Solutions to Final Exam

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1.1

Lagrange's Formula:

$$\begin{aligned} P(x) =& 0\frac{(x+0.5)(x-0.5)}{(-1+0.5)(-1-0.5)} - 1\frac{(x+1)(x-0.5)}{(-0.5+1)(-0.5-0.5)} + 1\frac{(x+1)(x+0.5)}{(0.5+1)(0.5+0.5)} \\ =& \frac{8}{3}x^2 + 2x - \frac{2}{3} \end{aligned} \tag{1}$$

Newton's Formula: We first calculate that $f[-1;-0.5]=-2, f[-0.5;0.5]=2$ $f[-1;-0.5;0.5]=\frac{8}{3}$. The by Newton's formula we calculate the interpolation polynomial:

$$\begin{aligned}P(x)=&f[-1]+f[-1;-0.5](x+1)+f[-1;-0.5;0.5](x+1)(x+0.5)\\=&0-2(x+1)+\frac{8}{3}(x+1)(x+0.5)\\=&(\frac{8}{3}x-\frac{2}{3})(x+1)\end{aligned} \tag{2}$$

1.2

The piecewise interpolation function

$$P(x)=\begin{cases}-2(x+1) & -1\le x\le -0.5\\ 2x & -0.5\le x\le 0.5\end{cases} \tag{3}$$

And interpolation value become vacuous when x is outside the interval $[-1,0.5]$.

1.3

The basis of spline function is

$$1, x+0.5, (x+0.5)^2, (x+0.5)^3, (x+0.5)^3_+.$$

Here $a_+=\max(a,0)$. We suppose the spline function has the form

$$P(x)=a_1(x+0.5)^3_++a_2(x+0.5)^3+a_3(x+0.5)^2+a_4(x+0.5)+a_5.$$

Then five conditions

$$P(-1)=0, P(-0.5)=-1, P(0.5)=1, P''(-1)=0, P''(0.5)=0$$

become

$$-\frac{1}{8}a_2+\frac{1}{4}a_3-\frac{1}{2}a_4+a_5=0 \tag{4}$$

$$a_5=-1 \tag{5}$$

$$a_1+a_2+a_3+a_4+a_5=1 \tag{6}$$

$$-3a_2+2a_3=0 \tag{7}$$

$$6a_1+6a_2+2a_3=0 \tag{8}$$

Solve the system we find

$$a_1=-4, a_2=\frac{8}{3}, a_3=4, a_4=-\frac{2}{3}, a_5=-1,$$

and therefore the spline function is

$$P(x)=-4(x+0.5)^3_++\frac{8}{3}(x+0.5)^3+4(x+0.5)^2+-\frac{2}{3}(x+0.5)-1.$$

Again, the function value is vacuous when x outside $[-1,0.5]$, as we indicated in previous subproblem.

1.4

The following least square problem is

$$\Phi(a,b) = (-a+b)^2 + (-0.5a+b+1)^2 + (0.5a+b-1)^2 = (3a^2)/2 - 2ab - 2a + 3b^2 + 2.$$

We aim to find a, b minimizing $\Phi(a,b)$. The first order condition yields

$$3a - 2b - 2 = 0, 6b - 2a = 0$$

The solution is $a = \frac{6}{7}, b = \frac{2}{7}$. The linear function is $\frac{6}{7}x + \frac{2}{7}$.

2

2.1

Taking $f(x) = 1, x, x^2$ we have

$$A_1 + 2 + A_3 = 3 \tag{9}$$

$$2x_1 + 3A_3 = \frac{9}{2} \tag{10}$$

$$2x_1^2 + 9A_3 = 9 \tag{11}$$

We have ((11) - 3*(10)):

$$2(x_1^2 - 3x_1) = -\frac{9}{2} \tag{12}$$

yielding that $x_1 = \frac{3}{2}$. Then we know $A_3 = \frac{1}{2}$ and $A_1 = \frac{1}{2}$. And the scheme is just Simpson scheme. For $f(x) = x^3$, we have

$$LHS = \frac{81}{4} = 2(\frac{3}{2})^3 + \frac{1}{2}3^3 = RHS$$

But for $f(x) = x(3-x)(x-\frac{3}{2})^2$, LHS is non-zero but RHS is zero. Hence the algebraic order of the given scheme is 3.

2.2

Taking $f(x) = 1, x, x^2, x^3$ we have

$$A_1 + A_2 + A_3 + A_4 = 3 \tag{13}$$

$$\frac{1}{2}A_2 + 2A_3 + 3A_4 = \frac{9}{2} \tag{14}$$

$$\frac{1}{4}A_2 + 4A_3 + 9A_4 = 9 \tag{15}$$

$$\frac{1}{8}A_2 + 8A_3 + 27A_4 = \frac{81}{4} \tag{16}$$

This linear system (13), (14), (15), (16) has a unique solution

$$A_1 = 0, A_2 = \frac{6}{5}, A_3 = \frac{3}{2}, A_4 = \frac{3}{10}$$

And it is self-evident that the scheme is of third order from our deduction.

3

Rewrite the equation as

$$\frac{1}{2}(x+\frac{a}{x})=x.$$

We aim to find the fixed point of

$$\frac{1}{2}(x+\frac{a}{x}).$$

We will show that

$$x_{k+1}=\frac{1}{2}(x_k+\frac{a}{x_k}) \tag{17}$$

is locally convergent. Direct calculation yields

$$\begin{aligned} x_{k+1}-\sqrt{a} &= \frac{1}{2}(x_k-\sqrt{a}+\frac{a}{x_k}-\sqrt{a}) \\ &= (x_k-\sqrt{a})(\frac{1}{2}-\frac{1}{2}\frac{\sqrt{a}}{x_k}) \\ &= \frac{1}{2x_k}(x_k-\sqrt{a})^2 \end{aligned} \tag{18}$$

Then if $|x_k-\sqrt{a}|<\frac{1}{4}\sqrt{a}$, we obtain that

$$|x_{k+1}-\sqrt{a}|\leq\frac{1}{6}|x_k-\sqrt{a}|$$

Hence the scheme converges locally.

From (18), we conclude that the scheme is of second order, and the error constant

$$\lim_{n\to\infty}\frac{|x_{k+1}-\sqrt{a}|}{|x_k-\sqrt{a}|^2}=\frac{1}{2\sqrt{a}}$$

4

4.1

We expand LHS and RHS resp. and for simplicity we use $f, f[x], f[y]$ to denote $f(x_n,y_n), f_x(x_n,y_n), f_y(x_n,y_n)$.

LHS

$$\begin{aligned} y_{n+1}=&y_n+hf+ \\ &+\frac{1}{2}h^2(f[x]+f[y]y') \\ &+\frac{1}{6}h^3(f[xx]+2f[xy]y'+f[yy]y'^2+f[y]y'')+O(h^4) \end{aligned} \tag{19}$$

Noticing that $y'=f(x,y(x))=f, y''=f[x]+f[y]y'=f[x]+ff[y]$, we have

$$\begin{aligned} y_{n+1}=&y_n+hf+ \\ &+\frac{1}{2}h^2(f[x]+f[y]f) \\ &+\frac{1}{6}h^3(f[xx]+2ff[xy]+f^2f[yy]+f[x]f[y]+ff[y]^2)+O(h^4) \end{aligned} \tag{20}$$

RHS

$$\begin{aligned}K_2 =& f+\\ &+h(ff[y]b_{21}+f[x]a_2)+\\ &+\frac{1}{2}h^2(f^2f[yy]b_{21}^2+2ff[xy]a_2b_{21}+f[xx]a_2^2)+O(h^3)\end{aligned} \tag{21}$$

$$\begin{aligned}K_3 =& f+\\ &+h(ff[y]b_{31}+ff[y]b_{32}+f[x]a_3)+\\ &+\frac{1}{2}h^2(f^2f[yy]b_{31}^2+2f^2f[yy]b_{31}b_{32}+f^2f[yy]b_{32}^2+\\ &+2ff[y]^2b_{21}b_{32}+2ff[xy]a_3b_{31}+2ff[xy]a_3b_{32}+2f[x]f[y]a_2b_{32}+f[xx]a_3^2)+O(h^3)\end{aligned} \tag{22}$$

Hence

1. Comparing h: $c_1+c_2+c_3=1$
2. Comparing h^2 term: $f[x]$ term:
$$c_2a_2+c_3a_3=\frac{1}{2}$$
$ff[y]$ term:
$$c_2b_{21}+c_3b_{31}+c_3b_{32}=\frac{1}{2}$$
3. Comparing h^3 term: $f[xx]$ term:
$$c_2a_2^2+c_3a_3^2=\frac{1}{3}$$
$ff[xy]$ term:
$$2c_2a_2b_{21}+2c_3a_3b_{31}+2c_3a_3b_{32}=\frac{2}{3}$$
$f^2f[yy]$ term:
$$c_2b_{21}^2+c_3b_{31}^2+2c_3b_{31}b_{32}+c_3b_{32}^2=\frac{1}{3}$$
$f[x]f[y]$ term:
$$2c_3a_2b_{32}=\frac{1}{3}$$
$ff[y]^2$ term:
$$2c_3b_{21}b_{32}=\frac{1}{3}$$

The eight equations are what we need.

4.2

We choose

$$a_2 = \frac{1}{2}, a_3 = \frac{3}{4}$$
$$c_1 = \frac{2}{9}, c_2 = \frac{1}{3}, c_3 = \frac{4}{9}$$
$$b_{21} = \frac{1}{2}, b_{31} = 0, b_{32} = \frac{3}{4}$$

And it is a solution to equations in 4.1.

4.3

From 4.1 and 4.2, we have already known that $y(x_{n+1}) - y(x_n) = O(h^4)$. Based on this, we prove the scheme is of third order. We suppose our scheme is written as

$$y_{n+1} = y_n + h\phi(x_n, y_n, h)$$

We first prove the result provided ϕ is Lipschitz on y.

Proof. We define $\bar{y}_{n+1} = y(x_n) + h\phi(x_n, y(x_n), h)$,then we have $|\bar{y}_{n+1} - y(x_{n+1})| \leq Ch^4$ for some constant C (independent on h).

Set $e_n = |y_n - y(x_n)|$, we found that

$$\begin{aligned} e_{n+1} =& |y_{n+1} - y(x_{n+1})| \\ \leq& |y_{n+1} - \bar{y}_{n+1}| + |\bar{y}_{n+1} - y(x_n)| \\ \leq& |y_n + h\phi(x_n, y_n, h) - y(x_n) - h\phi(x_n, y_n, h)| + Ch^4 \\ \leq& (1 + hL)e_n + Ch^4. \end{aligned} \tag{23}$$

Here L is the Lipschitz constant of ϕ.

We have

$$\begin{aligned} e_{n+1} \leq& (1 + hL)e_n + Ch^4 \\ \leq& (1 + hL)e_{n-1} + C[1 + (1 + hL)]h^4 \\ \leq& \cdots \\ \leq& (1 + hL)^n e_0 + C\frac{(1 + hL)^n - 1}{hL}h^4 \\ \leq& \frac{C}{L}h^3(e^{nhL} - 1) \end{aligned} \tag{24}$$

Since nh is the length of time interval and hence a fixed number, we conclude the result. □

It suffices to show that ϕ is Lipschitz on y, and the constant is not dependent on $h < 1$. In fact, we show this is correct for all Runge–Kutta formula, provided f is Lipschitz on y.

Suppose that $|f(x, y) - f(x, y')| \leq L_f|y - y'|$, we have

$$|K_1(x_n, y_n) - K_1(x_n, y_n')| \leq L_f|y - y'| \tag{25}$$

$$|K_2(x_n, y_n) - K_2(x_n, y_n')| \leq L_f|y - y'| + L_f|hb_{21}||K_1(x_n, y_n) - K_1(x_n, y_n')| \leq [L_f + L_f^2|b_{21}]|y - y'| \tag{26}$$

$$\begin{aligned} |K_3(x_n, y_n) - K_3(x_n, y_n')| \leq & L_f|y - y'| + L_f|b_{31}||K_1(x_n, y_n) - K_1(x_n, y_n')| + L_f|b_{32}||K_2(x_n, y_n) - K_2(x_n, y_n')| \\ \leq & \Big[L_f + L_f^2|b_{31}| + L_f^2|b_{32}| + L_f^3|b_{32}b_{21}|\Big]|y - y'| \end{aligned} \tag{27}$$

Hence we conclude that $\phi(x, y, h)$ is Lipschitz on y, with constant $3L_f + L_f^2|b_{21} + L_f^2|b_{31}| + L_f^2|b_{32}| + L_f^3|b_{32}b_{21}|$.