Population Growth in a Closed System

Volterra model for population growth in a closed system is

$$\frac{dp}{d\tilde{t}} = ap - bp^2 - cp\int_0^{\tilde{t}} p(x)\,dx \qquad p(0) = p_0$$

Here $a>0$ is the birth rate coefficient

$b>0$ is the intraspecies competition coefficient / crowding coefficient

$c>0$ is the toxicity coefficient

The term $cp\int p\,dx$ represents the effect of toxin accumulation on the species. In fact $\int p\,dx$ represents the total amount of toxin produced since time 0. The individual death rate is proportional to this integral so the population death rate due to toxicity includes a factor p. The constant c measures the sensitivity to toxins.

Non-dimensionalisation

Set $t = A\tilde{t}$ $\quad u(t) = Bp$

then should get

$$K\frac{du}{dt} = u - u^2 - u\int_0^t u(s)\,ds \qquad u(0) = u_0$$

where $K = \dfrac{c}{ab}$ (should have $B = b/a$, $A = c/b$)

Stationary solutions :

$$u(1-u-\int_0^t u(s)\,ds) = 0$$

So either $u=0$ or $u(t) = 1 - \int_0^t u(s)\,ds$

$$\Rightarrow \frac{du}{dt} = -u(t)$$

$$\Rightarrow u = Ae^{-t}$$

So in either case, steady state solⁿ is $u=0$.

Therefore expect that as $t\to\infty$, $u\to 0$.

Can also write

$$\frac{du}{dt} - \frac{u}{K}(1-u-\int_0^t u(s)\,ds) = 0$$

$$\Rightarrow u(t) = A\exp\left\{\frac{1}{K}\int_0^t \left(1-u(x)-\int_0^s u(s)\,ds\right)dx\right\}$$

$u(0) = u_0 \Rightarrow A = u_0$

So can write $u(t) = u_0 \exp\left\{\frac{1}{K}\int_0^t \left(1-u(x)-\int_0^s u(s)\,ds\right)dx\right\}$

So $u(t) > 0$ $\forall t$ if $u_0 > 0$

So population is non-negative for all time.

Numerical Solution

We have $\frac{du}{dt} = f(t,u)$ $\quad u(0) = u_0$

where $f(t,u) = \frac{u}{K}(1-u-\int_0^t u(s)\,ds)$

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Explicit Euler scheme:

let $t_n = nh$ for $n = 0, 1, 2, \dots$

let $U_n \approx u(t_n)$ then explicit Euler scheme is

$$\frac{U_{n+1} - U_n}{h} = f(t_n, U_n) \qquad n = 0, 1, 2, \dots$$

$$U_0 = u_0$$

ie $U_{n+1} = U_n + h f(t_n, U_n)$

$$= U_n + h U_n \left(1 - \frac{U_n}{K} - \int_0^{t_n} u(s)\,ds\right)$$

We can compute (approximate) the integral using composite trapezium rule so

$$\int_0^{t_n} u(s)\,ds \approx \frac{h}{2}\left(u(t_0) + 2\sum_{i=1}^{n-1} u(t_i) + u(t_n)\right)$$

$$\approx \frac{h}{2}\left(U_0 + 2\sum_{i=1}^{n-1} U_i + U_n\right)$$

$$= T_n$$

Also note $T_n = \frac{h}{2}\left(U_0 + 2\sum_{i=1}^{n-1} U_i + U_n\right)$

$$= \frac{h}{2}\left(U_0 + 2\sum_{i=1}^{n-2} U_i + U_{n-1}\right) + \frac{h}{2}\left(U_{n-1} + U_n\right)$$

$$= T_{n-1} + \frac{h}{2}\left(U_{n-1} + U_n\right)$$

and $T_0 = 0$.

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So we have

$$\left.\begin{aligned} U_{n+1} &= U_n + \frac{hU_n}{K}(1 - U_n - T_n) \\ T_{n+1} &= T_n + \frac{h}{2}(U_n + U_{n+1}) \end{aligned}\right\} n = 0, 1, 2, \ldots$$

$U_0 = u_0, \quad T_0 = 0.$

Implicit Euler:

$$\frac{U_{n+1} - U_n}{h} = f(t_{n+1}, U_{n+1})$$

$$\Rightarrow \left.\begin{aligned} U_{n+1} &= U_n + \frac{hU_{n+1}}{K}(1 - U_{n+1} - T_{n+1}) \\ T_{n+1} &= T_n + \frac{h}{2}(U_n + U_{n+1}) \end{aligned}\right\} n = 0, 1, 2 \ldots \quad (*)$$

$U_0 = u_0, \quad T_0 = 0$

Can think of $(*)$ as a set of coupled nonlinear equations for U_{n+1} and T_{n+1} which can be solved using Newton's method. Perhaps it is easier to substitute T_{n+1} from 2nd equation into 1st equation & to solve the quadratic for U_{n+1}.

<u>To Do</u>

1) Check the non dimensionalisation.
2) Code up explicit Euler scheme with trapezium rule
3) Code up implicit Euler scheme with trapezium rule (or code up θ-method with trapezium rule)
4) Instead of using composite trapezium rule could you use the more accurate composite Simpson's rule (which needs an even number of subintervals). Alternatively think why this may

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not be a good idea.

5) Run the codes for a range of values of K & u_0.
Try both $u_0 < 1$ and $u_0 \geq 1$. What is the effect of K?

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