Lab 5: Numerical ODE

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Contents

In this report we introduce several explicit numerical schemes of ODE, including both single step method (Runge–Kutta Scheme) and multi-step method (Prediction–Correction Scheme). We will introduce the algorithms, convergence analysis and stabilization analysis. In numerical experiments, we will use a toy example to check the correctness and show the numerical simulation of Lorentz model.

1 Explicit Runge–Kutta Scheme

For a given equation $\dot{y} = f(x, y)$, assume the selected time step is h, and initial value x_0, y_0. For step n, we consider the following scheme,

$$y_{n+1} = y_n + h(c_1 K_1 + \cdots c_m K_m) \tag{1}$$

where

$$K_j = f(x_j + a_j h, y_n + h \sum_{i=0}^{j-1} b_{ji} K_i)$$

This type of numerical ode scheme is called explicit Runge–Kutta Scheme, associated with Butcher Tableau $\{a, B, c\}$.

We list several RK scheme and its order.

1. Forward Euler.

$$a = [1]$$

$$b = [0]$$

,

$$c = [1]$$

its order is 1.

2. Improved Euler.

$$a = [0, 1]$$
$$b = \begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix}$$
$$c = [1/2, 1/2]$$

its order is 2.

3. Heun.

$$a = [0, 2/3]$$
$$b = \begin{bmatrix} 0 & 2/3 \\ 0 & 0 \end{bmatrix}$$
$$c = [1/4, 3/4]$$

its order is 2.

4. Kutta3

$$a = [0, 1/2, 1]$$
$$b = [[], [1/2], [-1, 2]]$$

(Here we only show its lower-triangular part)

$$c = [1/6, 2/3, 1/6]$$

its order is 3.

5. RK4

$$a = [0, 1/2, 1/2, 1]$$
$$b = [[], [1/2], [0, 1/2], [0, 0, 1]]$$
$$c = [1/6, 1/3, 1/3, 1/6]$$

its order is 4.

2 Prediction–Correction Scheme

In multistep method we usually use a explicit scheme to predict the value of y_{n+1} and use the predicted value to obtain a more accurate value by an implicit scheme, this is called prediction–correction scheme. We will introduce two popular scheme, improved euler and adams4.

To obtain the computational value of y_{n+1}, we consider the following scheme. For P step, consider

$$y^* = y_n + h \sum_{i=1}^{M} p_i f(x_{n+1-i}, y_{n+1-i})$$

. For C step, consider

$$y_{n+1} = y_n + h \sum_{i=2}^{N} c_i f(x_{n+2-i}, y_{n+2-i}) + h f(x_n, y^*)$$

. We consider PC scheme of this type throughout this paper. Two examples will be considered

1. Improved Euler.

$$p = [1]$$

$$c = [1/2, 1/2]$$

its order is 2.

$$p = [55/24, -59/24, 37/24, -9/24]$$

$$c = [9/24, 19/24, -5/24, 1/24]$$

3 Numerical Experiments

3.1 Test accuracy and order

We use simple test function $y(x) = e^x$, hence $f(x, y) = y$ and $y(0) = 1$. The result is shown in the following table.

method\stepsize	0.01	0.001	order
Euler	1.346800e-02	1.357896e-03	1.00
iEuler	6.764706e-03	6.792591e-04	1.00
Heun	4.496590e-05	4.527073e-07	2.00
Kutta3	1.123594e-07	1.131708e-10	3.00
RK4	2.246421e-10	2.042810e-14	4.04
Adams4	6.471996e-10	6.616929e-14	3.99

3.2 Lorentz system

In this subsection we show the numerical simulation of Lorentz system. Lorenz system is the following ODE system:

$$dx/dt = \sigma(y - x)$$

$$dy/dt = x(\rho - z) - y$$

$$dz/dt = xy - \beta z$$

We will simulate the evolution for various initial value and parameters σ, ρ and β.

We first consider a fixed parameter:$\sigma = 10, \rho = 8, \beta = 8/3$. And we choose different initial value to begin our simulation.

Figure 1: $x_0 = [-10, 10, 25]$

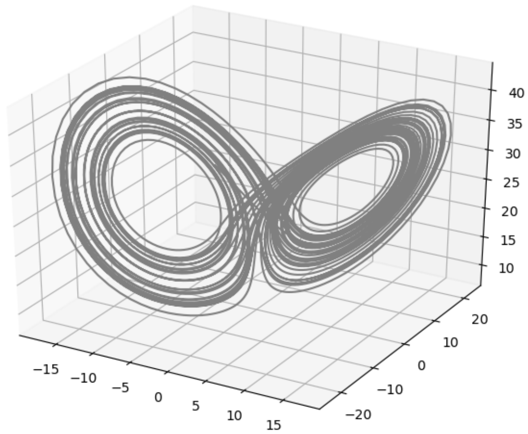


Figure 2: $x_0 = [6, -7, 3]$

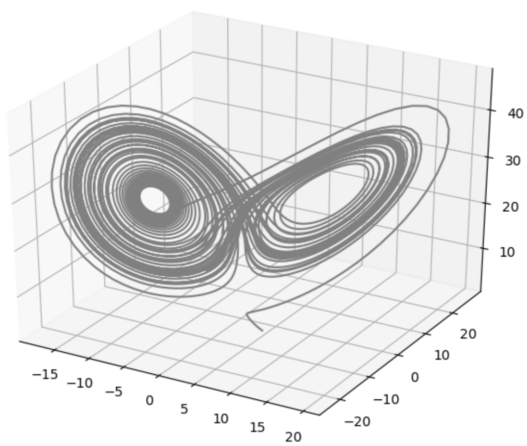


Figure 3: $x_0 = [0, 0, 1]$

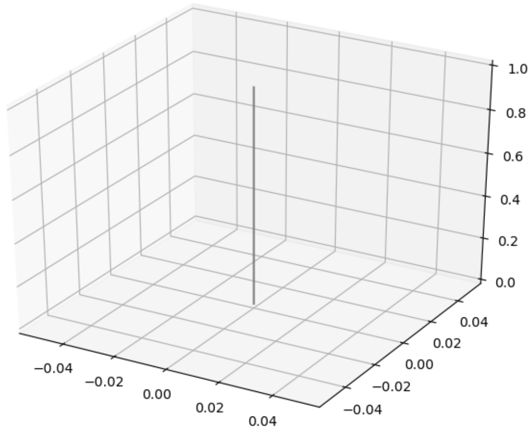
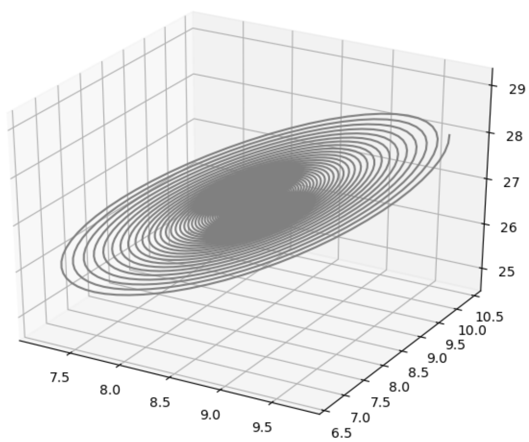


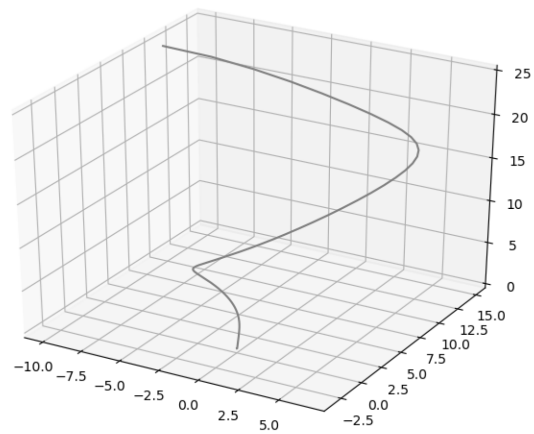
Figure 4: $x_0 = [8.5, 8.5, 27]$



From these figures we can know there are several types, dependent on the choice of initial value. 1) Converges to a unstable stationary point. 2) Around unstable stationary point 3)Lorentz attractor.

Next we examine the choice of parameters.

Figure 5: $x_0 = [-10, 10, 25], \sigma = 28, \rho = 0.5, \beta = 8/3$. When $\rho < 1$, we find the system will converge to $[0, 0, 0]$



When $\rho > 1$ we find that there are three stationary point $[0, 0, 0], [\pm\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1]$. Interesting behavior will be found if we choose different ρ.

Figure 6: $x_0 = [-10, 10, 25], \sigma = 20, \rho = 14, \beta = 8/3$.The system will converge to $[0, 0, 0]$

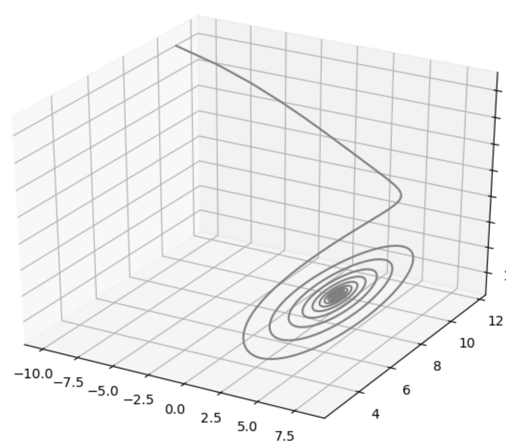
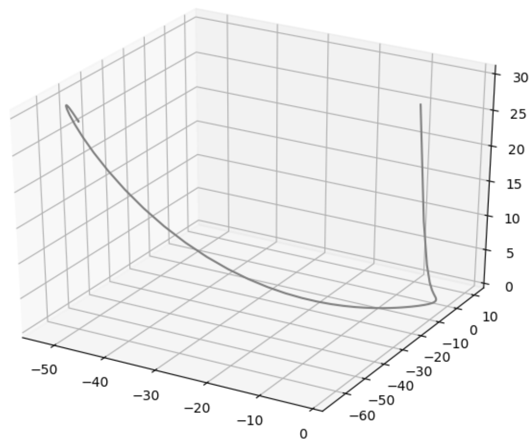


Figure 7: $x_0 = [-10, 10, 25], \sigma = 10, \rho = 28, \beta = 100$.The system will converge to $[\pm\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1]$



Extensive experiments show that when $\sigma < \beta + 1$, the convergence result is always correct.
When $\sigma > \beta + 1$ the result can be either convergent or chaotic.