Compact orthogonal drawings of rooted trees

Radial Tree Drawings: Examples

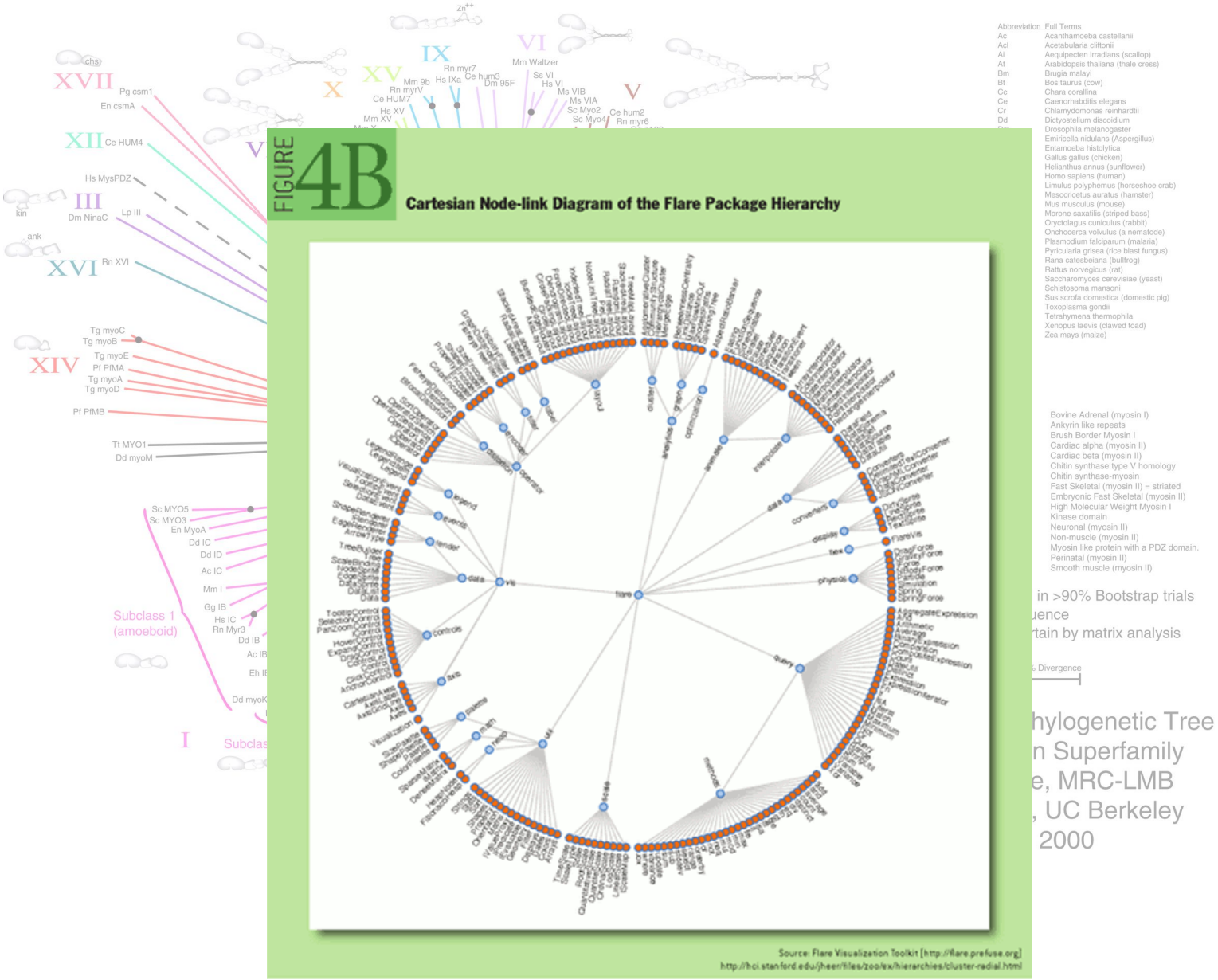


Abbreviation	Full Terms
Ac	Acanthamoeba castellanii
Acl	Acetabularia clifftonii
Ai	Aequipecten irradians (scallop)
At	Arabidopsis thaliana (thale cress)
Bm	Brugia malayi
Bt	Bos taurus (cow)
Cc	Chara corallina
Ce	Caenorhabditis elegans
Cr	Chlamydomonas reinhardtii
Dd	Dictyostelium discoidium
Dm	Drosophila melanogaster
En	Emericella nidulans (Aspergillus)
Eh	Entamoeba histolytica
Gg	Gallus gallus (chicken)
Ha	Helianthus annus (sunflower)
Hs	Homo sapiens (human)
Lp	Limulus polyphemus (horseshoe crab)
Ma	Mesocricetus auratus (hamster)
Mm	Mus musculus (mouse)
Ms	Morone saxatilis (striped bass)
Oc	Oryctolagus cuniculus (rabbit)
Ov	Onchocerca volvulus (a nematode)
Pf	Plasmodium falciparum (malaria)
Pg	Pyricularia grisea (rice blast fungus)
Rc	Rana catesbeiana (bullfrog)
Rn	Rattus norvegicus (rat)
Sc	Saccharomyces cerevisiae (yeast)
Sm	Schistosoma mansoni
Ss	Sus scrofa domestica (domestic pig)
Tg	Toxoplasma gondii
Tt	Tetrahymena thermophila
Xl	Xenopus laevis (clawed toad)
Zm	Zea mays (maize)

Adren	Bovine Adrenal (myosin I)
ank	Ankyrin like repeats
Bb	Brush Border Myosin I
CaA	Cardiac alpha (myosin II)
CaB	Cardiac beta (myosin II)
chs	Chitin synthase type V homology
csm	Chitin synthase-myosin
FSk	Fast Skeletal (myosin II) = striated
FSkE	Embryonic Fast Skeletal (myosin II)
HMWMI	High Molecular Weight Myosin I
kin	Kinase domain
neur	Neuronal (myosin II)
nm	Non-muscle (myosin II)
PDZ	Myosin like protein with a PDZ domain.
Peri	Perinatal (myosin II)
sm	Smooth muscle (myosin II)

An Unrooted Phylogenetic Tree
of the Myosin Superfamily
Tony Hodge, MRC-LMB
Jamie Cope, UC Berkeley
July 2000

Radial Tree Drawings: Examples



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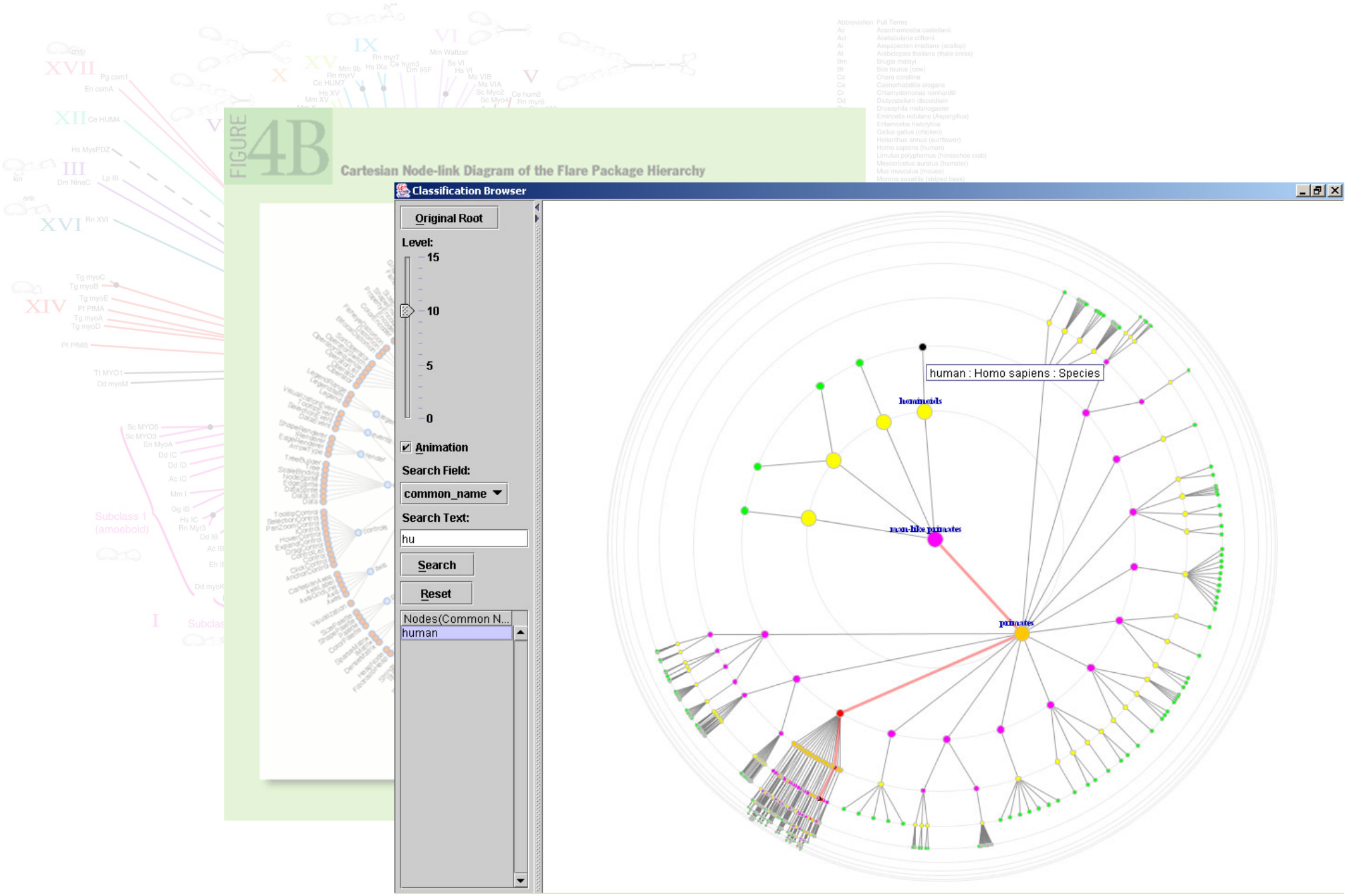
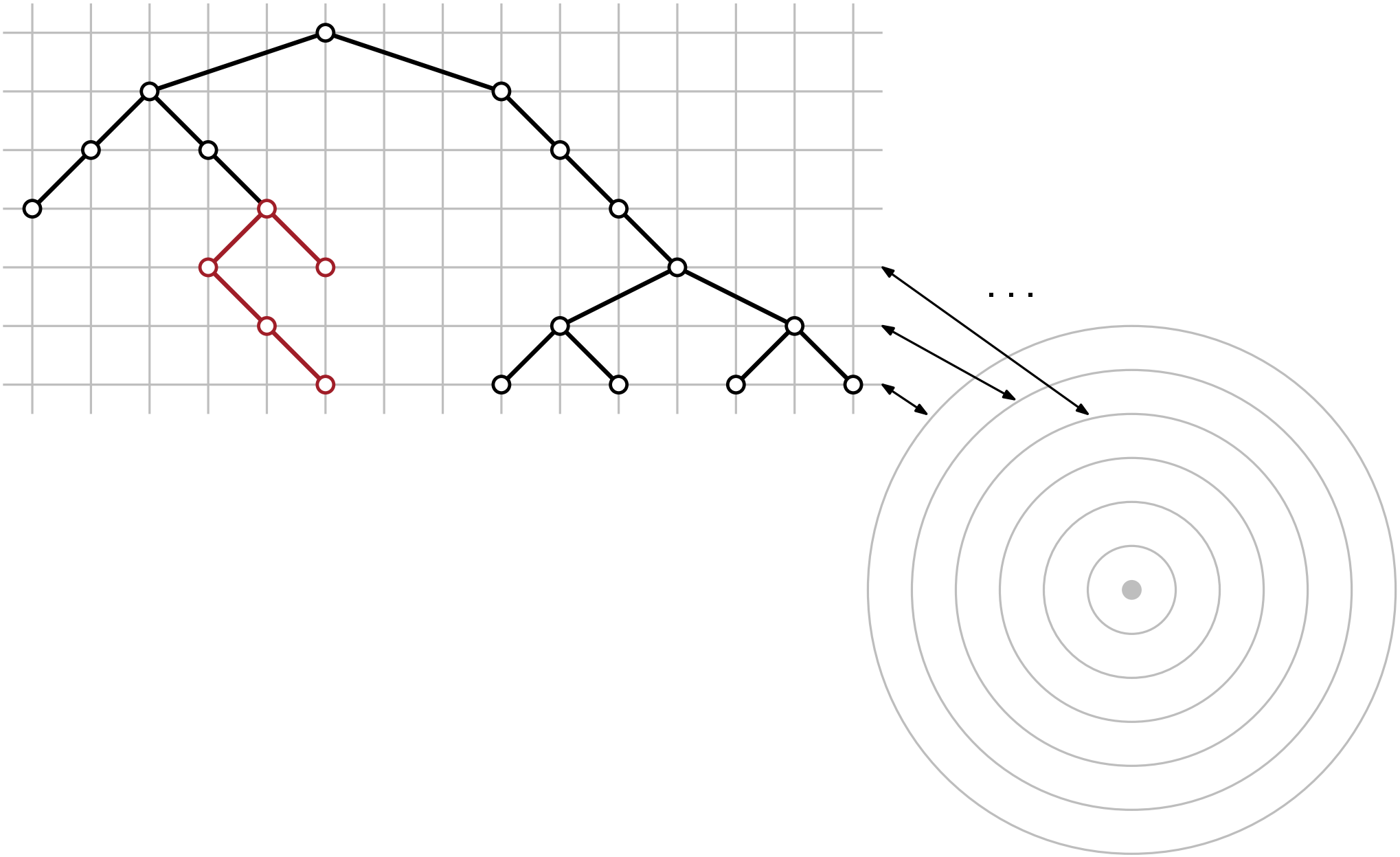


Radial Tree Drawings

Can we put layered drawings on rings?



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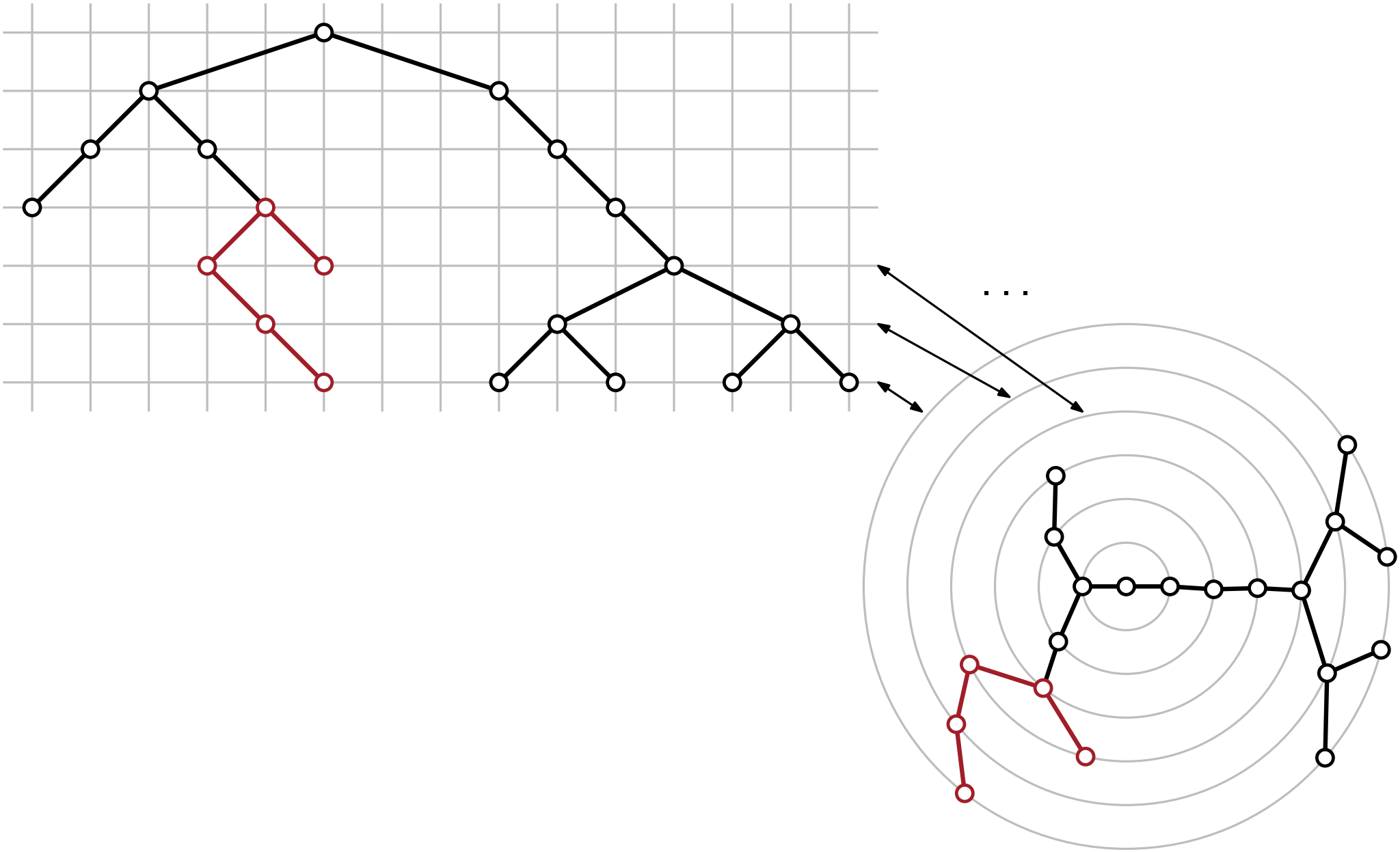

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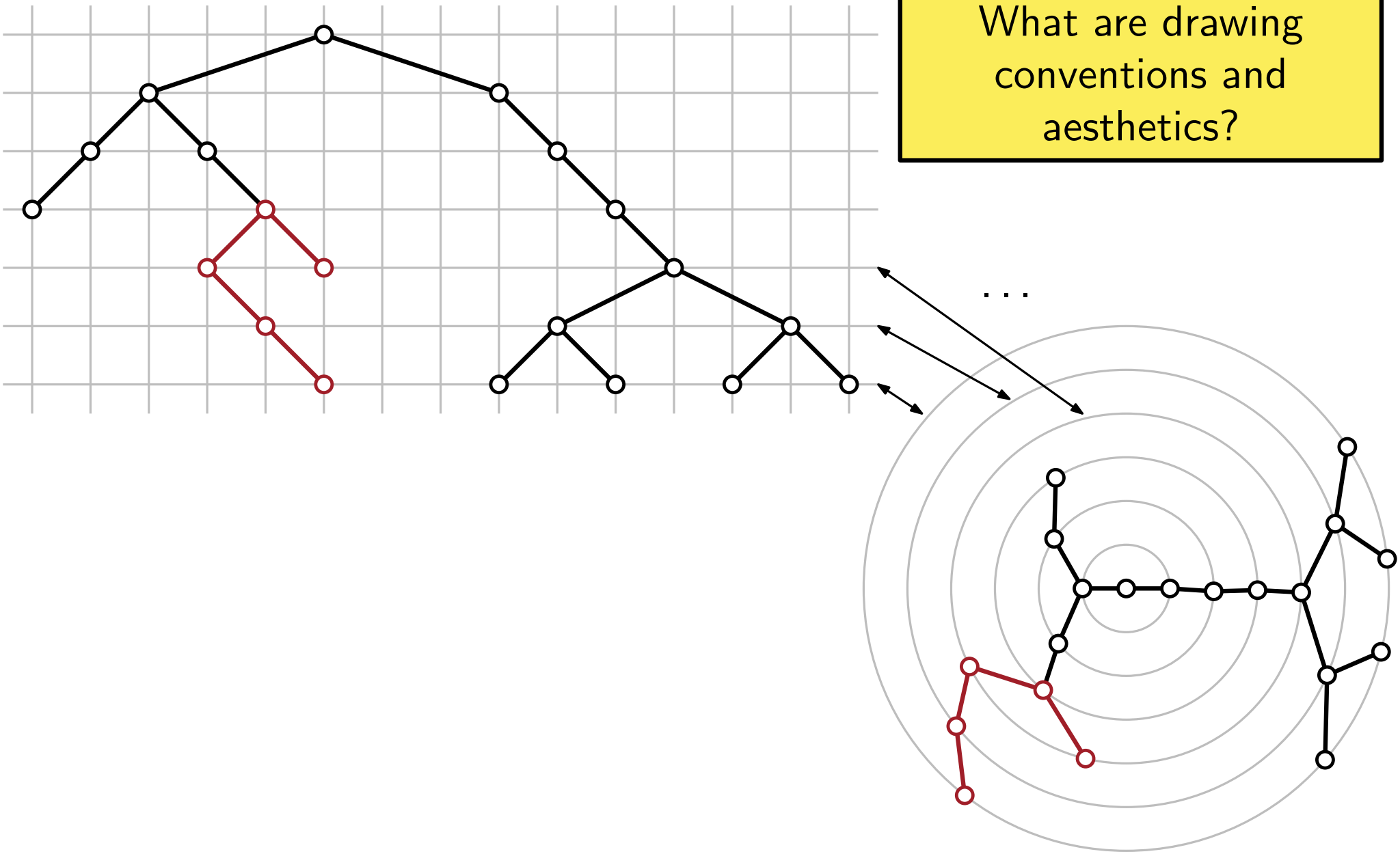

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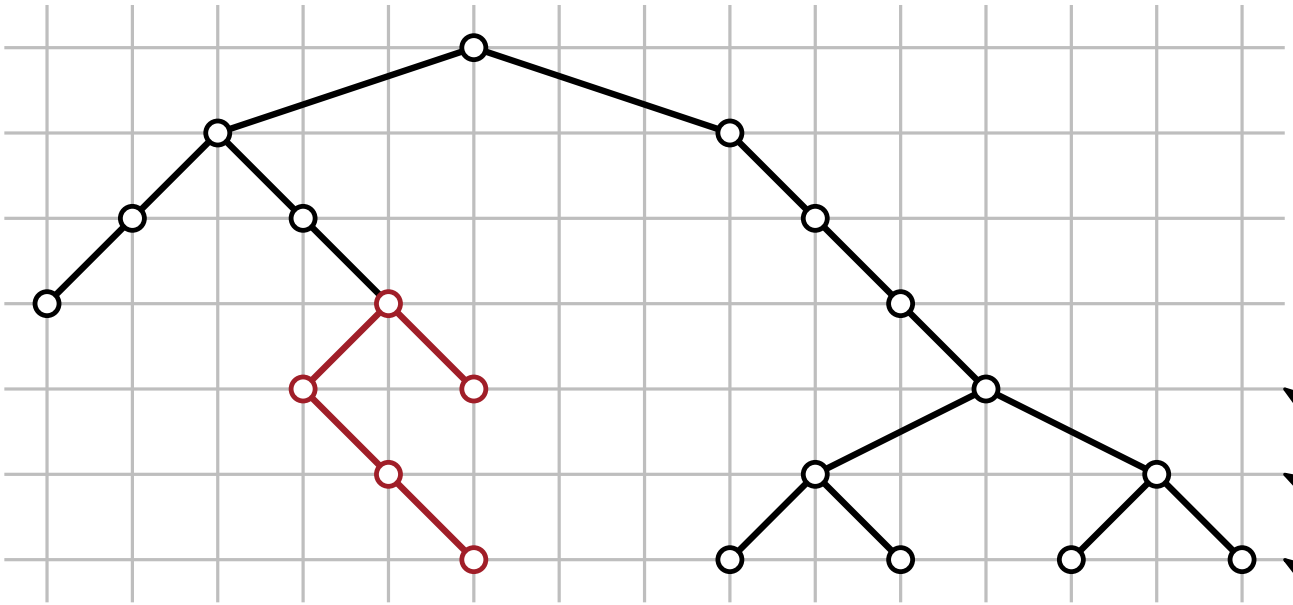
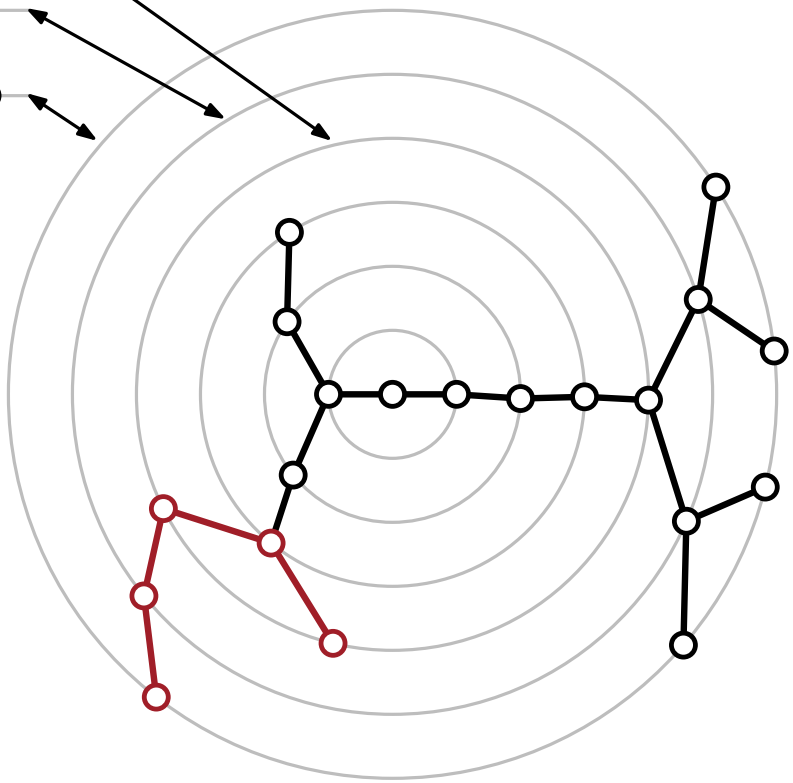


What are drawing conventions and aesthetics?



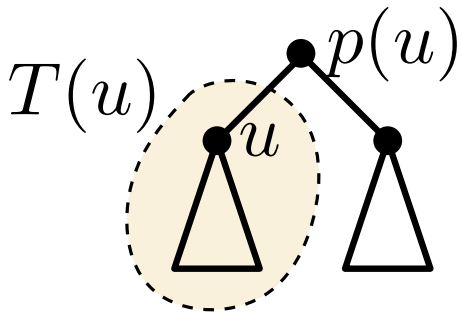
- layers are concentric circles
- straight-line edges
- no crossings
- balanced distribution of nodes
- often used for unrooted trees (pick central node as root)

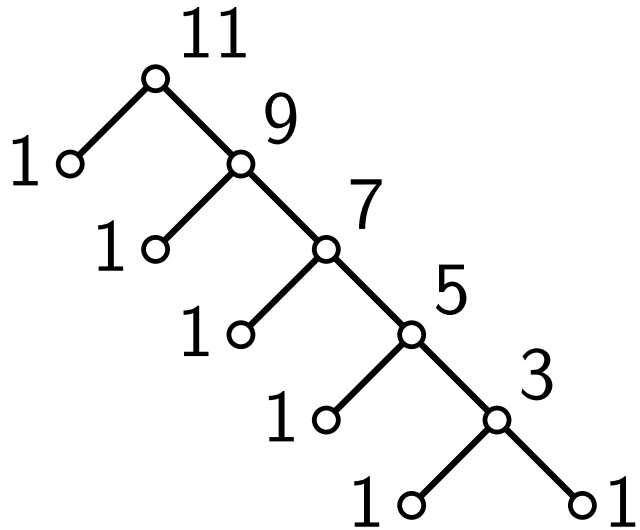
Radial Tree Drawings: Planarity

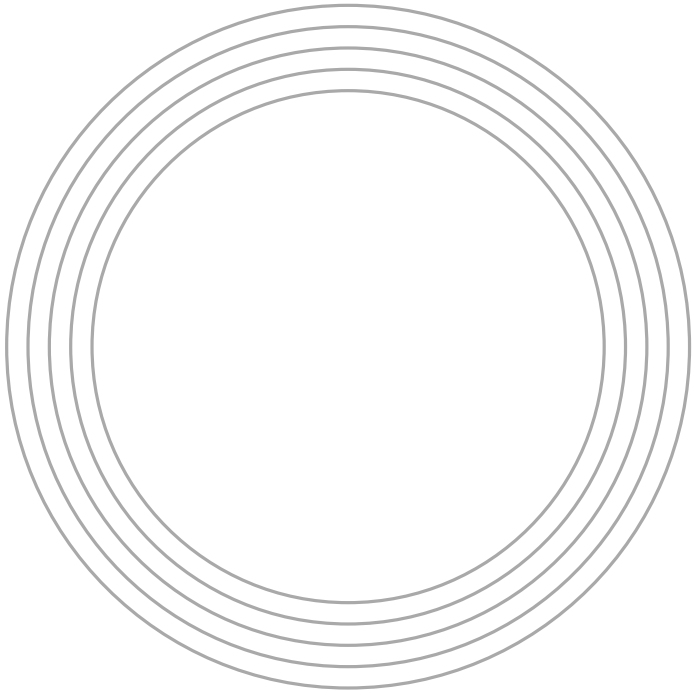
Distribute subtrees based on their node count (or leaf count)

Example: wedge angle τ_u for the subtree $T(u)$:

$$\tau_u = \frac{|T(u)|}{|T(p(u))|-1} \cdot \tau_{p(u)}; \text{ for root } r\text{: } \tau_r = 2\pi$$







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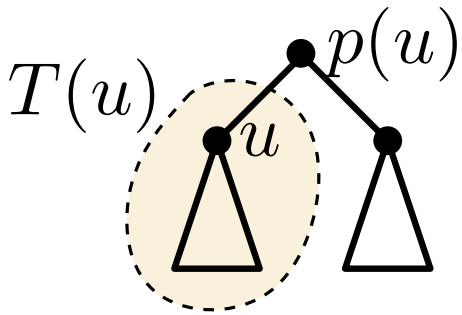

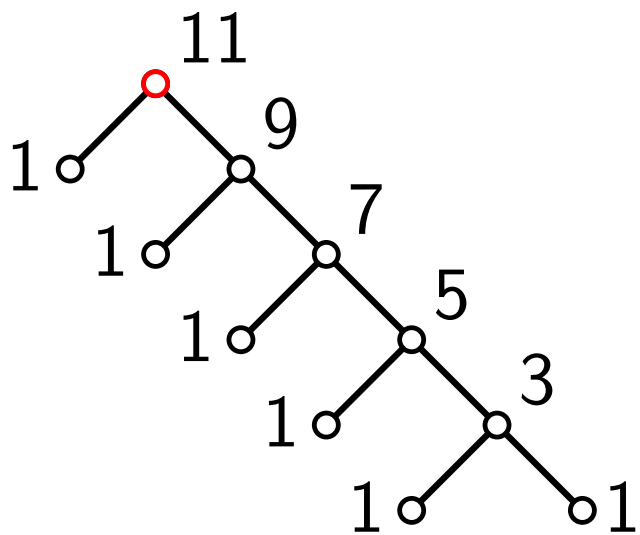

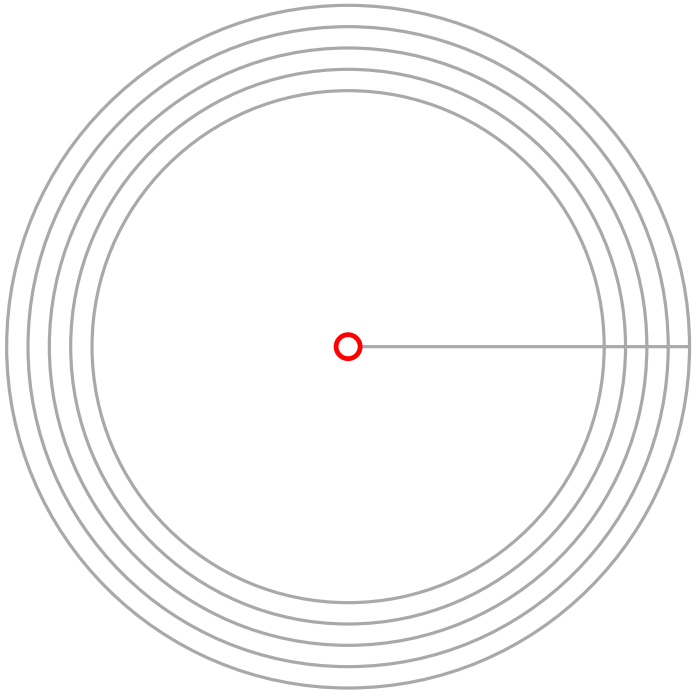

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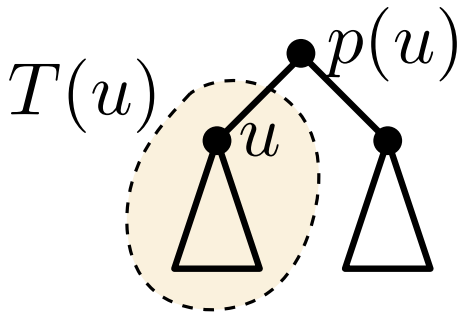

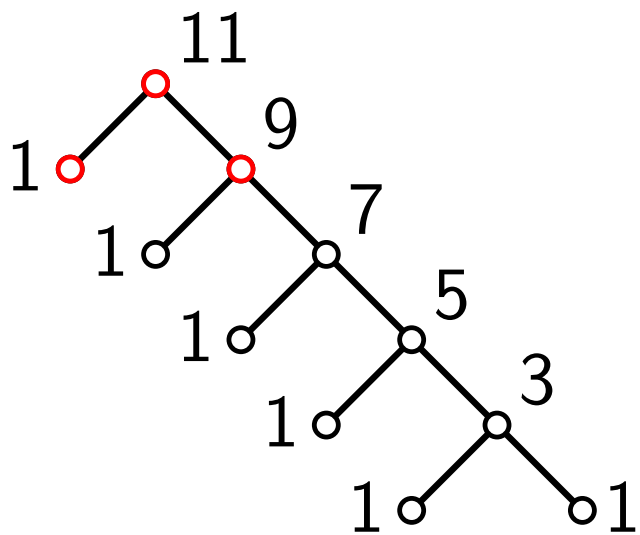

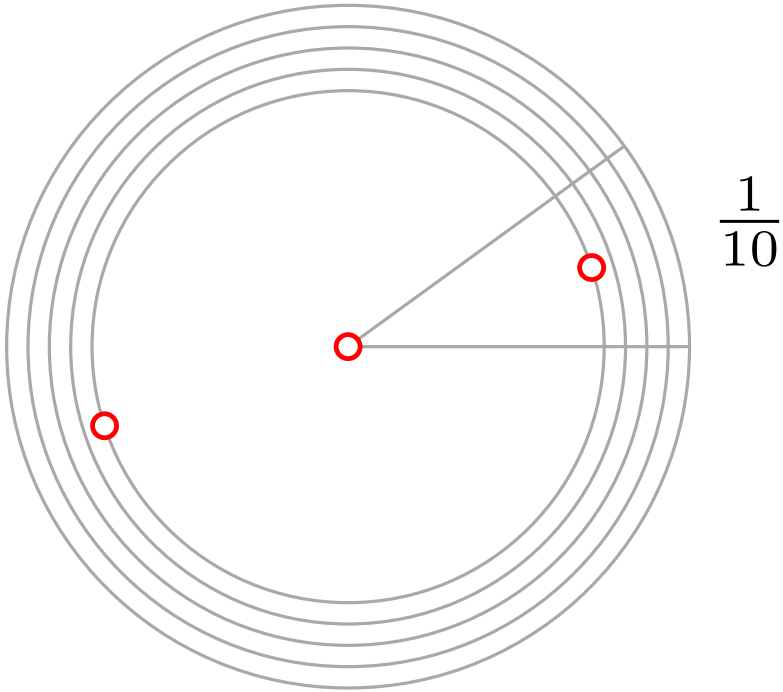

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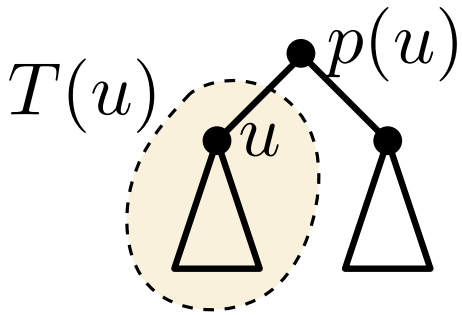

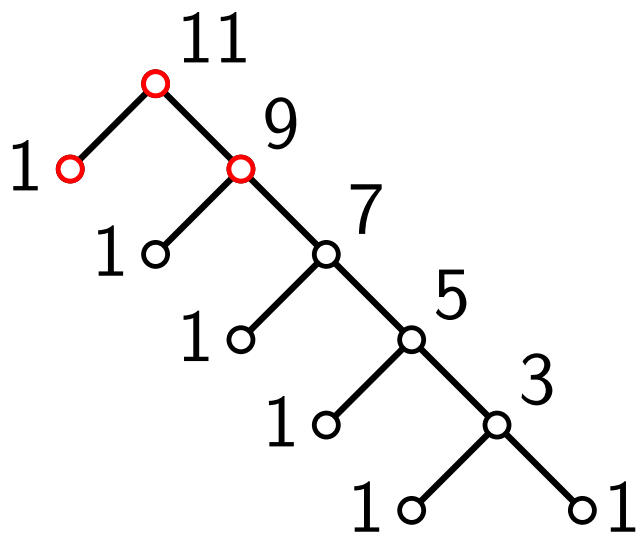

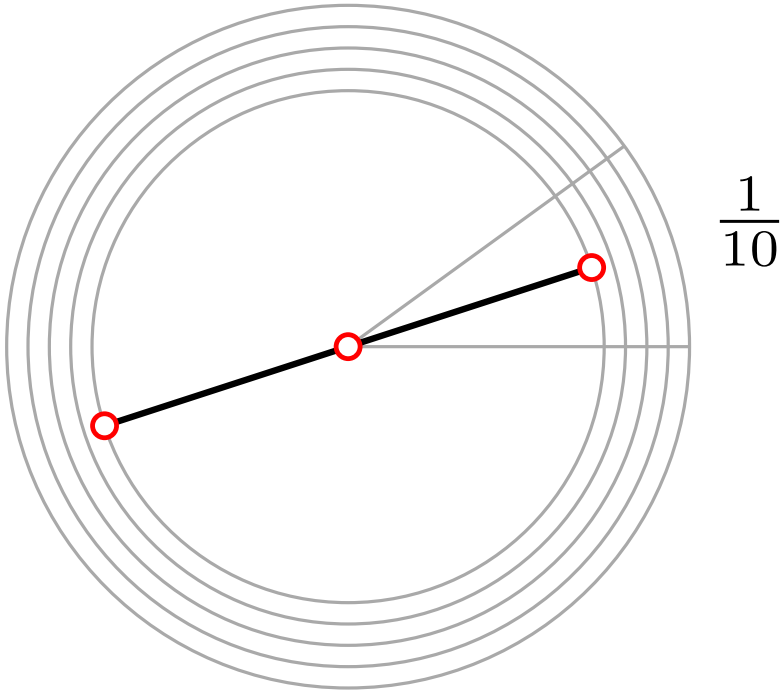

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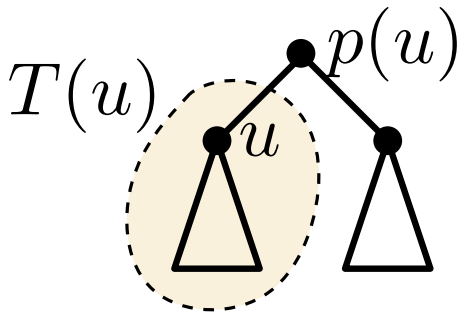

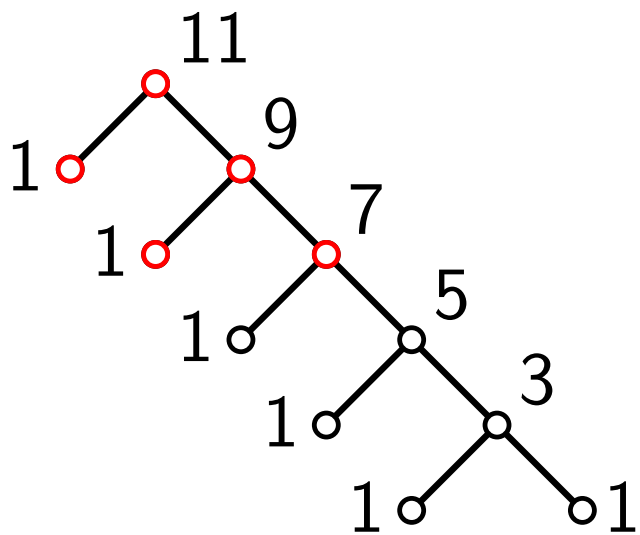

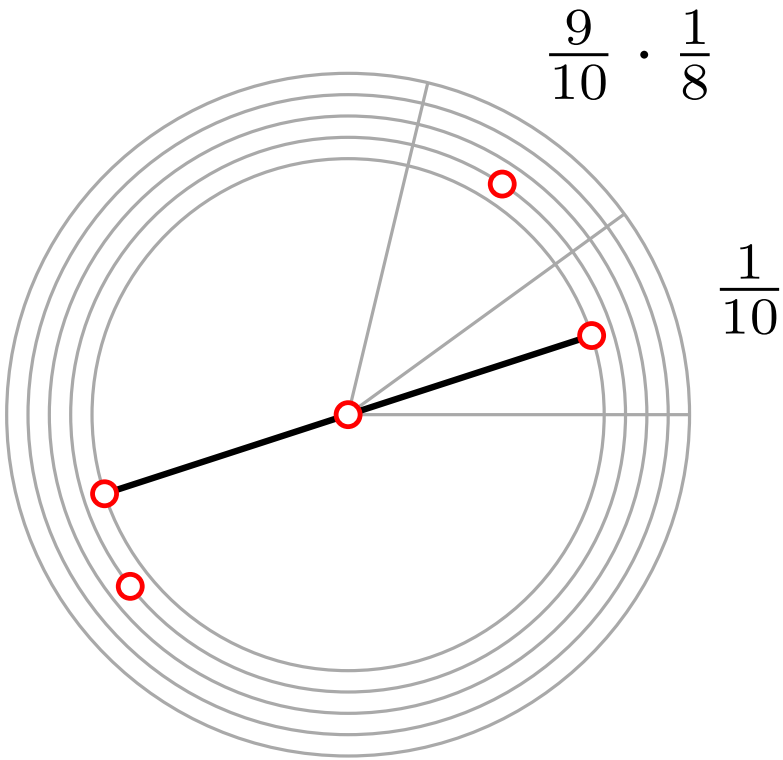

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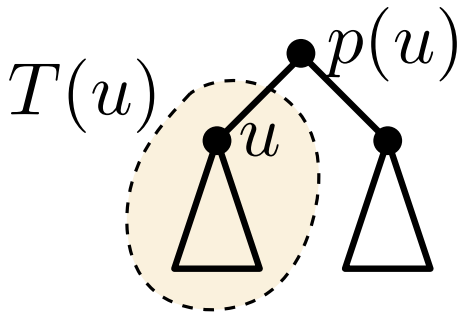

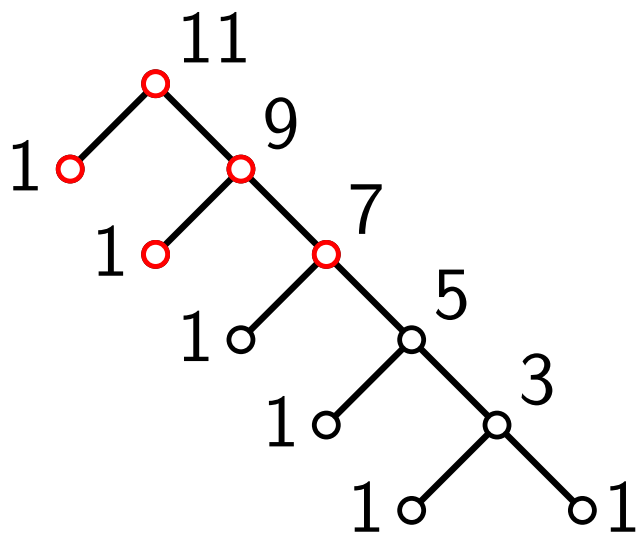

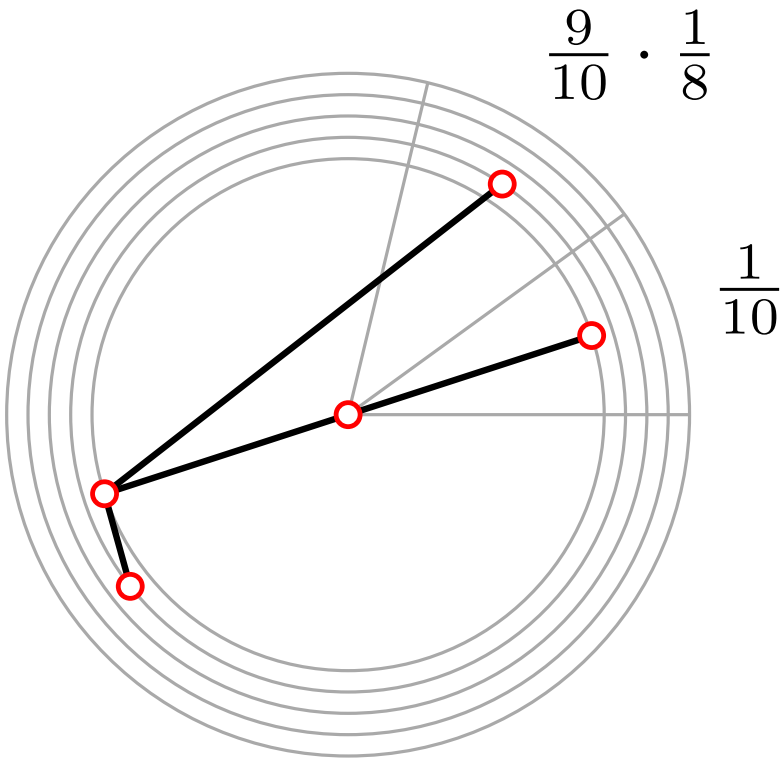

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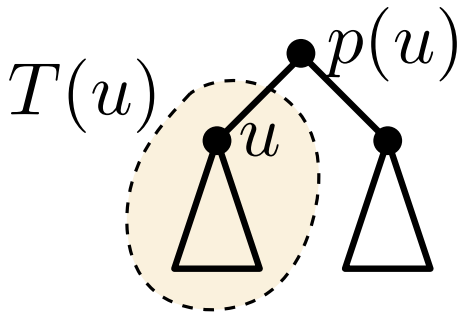

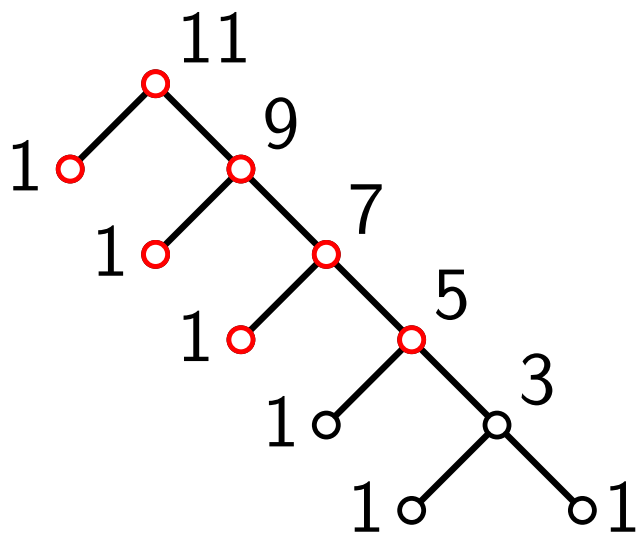

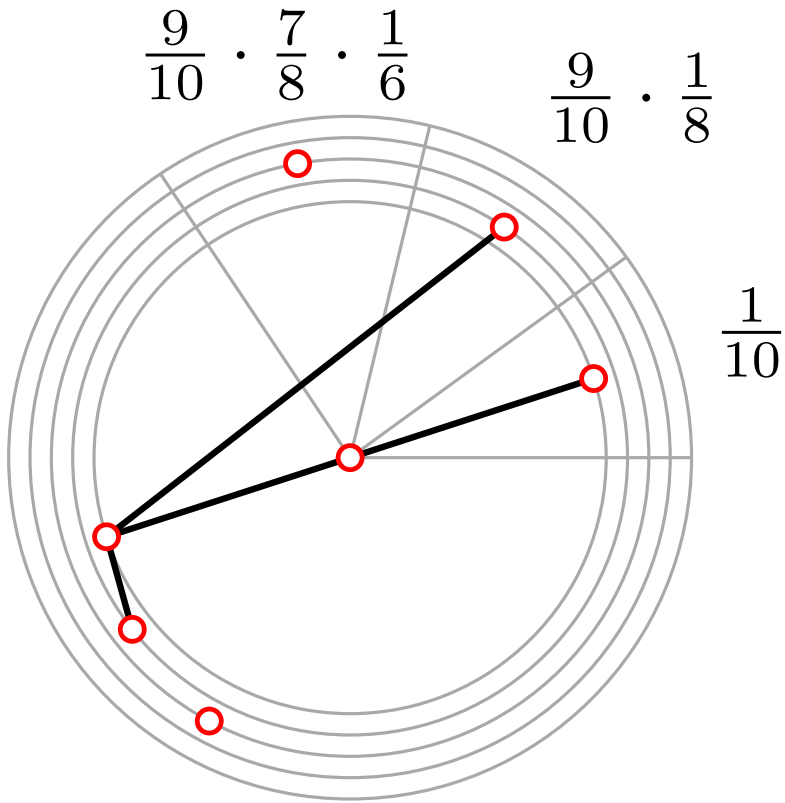

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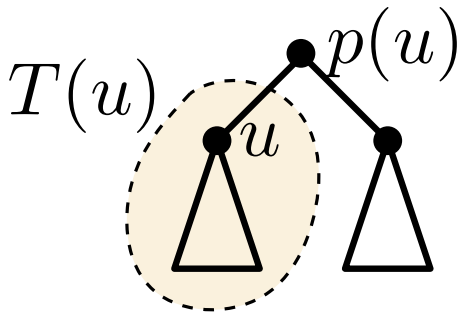
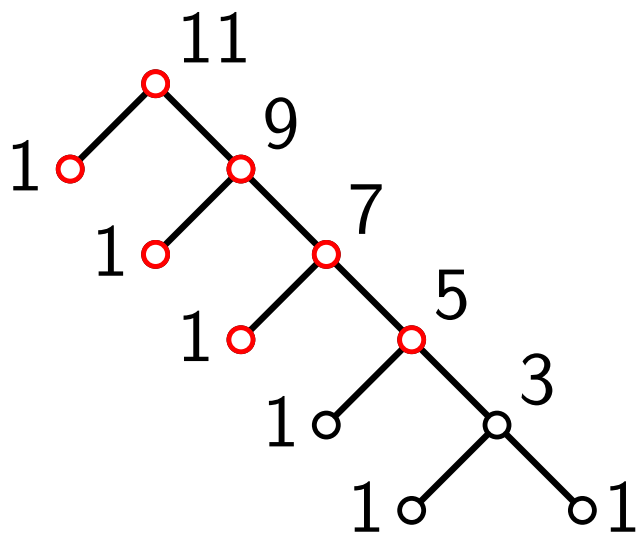
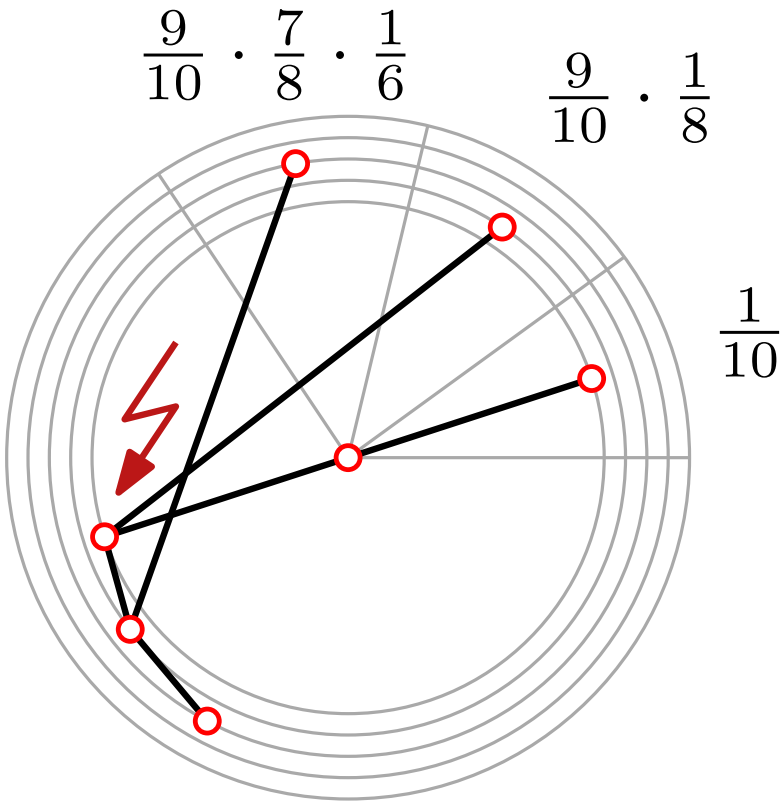
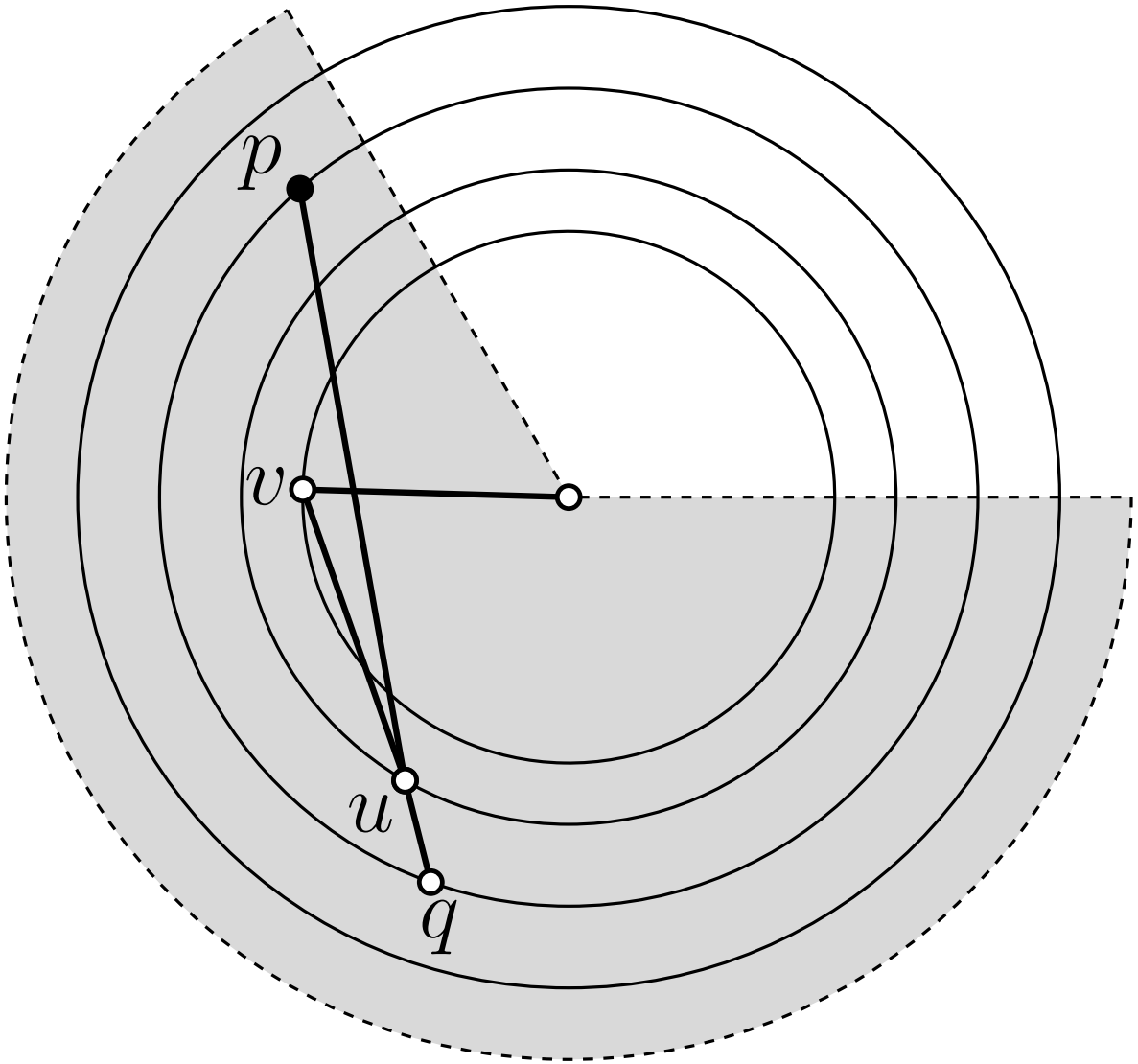






Radial Tree Drawings: Avoid Crossings



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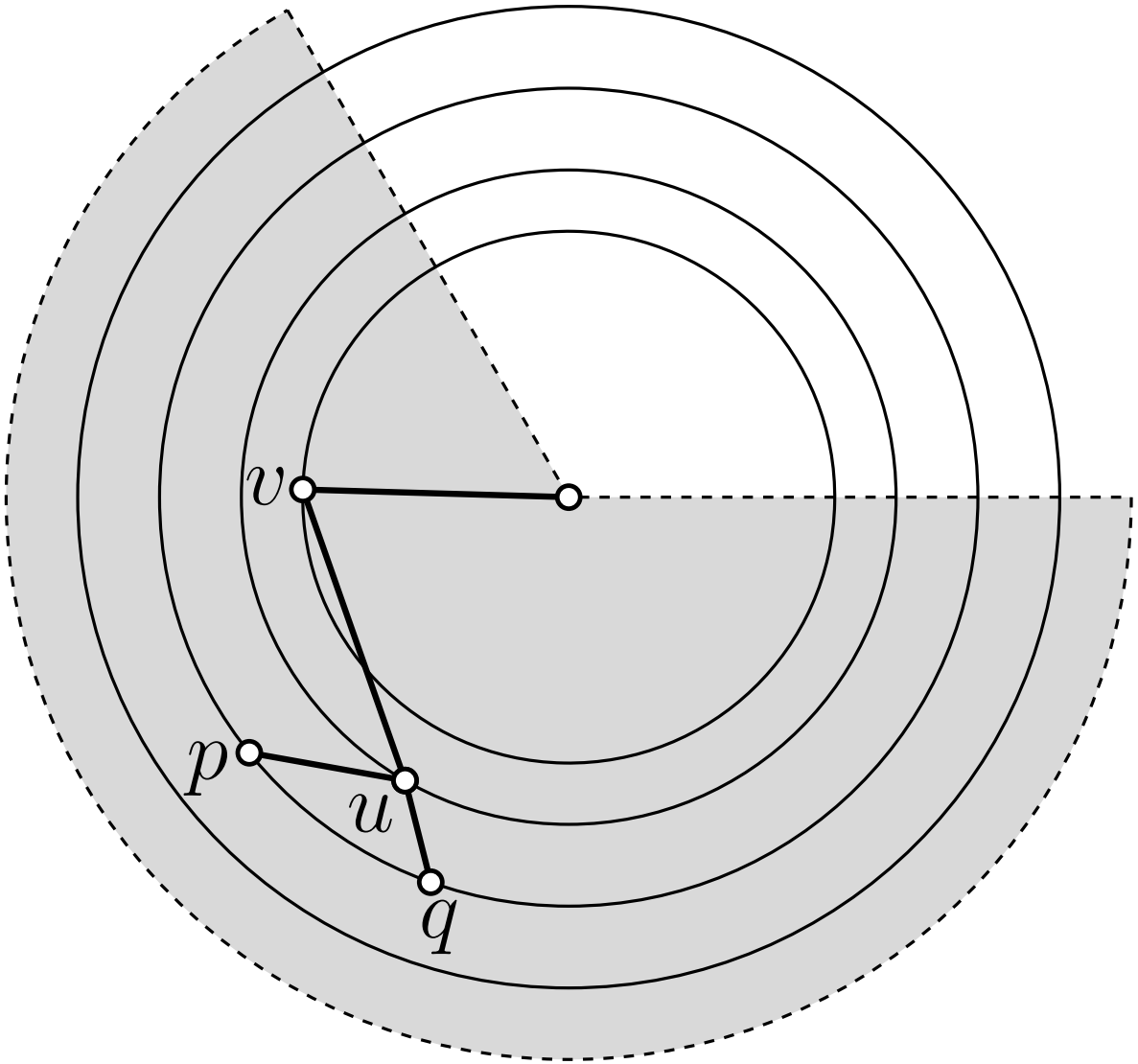

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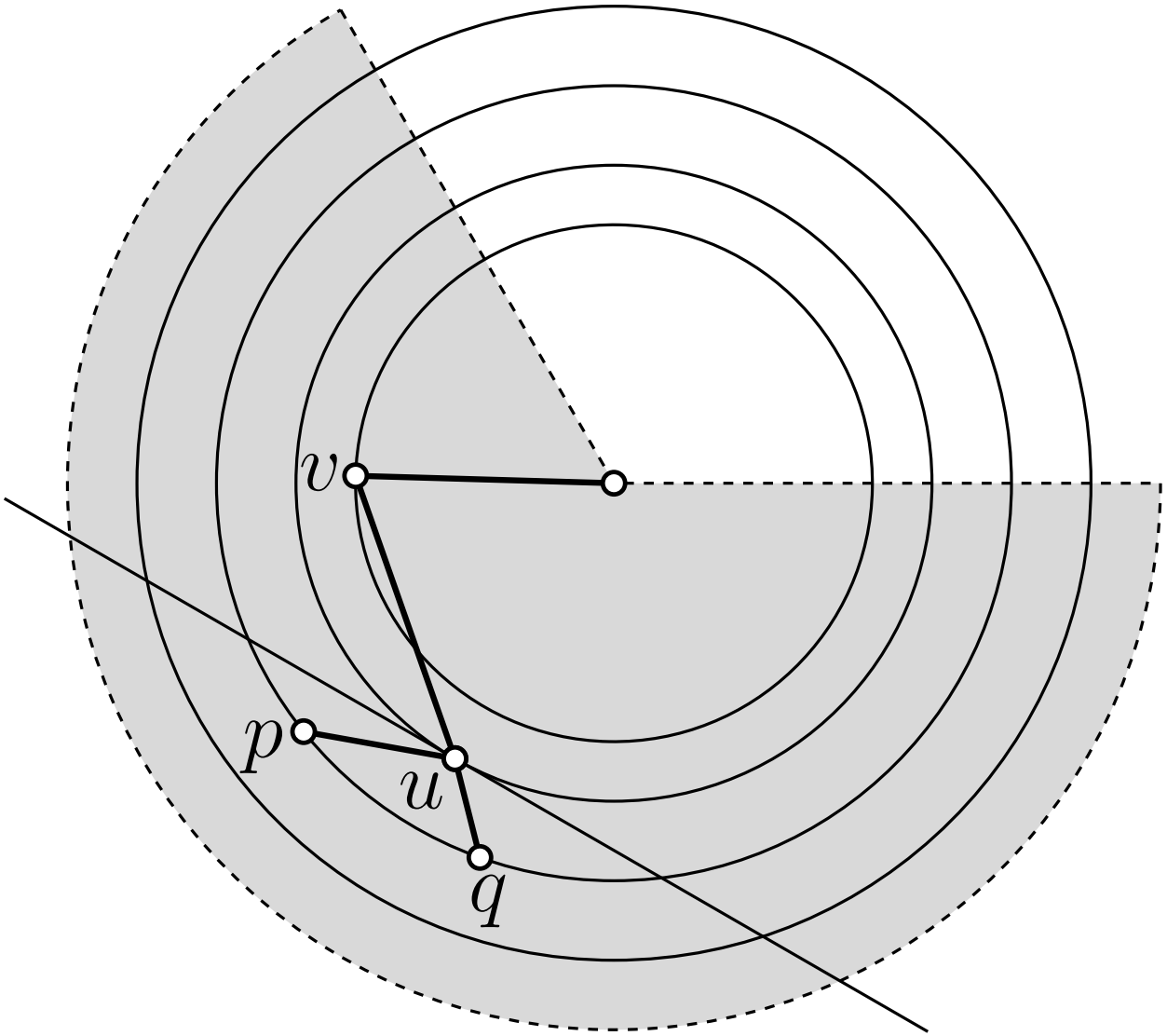

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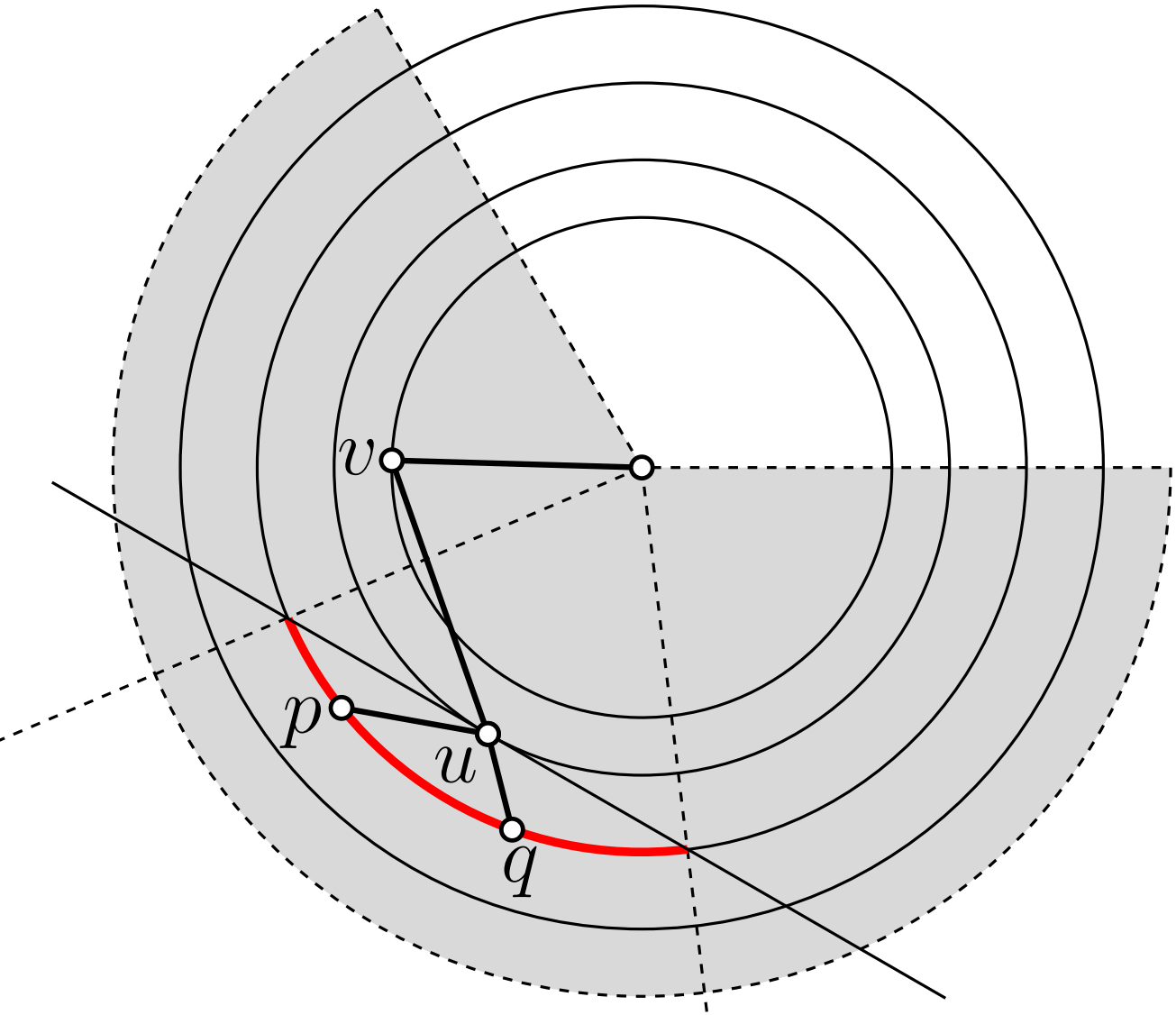

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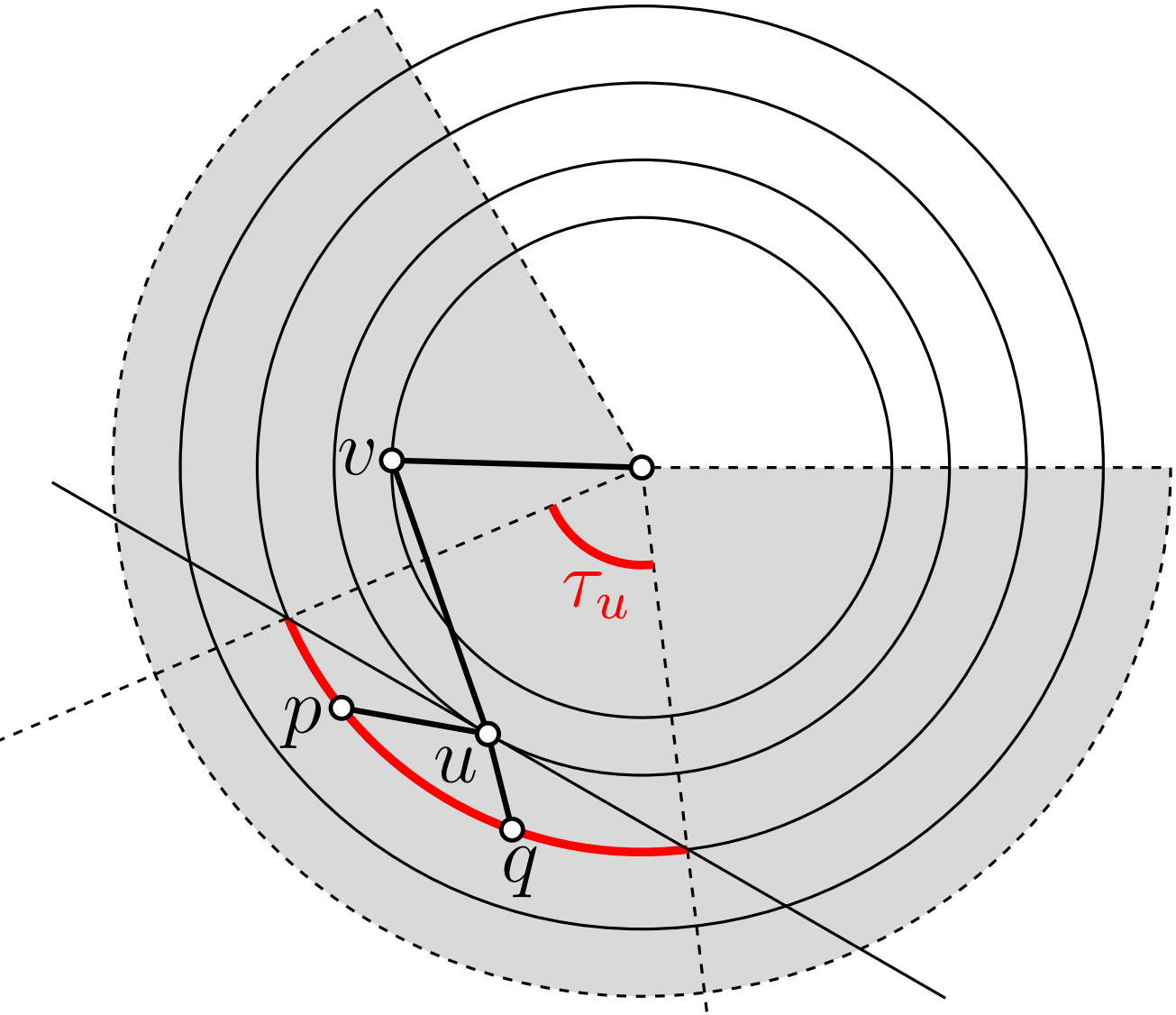

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- τ_u: wedge angle for $T(u)$
- ρ_i: radius of layer i
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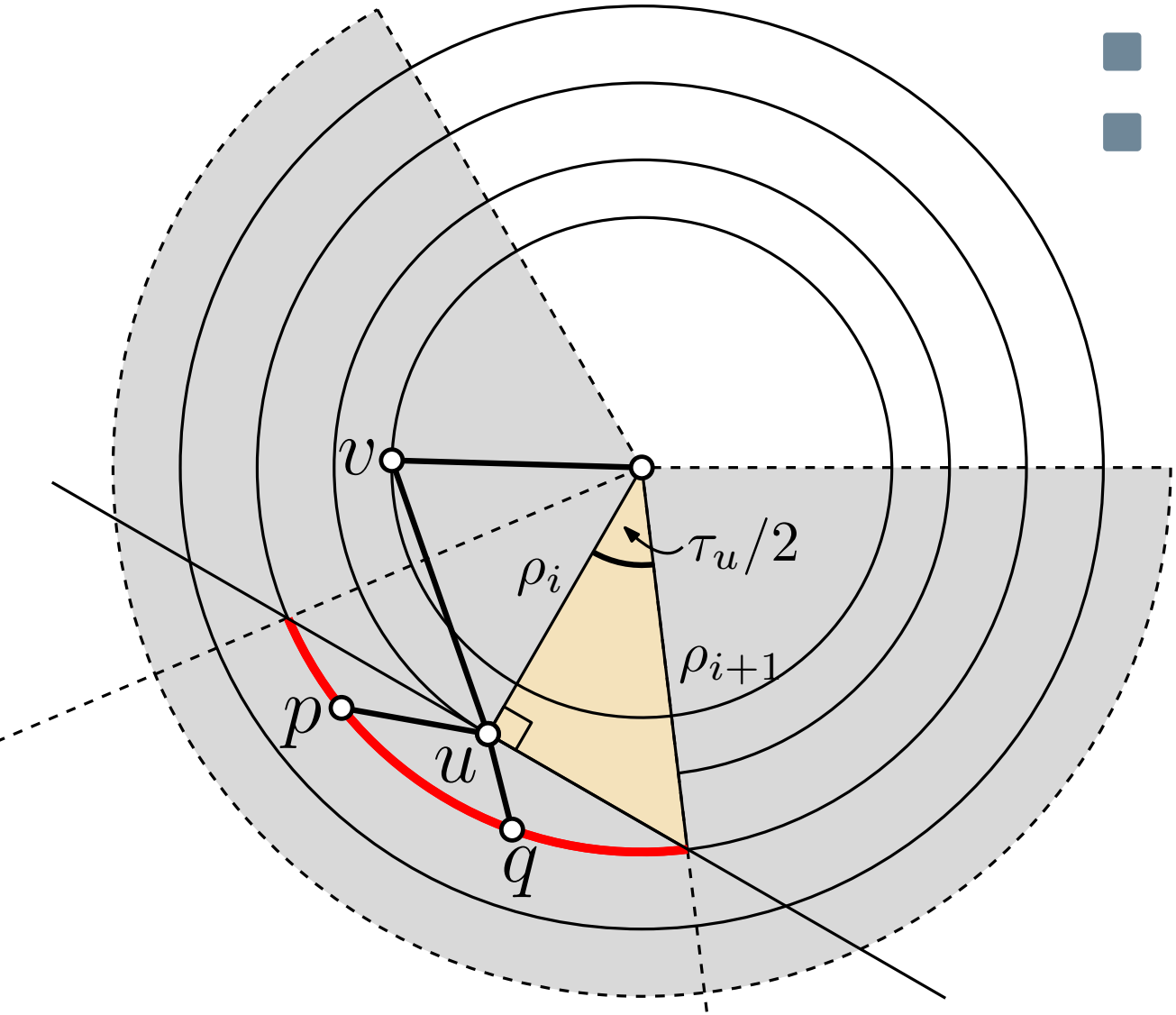

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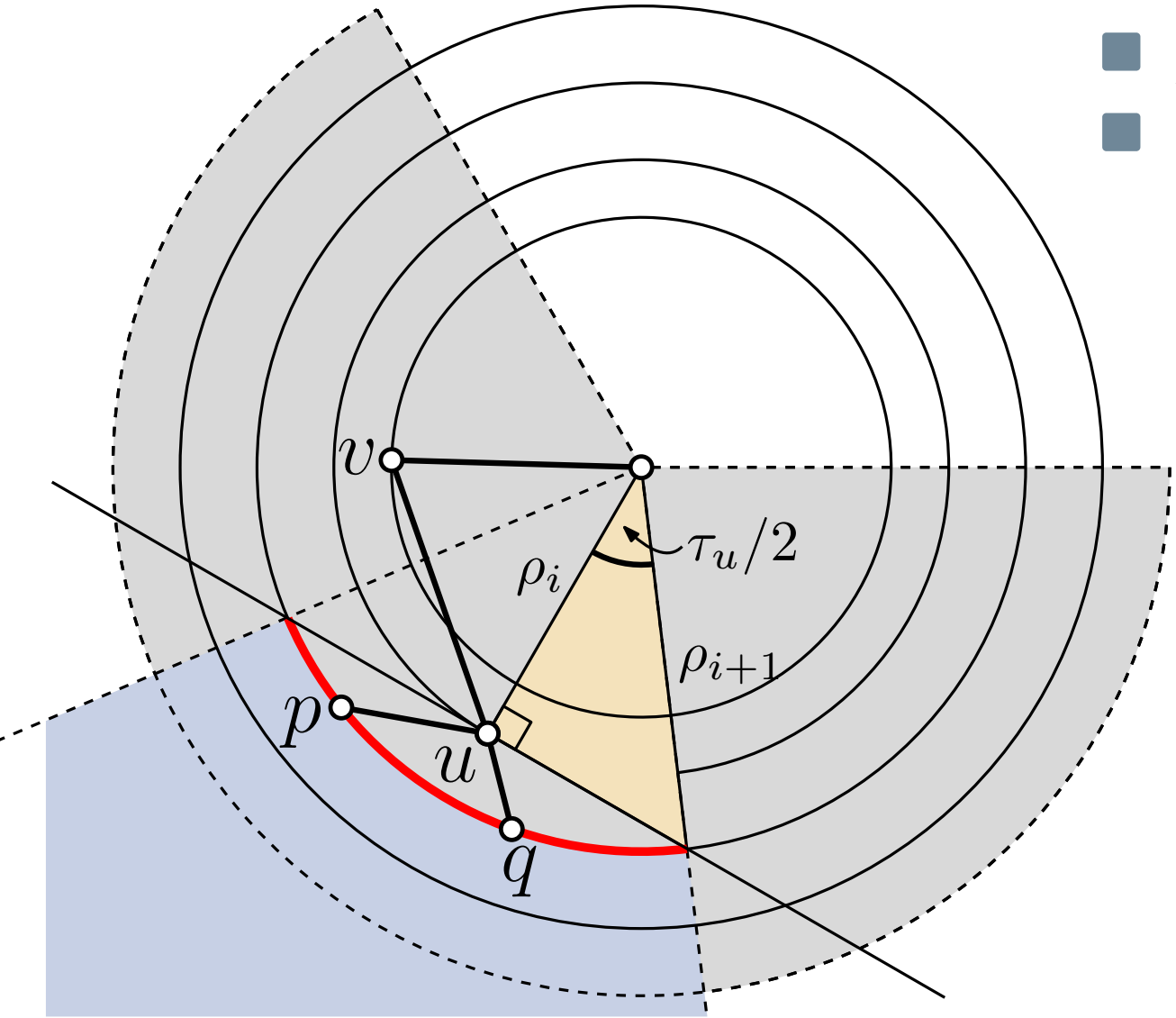

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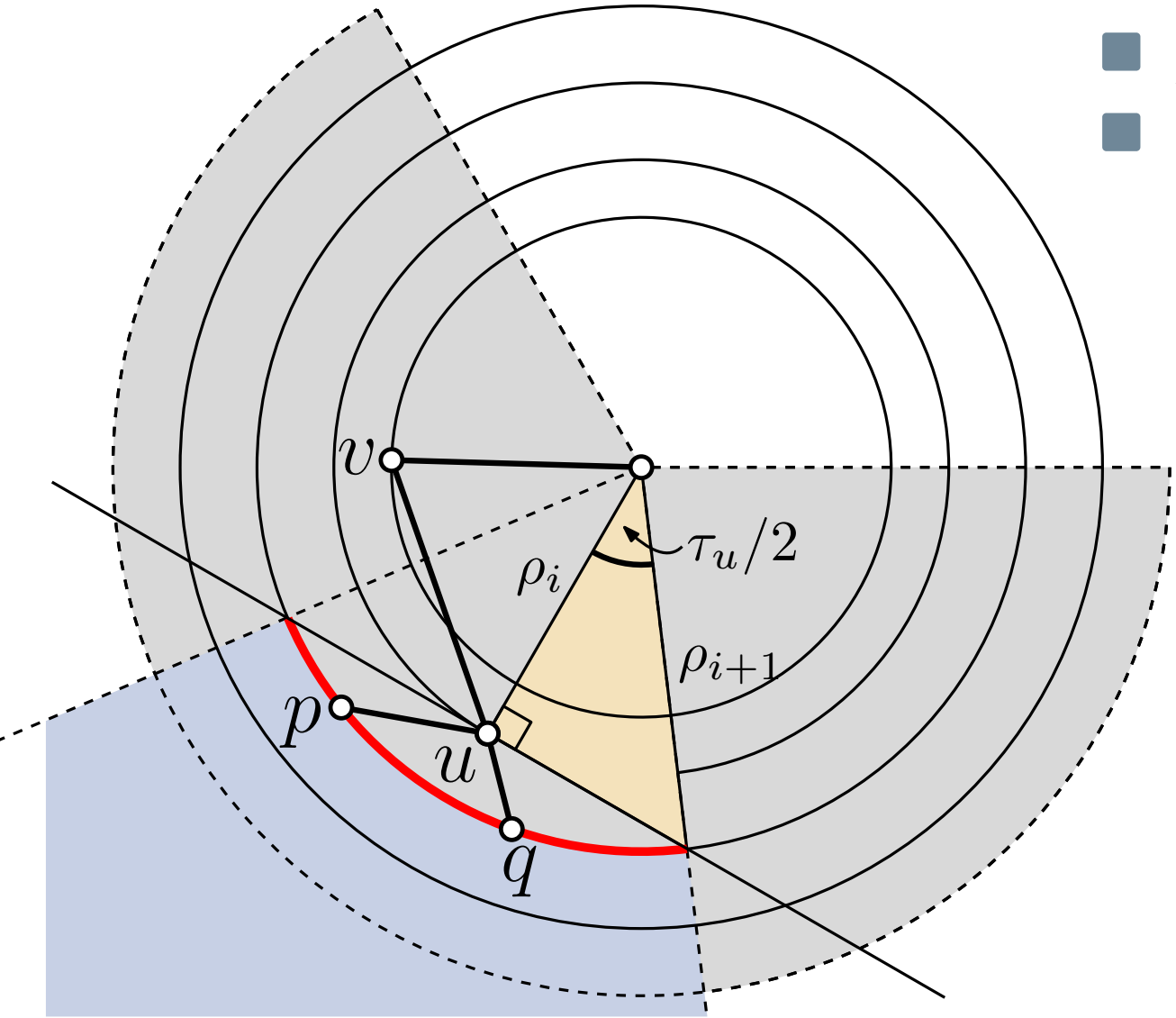

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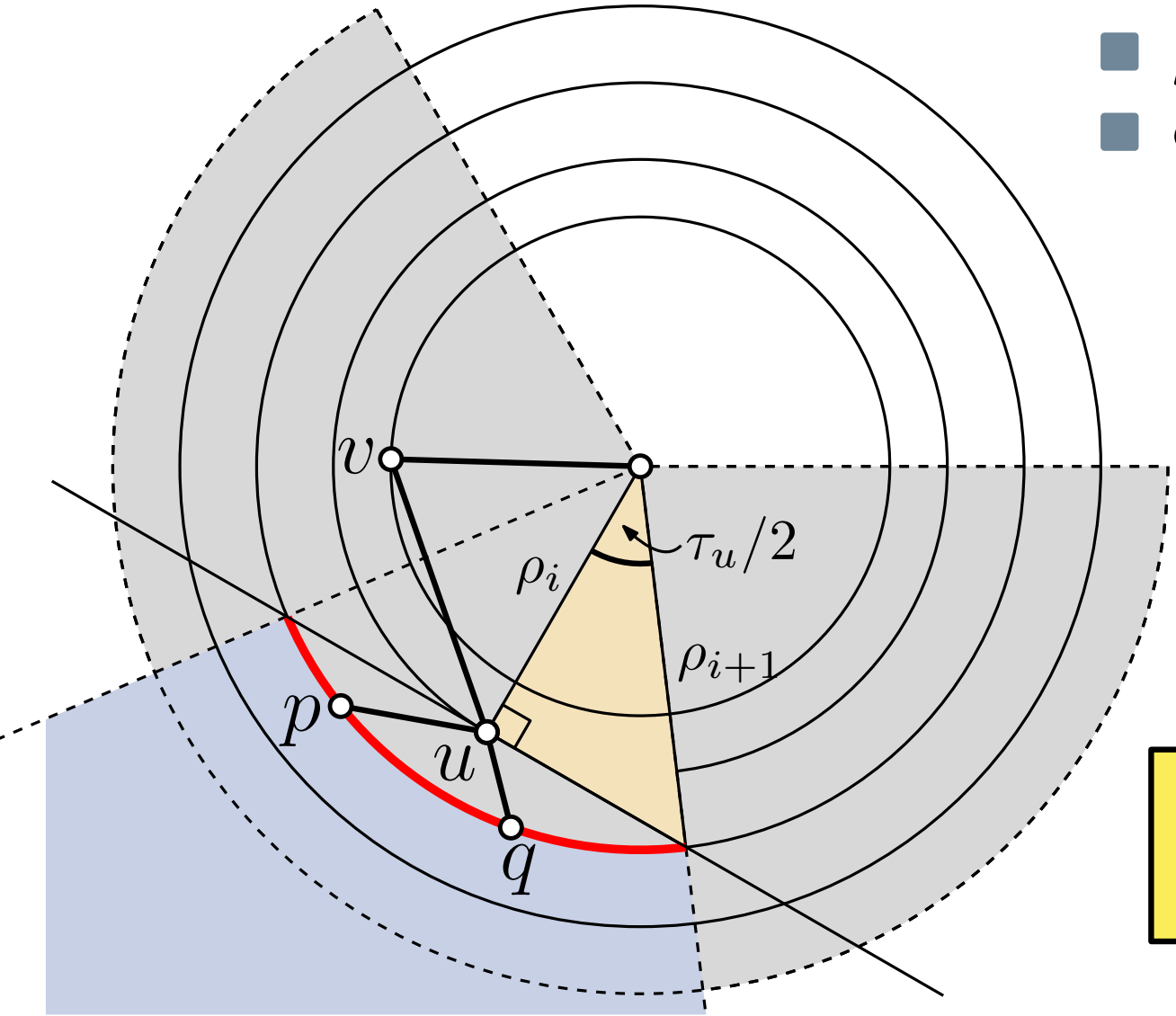

Why does this provide planarity?

- $\tau_u = \min\{\frac{|T(u)|}{|T(p(v))|-1} \cdot \tau_{p(u)}, 2\arccos\frac{\rho_i}{\rho_{i+1}}\}$ (correction)

Radial Tree Drawings: Algorithm

RadialTree(tree T, root $r \in T$, radii $\rho_1 < \cdots < \rho_{\mathsf{height}(T)}$)

begin
- $postorder(r)$
- $preorder(r, 0, 0, 2\pi)$
- **return** $(d_v, \alpha_v)_{v \in T}$
- `// polar coordinates`

postorder(vertex v)
- Compute all subtree sizes – recursively!

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subtree size $|T(v)|$

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- $d_v \leftarrow \rho_t$
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- **if** $t > 0$ **then**
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- $left \leftarrow \alpha_{\min}$
- **foreach** child w of v **do**
  - $right \leftarrow left + \frac{n_w}{n_v - 1} \cdot (\alpha_{\max} - \alpha_{\min})$
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Theorem: Algorithm RadialTree computes a planar radial drawing in $O(n)$ time.

Overview

Layered drawings of rooted trees

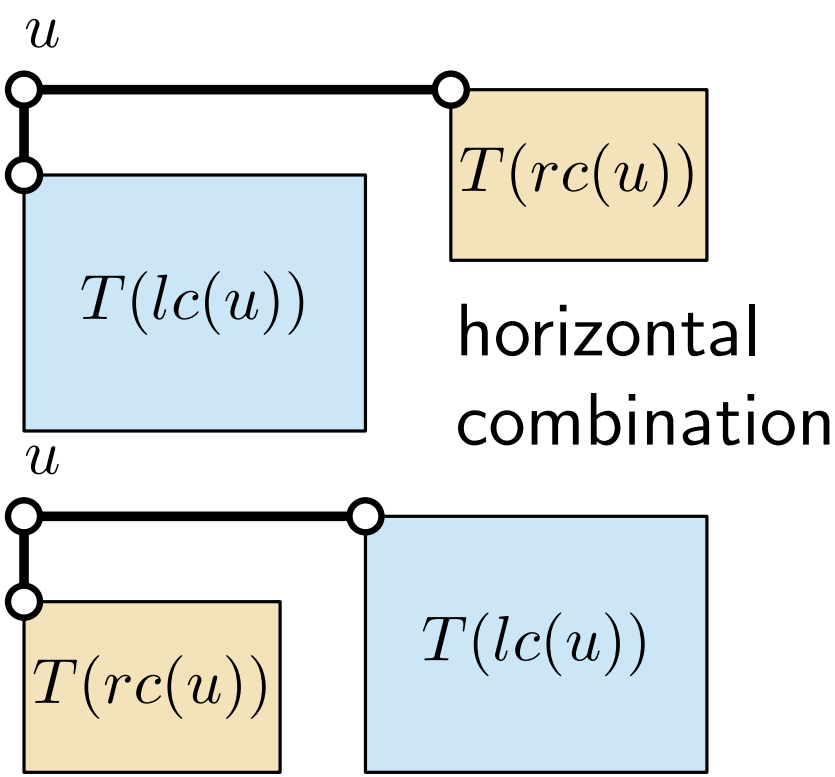
Radial tree drawings

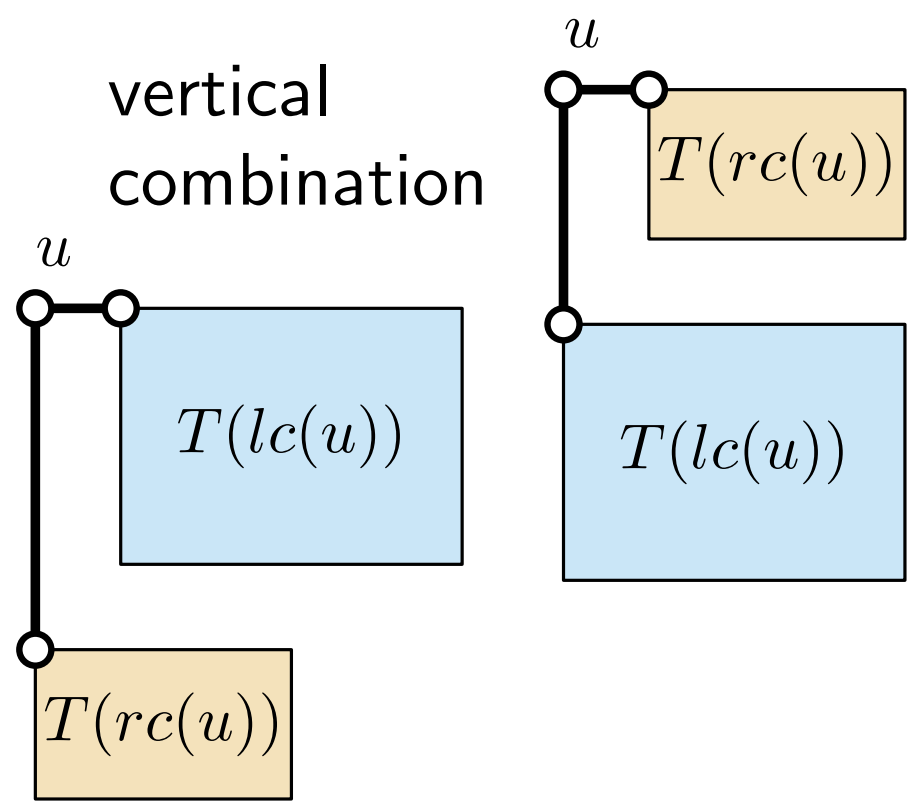
Compact orthogonal drawings of rooted trees

Compact Orthogonal Tree Drawings: HV Layout

Def: HV layout is a planar, straight-line grid drawing of a rooted binary tree with

- children either vertically aligned and below the parent or horizontally aligned and to the right
- recursively defined by placing bounding boxes of subtrees without intersections





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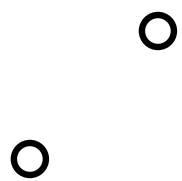

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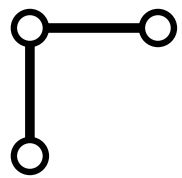

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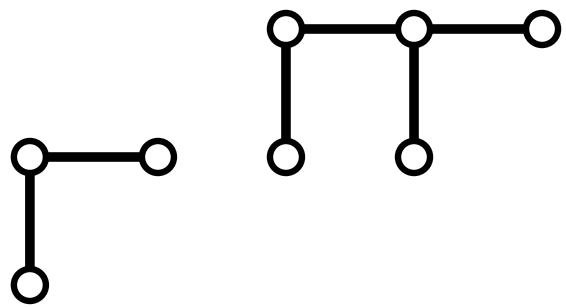

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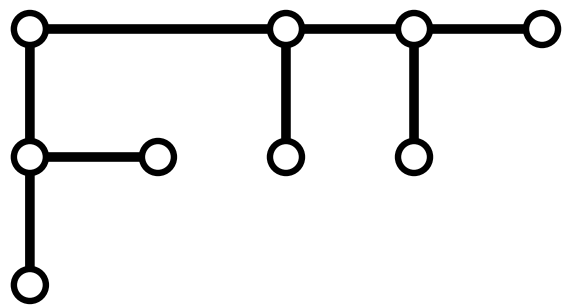

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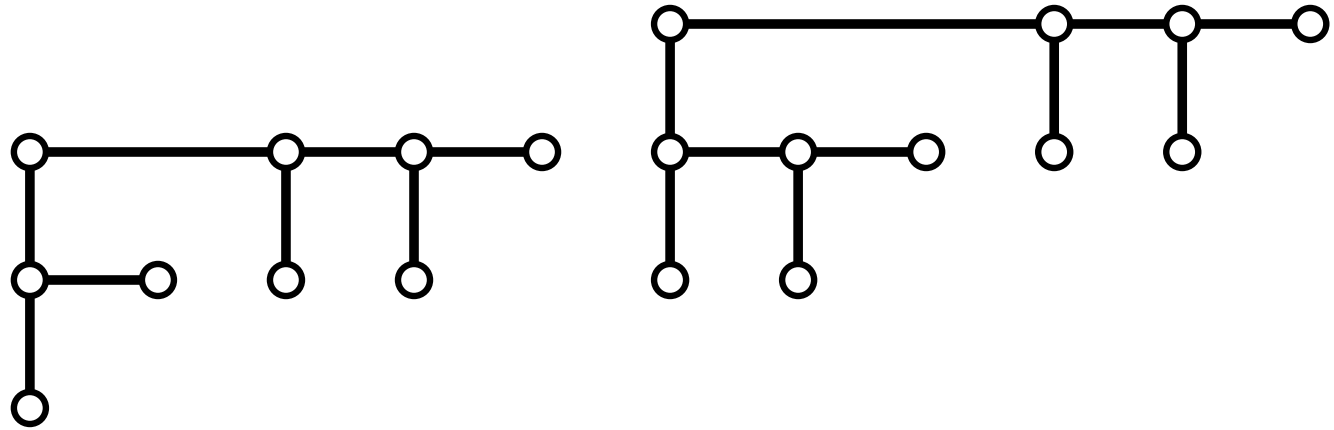

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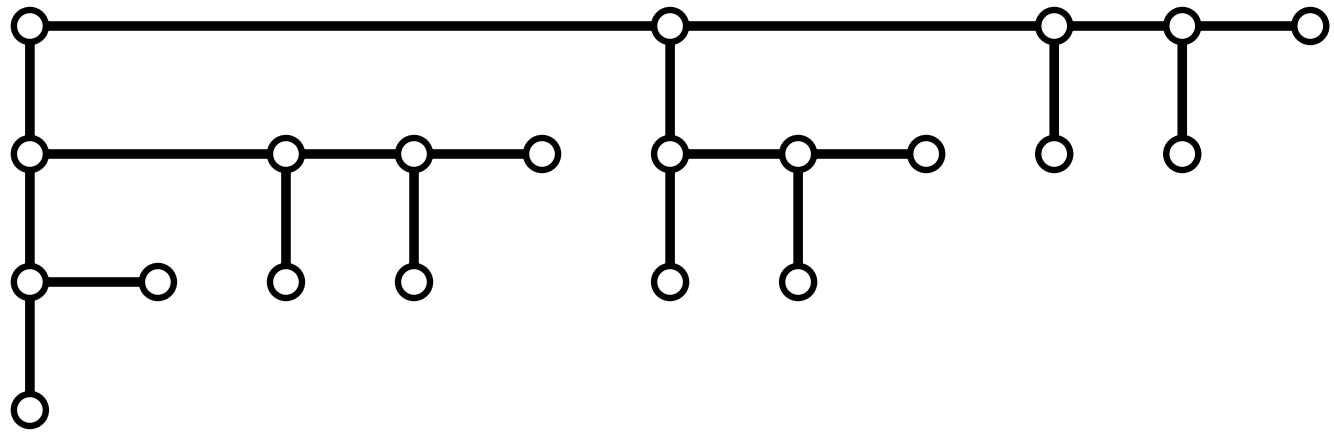

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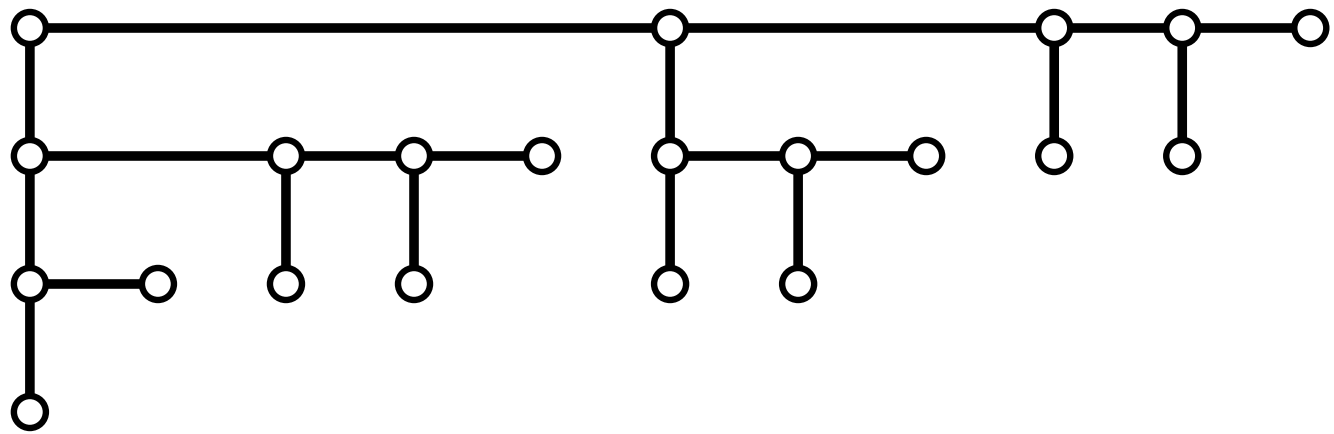

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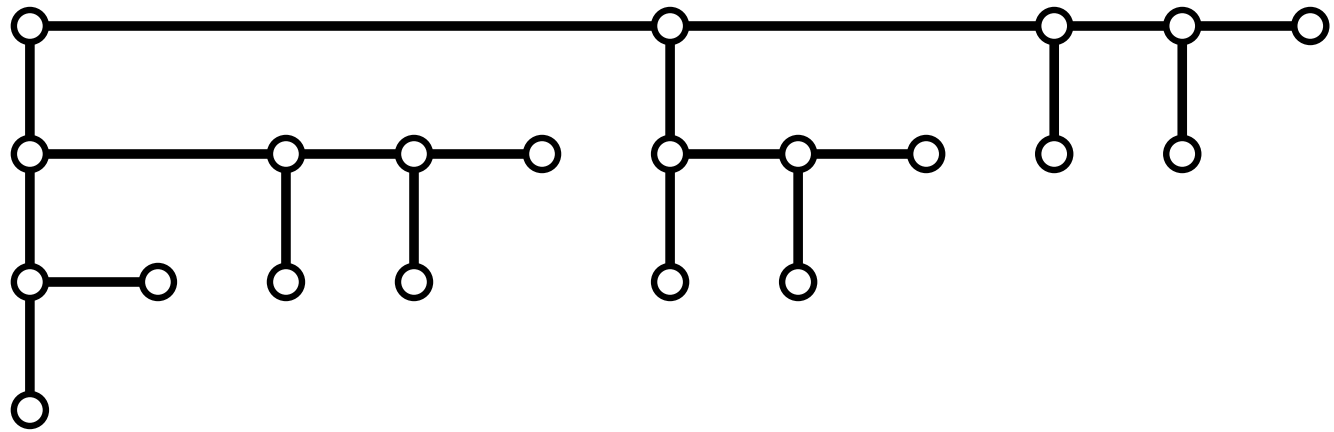

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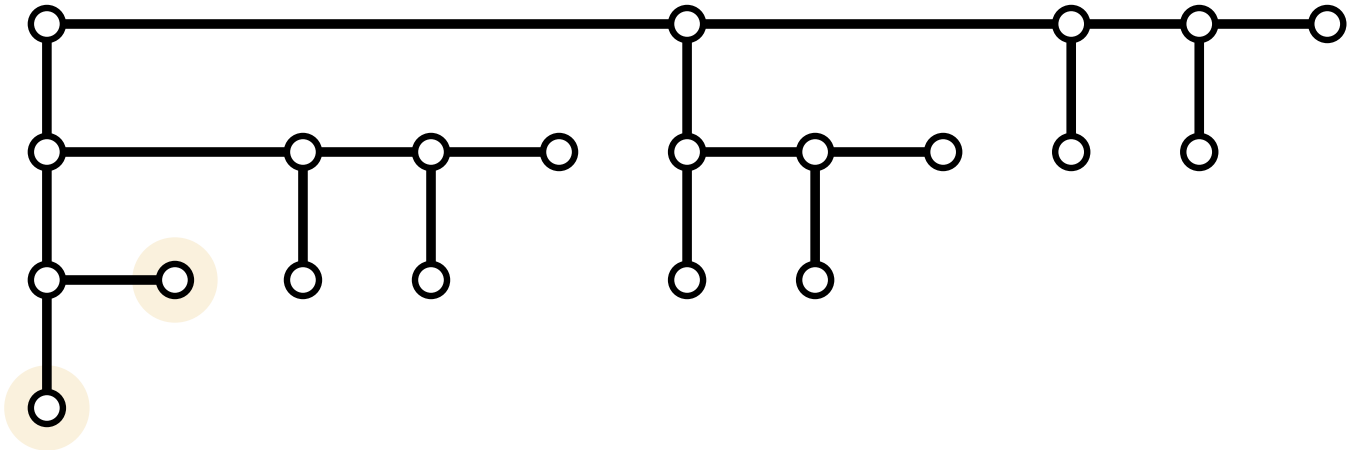

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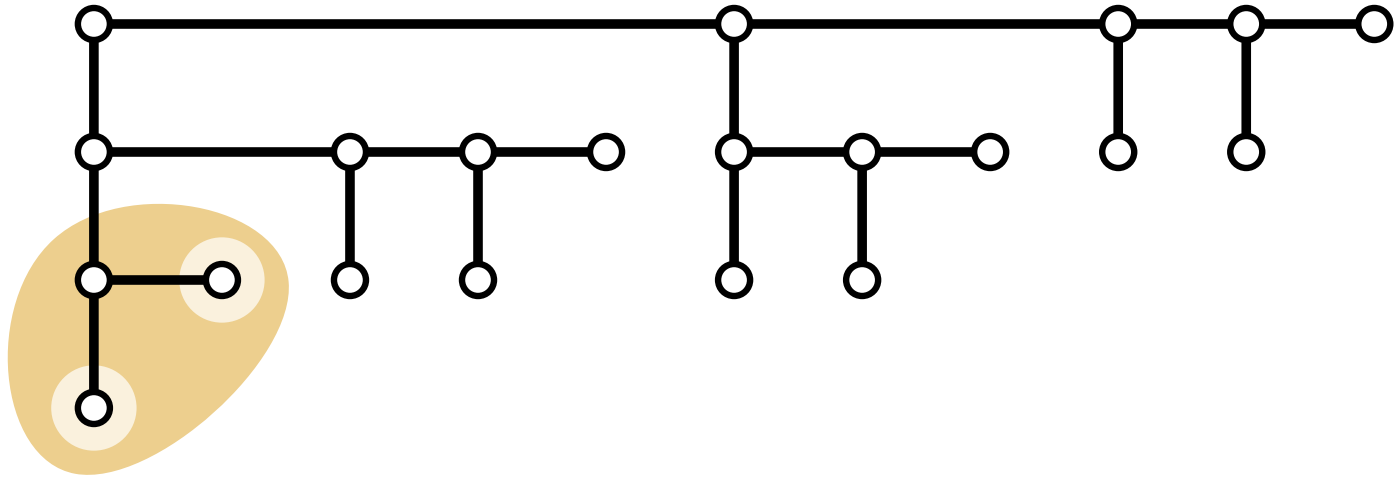

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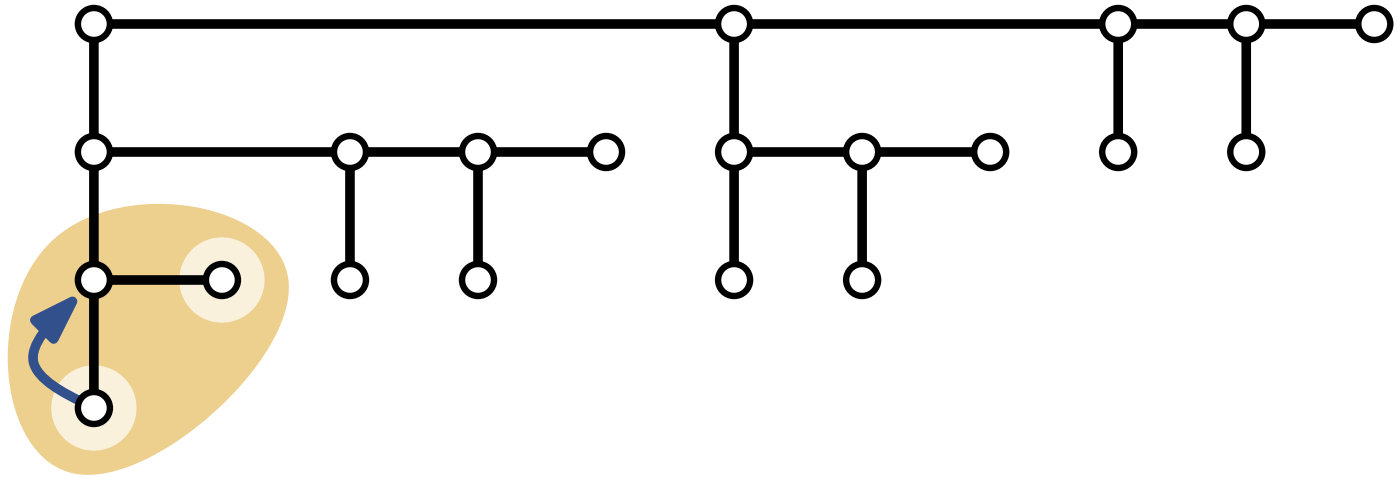

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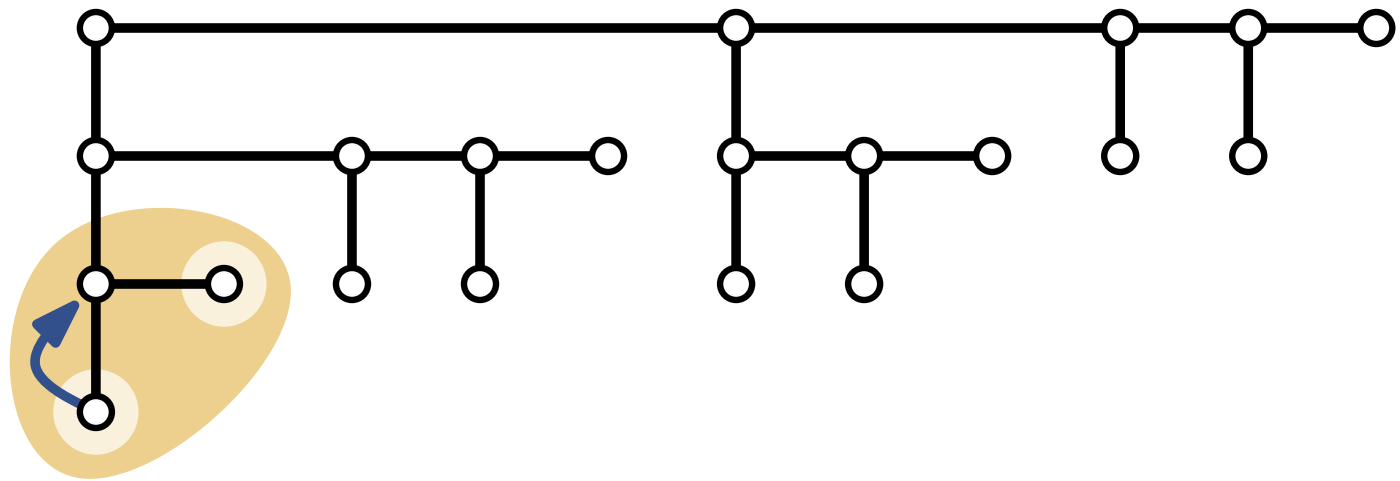

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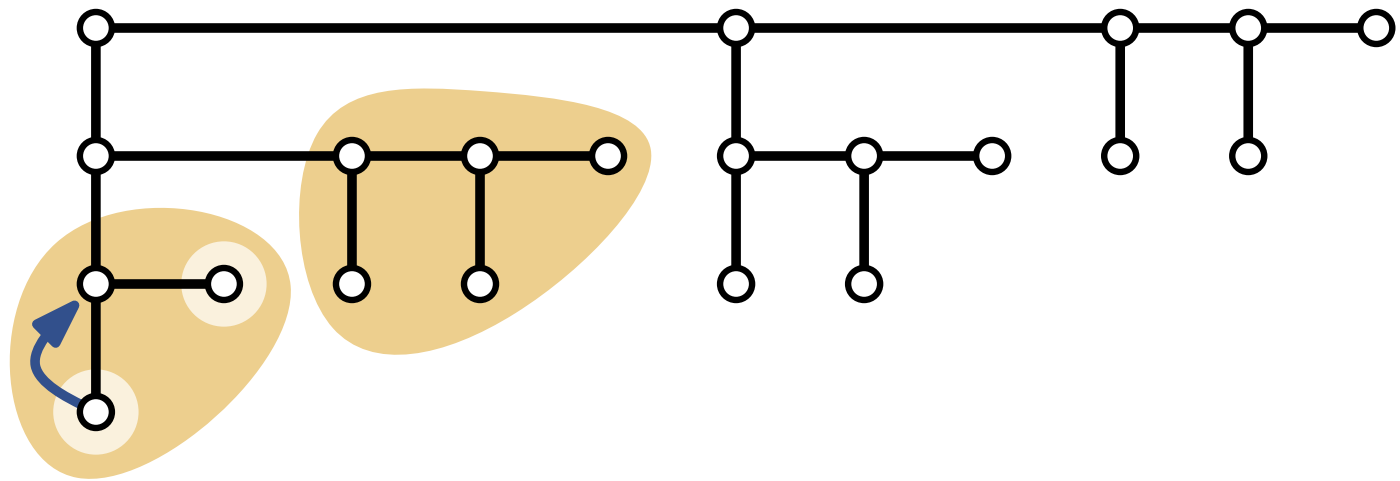

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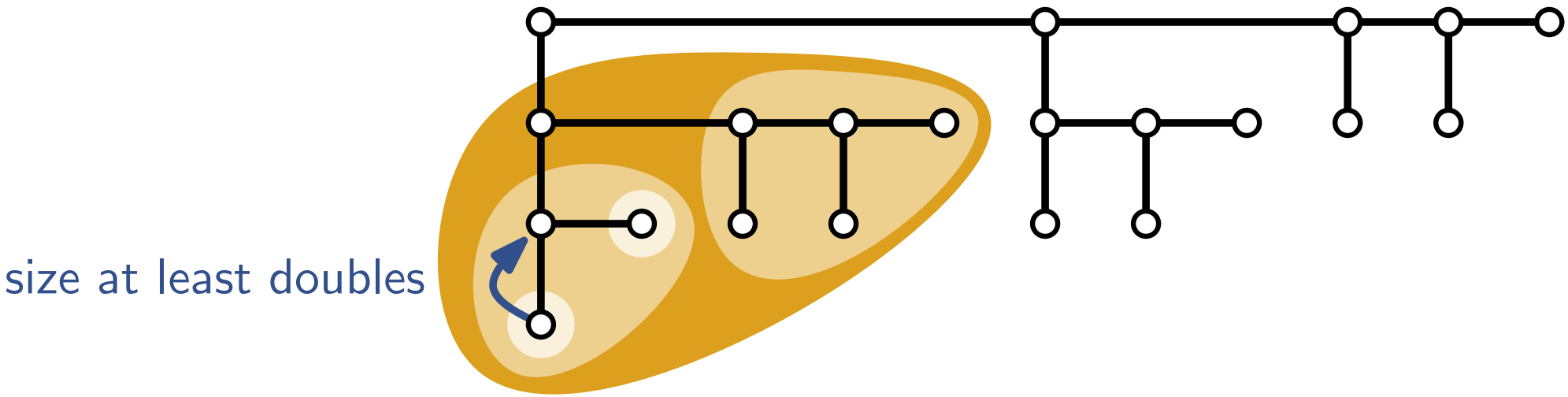

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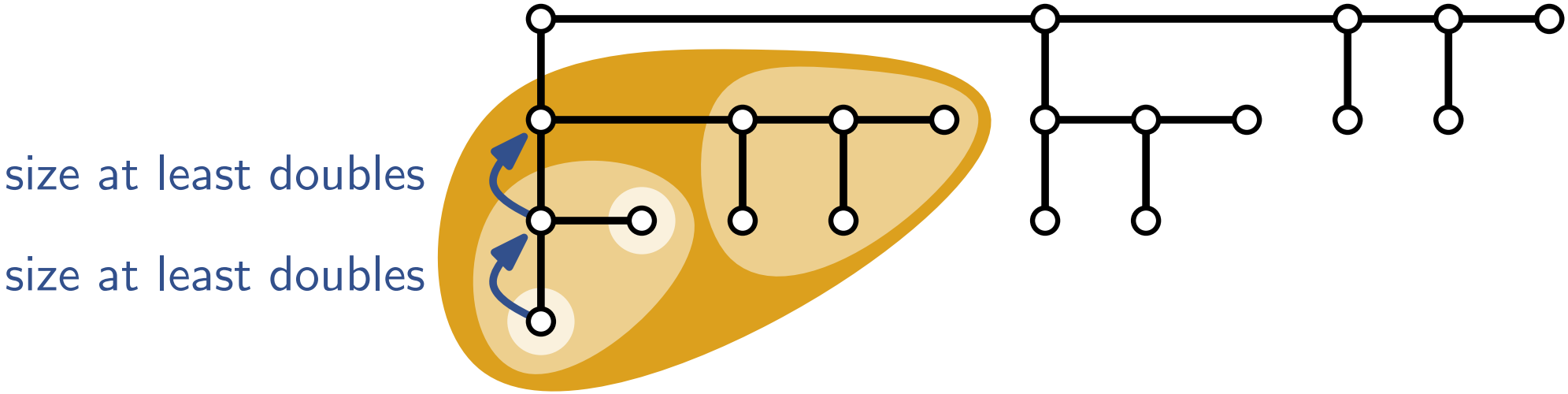

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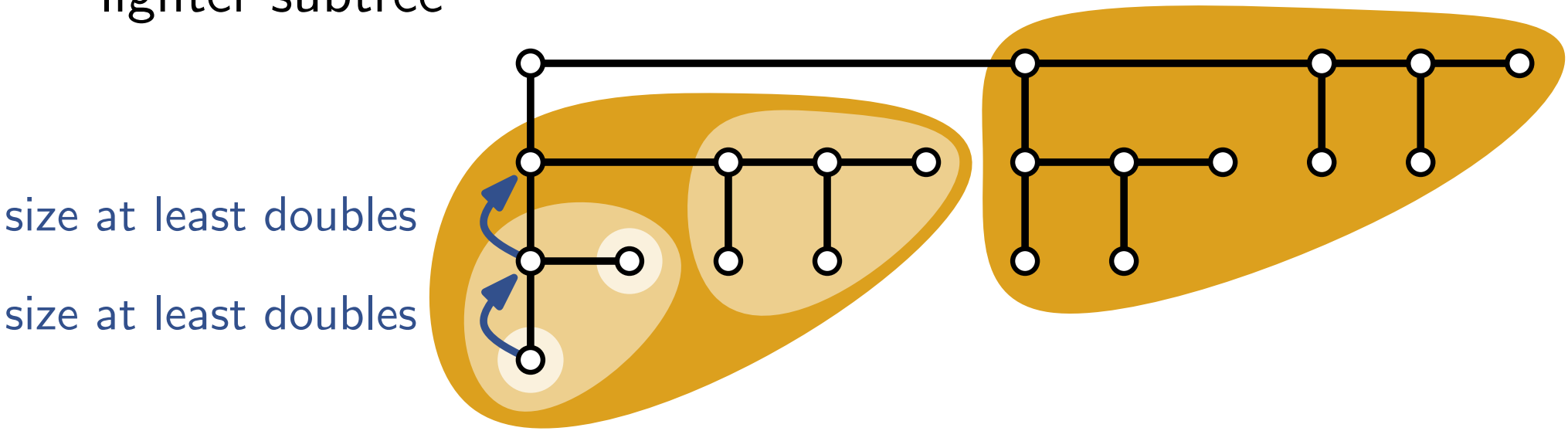

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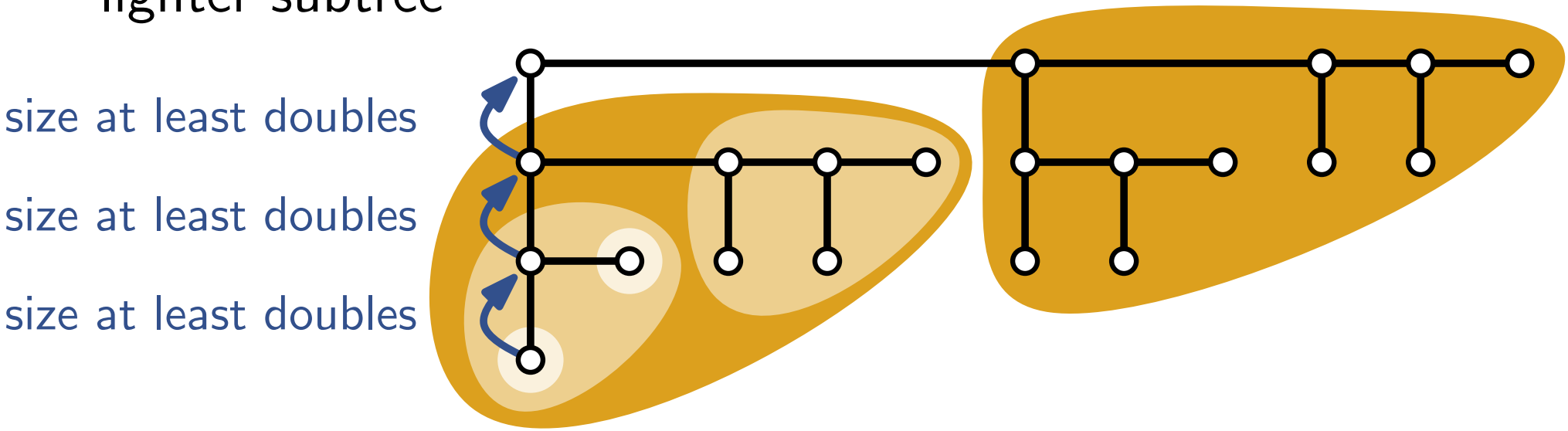

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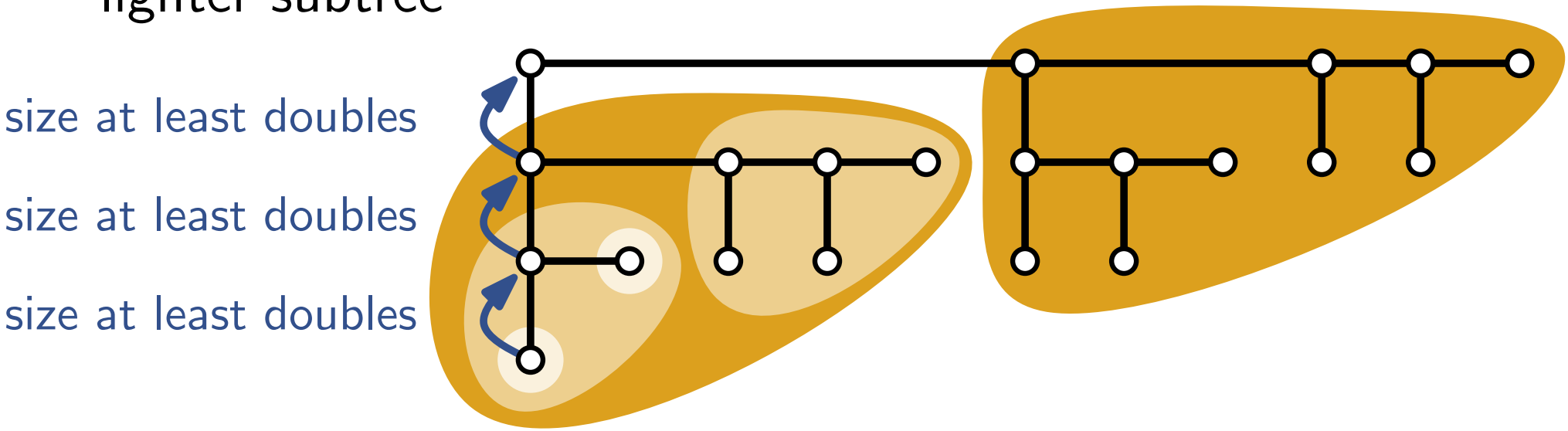

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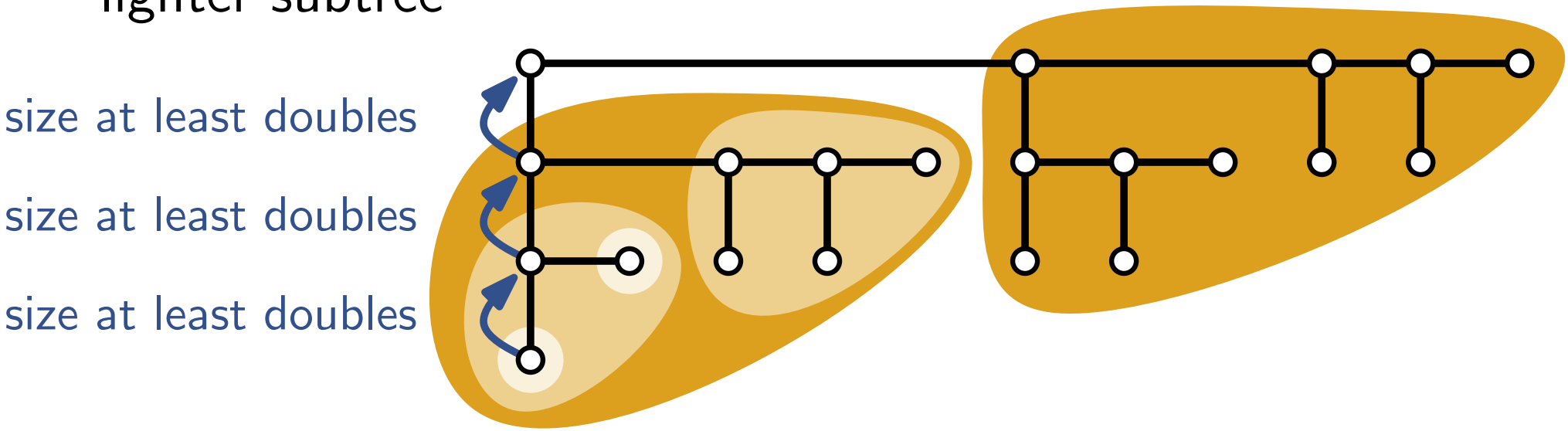

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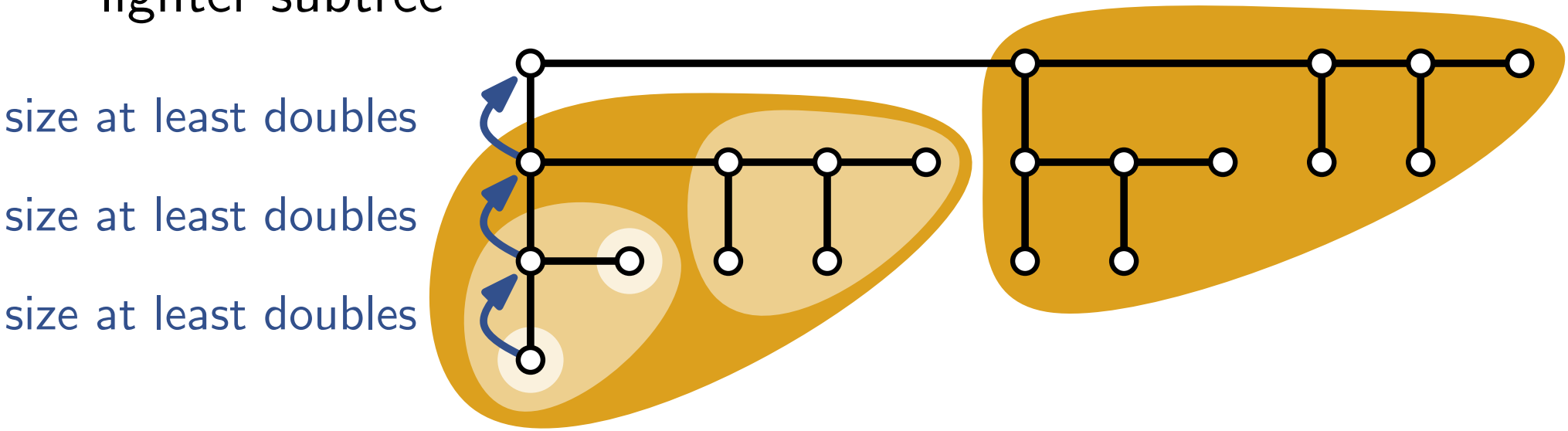

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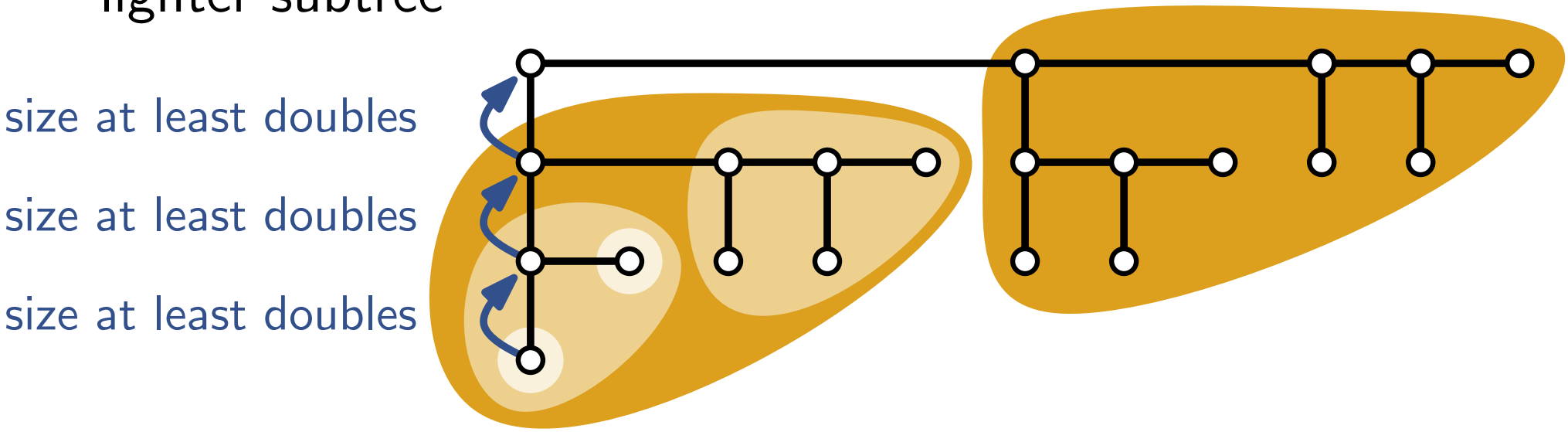

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But: we might have changed child order

Recall: Layout Problem

Graph visualization problem

given: graph $G = (V, E)$
find: drawing Γ of G that

- complies with the given drawing conventions
- optimizes the given aesthetics
- satisfies the partial/local constraints

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today:

- (binary/ordered) trees
- planar, straight-line (grid) drawings
- optional: centered parent, layered, upward, ordered, isomorphic subtrees
- optimization: area
- NP-hard: width minimization
- competing: width vs. isomorphic subtrees, area vs. ordered

Summary and Outlook

- layered tree drawings
- radial tree drawings
- HV tree drawings

Summary and Outlook

- layered tree drawings
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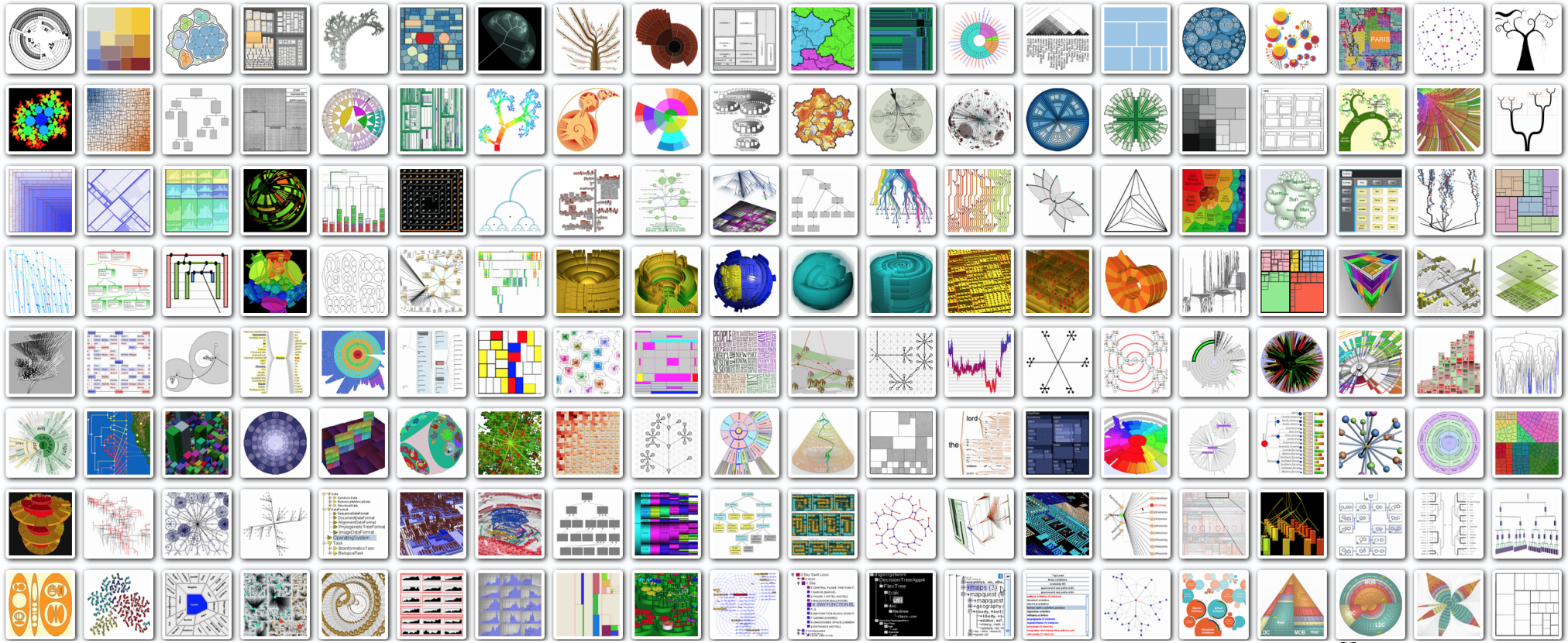
Trees are practically important graph class! Many layout algorithms exist and new ones are developed.



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FYI: Area Minimization for Trees

Area minimization and area bounds of tree drawings in many flavors is still an active (theoretical) research topic.

Some examples:

- unordered, upward, binary: $O(n \log \log n)$ (1996)
- ordered, upward, binary: $O(n \log n)$ (2003)
- ordered, non-upward, orthogonal, binary: $O(nc^{\sqrt{\log n}})$ (2017)
- unordered, non-upward, binary: $\Theta(n)$ (2003)
- unordered, upward, general: $O(n\sqrt{\log n}(\log \log n)^k)$ (2017)
- ordered, upward, general: $O(nc^{\sqrt{\log n}})$ (1999)
- ...

see [Chan, 2017] for more

$\rightarrow$ unknown, whether most bounds are tight