A Construction of C^r Conforming Finite Element Space in Any Dimension

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Main Result

The H^m Problem

Find $u \in H_0^m(\Omega)$ s.t.

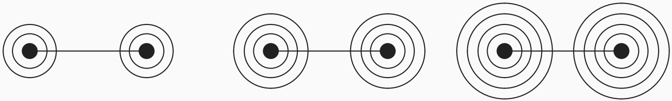
$$(\nabla^m u, \nabla^m v) = (f, v) \quad \forall v \in H_0^m(\Omega).$$

The (conforming) finite element method chooses a (finite-dimensional) subspace V_h of $H_0^m(\Omega)$, and find $u_h \in V_h$ s.t.

$$(\nabla^m u_h, \nabla^m v_h) = (f, v_h) \quad \forall v_h \in V_h.$$

For piecewise smooth functions, H^m conforming $\Leftrightarrow C^{m-1}$ continuity.

In 1D, we can choose V_h as (generalized) Hermite element. It is much more challenging when consider higher dimensions.



Main Result

Hu, J., Lin, T., & Wu, Q. (2021). A Construction of C^r Conforming Finite Element Spaces in Any Dimension. arXiv preprint arXiv:2103.14924.

Theorem (Hu-L.-Wu, 2021.)

Given $u \in \mathcal{P}_k(K)$, $k \geq 2^d r + 1$. Define the DOF for a simplex (codimension s)F of K, (the space $\mathcal{B}_{F,n,k}$ will be introduced later)

$$\frac{1}{|F|}\int_F (D^\theta u) \cdot v \quad \forall |\theta| = n \leq r2^s, \quad \forall v \in \mathcal{B}_{F,n,k}.$$

Then this set of degrees of freedom is unisolvent for the shape function space $\mathcal{P}_k(K)$, and the resulting finite element space is C^r.

	$\mathbb{R}^1$	$\mathbb{R}^2$	$\mathbb{R}^3$	$\mathbb{R}^4$	$\mathbb{R}^{n\geq 5}$
H^1	Lagrange P_1	Lagrange P_1	Lagrange P_1	Lagrange P_1	Lagrange P_1
H^2	Hermite P_3	Argyris P_5	Ženíšek P_9	Zhang P_{17}	HLW P_{2^n+1}
H^3	gHermite P_5	Ženíšek P_9	Zhang P_{17}	HLW P_{33}	HLW $P_{2^{n+1}+1}$
H^4	gHermite P_7	BZ P_{13}	HLW P_{25}	HLW P_{49}	HLW $P_{3\cdot 2^n+1}$
$H^{m\geq 5}$	gHermite P_{2m-1}	BZ P_{4m-3}	HLW P_{8m-7}	HLW P_{16m-15}	HLW $P_{(m-1)2^n+1}$

DEFINITION OF $\mathcal{B}_{F,n,k}$

Define the set of multi-indices

$$\Sigma(d,k) := \{(\alpha_0, \alpha_1, \cdots, \alpha_d) \in \mathbb{N}_{\geq 0}^{d+1} : \sum_{i=0}^{d} \alpha_i = k\}$$

Given an array $\boldsymbol{r} = r_1, \cdots, r_d$, let

$$\Sigma_0^{(\boldsymbol{r})}(d,k) = \big\{(\alpha_0, \cdots, \alpha_d) \in \Sigma(d,k) : \alpha_{i_1} + \cdots + \alpha_{i_s} > r_s, \\ \forall i_1, \cdots, i_s, s = 1, 2, \cdots, d\big\}.$$

For simplex F with co-dimension s, define the bubble function space

$$\mathcal{B}_{F,n,k} := \operatorname{span}\{\lambda_0^{\alpha_0} \cdots \lambda_{d-s}^{\alpha_{d-s}}, \alpha \in \Sigma_0^{(r_s - n, \cdots, r_d - n)}(d-s, k-n)\}.$$

Note here r_s indicates the C^{r_s} continuity on codimension s simplices.

For example, in two dimensions, let $r_1 = 1$, $r_2 = 2$, then $\mathcal{B}_{f,0,k} = (\lambda_0\lambda_1\lambda_2)^2 P_{k-6}$. This is the interior DOF of the Argyris element. But in general, it is **NOT** a complete bubble form.

Tool: Intrinsic Decomposition

An Intrinsic Decomposition of Multi-indices

Define the continuity vector $\boldsymbol{r} = (r_1, \cdots, r_d)$ and degree k such that

Assumption: $r_d \geq 2r_{d-1} \geq \cdots \geq 2^{d-1} r_1$ **and** $k \geq 2r_d + 1$.

$$\Sigma_d^{(\boldsymbol{r})}(d,k) := \{(\alpha_0, \cdots, \alpha_d) \in \Sigma(d,k) : \text{ There exists a subset } \boldsymbol{N}_d \subseteq [d], \text{ such that } \mathrm{Card}(\boldsymbol{N}_d) = d \text{ and } \sum_{i \in \boldsymbol{N}_d} \alpha_i \leq r_d\},$$

and then,

$$\Sigma_s^{(\boldsymbol{r})}(d,k) := \{(\alpha_0, \cdots, \alpha_d) \in \Sigma(d,k) : \text{ There exists a subset } \boldsymbol{N}_s \subseteq [d], \text{ such that } \mathrm{Card}(\boldsymbol{N}_s) = s \text{ and } \sum_{i \in \boldsymbol{N}_s} \alpha_i \leq r_s\} \setminus \bigcup_{s' \geq s+1} \Sigma_{s'}^{(\boldsymbol{r})}(d,k),$$

for $s = d-1, \cdots, 1$ sequentially. Finally, we find that

$$\Sigma_0^{(\mathbf{r})}(d,k) := \Sigma(d,k) \setminus \bigcup_{s'=1}^{d} \Sigma_{s'}^{(\boldsymbol{r})}(d,k),$$

which is coincident with the definition of $\Sigma_0^{(\boldsymbol{r})}(d,k)$.

Refined Intrinsic Decomposition

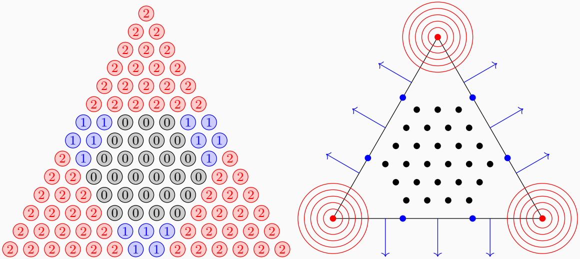


Figure 1: The decomposition of $\Sigma(2, 13)$ with the continuity vector $\boldsymbol{r} = (1, 5)$ and polynomial degree $k = 13$.

This correspondence leads to the following definition. By assumption, the all the following definition are well-defined.

Let $\boldsymbol{N}(\alpha)$ be the $\boldsymbol{N}_s$ in the aforementioned definition, and $n(\alpha) := \sum_{i \in \boldsymbol{N}(\alpha)} \alpha_i$. Define the **refined intrinsic decomposition** is defined as

$$\Sigma_{\boldsymbol{N},n}(d, k) = \{\alpha \in \Sigma(d, k) : \boldsymbol{N}(\alpha) = \boldsymbol{N} \text{ and } n(\alpha) = n\}.$$

BIJECTION LEMMA

Lemma (Bijection Lemma)

For general pair $\mathbf{N}, n$, let $\Delta = [d] \setminus \mathbf{N}$, then the mapping

$$\mathcal{R}_{\mathbf{N},\Delta} : \alpha \mapsto (\theta, \sigma), \ \ \theta_i = \alpha_i, i \in \mathbf{N} \ \ \textit{and} \ \ \sigma_i = \alpha_i, i \in \Delta,$$

is a bijection between $\Sigma_{\mathbf{N},n}(d,k)$ and $\Sigma(\mathbf{N}, n) \times \Sigma_0^{(\boldsymbol{q})}(\Delta, k-n)$ with $\boldsymbol{q} := (r_s - n, \cdots, r_d - n)$.

Given a subsimplex F, let $\boldsymbol{N} = \mathbb{I}_d \setminus \langle F \rangle$, then

$$\Sigma(\boldsymbol{N}, n) \iff \textbf{a partial derivative } D^{\theta}$$

$$\Sigma_0^{\boldsymbol{q}}(\Delta, k-n) \iff \textbf{a polynomial, which forms a basis of } \mathcal{B}_{F,n,k}$$

Since $\Sigma_{\boldsymbol{N},n}$ forms a partition of $\Sigma(d,k)$, then the total number of the degrees of freedom are equal to the cardinality of $\Sigma(d,k)$.

To unify the notation, let φ_α be such degrees of freedom $\int_\Delta D^{\theta}(\cdot)\boldsymbol{\lambda}^{\sigma}$.

Proof

Index Order

Now it suffices to show when all degrees of freedom vanishes for $u \in P_k(K)$, then $u = 0$.

Set $\mathcal{P}_{\mathbf{N},n} := \text{span}\left\{\lambda_0^{\alpha_0}\cdots\lambda_d^{\alpha_d} : \alpha \in \Sigma_{\mathbf{N},n}(d,k)\right\}$. For any $u \in \mathcal{P}_k(K)$, there is **a unique decomposition** $u = \sum_{\mathbf{N},n} u_{\mathbf{N},n}$ with $u_{\mathbf{N},n} \in \mathcal{P}_{\mathbf{N},n}$.

We introduce the following order first.

Definition (Order of the index)

For all the pairs $(\mathbf{N}, n)$, introduce the following order: Say $(\mathbf{N}', n') \preceq (\mathbf{N}, n)$ if

$$\mathbf{N}' \supsetneq \mathbf{N} \qquad \text{or} \qquad \mathbf{N}' = \mathbf{N} \text{ and } n' \leq n.$$

Say $(\mathbf{N}', n') \prec (\mathbf{N}, n)$ if $(\mathbf{N}', n') \preceq (\mathbf{N}, n)$ and $(\mathbf{N}', n') \neq (\mathbf{N}, n)$.

We can show that if $(\mathbf{N}', n') \npreceq (\mathbf{N}, n)$, then for all $\beta \in \Sigma_{\mathbf{N}',n'}$, $\sum_{i\in\mathbf{N}} \beta_i > n$.

Induction Lemma

Lemma (Induction Lemma)

If the condition $(\mathbf{N}', n') \preceq (\mathbf{N}, n)$ **DOES NOT** *hold, then* $\varphi_\alpha(u_{\mathbf{N}',n'}) = 0$ *for all* $\alpha \in \Sigma_{\mathbf{N},n}$.

Proof.

WLOG, we assume that $\boldsymbol{N} = \{0, 1, \cdots, s-1\}$. For $\beta \in \Sigma_{\mathbf{N}',n'}$, it suffices to show that $\varphi_\alpha(\boldsymbol{\lambda}^\beta) = 0$. Since $\sum_{i=0}^{s-1} \beta_i > n$ and $\sum_{i=0}^{s-1} \alpha_i = n$, it follows from $\lambda_{<s}$ vanishes on $\Delta = \langle [d] \setminus \boldsymbol{N} \rangle$ that

$$\partial_{\boldsymbol{n}_0}^{\alpha_0} \cdots \partial_{\boldsymbol{n}_{s-1}}^{\alpha_{s-1}} (\lambda_0^{\beta_0} \cdots \lambda_{s-1}^{\beta_{s-1}} \cdots) = 0$$

on Δ. $\square$

Proof

Proof of the Unisolvency

The unisolvency comes from the above two lemmas.

Induction Lemma: $(\mathbf{N}', n') \not\preceq (\mathbf{N}, n) \Longrightarrow \varphi_\alpha(u_{\mathbf{N}',n'}) = 0$ for some $\alpha \in \Sigma_{\mathbf{N},n}$.

Partial Unisolvency: $\varphi_\alpha(u_{\mathbf{N},n}) = 0$ for all $\alpha \in \Sigma_{\mathbf{N},n}$, then $u_{\mathbf{N},n} = 0$.

- First, consider an element $(\mathbf{N}, n)$ which is smallest under the partial order $\preceq$, then by Induction Lemma, $0 = \varphi_\alpha(u_{\mathbf{N},n}) + \varphi_\alpha(\sum_{\mathbf{N}',n' \neq \mathbf{N},n} u_{\mathbf{N}',n'}) = \varphi_\alpha(u_{\mathbf{N},n})$ for all $\alpha \in \Sigma_{\mathbf{N},n}$. Then by partial unisolvency, $u_{\mathbf{N},n} = 0$.
- Then, suppose that for all $u_{\mathbf{N}'',n''}$ such that $(\mathbf{N}'', n'') \prec (\mathbf{N}, n)$ are zero. By induction lemma, for $\alpha \in \Sigma_{\mathbf{N},n}$, $0 = \varphi_\alpha(u_{\mathbf{N},n}) + \varphi_\alpha(\sum_{\mathbf{N}',n' \not\preceq \mathbf{N},n} u_{\mathbf{N}',n'}) = \varphi_\alpha(u_{\mathbf{N},n})$. Again by partial unisolvency, $u_{\mathbf{N},n} = 0$.
- Then induction lemma yields that $u = 0$.

Lemma (Unisolvency for $\mathcal{P}_{\mathbf{N},n}$)

If $\varphi_\alpha(u_{\mathbf{N},n}) = 0$ for all $\alpha \in \Sigma_{\mathbf{N},n}(d,k)$, then $u_{\mathbf{N},n} = 0$.

Proof.

WLOG, we assume that $\boldsymbol{N} = \{0, 1, 2, \cdots, s-1\}$.
Recall that

$$u \in \mathcal{P}_{\mathbf{N},\boldsymbol{n}} = \sum_{\theta,\sigma} c_{\sigma,\theta}(\lambda_0^{\theta_0} \cdots \lambda_{s-1}^{\theta_{s-1}})(\lambda_s^{\sigma_0} \cdots \lambda_d^{\sigma_{d-s}}) \to \sum_{\theta,\sigma} c_{\theta,\sigma} \boldsymbol{\lambda}_{<s}^{\theta} \boldsymbol{\lambda}_{\geq s}^{\sigma}.$$

Here $\theta \in \Sigma(\boldsymbol{N}, n), \sigma \in \Sigma_0^{\boldsymbol{q}}(\Delta, k-n)$.

- Since $\lambda_{\leq s}$ vanishes on Δ, we have $D^{\theta'} \boldsymbol{\lambda}_{<s}^{\theta} \boldsymbol{\lambda}_{\geq s}^{\sigma} = [D^{\theta'} \boldsymbol{\lambda}_{\leq s}^{\theta}] \boldsymbol{\lambda}_{>s}^{\sigma}$ on Δ, for $\theta' \in \Sigma(\boldsymbol{N}, n)$.
- Find a linear combination of $\widetilde{D}^{\theta'}$ of $D^{\theta'}$, such that $\widetilde{D}^{\theta'} \boldsymbol{\lambda}_{\leq s}^{\theta} = \delta_{\theta,\theta'}$.
- Find a polynomial $\widetilde{p}^{\sigma'} \in \boldsymbol{\lambda}\Sigma_0^{\boldsymbol{q}}$ such that $(\boldsymbol{\lambda}_{>s}^{\sigma}, \widetilde{p}^{\sigma'})_\Delta = \delta_{\sigma,\sigma'}$.
- By bijection lemma, $\int_\Delta \widetilde{D}^{\theta'} u \widetilde{p}^{\sigma'}$ is a linear combination of φ_α.
- Finally, $c_{\theta,\sigma} = \int_\Delta \widetilde{D}^{\theta} u \widetilde{p} = 0$, by assumption, therefore, $u = 0$.

$\square$

Discussions

Some are known to us:

- A geometric view, by **[Chen & Huang, 2021]**;
- Explicit Basis Functions in 3D and 4D, by **[Zhang, 2022]**;

Much more are unknown to us:

- Is the assumption necessary to construct some finite elements?
- Lower the continuity assumption in some macro-element, e.g. Alfeld split or Powell–Sabin split.
- Nonconforming finite elements.

Thank you for listening!