

Solutions to Final Exam

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1

1.1

Lagrange's Formula:

$$\begin{aligned}
 P(x) &= 0 \frac{(x+0.5)(x-0.5)}{(-1+0.5)(-1-0.5)} - 1 \frac{(x+1)(x-0.5)}{(-0.5+1)(-0.5-0.5)} + 1 \frac{(x+1)(x+0.5)}{(0.5+1)(0.5+0.5)} \\
 &= \frac{8}{3}x^2 + 2x - \frac{2}{3}
 \end{aligned} \tag{1}$$

Newton's Formula: We first calculate that $f[-1; -0.5] = -2$, $f[-0.5; 0.5] = 2$, $f[-1; -0.5; 0.5] = \frac{8}{3}$. The by Newton's formula we calculate the interpolation polynomial:

$$\begin{aligned} P(x) &= f[-1] + f[-1; -0.5](x+1) + f[-1; -0.5; 0.5](x+1)(x+0.5) \\ &= 0 - 2(x+1) + \frac{8}{3}(x+1)(x+0.5) \\ &= \left(\frac{8}{3}x - \frac{2}{3}\right)(x+1) \end{aligned} \quad (2)$$

1.2

The piecewise interpolation function

$$P(x) = \begin{cases} -2(x+1) & -1 \leq x \leq -0.5 \\ 2x & -0.5 \leq x \leq 0.5 \end{cases} \quad (3)$$

And interpolation value become vacuous when x is outside the interval $[-1, 0.5]$.

1.3

The basis of spline function is

$$1, x+0.5, (x+0.5)^2, (x+0.5)^3, (x+0.5)_+^3.$$

Here $a_+ = \max(a, 0)$. We suppose the spline function has the form

$$P(x) = a_1(x+0.5)_+^3 + a_2(x+0.5)^3 + a_3(x+0.5)^2 + a_4(x+0.5) + a_5.$$

Then five conditions

$$P(-1) = 0, P(-0.5) = -1, P(0.5) = 1, P''(-1) = 0, P''(0.5) = 0$$

become

$$-\frac{1}{8}a_2 + \frac{1}{4}a_3 - \frac{1}{2}a_4 + a_5 = 0 \quad (4)$$

$$a_5 = -1 \quad (5)$$

$$a_1 + a_2 + a_3 + a_4 + a_5 = 1 \quad (6)$$

$$-3a_2 + 2a_3 = 0 \quad (7)$$

$$6a_1 + 6a_2 + 2a_3 = 0 \quad (8)$$

Solve the system we find

$$a_1 = -4, a_2 = \frac{8}{3}, a_3 = 4, a_4 = -\frac{2}{3}, a_5 = -1,$$

and therefore the spline function is

$$P(x) = -4(x+0.5)_+^3 + \frac{8}{3}(x+0.5)^3 + 4(x+0.5)^2 + -\frac{2}{3}(x+0.5) - 1.$$

Again, the function value is vacuous when x outside $[-1, 0.5]$, as we indicated in previous subproblem.

1.4

The following least square problem is

$$\Phi(a, b) = (-a + b)^2 + (-0.5a + b + 1)^2 + (0.5a + b - 1)^2 = (3a^2)/2 - 2ab - 2a + 3b^2 + 2.$$

We aim to find a, b minimizing $\Phi(a, b)$. The first order condition yields

$$3a - 2b - 2 = 0, 6b - 2a = 0$$

The solution is $a = \frac{6}{7}, b = \frac{2}{7}$. The linear function is $\frac{6}{7}x + \frac{2}{7}$.

2

2.1

Taking $f(x) = 1, x, x^2$ we have

$$A_1 + 2 + A_3 = 3 \quad (9)$$

$$2x_1 + 3A_3 = \frac{9}{2} \quad (10)$$

$$2x_1^2 + 9A_3 = 9 \quad (11)$$

We have $((11) - 3*(10))$:

$$2(x_1^2 - 3x_1) = -\frac{9}{2} \quad (12)$$

yielding that $x_1 = \frac{3}{2}$. Then we know $A_3 = \frac{1}{2}$ and $A_1 = \frac{1}{2}$. And the scheme is just Simpson scheme. For $f(x) = x^3$, we have

$$LHS = \frac{81}{4} = 2(\frac{3}{2})^3 + \frac{1}{2}3^3 = RHS$$

But for $f(x) = x(3-x)(x-\frac{3}{2})^2$, LHS is non-zero but RHS is zero. Hence the algebraic order of the given scheme is 3.

2.2

Taking $f(x) = 1, x, x^2, x^3$ we have

$$A_1 + A_2 + A_3 + A_4 = 3 \quad (13)$$

$$\frac{1}{2}A_2 + 2A_3 + 3A_4 = \frac{9}{2} \quad (14)$$

$$\frac{1}{4}A_2 + 4A_3 + 9A_4 = 9 \quad (15)$$

$$\frac{1}{8}A_2 + 8A_3 + 27A_4 = \frac{81}{4} \quad (16)$$

This linear system (13), (14), (15), (16) has a unique solution

$$A_1 = 0, A_2 = \frac{6}{5}, A_3 = \frac{3}{2}, A_4 = \frac{3}{10}$$

And it is self-evident that the scheme is of third order from our deduction.

3

Rewrite the equation as

$$\frac{1}{2}\left(x + \frac{a}{x}\right) = x.$$

We aim to find the fixed point of

$$\frac{1}{2}\left(x + \frac{a}{x}\right).$$

We will show that

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right) \quad (17)$$

has local convergence. Direct calculation yields

$$\begin{aligned} x_{k+1} - \sqrt{a} &= \frac{1}{2}\left(x_k - \sqrt{a} + \frac{a}{x_k} - \sqrt{a}\right) \\ &= (x_k - \sqrt{a})\left(\frac{1}{2} - \frac{1}{2}\frac{\sqrt{a}}{x_k}\right) \\ &= \frac{1}{2x_k}(x_k - \sqrt{a})^2 \end{aligned} \quad (18)$$

Then if $|x_k - \sqrt{a}| < \frac{1}{4}\sqrt{a}$, we obtain that

$$|x_{k+1} - \sqrt{a}| \leq \frac{1}{6}|x_k - \sqrt{a}|$$

Hence the scheme converges locally.

From (18), we conclude that the scheme is of second order, and the error constant

$$\lim_{n \rightarrow \infty} \frac{|x_{k+1} - \sqrt{a}|}{|x_k - \sqrt{a}|^2} = \frac{1}{2\sqrt{a}}$$

4

4.1

We expand LHS and RHS resp. and for simplicity we use $f, f[x], f[y]$ to denote $f(x_n, y_n), f_x(x_n, y_n), f_y(x_n, y_n)$.

LHS

$$\begin{aligned} y_{n+1} &= y_n + hf + \\ &\quad + \frac{1}{2}h^2(f[x] + f[y]y') \\ &\quad + \frac{1}{6}h^3(f[xx] + 2f[xy]y' + f[yy]y'^2 + f[y]y'') + O(h^4) \end{aligned} \quad (19)$$

Noticing that $y' = f(x, y(x)) = f, y'' = f[x] + f[y]y' = f[x] + ff[y]$, we have

$$\begin{aligned} y_{n+1} &= y_n + hf + \\ &\quad + \frac{1}{2}h^2(f[x] + f[y]f) \\ &\quad + \frac{1}{6}h^3(f[xx] + 2ff[xy] + f^2f[yy] + f[x]f[y] + ff[y]^2) + O(h^4) \end{aligned} \quad (20)$$

RHS

$$\begin{aligned}
K_2 = & f + \\
& + h(ff[y]b_{21} + f[x]a_2) + \\
& + \frac{1}{2}h^2(f^2f[yy]b_{21}^2 + 2ff[xy]a_2b_{21} + f[xx]a_2^2) + O(h^3)
\end{aligned} \tag{21}$$

$$\begin{aligned}
K_3 = & f + \\
& + h(ff[y]b_{31} + ff[y]b_{32} + f[x]a_3) + \\
& + \frac{1}{2}h^2(f^2f[yy]b_{31}^2 + 2f^2f[yy]b_{31}b_{32} + f^2f[yy]b_{32}^2 + \\
& + 2ff[y]^2b_{21}b_{32} + 2ff[xy]a_3b_{31} + 2ff[xy]a_3b_{32} + 2f[x]f[y]a_2b_{32} + f[xx]a_3^2) + O(h^3)
\end{aligned} \tag{22}$$

Hence

1. Comparing h : $c_1 + c_2 + c_3 = 1$
2. Comparing h^2 term: $f[x]$ term:

$$c_2a_2 + c_3a_3 = \frac{1}{2}$$

 $ff[y]$ term:

$$c_2b_{21} + c_3b_{31} + c_3b_{32} = \frac{1}{2}$$

3. Comparing h^3 term: $f[xx]$ term:

$$c_2a_2^2 + c_3a_3^2 = \frac{1}{3}$$

 $ff[xy]$ term:

$$2c_2a_2b_{21} + 2c_3a_3b_{31} + 2c_3a_3b_{32} = \frac{2}{3}$$

 $f^2f[yy]$ term:

$$c_2b_{21}^2 + c_3b_{31}^2 + 2c_3b_{31}b_{32} + c_3b_{32}^2 = \frac{1}{3}$$

 $f[x]f[y]$ term:

$$2c_3a_2b_{32} = \frac{1}{3}$$

 $ff[y]^2$ term:

$$2c_3b_{21}b_{32} = \frac{1}{3}$$

The eight equations are what we need.

4.2

We choose

$$\begin{aligned} a_2 &= \frac{1}{2}, a_3 = \frac{3}{4} \\ c_1 &= \frac{2}{9}, c_2 = \frac{1}{3}, c_3 = \frac{4}{9} \\ b_{21} &= \frac{1}{2}, b_{31} = 0, b_{32} = \frac{3}{4} \end{aligned}$$

And it is a solution to equations in 4.1.

4.3

From 4.1 and 4.2, we have already known that $y(x_{n+1}) - y(x_n) = O(h^4)$. Based on this, we prove the scheme is of third order. We suppose our scheme is written as

$$y_{n+1} = y_n + h\phi(x_n, y_n, h)$$

We first prove the result provided ϕ is Lipschitz on y .

Proof. We define $\bar{y}_{n+1} = y(x_n) + h\phi(x_n, y(x_n), h)$, then we have $|\bar{y}_{n+1} - y(x_{n+1})| \leq Ch^4$ for some constant C (independent on h).

Set $e_n = |y_n - y(x_n)|$, we found that

$$\begin{aligned} e_{n+1} &= |y_{n+1} - y(x_{n+1})| \\ &\leq |y_{n+1} - \bar{y}_{n+1}| + |\bar{y}_{n+1} - y(x_n)| \\ &\leq |y_n + h\phi(x_n, y_n, h) - y(x_n) - h\phi(x_n, y_n, h)| + Ch^4 \\ &\leq (1 + hL)e_n + Ch^4. \end{aligned} \tag{23}$$

Here L is the Lipschitz constant of ϕ .

We have

$$\begin{aligned} e_{n+1} &\leq (1 + hL)e_n + Ch^4 \\ &\leq (1 + hL)e_{n-1} + C[1 + (1 + hL)]h^4 \\ &\leq \dots \\ &\leq (1 + hL)^n e_0 + C \frac{(1 + hL)^n - 1}{hL} h^4 \\ &\leq \frac{C}{L} h^3 (e^{nhL} - 1) \end{aligned} \tag{24}$$

Since nh is the length of time interval and hence a fixed number, we conclude the result. \square

It suffices to show that ϕ is Lipschitz on y , and the constant is not dependent on $h < 1$. In fact, we show this is correct for all Runge-Kutta formula, provided f is Lipschitz on y .

Suppose that $|f(x, y) - f(x, y')| \leq L_f |y - y'|$, we have

$$|K_1(x_n, y_n) - K_1(x_n, y'_n)| \leq L_f |y - y'| \tag{25}$$

$$|K_2(x_n, y_n) - K_2(x_n, y'_n)| \leq L_f |y - y'| + L_f |h b_{21}| |K_1(x_n, y_n) - K_1(x_n, y'_n)| \leq [L_f + L_f^2 |b_{21}|] |y - y'| \quad (26)$$

$$\begin{aligned} |K_3(x_n, y_n) - K_3(x_n, y'_n)| &\leq L_f |y - y'| + L_f |b_{31}| |K_1(x_n, y_n) - K_1(x_n, y'_n)| + L_f |b_{32}| |K_2(x_n, y_n) - K_2(x_n, y'_n)| \\ &\leq [L_f + L_f^2 |b_{31}| + L_f^2 |b_{32}| + L_f^3 |b_{32} b_{21}|] |y - y'| \end{aligned} \quad (27)$$

Hence we conclude that $\phi(x, y, h)$ is Lipschitz on y , with constant $3L_f + L_f^2 |b_{21}| + L_f^2 |b_{31}| + L_f^2 |b_{32}| + L_f^3 |b_{32} b_{21}|$.