Lab 2: Oliker–Prüssner Method Solving Alexandroff Solution for Monge–Ampère Equation

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Contents

1 Problem Setting and Finite Difference

1.1 Monge–Ampère Equation

We solve the following Monge–Ampère Equation in 2D.

$$\det(D^2u) = f \text{ in } \Omega \tag{1}$$

$$u|_{\partial\Omega} = g \text{ on } \partial\Omega \tag{2}$$

Define the subgradient of a convex function u on some interior point x:

$$\partial u(x) = \{p \in \mathbb{R}^2 | u(y) \leq u(x) + p \cdot (y - x) \text{ for all } y\}.$$

The alexandroff solution satisfies

$$|\partial u| = f$$

for all interior points. The concept can be generalized to the case when f is a measure, means

$$|\cup_{x\in E}\partial u(x)| = |f(E)|$$

For $f > 0$, we can show the existence and uniqueness of Alexandroff solution. Moreover, the Alexandroff solution and the viscosity solution are the same. We omit these concept and only call the solution of MA equation hereafter.

1.2 Oliker–Prüssner Method

For nodal function u_h on point set $X_h = \{x_i\}$, the OP method seek for a solution

$$|\partial u_h(x_i)| = f_h(x_i) x_i \in \mathcal{N}_\mathrm{h}^\mathrm{I} \tag{3}$$

$$u|_{\partial\Omega} = g_h(x_i) x_i \in \mathcal{N}_\mathrm{h}^\partial \tag{4}$$

Here the subgradient of a nodal function is defined as

$$\partial u_h(x_i) = \{p \in \mathbb{R}^2 | u_h(x_j) \le u_h(x_i) + p\cdot(x_j - x_i) \text{ for all } x_j\}.$$

We now bridge the connection between subgradients that of function and nodal function. Consider the convex envelope

$$\Gamma u_h(x) = \sup\{L(x) : L \text{ is affine and } L(x_i) \le u_h(x_i)\}.$$

A nodal function is convex if and only if $u_h(x_i) = \Gamma u_h(x_i)$. The connection is

Proposition 1.

$$\partial u_h(x_i) = \partial\Gamma u_h(x_i).$$

The existence of OP solution is guaranteed by Perron's iteration. Suppose we have a good initial value such that $u_h^0(x_i) = g_h(x_i)$ for $x_i \in \mathcal{N}_\mathrm{h}^\partial$ and $|\partial u_h^0(x_i)| \ge x_i$ for $x_i \in \mathcal{N}_\mathrm{h}^\mathrm{I}$. Then consider the following algorithm:

Algorithm 1 Framework of Perron's Iteration

```
1: while not converge do
2:   for $x_i \in \mathcal{N}_\mathrm{h}^\mathrm{I}$ do
3:     Find $u$ such that $|\partial\tilde{u}_h(x_i)| = f_h(x_i)$, where $\tilde{u}_h$ modify the value of $u_h$ at $x_i$ to $u$.
4:     Update $U_h(x_i) = u$.
5:   end for
6: end while
```

Algorithm 1 generates a sequence u_h^k, we can prove the following things by induction and the monotonicity of subgradient.

Proposition 2. *$u_h^{k+1}(x_i) \ge u_h^k(x_i)$, and each u_h^k is convex. Moreover,*

$$|\partial\tilde{u}_h^k(x_i)| \ge f_h(x_i)$$

After this, the existence of OP solution is clear: Since u_h^k is convex, we have

$$u_h(x_i) \le \sup\{L(x) : L \text{ is affine and } L(x_i) \le u_h(x_i) \text{ for all } x_i \in nhp\}.$$

Hence $u_h^k(x_i)$ is bounded and increasing thus convergent. Denote the limit as $u_h^\infty(x_i)$, argument based on continuous dependency yields that u_h^∞ is our desired OP solution. In practice we choose the convergence criterion as $u_h^k(x_i) < u_h^{k-1}(x_i) + tol$ for all interior points.

1.3 Computations of subgradient

However, the subgradient is not computable, hence we need a little more work to derive our algorithm. Consider a convex nodal function u_h. Since Γu_h is clearly piecewise linear function, thus we can define a mesh $\mathcal{T}$ such that the Lagrange interpolation of $u_h(x_i)$ under $\mathcal{T}$ is exactly Γu_h:

$$\mathcal{I}_{\mathcal{T}} u_h(x) = \Gamma u_h(x).$$

We call $\mathcal{T}$ be the induced mesh of u_h. The key point is that

Proposition 3. *Suppose u_h is convex. Then $\mathcal{T}$ is the induced mesh if and only if $\mathcal{I}_{\mathcal{T}} u_h$ is convex.*

Proof. Clearly Γu_h has the following equivalent definition:

$$\Gamma u_h(x_i) = \sup\{ \ \mathcal{I}_{\mathcal{T}} u_h(x) : \mathcal{T} \text{ over all possible mesh } \}.$$

We only need to verify that $\mathcal{I}_{\mathcal{T}} u_h(x_i) \geq \Gamma u_h(x_i)$. Since for any simplex $T \in \mathcal{T}'$ containing x_i, the convexity of $\mathcal{I}_{\mathcal{T}} u_h$ yields that $\mathcal{I}_{\mathcal{T}'} u_h(x_i) \leq \mathcal{I}_{\mathcal{T}} u_h(x_i)$. Take supremum on $\mathcal{T}'$ we conclude the result. ☐

Notice that $\mathcal{I}_{\mathcal{T}} u_h$ is equivalent to all jump of interior face $F = T^+ \cap T^-$:

$$[\mathcal{I}_{\mathcal{T}} u_h]_F = -(\nabla \mathcal{I}_{\mathcal{T}} u_h)|_{T^+} \cdot \mathbf{n}^+ - (\nabla \mathcal{I}_{\mathcal{T}} u_h)|_{T^-} \cdot \mathbf{n}^-$$

is nonnegative. This will be a good criterion in practice to check whether $\mathcal{T}$ is the induced mesh. The computation of subgradient are made easy once the induced mesh is obtained, according to the following proposition.

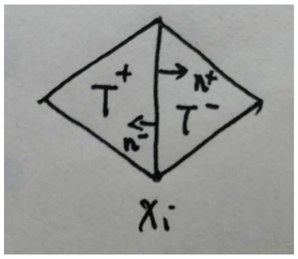


Figure 1: Jump of an edge

Proposition 4. *Suppose u_h is convex and $\mathcal{T}$ is its induced mesh. Then the subgradient of $u_h(x_i)$ is the area of*

$$ConvexHull(\{\nabla \mathcal{I}_{\mathcal{T}} u_h : x_i \text{ is attached to} T \in \mathcal{T}.\}.)$$

Moreover, if we swap all T around x_i clockwisely, then the gradient we obtained is clockwise. Moreover, all gradient lie in the convex hull. Hence the convex hull is obtained automatically.

2 Implementation Details

In this section we deal with two issues. The first is how to get a suitable initial value (together with the initial induced mesh). The second is how to update the value in perron's iteration and update the induced mesh. We first introduce the data structure used in implementation.

2.1 Data Structure

X,Y : N array, stores the position of nodes.
U: N array, stores the value of nodes.
G: N array, stores the exact solution (and boundary value therefore) of nodes.
F: N array, stores RHS.
elem: $NT \times 3$ array, stores which three nodes form an element. (in counterclockwise order as usual).
adj: (adjacent linked list)$N \times 40 \times 3$ array, stores the elements around node i (in clockwise order). Here we use array to simulate the linked list since many operation we encounter is add and delete. Each $adj(i,:,:)$ is a linked list, with $adj(i,x,1)$ is the corresponding element, $adj(i,x,2)$ indicates the location in element, i.e. satisfying the equation

$$elem(adj(i,x,1), adj(i,x,2)) = i$$

and $adj(i,x,3)$ indicates the entry next to x in the linked list.
elemind: (element indexs) $NT \times 3$ array. $elemind(T,i)$ tells us where T is on the linked list $adj(i,:,:)$.
bdadj: $N \times 2$. For boundary points, $bdadj(i,1), bdadj(i,2)$ is the neighborhood of i. Used in lifting the boundary.

Here are several comments.
1. From only one **elem** array, we can recover all the auxiliary arrays, the implementation is in **meshinit.m** and we omit it here.
2. There are slight difference of adjacent linked list for interior point and boundary point. The former is looped while thee latter is not. However the implementation will treat both as looped linked list, and once we need lift the boundary value (as we will see), we need to drop the virtual link by some special judge. (See Figure 2)

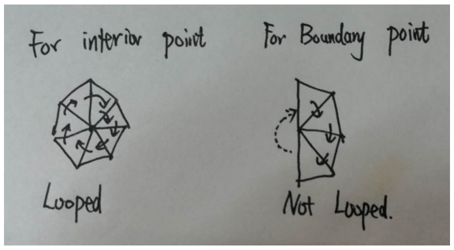


Figure 2: Linked list for interior point and boundary point.

2.2 Mesh Flipping

In this subsection we deal with how to get the induced mesh if we lift value at x_i. When lifting x_i, two kinds of edge will change their jump value. 1) edges adjacent to x_i; 2) edges opposite to x_i. The next result shows we only need to consider the adjacent ones.

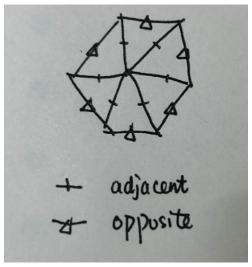


Figure 3: Two kinds of edges around x_i

Proposition 5. *Suppose u_h is convex and $\mathcal{T}$ is its induced map. Suppose F is opposite to x_i, then $[\mathcal{I}_{\mathcal{T}} u_h]_F$ is increasing w.r.t x_i. Suppose F is adjacent tto x_i, then $[\mathcal{I}_{\mathcal{T}} u_h]_F$ is decreasing w.r.t x_i.*

Proof. WLOG we assume ABC, ABD is two elements, with $A = (-1, 0), B = (1, 0), C = (x_1, y_1), D = (x_2, -y_2)$, where $y_1, y_2 > 0$. Then the jump is $u_h(C)/y_1 + u_h(D)/y_2$, increasing w.r.t two opposite nodes. The second statement can be proved in a similar way. □

Hence we only need to consider the adjacent edges. For each adjacent edge F_t, suppose $F_t = T_t \cap T_{t+1}$. We first notice that since $(\nabla \mathcal{I}_{\mathcal{T}} u_h)|_{T_t}$ is linear on $u_h(x_i)$, we can find $P_t, Q_t \in \mathbb{R}^2$ such that $(\nabla \mathcal{I}_{\mathcal{T}} u_h)|_{T_t} = P_t - u_h(x_i)Q_t$. Suppose the normal vector of F_t on T_t is N_t, then the jump is

$$P_t \cdot N_t - u_h(x_i)Q_t \cdot N_t - P_{t+1} \cdot N_{t+1} + u_h(x_i)Q_{t+1} \cdot N_{t+1}.$$

By Proposition 5, we know if $u_h(x_i) \le \theta_t$, then $[\mathcal{I}_{\mathcal{T}} u_h]_{F_t} \le 0$. Here

$$\theta_t = \frac{P_t \cdot N_t - P_{t+1} \cdot N_{t+1}}{Q_t \cdot N_t - Q_{t+1} \cdot N_{t+1}}. \tag{5}$$

Hence $u_h(x_i) = U_i$ makes $|\partial u_h(x_i)| = f_h(x_i)$. We conclude that if $U_i < \min_t \theta_t$ then we can use the current $\mathcal{T}$ to compute subgradient. Otherwise we should flip the edge $t_0 := \arg\min \theta_t$, and recalculate all things around x_i.

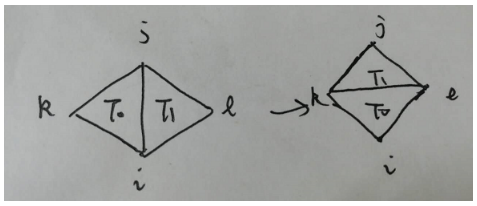


Figure 4: Edge flipping

The flip procedure is simple (see Figure 4): we just change *elem*$(T_0, :)$ and *elem*$(T_1, :)$, then update the adjacent linked list about i, j, k, l. At last we update the *elemind* array. It is noticeable that flipping has very nice property.

Proposition 6. *Suppose $u_h(x_i) = \theta_{t_0}$ and we have flipped the edge $F_{t_0} = T_0 \cap T_1$ into $F' = T_0' \cap T_1'$. Then the jump of other edges do not change, while the jump of new value is increasing w.r.t. x_i.*

Proof. The only thing need to verify is edge like ik, lk, jk, lk. This is because in the threshold, the values in x_i, x_j, x_k, x_l are in the same plane. Hence the jump of ik, for example, will not change. The latter statement is because F' is opposite to x_i. $\square$

In practice, we use subfunction *getdis* to compute P_t, Q_t, N_t.

2.3 Lift the Boundary

Now we are ready to face the first problem: how to get a suitable initial value for Perron's iteration. We first choose

$$u_h(x_i) = \frac{A}{2}(x_i)^2 - R^2$$

such that u_h is negative on $\mathcal{N}_h^\partial$. The A is chosen sufficiently large such that $|\partial u_h(x_i)| \geq f_h(x_i)$. We have proved that for the quadratic function, Delaunay triangulation gives the induced mesh $\mathcal{T}_0$. Thus we can compute the subgradient using $\mathcal{T}_0$ and then choose a suitable A.

Now we use following algorithm to lift the boundary value from $u_h(x_i)$ to $g_h(x_i)$, see Algorithm 2.

We now show the algorithm will stop and give the correct $u_h(x_i)$. The value of $u_h(x_i)$ will increase until either reaches $g_h(x_i) u_h$ is not convex at x_i. The only possible case is $u_h(x_i) = \theta_{bd}$. Thus each step we can lift $u_h(x_i)$ to $\min(g_h(x_i), \theta_{bd})$.

Next we show for each step if we assign $u_h(x_i)$ to $\min(g_h(x_i), \theta_{bd})$, then it will converge. We have obtained a sequence $u_h^k(x_i)$, which is increasing and bounded. Thus it has a limit v_h.

Proposition 7. $v_h(x_i) = g_h(x_i)$.

Proof. Routine argument gives us that if x_j, x_k is the neighborhood of x_i, then

$$v_h(x_i) = \min(g_h(x_i), \frac{1}{2}(v_h(x_j) + v_h(x_k))).$$

For corner point, we certainly have $v_h(x_i) = g_h(x_i)$. Denote $S = \{i | v_h(x_i) = g_h(x_i)\}$. We prove by contradiction, suppose for some x_i, $v_h(x_i) < g_h(x_i)$, we choose x_i such that $v_h(x_i)$ is smallest and the distance from x_i to four corners is smallest. Then $v_h(x_i) = \frac{1}{2}(v_h(x_j) + v_h(x_k)))$ tells us that either j or k is in S. Suppose $j \in S$, then $k \notin S$, otherwise $v_h(x_i) \geq g_h(x_i)$.

Suppose $l_{-1} = j, l_0 = i, l_1 = k, \cdots, l_m$ are a series such that l_i and l_{i+1} are neighborhood, with $l_{-1}, l_m \in S$ while others $\notin S$. Then

$$v_h(x_{l_s}) = \frac{1}{2}(v_h(x_{l_{s-1}}) + v_h(x_{l_{s+1}}))$$

for $s = 0, 1, \cdots m-1$. Again we deduce that $v_h(x_i) \geq g_h(x_i)$ since v_h is linear, g_h is convex, with the same boundary value. Therefore all boundary points are in S. $\square$

2.4 Perron's Update

We deal with Perron's update for some interior point x_i, notice that the area of subgradient is a piecewise quadratic function. We have the following algorithm, see Algorithm 3.

Here subgrad is a function compute the area of polygon formed by all discrete gradient over T_t.

Algorithm 2 Lift the Boundary

```
 1: inputs: $u_h, g_h$.
 2: while $\max_{x_i \in \mathcal{N}_\mathrm{h}^\partial} g_h(x_i) - u_h(x_i) > tol$ do
 3:   for $x_i \in \mathcal{N}_\mathrm{h}^\partial$ do
 4:     if $g_h(x_i) - u_h(x_i) < tol$ then
 5:       continue
 6:     end if
 7:     while true do
 8:       for $t = 1 : size(adj(x_i)) + 1$ do
 9:         if $T_t$ and $T_{t+1}$ are not adjacent (the virtual link) then
10:           continue
11:         end if
12:         Compute $\theta_t$ by (5).
13:       end for
14:       $t_0 = \arg\min \theta_t$.
15:       if $x_i$ is not corner point then
16:         $\theta_{bd} = \frac{1}{2}(u_h(x_j) + u_h(x_k))$ (average of two neighborhood).
17:       else
18:         $\theta_{bd} = \infty$
19:       end if
20:       $\theta_0 = \min(\theta_{t_0}, \theta_{bd})$
21:       if $\theta_0 > g_h(x_i)$ then
22:         $u_h(x_i) := g_h(x_i)$
23:         break
24:       end if
25:       if $\theta_0 > \theta_{bd} - \varepsilon$ then
26:         $u_h(x_i) := \theta_{bd}$
27:         break
28:       else
29:         Flip $F_{t_0}$
30:         $u_h(x_i) = \theta_0$.
31:         continue
32:       end if
33:     end while
34:   end for
35: end while
```

Algorithm 3 Perron's Update

```
1: inputs: $u_h, f_h, x_i$.
2: while true do
3:   for $t = 1 : size(adj(x_i)) + 1$ do
4:     if $T_t$ and $T_{t+1}$ are not adjacent (the virtual link) then
5:       continue
6:     end if
7:     Compute $\theta_t$ by (5).
8:   end for
9:   $t_0 = \arg\min \theta_t$.
10:  if $subgrad(\theta_{t_0}) > f_h(x_i)$ then
11:    Flip $F_{t_0}$
12:    $u_h(x_i) := \theta_{t_0}$
13:    continue
14:  else
15:    Compute $C = subgrad(u_h(x_i))/(u_h(x_i) - \theta_{t_0})^2$
16:    $u_h(x_i) := \theta_{t_0} - \sqrt{f_h(x_i)/C}$
17:  end if
18: end while
```

3 Numerical Result

In this section we test OP method for several result: 1) smooth solution. 2) piecewise smooth solution. 3) Singular solution. Please see **loadfunction.m** for details.

We show the L^∞ error and discrete W_2^p error on $h = 1/2, 1/4, 1/8$, here the tolerance of both lift the boundary algorithm and Perron's iteration is chosen as $\varepsilon = 1e-10, 1e-6$, and the quadrate rule we use $quadpts(1)$ as low order and $quadpts(4)$ as high order.

3.1 tol = 1e-10, low order quadrate rule

Table 1: Problem 1

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	2	0.00	1.12E-01	0.00	6.15E-01	0.00	4.24E-01	0.00
1/2	568	0.06	4.78E-02	1.23	5.39E-01	0.19	3.42E-01	0.31
1/4	4093	2.16	1.37E-02	1.80	2.51E-01	1.10	1.66E-01	1.04
1/8	20428	48.53	3.55E-03	1.95	8.35E-02	1.59	5.66E-02	1.55

Table 2: Problem 2

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	2	0.00	4.02E-01	0.00	2.20E+00	0.00	1.52E+00	0.00
1/2	326	0.04	4.19E-02	3.26	8.23E-01	1.42	5.83E-01	1.38
1/4	2196	1.20	2.90E-02	0.53	7.89E-01	0.06	6.21E-01	-0.09
1/8	18374	44.43	1.27E-02	1.19	6.25E-01	0.34	5.04E-01	0.30

Table 3: Problem 3

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	26	0.00	8.58E-01	0.00	4.70E+00	0.00	3.24E+00	0.00
1/2	326	0.04	2.37E-01	1.86	5.57E+00	-0.24	3.66E+00	-0.17
1/4	2196	1.42	1.87E-01	0.34	5.61E+00	-0.01	4.26E+00	-0.22
1/8	18374	29.20	8.54E-02	1.13	5.50E+00	0.03	4.77E+00	-0.16

3.2 tol = 1e-6, low order quadrate rule

Table 4: Problem 1

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	2	0.00	1.12E-01	0.00	6.15E-01	0.00	4.24E-01	0.00
1/2	241	0.03	4.79E-02	1.23	5.40E-01	0.19	3.42E-01	0.31
1/4	911	0.63	1.41E-02	1.77	2.51E-01	1.10	1.66E-01	1.04
1/8	2614	9.41	4.27E-03	1.72	8.45E-02	1.57	5.65E-02	1.55

Table 5: Problem 2

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	2	0.00	4.02E-01	0.00	2.20E+00	0.00	1.52E+00	0.00
1/2	170	0.03	4.18E-02	3.26	8.23E-01	1.42	5.82E-01	1.38
1/4	949	0.70	2.91E-02	0.53	7.90E-01	0.06	6.21E-01	-0.09
1/8	4149	13.37	1.35E-02	1.10	6.28E-01	0.33	5.06E-01	0.30

Table 6: Problem 3

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	15	0.00	8.58E-01	0.00	4.70E+00	0.00	3.24E+00	0.00
1/2	176	0.03	2.37E-01	1.86	5.57E+00	-0.24	3.66E+00	-0.17
1/4	841	0.66	1.86E-01	0.35	5.61E+00	-0.01	4.26E+00	-0.22
1/8	2196	8.25	8.49E-02	1.13	5.51E+00	0.03	4.77E+00	-0.17

3.3 tol = 1e-10, high order quadrate rule

Table 7: Problem 1

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	2	0.00	8.10E-02	0.00	4.43E-01	0.00	3.06E-01	0.00
1/2	481	0.05	2.59E-02	1.64	3.14E-01	0.50	2.04E-01	0.58
1/4	3268	1.89	7.02E-03	1.88	1.43E-01	1.13	9.76E-02	1.06
1/8	17112	46.10	1.79E-03	1.97	4.79E-02	1.58	3.35E-02	1.54

Table 8: Problem 2

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	2	0.00	1.92E-01	0.00	1.05E+00	0.00	7.24E-01	0.00
1/2	369	0.04	3.50E-02	2.45	7.07E-01	0.57	4.63E-01	0.65
1/4	538	2.91	1.47E-02	1.25	3.95E-01	0.84	2.84E-01	0.70
1/8	97076	259.91	4.93E-03	1.58	3.01E-01	0.39	2.57E-01	0.15

Table 9: Problem 3

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	78	0.00	2.23E-01	0.00	1.22E+00	0.00	8.45E-01	0.00
1/2	288	0.04	1.31E-01	0.77	2.82E+00	-1.20	1.90E+00	-1.17
1/4	1067	0.72	4.74E-02	1.46	2.72E+00	0.05	1.89E+00	0.01
1/8	19054	51.78	1.55E-02	1.61	2.37E+00	0.20	1.84E+00	0.04

3.4 tol = 1e-6, high order quadrate rule

Table 10: Problem 1

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	2	0.00	8.10E-02	0.00	4.43E-01	0.00	3.06E-01	0.00
1/2	211	0.03	2.60E-02	1.64	3.14E-01	0.50	2.04E-01	0.58
1/4	893	0.66	7.21E-03	1.85	1.44E-01	1.13	9.74E-02	1.07
1/8	2544	9.18	2.70E-03	1.42	4.95E-02	1.54	3.35E-02	1.54

Table 11: Problem 2

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	2	0.00	1.92E-01	0.00	1.05E+00	0.00	7.24E-01	0.00
1/2	188	0.02	3.50E-02	2.45	7.08E-01	0.57	4.63E-01	0.65
1/4	1527	1.18	1.49E-02	1.23	3.97E-01	0.84	2.86E-01	0.69
1/8	2447	9.29	6.22E-03	1.26	3.12E-01	0.35	2.65E-01	0.11

Table 12: Problem 3

	iter	time(s)	L_∞	order	W_1^2	order	W_2^2	order
1	44	0.00	2.23E-01	0.00	1.22E+00	0.00	8.45E-01	0.00
1/2	149	0.02	1.31E-01	0.77	2.82E+00	-1.20	1.90E+00	-1.17
1/4	496	0.43	4.74E-02	1.46	2.72E+00	0.05	1.89E+00	0.01
1/8	1927	7.43	1.57E-02	1.59	2.37E+00	0.20	1.83E+00	0.04

We also plot the converge history for small cases.

Figure 5: Convergence History of three problems, x-axis: iter, y-axis: $\log_{10}(\|U - U_0\|_\infty)$. $h = 1/2$.

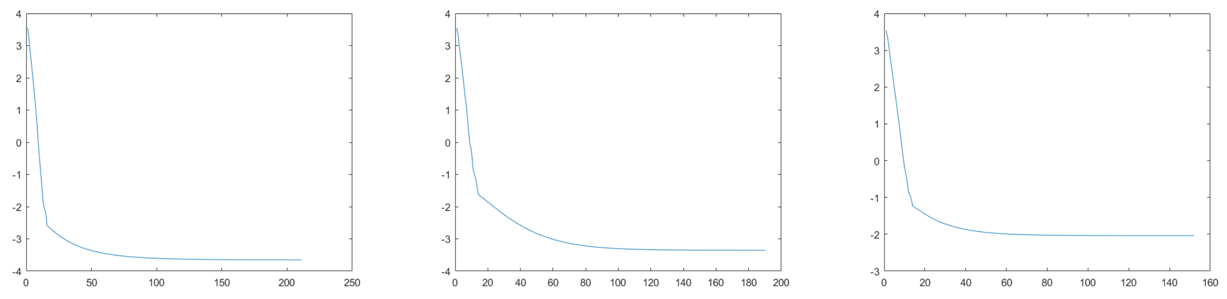
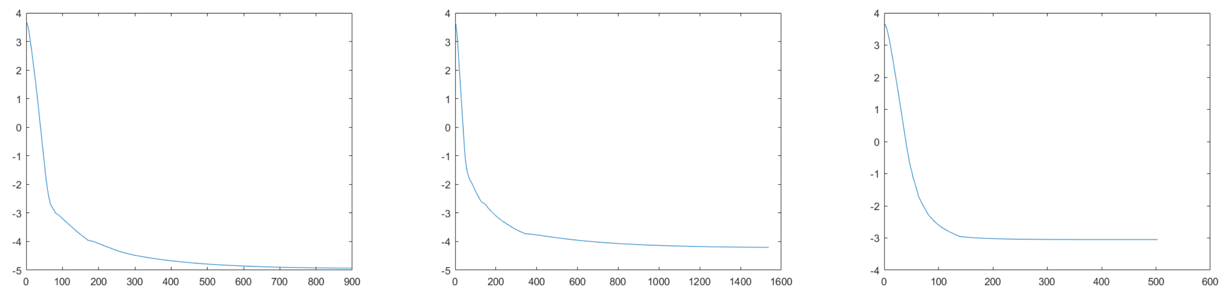


Figure 6: Convergence History of three problems, x-axis: iter, y-axis: $\log_{10}(\|U - U_0\|_\infty)$. $h = 1/4$.



3.5 Discussion

When u is smooth enough, we can observe the second order convergence, both W_p^2 and L_∞ norm. But when u is piecewise smooth or singular, the order cannot be observed so clearly.

Also, the error seems heavily dependent on the numerical integration, as we see in the table. Early stop will not damage the structure.

In experiment, the induced mesh will change usually reflects a big decreases of error. (In the figures above, mesh only changes in first circa 1/10 iterations.) This may enlighten us to develop an accelerated version of Perron's iteration, either using parallel techniques or using a nonlinear solver.