Population Growth week 2

$$\frac{du}{dt} = \frac{u}{K}\left(1 - u - \int_0^t u(s)\,ds\right) \qquad u(0) = u_0$$

Set $y(t) = \int_0^t u(s)\,ds$

$$\Rightarrow \frac{dy}{dt} = u(t) \qquad y(0) = 0$$

So we have a system

$$\frac{dy}{dt} = u \qquad y(0) = 0$$

$$K\frac{du}{dt} = u(1 - u - y) \qquad u(0) = u_0$$

We know that $u(t) > 0$ provided $u_0 > 0$

Hence $y(t)$ is monotonic increasing.

$\frac{du}{dt} = 0$ if $u = 0$ or $1 - u - y = 0$

For states (y, u) such that $1 - u - y > 0$, $\frac{du}{dt} > 0$

so u is increasing

For states (y, u) such that $1 - u - y < 0$, $\frac{du}{dt} < 0$

so u is decreasing.

Thus since $y(0) = 0$ we have that if $u_0 < 1$ then $\frac{du}{dt} > 0$ initially so $u(t)$ (and $y(t)$) increase until $1 - u - y = 0$. The max of u is attained at this point then $u(t)$ decreases monotonically.

If $u_0 \geq 1$ then $\frac{du}{dt} \leq 0$ initially so u_0 is the max of $u(t)$ and $u(t)$ decreases monotonically.

We have $\frac{dy}{dt} = u$ $\quad \frac{du}{dt} = \frac{u}{k}(1-u-y)$

So $\frac{du}{dy} = \frac{1-u-y}{k}$

Also when $t=0$, $y=0$ and $u=u_0$

$\Rightarrow u(y=0) = u_0$

$\Rightarrow u(y) = (1+k-y) - (1+k-u_0)e^{-y/k}$

For $u_0 < 1$ the max of u occurs at a turning point, ie when $\frac{du}{dy} = 0$

$\Rightarrow y_c = -k \log \frac{k}{1+k-u_0}$

Then max of u is $u_{max} = u(y_c) = 1 + k \log \frac{k}{1+k-u_0}$

Note that there is a 1-1 map between t & y, so since $u(y)$ has only 1 turning point (when u_0) so does $u(t)$.

Numerical Solution

$$\frac{dy}{dt} = u \qquad y(0) = 0$$

$$\frac{du}{dt} = \frac{u}{K}(1-u-y) \qquad u(0) = u_0$$

Explicit Euler: let $U^n \approx u(t_n)$, $Y^n \approx y(t_n)$

$$\frac{Y^{n+1} - Y^n}{h} = U^n \qquad n = 0, 1, 2 \dots$$

$$\frac{U^{n+1} - U^n}{h} = \frac{U^n}{K}(1 - U^n - Y^n)$$

$U^0 = u_0$, $Y^0 = 0$.

Implicit Euler:

① $$\frac{Y^{n+1} - Y^n}{h} = U^{n+1} \qquad n = 0, 1, 2 \dots$$

② $$\frac{U^{n+1} - U^n}{h} = \frac{U^{n+1}}{K}(1 - U^{n+1} - Y^{n+1})$$

$U^0 = u_0$, $Y^0 = 0$.

Either re-arrange ① to get Y^{n+1} in terms of Y^n, U^{n+1}, h then subst in ② & solve a quadratic

Or use Newton's method for a nonlinear system.

To Do

1) Code explicit scheme for the system
2) Code θ-method for the system.
3) Check results are same as last week.

4) Run code for different values of u_0, in particular, $u_0 < 1$ & compare theoretical max of u with the maximum achieved by the numerical method.

Compare y_c with y^k where k is s.t. $U^k = \max U^n$

5) Code the 2 step Adams Bashforth scheme (AB2)

For a problem of the form $\frac{d\underline{u}}{dt} = \underline{f}(t, \underline{u})$ this is

$$\frac{\underline{U}^{n+1} - \underline{U}^n}{h} = \frac{1}{2}\left(3\,\underline{f}(t_n, \underline{U}^n) - \underline{f}(t_{n-1}, \underline{U}^{n-1})\right) \qquad n \geq 1$$

(this is a 2nd order scheme)