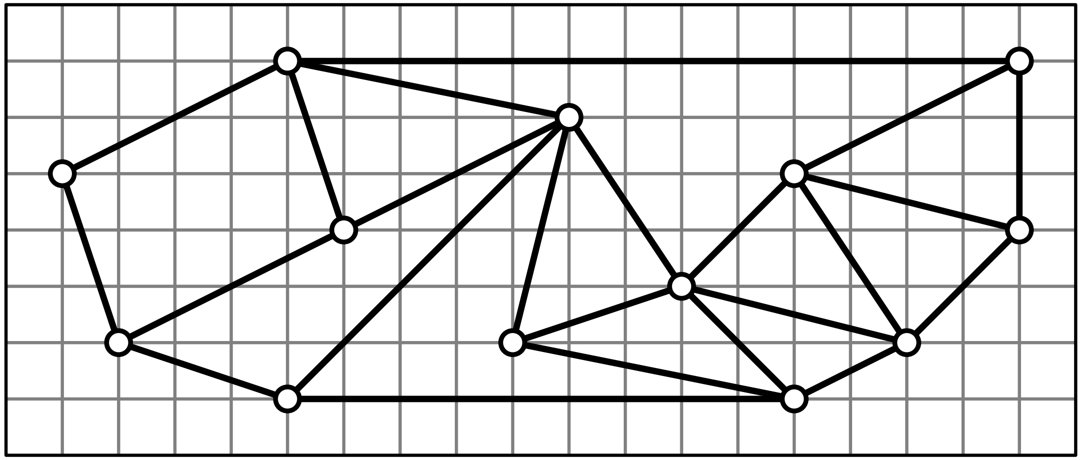
Lower Bounds for Planar Grid Drawings

Lecture Graph Drawing Algorithms · 192.053

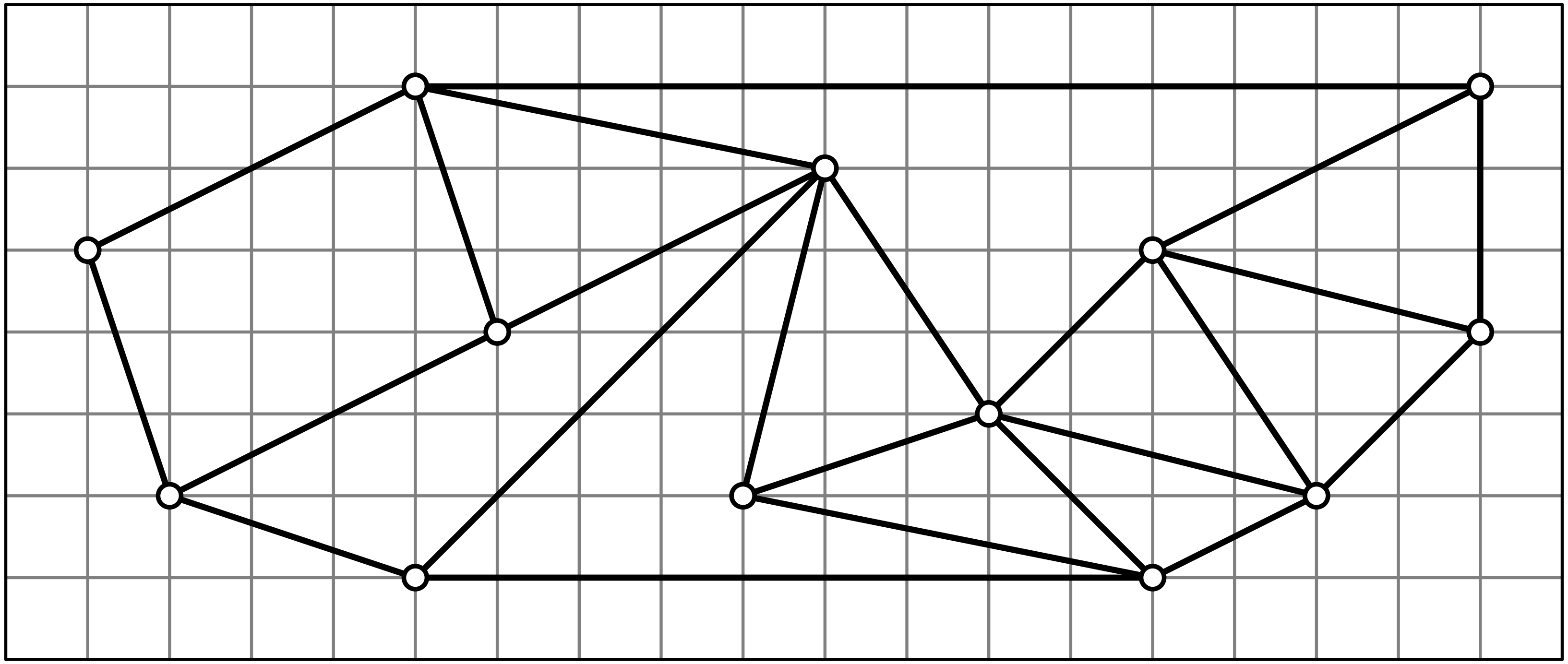
Martin Nöllenburg

19.06.2018



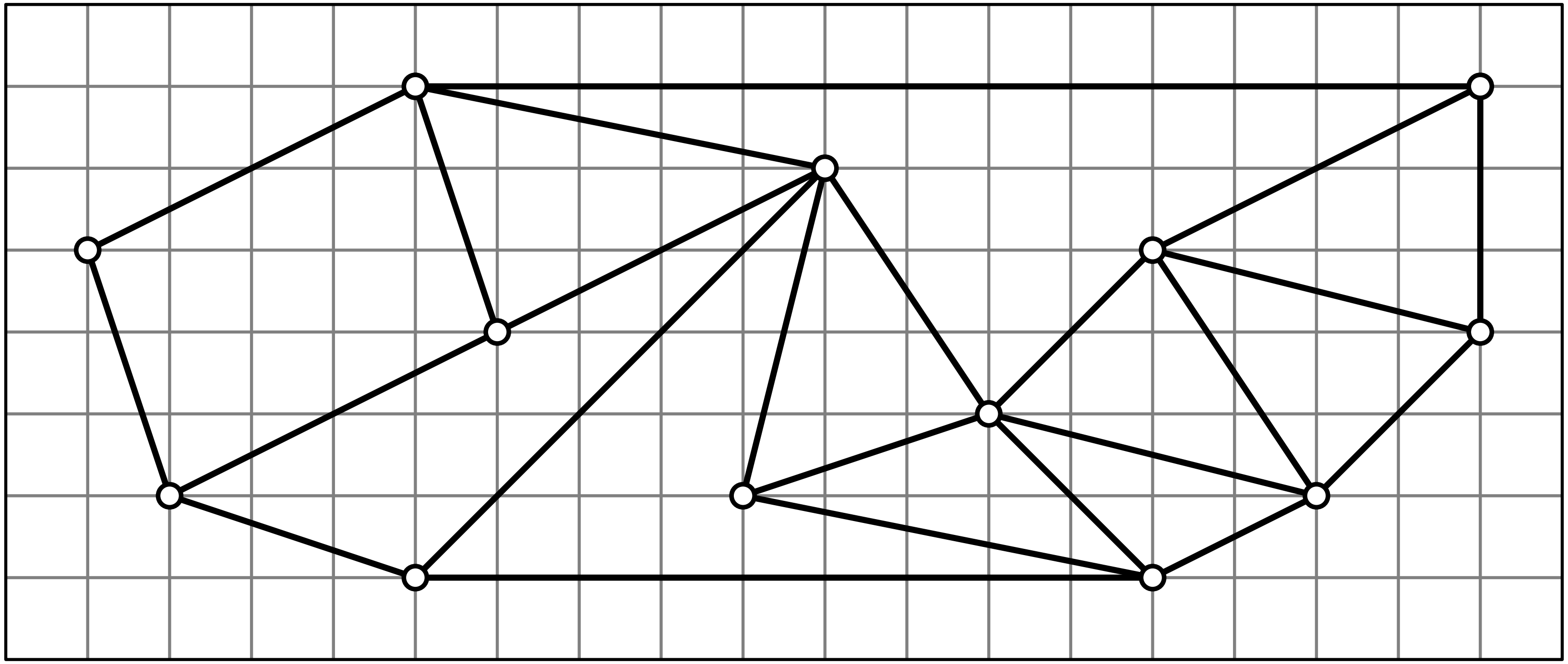
ALGORITHMS AND
COMPLEXITY GROUP

Area of Planar Grid Drawings



One common aesthetic of planar grid drawings is the drawing area. We aim to determine tight upper and lower bounds.

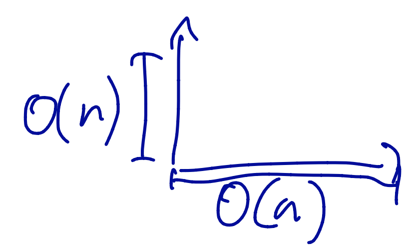
Area of Planar Grid Drawings



One common aesthetic of planar grid drawings is the drawing area. We aim to determine tight upper and lower bounds.

What do we know?

area: $O(n) \times O(n)$



Upper Bounds

- Every planar graph G has a planar grid drawing of area $(2n-4) \times (n-2)$. Lecture 4, [de Fraysseix, Pach, Pollack '90]
- Every planar graph G has a planar grid drawing of area $(n-2) \times (n-2)$. Lecture 5, [Schnyder '90]
- Every planar graph G has a planar grid drawing of area $2n/3 \times 4n/3$. [Brandenburg '08]

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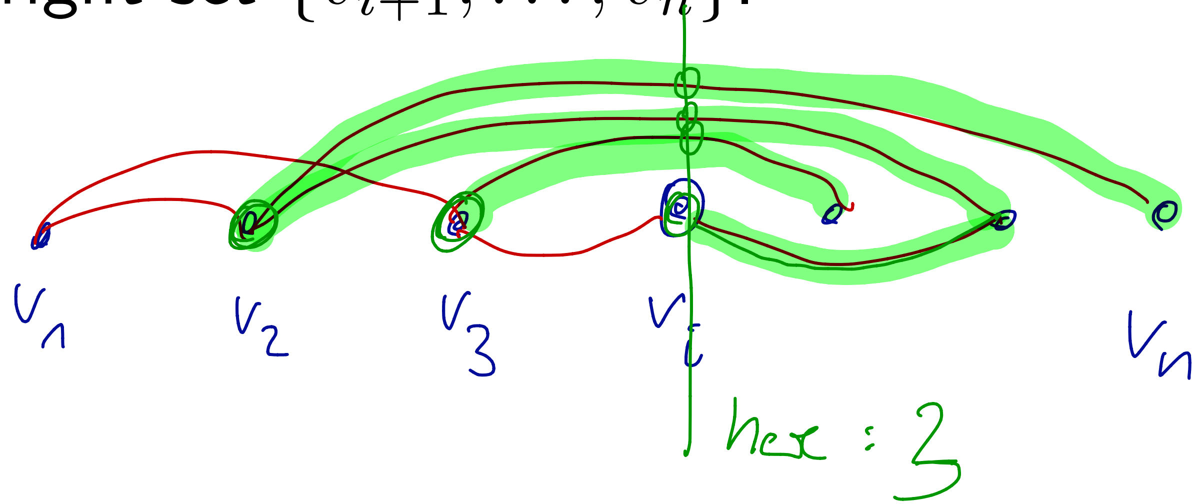
Today: lower bounds

Theorem 1: Let G be a planar graph of **pathwidth** $\mathrm{pw}(G)$. Then every planar grid drawing of G requires height $h \geq \mathrm{pw}(G)$.

Pathwidth

Def: A vertex ordering $v_1, v_2, \ldots, v_n$ of vertex set V of a graph $G = (V, E)$ has **search width** $\leq k$ if for each $1 \leq i \leq n$ at most k vertices of the left set $\{v_1, \ldots, v_i\}$ have neighbors in the right set $\{v_{i+1}, \ldots, v_n\}$.



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Def: A graph $G = (V, E)$ has **pathwidth** $\mathrm{pw}(G) \leq k$ if it has a vertex ordering of search width $\leq k$.

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Def: A graph $G = (V, E)$ has **pathwidth** $\mathrm{pw}(G) \leq k$ if it has a vertex ordering of search width $\leq k$.

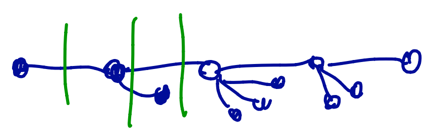
Testing if a graph has pathwidth k
is NP-hard and APX-hard.

[Bodlaender et al. '95]

"hard to approximate within constant factor"

pw = 0:

pw = 1

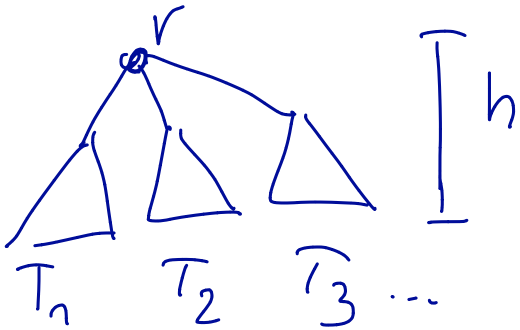


„caterpillar"

pw = 2: try to find example!

Special Case: Trees

Obs: For a tree T with root r and height h we have $\mathrm{pw}(T) \le h$.

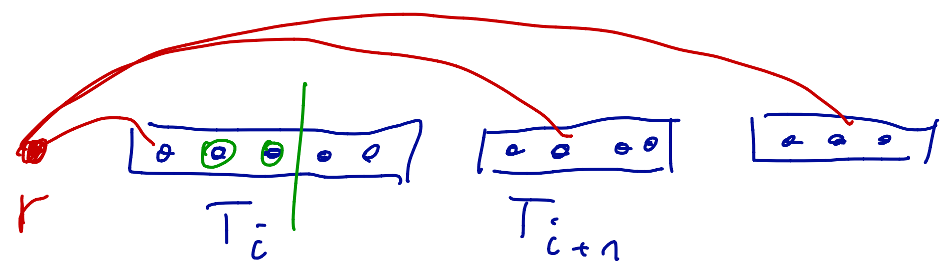


Proof: Induction on h

$h = 0$ $\quad$ $\mathrm{pw}(T) = 0$ ✓

$h > 0$: each T_i has height $\le h-1$

so $\mathrm{pw}(T_i) \le h-1$ by ind. hyp.

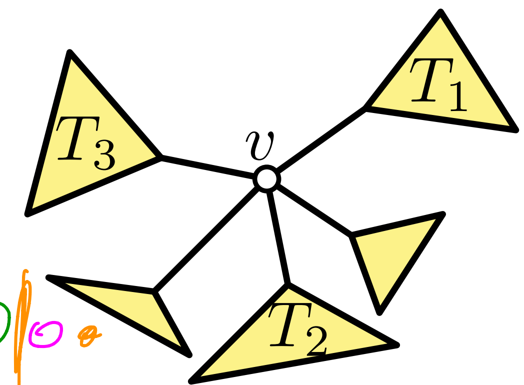


This has search width $\le h$

Special Case: Trees

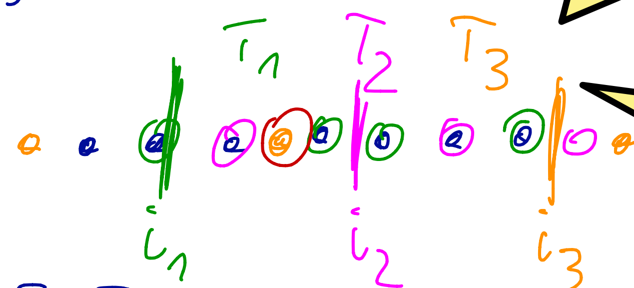
Obs: For a tree T with root r and height h we have $\mathrm{pw}(T) \leq h$.

Lemma 1: Let T be a tree and v a vertex of T such that the forest $T - v$ after removal of v decomposes into at least three subtrees T_1, T_2, T_3 with $\mathrm{pw}(T_i) \geq k$ for $i = 1, 2, 3$. Then $\mathrm{pw}(T) \geq k + 1$.



Proof: assume $pw(T) \leq k$ $(=k)$

$\Rightarrow \exists$ vertex ordering with s.w. k

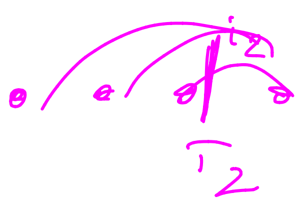


i_1, i_2, i_3 are positions of T_1, T_2, T_3 with searchwidth k, assume $i_1 < i_2 < i_3$

$\Rightarrow$ i_2 separates T_1 and T_3: T_1 is left of i_2

T_3 is right of i_2

(otherwise searchwidth $> k$)



no matter where v is in the ordering edge from v to T_1 or T_3 must cross i_2

Special Case: Trees

Obs: For a tree T with root r and height h we have $\mathrm{pw}(T) \leq h$.

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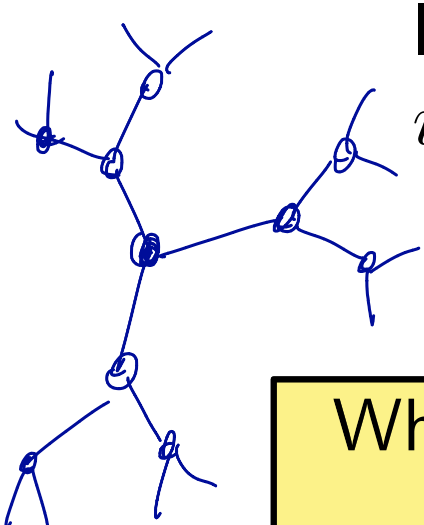
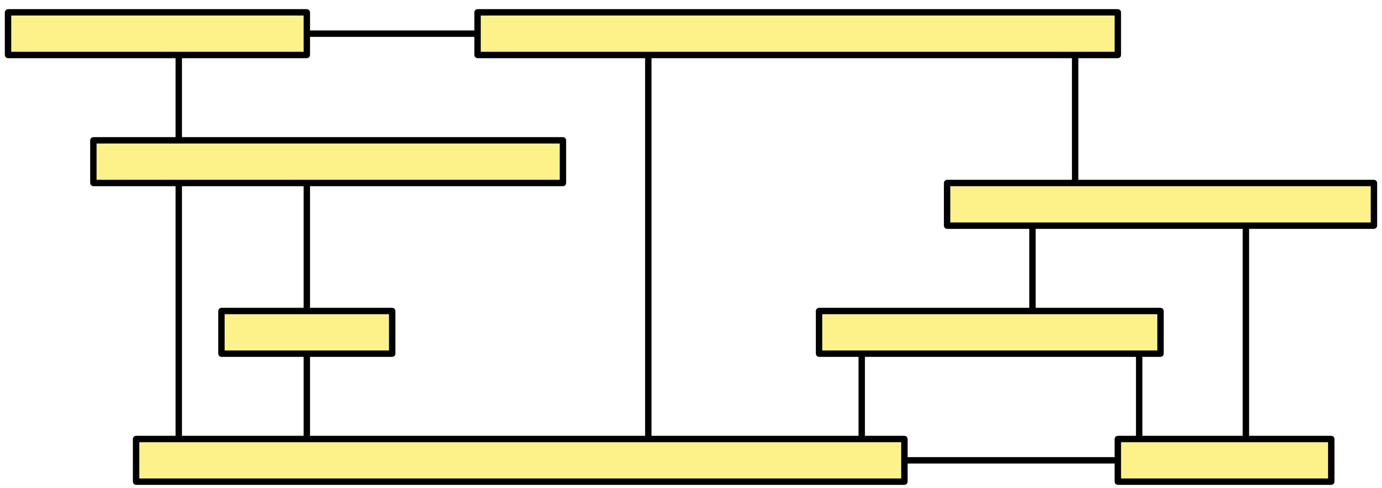


What does this mean for the pathwidth of a complete ternary tree T of height k?

For complete ternary trees we have equality $\mathrm{pw}(T) = $ height of T

Visibility Representation

Def: In a **visibility representation** of a graph $G = (V, E)$ every vertex $v \in V$ is drawn as an axis-parallel box and every edge $e \in E$ as a horizontal or vertical segment between the boxes of its end-vertices. No edge intersects other boxes or edges.



Height of Visibility Representations

Lemma 2: If a graph $G = (V, E)$ has a planar grid drawing of height h then it also has a visibility representation of height h.

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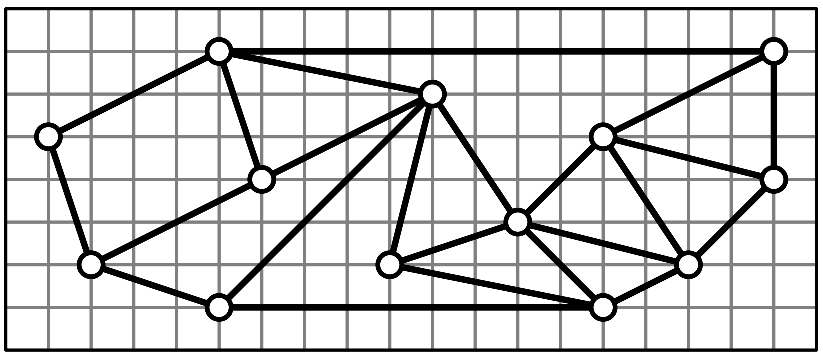

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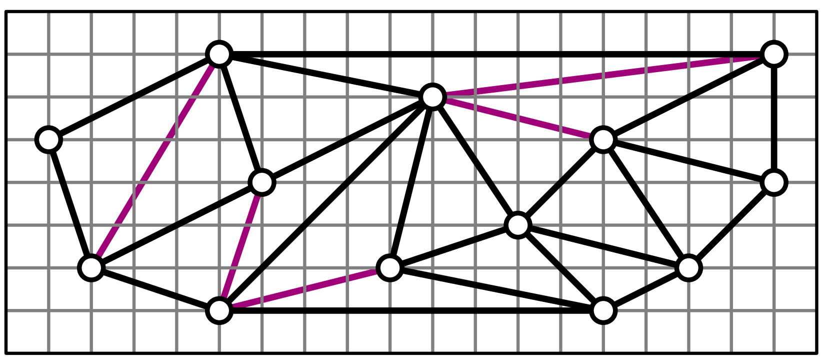

triangulate inner faces

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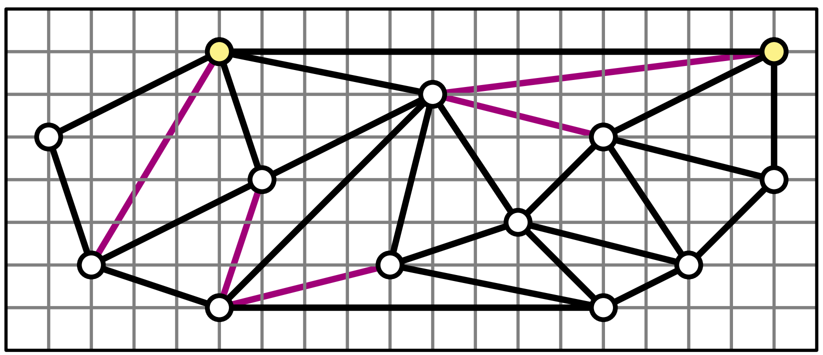

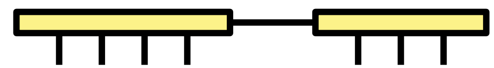

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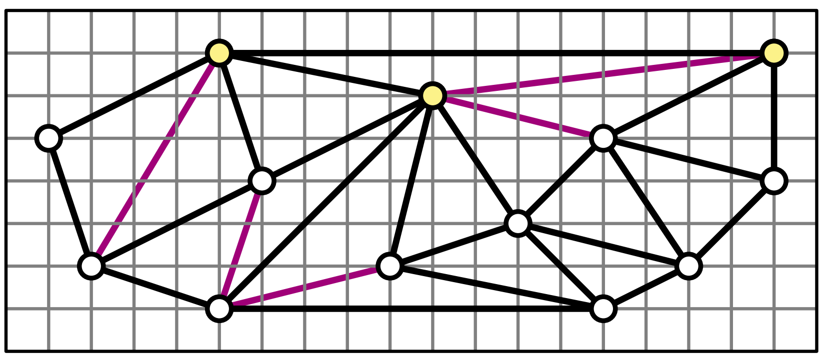
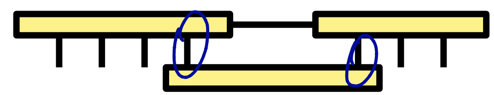
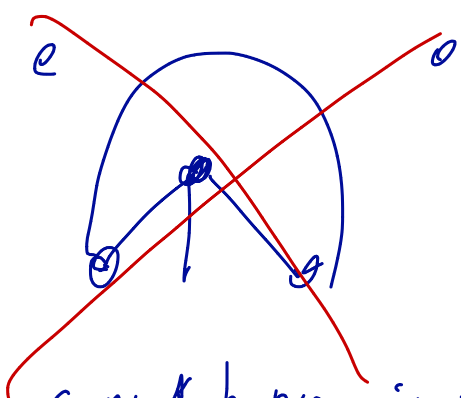




every face is a triangle, so every vertex has ≥ 1 previously placed neighbor



Cannot happen in triangulated drawing

Height of Visibility Representations

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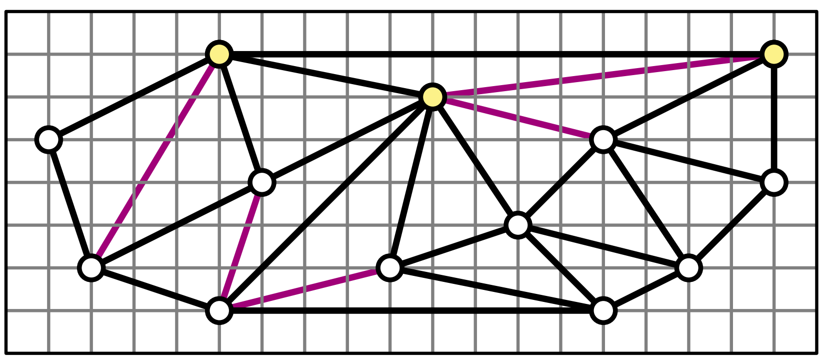

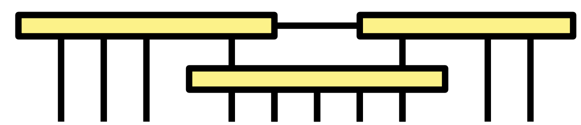

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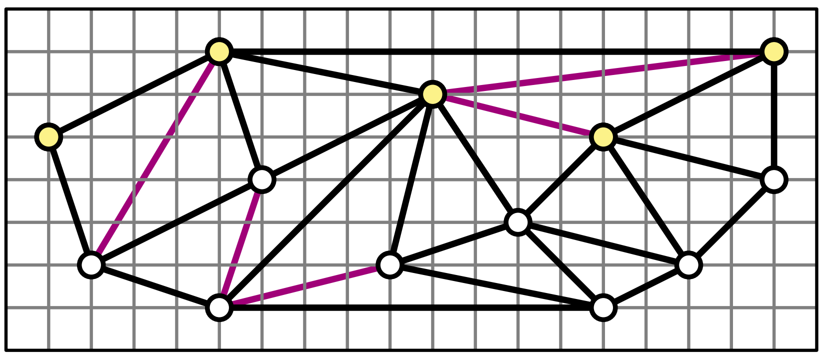

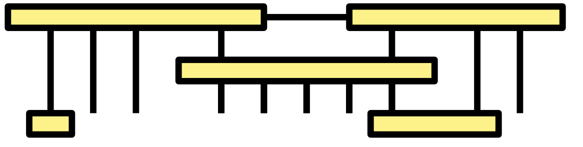

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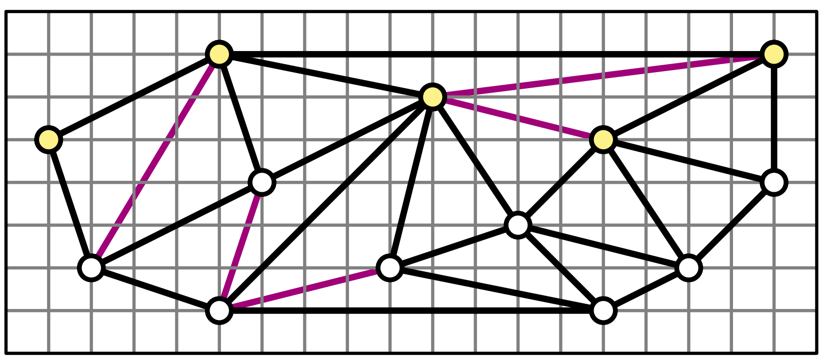

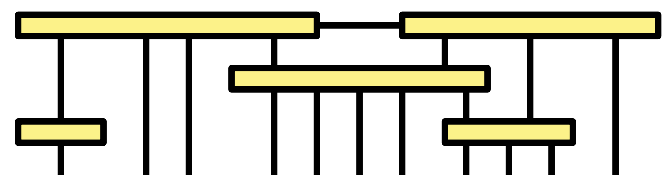

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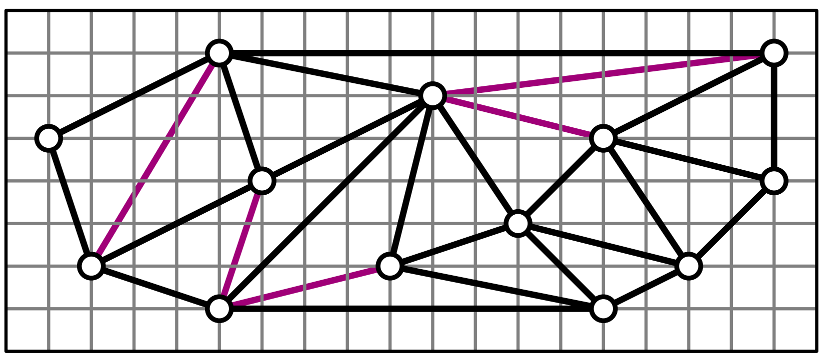
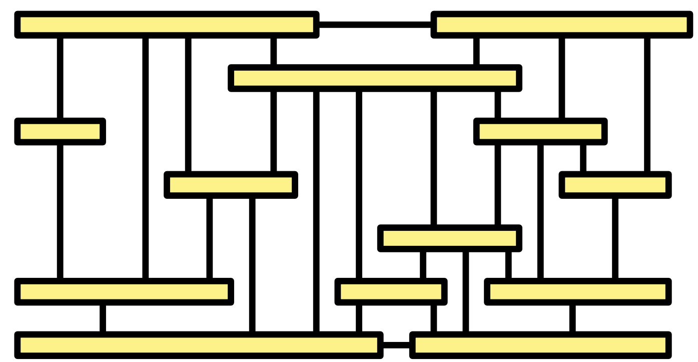




height of this construction remains the same!

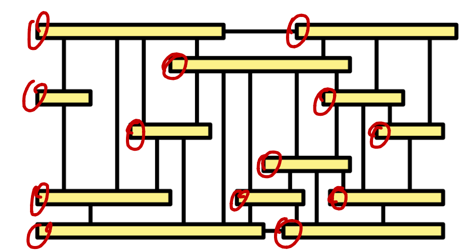
Height of Visibility Representations

Lemma 2: If a graph $G = (V, E)$ has a planar grid drawing of height h then it also has a visibility representation of height h.

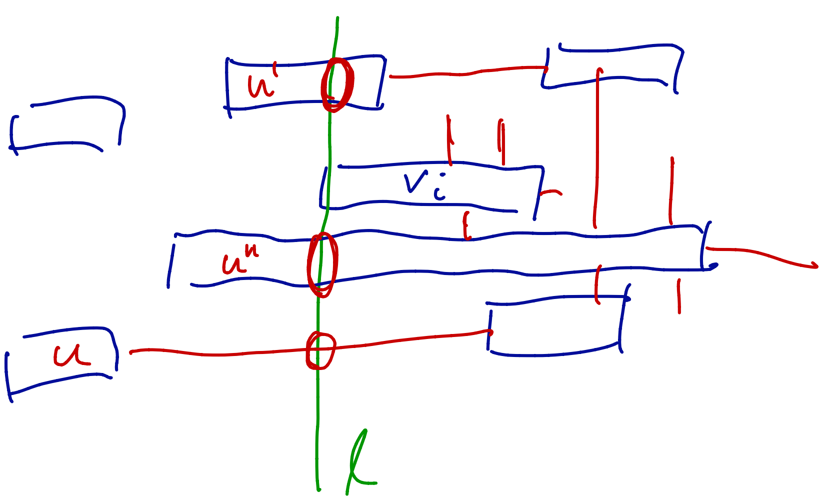
Lemma 3: If a graph $G = (V, E)$ has a visibility representation of height h then $\mathrm{pw}(G) \leq h$.

goal: vertex ordering of s.w. $\leq h$

use left endpoints of the boxes (ties: top to bottom)



consider any v_i in this ordering



any u left of ℓ with neighbors right of ℓ intersects ℓ on one exclusive row (or has horizontal edge)

$\Rightarrow sw \leq h$

Height of Visibility Representations

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Lemma 3: If a graph $G = (V, E)$ has a visibility representation of height h then $\mathrm{pw}(G) \leq h$.

This yields the desired lower bound

[Dujmovic et al. '01/'08], [Felsner, Liotta, Wismath '03]

Theorem 1: Let G be a planar graph of pathwidth $\mathrm{pw}(G)$. Then every planar grid drawing of G requires height $h \geq \mathrm{pw}(G)$.

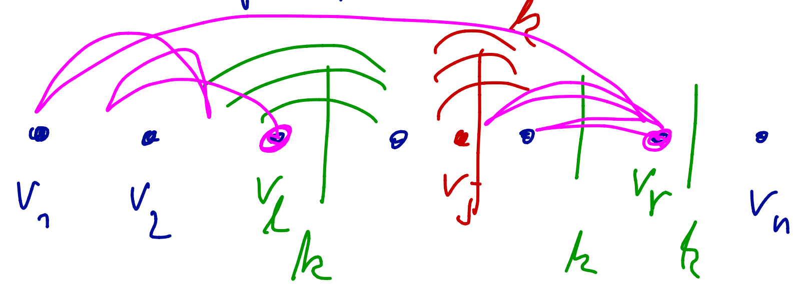
follows by contradiction using Lemma 2 + 3

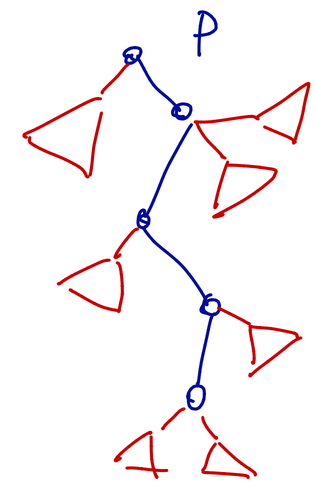
Drawings of Height $O(\mathrm{pw}(T))$

Theorem 2: A tree T has pathwidth $\mathrm{pw}(T) \leq k$ if and only if there is a path P in T such that all trees in forest $T - P$ have pathwidth at most $k-1$.

Proof „⇒"

- let $v_1, v_2, v_3, \dots, v_n$ be ordering of V with search width $\leq k$





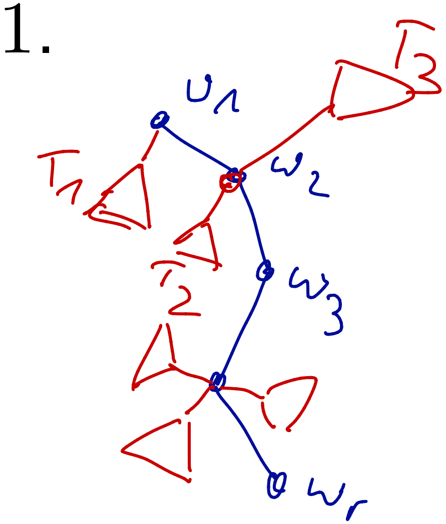
- let l, r be indices of leftmost and rightmost positions with $sw = k$
- if $l = r$ define $P = v_l$ ✓
- if $l < r$: define P as unique path in T from v_l to v_r
- let T' be subtree in $T - P$, consider ordering of T' in vertex sequence

assume one vertex of T' has $sw = k$, e.g. v_j, but then one edge of P "crosses" v_j, which means $pw(T) = k+1$ ↯

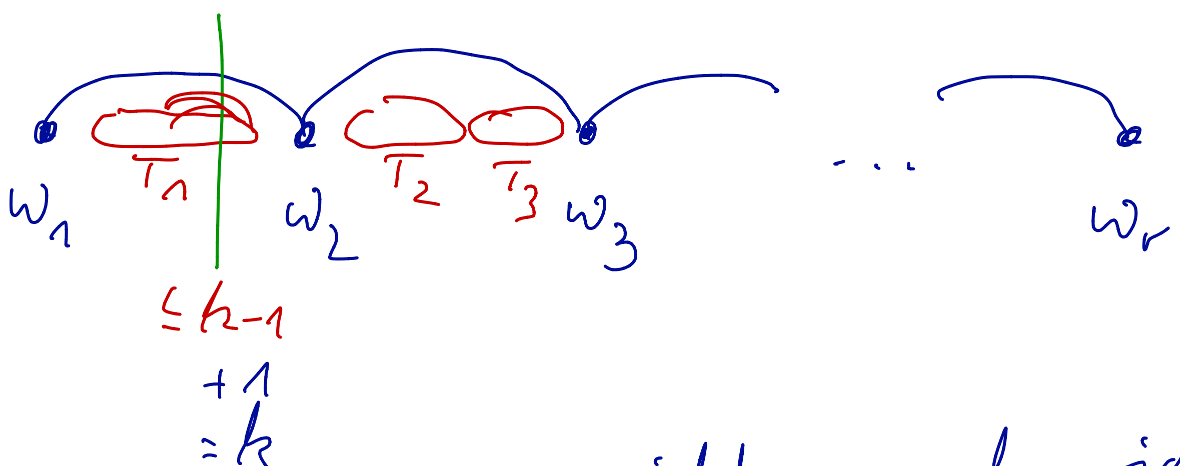
Drawings of Height $O(\mathrm{pw}(T))$

Theorem 2: A tree T has pathwidth $\mathrm{pw}(T) \leq k$ if and only if there is a path P in T such that all trees in forest $T - P$ have pathwidth at most $k - 1$.

"⇐" Let P this path $P = w_1, w_2, w_3, \ldots, w_r$



construct ordering:



$\Rightarrow$ yields search width k $\square$

Drawings of Height $O(\mathrm{pw}(T))$

Theorem 2: A tree T has pathwidth $\mathrm{pw}(T) \leq k$ if and only if there is a path P in T such that all trees in forest $T - P$ have pathwidth at most $k - 1$.

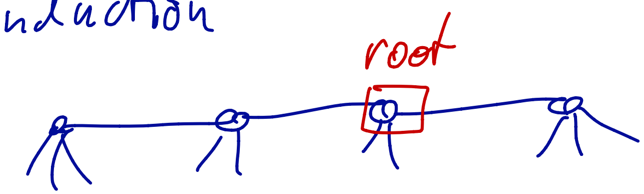
Such a path P is called a **main path** of T.

Drawings of Height $O(\mathrm{pw}(T))$

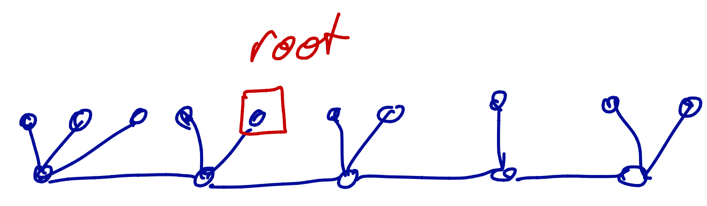
Theorem 3: Let T be a tree with root r. Then T has a planar grid drawing of height $2\,\mathrm{pw}(T)$ with r in the topmost row. If r is part of a main path of T then the height is $\max\{2\,\mathrm{pw}(T)-1, 2\}$.

Proof: by induction

$\mathrm{pw}(T) = 1$



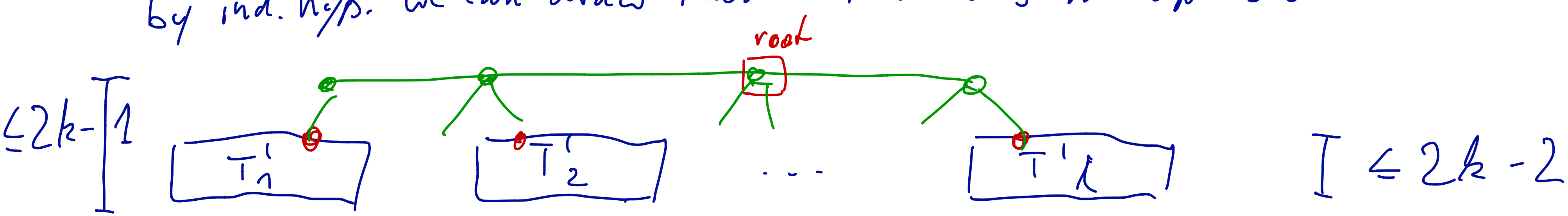
or



$\mathrm{pw}(T) = k > 1$

case 1 root is part of main path $\overset{\text{Thm 2}}{\Longrightarrow}$ all T' in $T-P$ have $\mathrm{pw}(T') \leq k-1$

by ind. hyp. we can draw them with roots in top row



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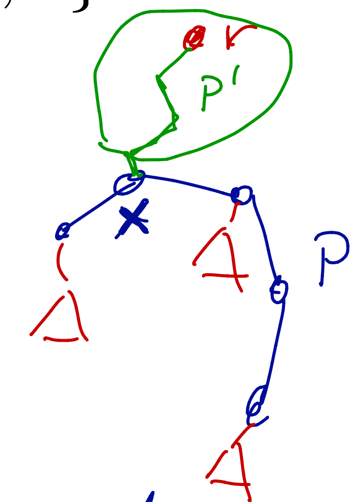


case 2 root is not part of main path P

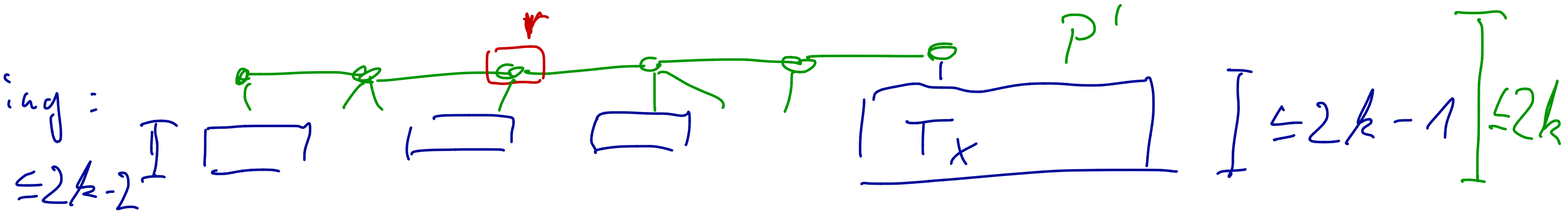
let x be topmost vertex of P

and P' path from r to parent(x)

consider T − P'

- tree T_x with root x has drawing of height $\leq 2k-1$
- all other trees T' in T − P' have $\mathrm{pw}(T') \leq k-1$

drawing:



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But: There are graphs of small pathwidth that require linear height in every planar grid drawing.

Drawings of Height $O(\mathrm{pw}(T))$

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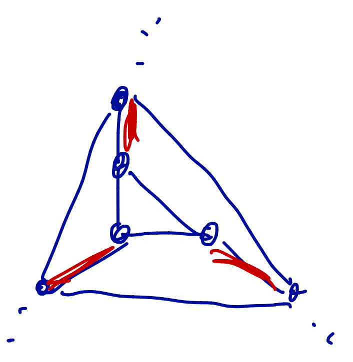
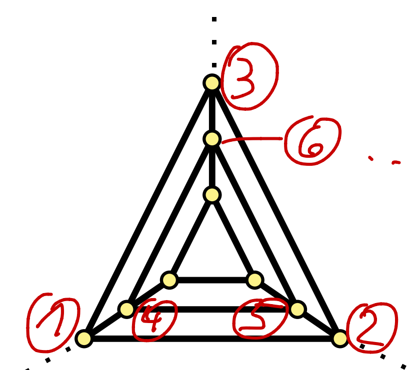


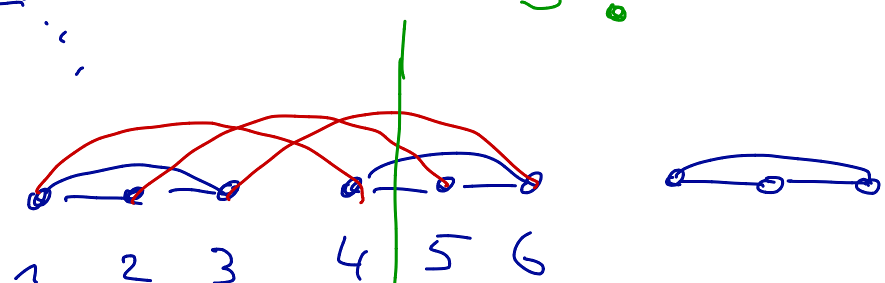
needs height $2 \cdot \frac{n}{3}$



G_n with $3n$ vertices

sw = 3 !

pathwidth?



Further Results

- Every maximal outerplanar graph G can be drawn with height $4\,\mathrm{pw}(G)$. [Biedl '13]
- Every outerplanar graph G can be drawn with height $64\,\mathrm{pw}(G)$. [Babu et al. '13]
- There are series-parallel graphs with pathwidth $O(\log n)$ and height $\Omega(2^{\sqrt{\log n}})$ in every planar grid drawing. [Frati '10]
- For a given integer h and a graph G one can test in time $O(2^{32h^3} n)$, whether a drawing of height h exists. Hence this problem is fixed-parameter tractable (FPT). [Dujmovic et al. '01/'08]
- For small graphs G there is an ILP/SAT model to compute the pathwidth $\mathrm{pw}(G)$. [Biedl et al. '13]