PHYS250 Formal Report

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Coupled oscillators

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Abstract

The aim of this experiment is to study the relationship of systems of coupled oscillators. This will be shown through testing three modes of oscillation of two coupled oscillators. In one mode, there will be no coupling and the oscillators will move in sync. In the other two modes, the oscillators will be tested at a maximum and minimum coupling. The results show that an uncoupled oscillator will oscillate at one frequency, the natural frequency, while the coupled systems will oscillate as a function of the sum of two oscillating systems.

Introduction

This experiment explores the relationship of an uncoupled oscillator and two other modes of oscillation for coupled oscillators. The aim of this experiment is to study the motion of interacting systems, and the effect of oscillations between a coupled system of masses. In order to do this, three systems of oscillators will be studied, uncoupled, coupled at +100 and coupled at -100.

Uncoupled oscillators will have one degree of freedom and act much like a regular oscillator. As there will be no opposing force from the current generated by the coil, both masses will oscillate at the natural frequency. The equation of motion for this system is simply

$$m\ddot{x} = kx \tag{1}$$

On the other hand, coupled oscillators are more complex, having more than one degree of freedom. The system in this experiment will consist of two individual systems of oscillating masses placed in parallel. An induced current in a coil will create a repulsive force between the two masses, allowing them to oscillate with respect to each other.

Theory

The equation of motion for the uncoupled oscillator is given in equation 1. The frequency of the system is then given by

$$\omega = \sqrt{\frac{k}{m}} \tag{2}$$

The equations of motion for coupled oscillators[2] of m_1 and m_2 respectively are

$$m\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2x_2 \tag{3}$$

$$m\ddot{x}_2 = k_2x_1 - (k_2 + k_3)x_2 \tag{4}$$

The potential energy of the system is given by

$$U = \frac{1}{1}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k(x_1 - x_2)^2 \tag{5}$$

and the Kinetic energy is

$$K = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 \tag{6}$$

The equations of motion can be rewritten as a matrix

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_1 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Where the first matrix is the matrix $\mathbf{M}$ and the second matrix is $\mathbf{K}$ and can be expressed as $\mathbf{M}\ddot{\vec{x}} + \mathbf{K}\vec{x} = 0$. If the system is assumed to oscillate at the same eigenfrequency, a solution of $\ddot{\vec{x}} = \vec{u}e^{i\omega t}$ can be assumed. This gives $\ddot{\vec{x}} = -\omega\vec{x}$ and the equation of the matrix is

$$(\mathbf{K} - \omega^2\mathbf{M})\vec{x} = 0$$

By solving for the eigenvalues of the matrix, the corresponding frequencies are found. In the case of this experiment, the mass of the two masses are assumed to be the same as well as the spring constants of the attached springs. The spring between the masses is the only spring assumed to have a different constant. The determinant of the matrix $[\mathbf{K} - \omega^2 m\mathbf{I}]^2$ is

$$(k_1 + k_2 - \omega^2 m)^2 - k_2^2 = 0$$

whose roots are

$$\omega_1 = \omega_n \tag{7}$$

$$\omega_2 = \omega_n\sqrt{1 + \frac{k_2}{k_1}}$$

The general equation of motion for a system of equations[2] is given by

$$\mathbf{x}(t) = A_1\cos(\omega_1 t + \phi_1)\mathbf{u}_1 + A_2\cos(\omega_2 t + \phi_2)\mathbf{u}_2 \tag{8}$$

where the vector $\mathbf{u}_i$ is given by the matrix

$$(\mathbf{K} - \omega_i^2\mathbf{M})\vec{u}_i = 0 \tag{9}$$

In a coupled system, there will be a superposition of modes. In the case studied in this experiment, the two modes will be coupled at +100 and -100, and as a result will have two frequencies. Therefore, for these systems, there will be two frequencies, ω_1 and ω_2 and two amplitudes A_1 and A_2.

The relationship of the two spring constants can be defined as the constant $\epsilon = \frac{k_2}{k_1}$ where

$$\epsilon = \frac{1}{2}[(\frac{f_2}{f_1})^2 - 1] \tag{10}$$

While it will not be studied in this experiment, the resonance of the system depends on the quality factor of the spring[3], and is found through the equation

$$\frac{\omega_r}{\omega_0} = \sqrt{1 - \frac{1}{2Q^2}} \tag{11}$$

The resonant frequency is known to be near the natural frequency, and tends towards the natural frequency at large quality factors. Amplitude resonance also occurs when the Quality factor of the system is $Q > \frac{1}{\sqrt{2}}$[3].

Experimental Procedure

1. The mass of the permanent magnet block was measured to be 16.1248 ± 0.0001 and the mass of the electromagnetic block was measured to be 16.1313 ± 0.0001.
2. To measure the natural frequency of the oscillator, the coupling was set to zero. Several bumps were given before starting the program.
3. The program was started and data was collected for 200 seconds at 100 Hz.
4. The data was saved in a .csv file.
5. The natural frequency of the uncoupled system was approximated at 3.7 Hz.
6. Steps 1-3 was repeated for a coupled system at +100 and -100.
7. The frequencies of the system coupled at +100 was approximated at 2.9 Hz and 3.7 Hz.
8. The frequencies of the system coupled at -100 was approximated at 3.7 Hz and 4.00 Hz.

Results and Analysis

	Uncoupled	**Coupled at +100**	**Coupled at -100**
f_1(Hz)	3.6600 ± 0.0002	2.9356 ± 0.0004	3.6956 ± 0.0003
ω_1	22.9965 ± 0.0013	18.4449 ± 0.0025	23.2201 ± 0.0019
f_2(Hz)	–	3.6693 ± 0.0003	4.0202 ± 0.0002
ω_2	–	23.0548 ± 0.0019	25.2597 ± 0.0013

Table 1: Frequencies and amplitudes of the three systems

The frequencies of the oscillators were found by fitting the function to a Lorentzian model. By doing this, the natural frequency of the uncoupled system was found to be 3.66Hz.

Knowing the natural frequency of the system and the masses of the two blocks, the spring constant, k_1, is found using the equation

$$\omega_1 = \sqrt{\frac{k_1}{m}}$$

where the values of m and ω_1 are m = 0.0161 and $\omega_1 = 22.9965$.

$k_1 = \omega_1^2 m$

$k_1 = (22.9965)^2(0.0161)$

$k_1 = 8.5143$

+100 coupling

For the oscillator coupled at +100 coupling, the eigenmode frequencies given by the Lorentzian fit of the model are $f_1 = 2.9356\text{Hz}$ and $f_2 = 3.6693\text{Hz}$. The frequency splitting given by $\Delta f = f_2 - f_1$ is $\Delta f = 3.6693 - 2.9356 = 0.7337$ Hz.

The spring constant k_2 using the relationship $\epsilon = \frac{k_2}{k_1}$ where ϵ is

$$\epsilon = \frac{1}{2}[(\frac{f_2}{f_1})^2 - 1]$$

$\epsilon = \frac{1}{2}[(\frac{3.6693}{2.9356})^2 - 1]$
$\epsilon = 0.2812$
k_2 can be found using the relationship $\epsilon = \frac{k_2}{k_1}$.
$k_2 = \epsilon k_1$
$k_2 = 0.2812 \times 8.5143$
$k_2 = 2.39$

-100 coupling

For the oscillator coupled at -100 coupling, the eigenmode frequencies given by the Lorentzian fit of the model are $f_1 = 3.67\text{Hz}$ and $f_2 = 4.02\text{Hz}$. The frequency splitting for this system is $\Delta f = 4.02 - 3.67 = 0.35$ Hz.
$\epsilon = \frac{1}{2}[(\frac{4.0202}{3.6956})^2 - 1]$
$\epsilon = 0.09$
k_2 is found using the relationship $\epsilon = \frac{k_2}{k_1}$.
$k_2 = \epsilon k_1$
$k_2 = 0.09 \times 8.5143$
$k_2 = 0.7663$

Graphs

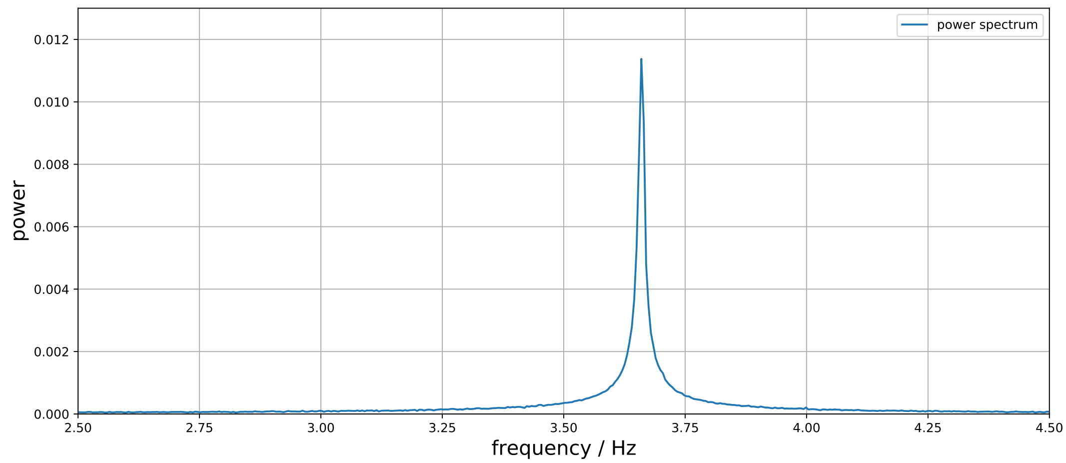

Figure 1: Power spectrum for uncoupled system

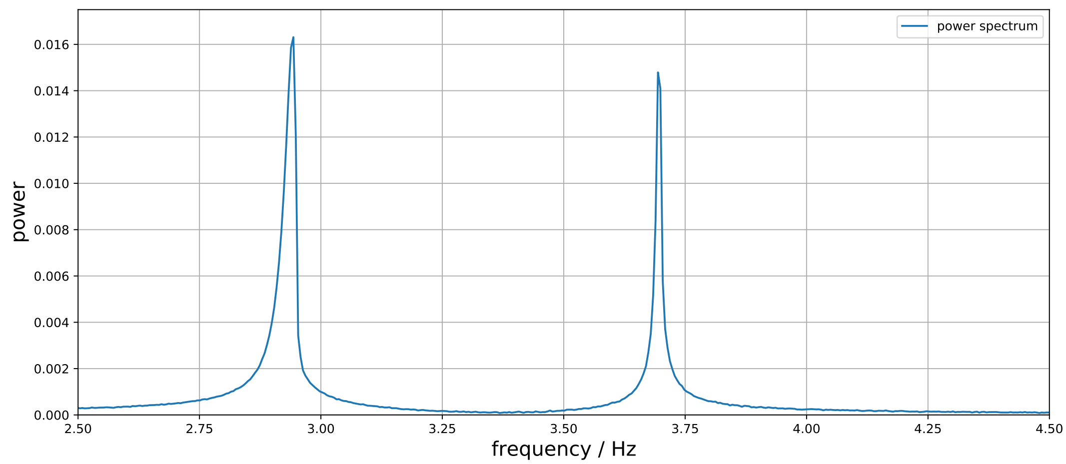

Figure 2: Power spectrum for coupled system at +100 coupling

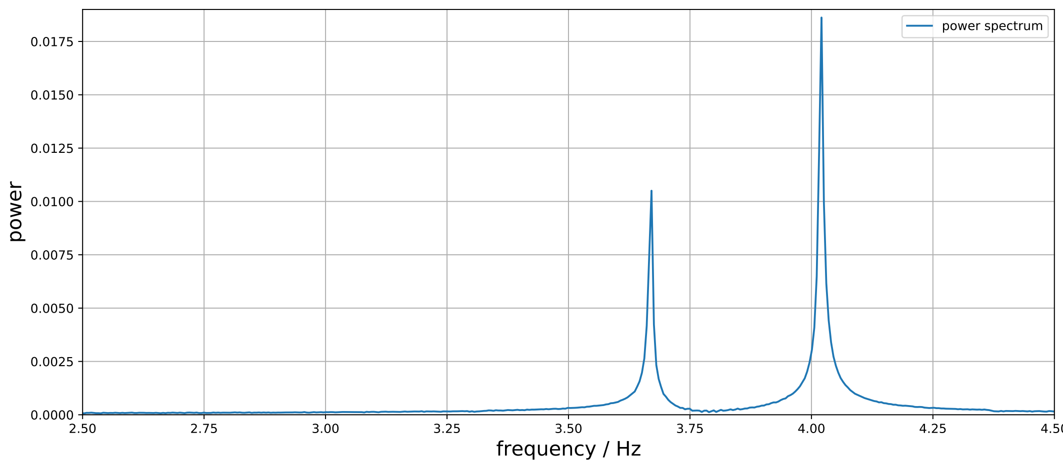

Figure 3: Power spectrum for coupled system at -100 coupling

Graphical tests of models against data

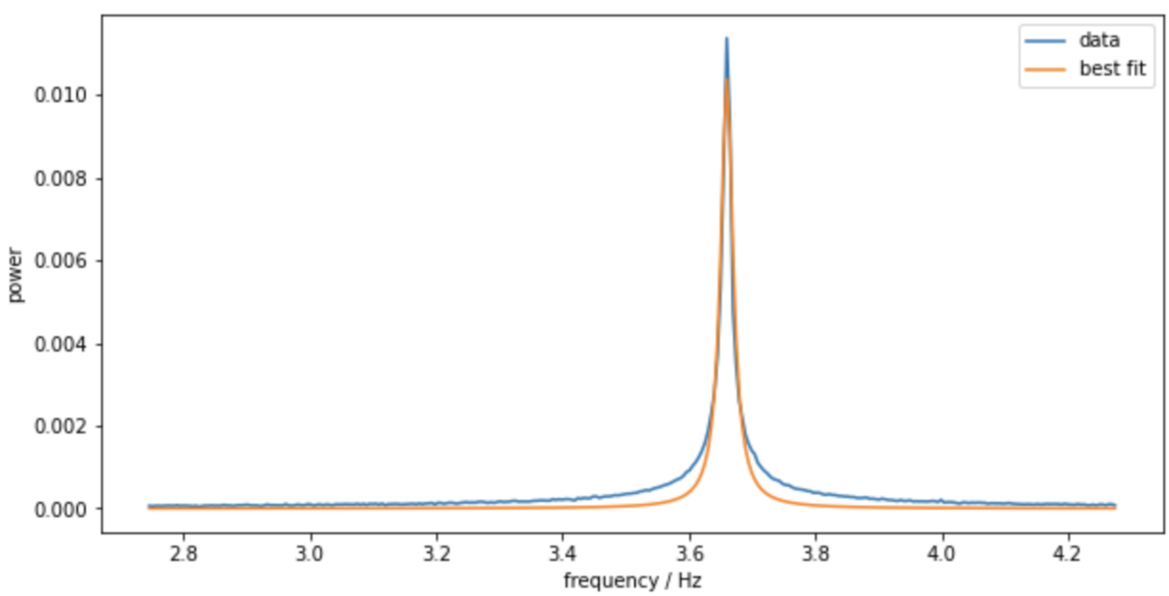

Figure 4: Model against data for uncoupled system

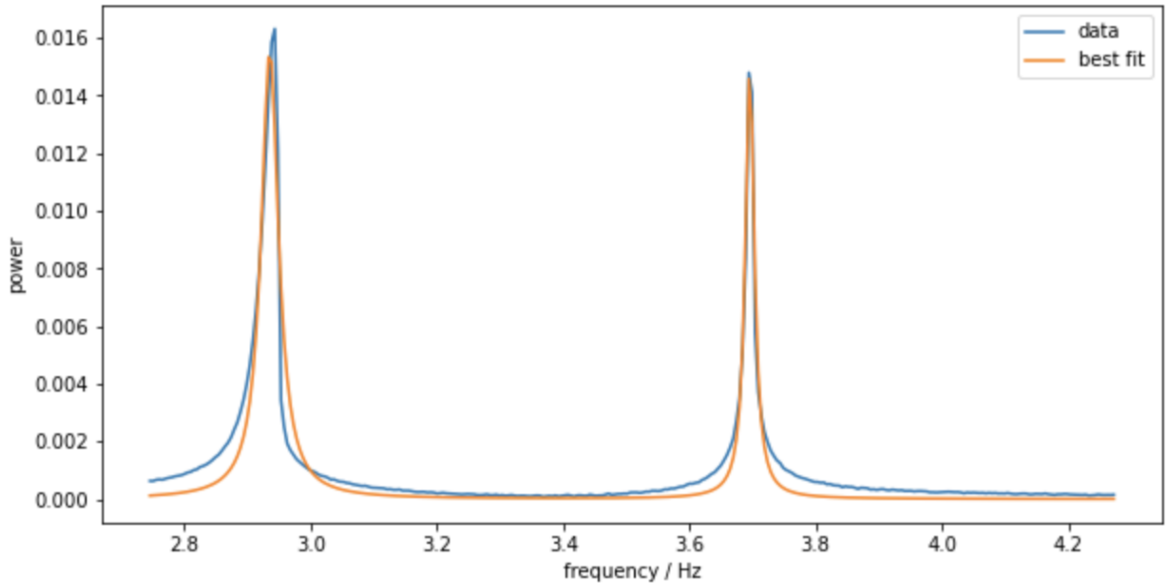

Figure 5: Model against data for system coupled at +100

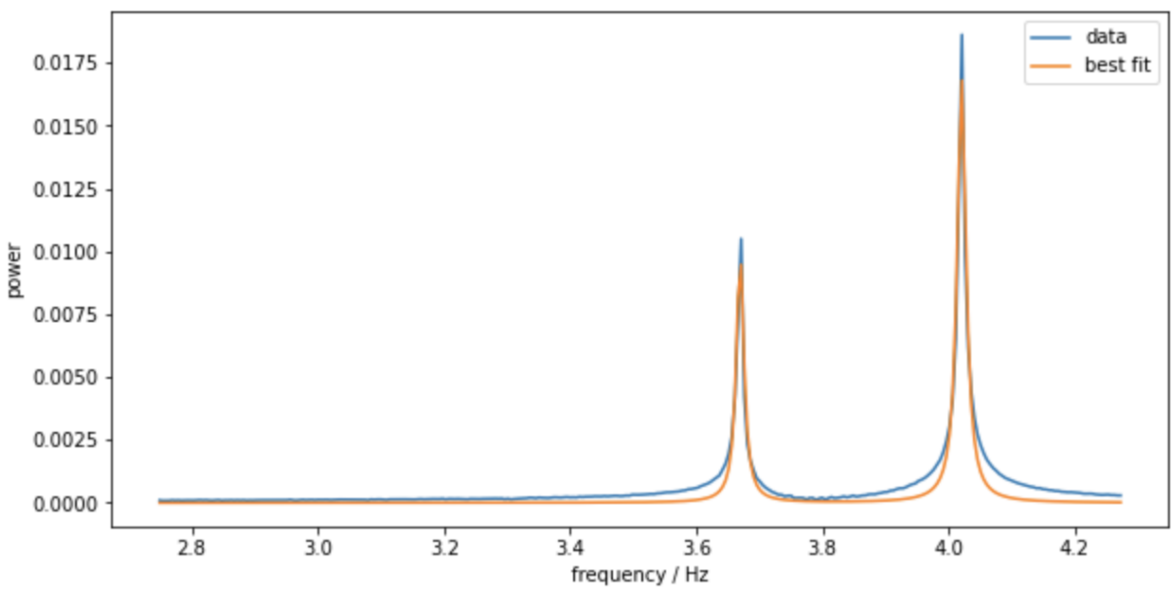

Figure 6: Model against data for system coupled at -100

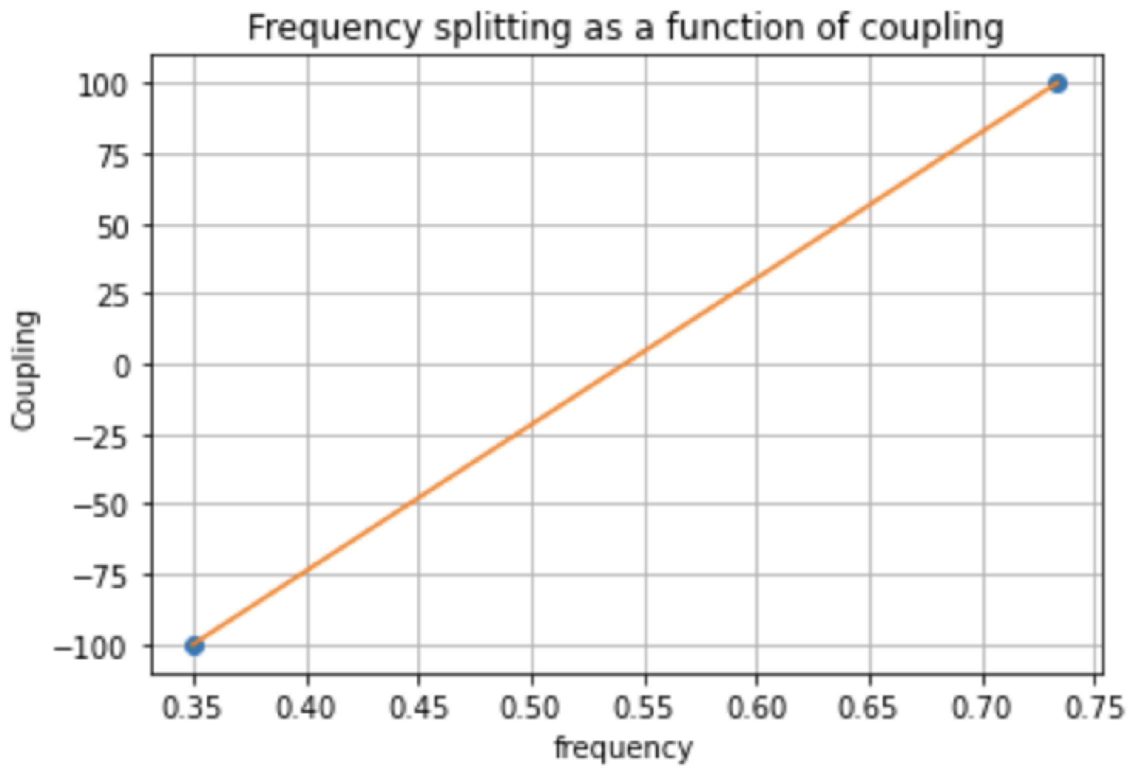

Figure 7: Frequency splitting as a function of coupling

Discussion and Conclusions

Sources of experimental uncertainty can result from the uncertainty in the measurements of the masses and the uncertainty when fitting the curve to the Lorentzian model. The system is also modeled with the assumption that it moves one dimensionally.If the system moved perpendicular to the measured motion, there would be loss of energy and would thus affect the measured frequency.

In the case of the uncoupled system, the frequency found was that of the natural frequency. In the cases of +100 and -100 coupling, one of the two frequencies of the systems oscillated close to the natural frequency, and was the resonant frequency. In the case of the +100 coupling, the other frequency was lower than the resonant frequency, and in the case of -100 coupling, the other frequency was higher than the resonant frequency. By comparing the frequency splitting as a function of coupling, it is shown that as coupling increases, so does the difference between the two frequencies. As shown, the two coupled systems can be modeled as a superposition of two systems, and therefore coupled systems can be modeled as having two frequencies at any given time.

References

1. McCarthy.L.(2002). On coupled mechanical harmonic oscillators, transients, and isolated oscillating systems. *American Journal of Physics,* 590-598.

2. B. McLean, Alastair. Coupled Oscillators - Remote Version. Experimental Outline, Queens University, 2021.

3. Di Stefano, Phillipe. Oscillations and Waves. 2021.