The Response Method

Thomas Monahan

Engineering Science
University of Oxford

November 9, 2022

Background and Motivation

- Devised by Munk and Cartwright in 1966, the response method seeks to quantify the ocean's "response" to external forces.
- By treating our system of interest as a black box we can evaluate its response to a given force through a direct comparison of input and output functions.
- Through this treatment, we assume that we can predict a sea state at a given time t using a weighted sum of the values of the potential such that

$$\hat{\zeta} = \sum_{s} w(s) V(\tau - t) \tag{1}$$

Input Functions

- Munk and Cartwright build on this treatment through a clever expansion of $V(t)$ in terms of spherical harmonics.
- Due to the elliptic (potatoid) nature of the Earth, spherical harmonics provide a convenient basis with which to describe the tidal potentials.

$$Y_n^m(\theta, \lambda) = U_n^m + iV_n^m \tag{2}$$

- Rapidly Convergent.
- Tidal species can be separated and investigated individually by changing the order m.

- Potentials are expanded at the surface of the Earth using a combination of complex spherical harmonics such that

$$V(\theta,\lambda;t) = g\sum_{n=0}^{m}\sum_{m=0}^{m}[a_n^m(t)U_n^m(\theta,\lambda) + b_n^m(t)V_n^m(\theta,\lambda)] \quad (3)$$

  where $U_n^m + iV_n^m$ are complex spherical harmonics and a_n^m and b_n^m are complex coefficients calculated beforehand from the global tide function $c_n^m = a_n^m + ib_n^m$.

- Additional potential functions are necessary to adequately describe the tidal movements. Radiational forces** as well as non-linear terms, were empirically found to reduce these residuals.

- Let us consider our prediction formalism within a continuous time domain. We find that the **Impulse Response Relation** is

$$\hat{\zeta} = \int_0^{\infty} c^*(t-\tau) w d\tau \tag{4}$$

where $c^* = a - ib$ and $w(t) = u(t) + iv(t)$ is the sea-level response to c^* at time $t = 0$.

- The **Admittance** is defined as the Fourier transform of the impulse response such that

$$Z(f) = \int_0^{\infty} w(\tau) e^{-2\pi i f \tau} d\tau \tag{5}$$

Admittances and the Impulse Response Relation

- Smoothed estimates of the admittance are given by the Fourier transform of the response weights.
- Predictions can be achieved through the direct convolution of response weights with the discrete input functions.
- Physically we can consider the response weights $w_n^m(t) = u_n^m(t) + v_n^m(t)$ to be the sea level response at a point to a unit impulse that varies according to the mutually orthogonal spherical harmonics $U_n^m(\theta, \lambda)$, $V_n^m(\theta, \lambda)$

Sequential Analysis

- The ability to separate out the oceanic responses to various forcing inputs lends itself to the development of a sequential analysis framework.
- Sequential analysis is an iterative method that consists of calculating the second-degree harmonics for a given potential as well as the associated smoothed admittances and then subtracting vectorially from the input sea level spectra.
- By iteratively calculating and subtracting the contributions of progressively smaller potentials we systematically reduce the residual error of our prediction.

Credo of Smoothness

- It is necessary to impose a limit on the number of lag intervals τ_s that we consider. Munk and Cartwright choose to restrict τ_s to arithmetic sequences of the form $\tau_s = s\Delta\tau$.
- The optimal lag interval was empirically found to be $\Delta\tau_s \approx 2days$.
- The Maximum lag is determined based on the "wiggliness" of the admittance function. We define the maximum lag $S\Delta\tau = \frac{1}{F}$ where F is the smallest wavelength of $Z(f)$.

Credo of Smoothness

- Munk and Cartwright found no evidence of wiggles shorter than $\frac{1}{6}c/d$. Thus, convolution lags are terminated at 6 days (S=3). (Further works have shown that S=2 is sufficient for most cases.)**
- This assumption will smooth out wiggles smaller than $\frac{1}{6}c/d$ and as such prevents the consideration of and rejects the existence of sharp resonance peaks.
- This Credo of Smoothness requirement makes the realistic generation of input functions critical to model performance.

- As a result of the credo of smoothness, it is critical that we adequately filter our data prior to conducting our analysis.
- Input and lagged data is passed through 6 bandpass filters corresponding to the 6 species of interest.

	Cycles per day	±	cycles per month	Harmonics
Species 1	1	±	4½	285-401
Species 2	2	±	4½	628-744
Species 3	3	±	5½	958-1100
Species 4	4	±	6½	1288-1456
Species 5	5	±	7½	1618-1812
Species 6	6	±	8½	1948-2168

Overview of Python Implementation

Computing Spectral Estimates

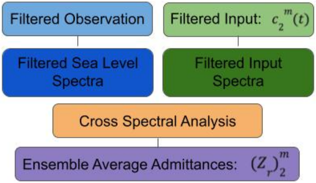
- The spectral estimates G_r, H_r for $a_n^m(t)$ and $\zeta(t)$ are given by

$$\begin{cases} G_r = 2n^{-1}\sum_t \psi(t)a(t)e^{2\pi irt/355} \\ H_r = 2n^{-1}\sum_t \psi(t)\zeta(t)e^{2\pi irt/355} \end{cases}$$

  where $\psi(t) = 1 + \cos(\pi t/355)$.
- The inclusion of the cosine taper function allows for rapid convergence of side-band effects.

Overview of Python Implementation

Cross Spectral Analysis



- The cross-spectrum between the observation and input spectra is defined as

$$C_r + iQ_r =< \frac{1}{2} G_r H_r^* > \tag{6}$$

with ensemble averages produced by averaging the quantities produced by the same r over different times.

- Since the input energy is noise free our cross-spectrum can be expanded to

$$C_r + iQ_r =< G_r G_r^* Z_r + |N_r||G_r|e^{i\nu r} > \tag{7}$$

where N_r is the noise from H_r and ν is its phase relative to G_r

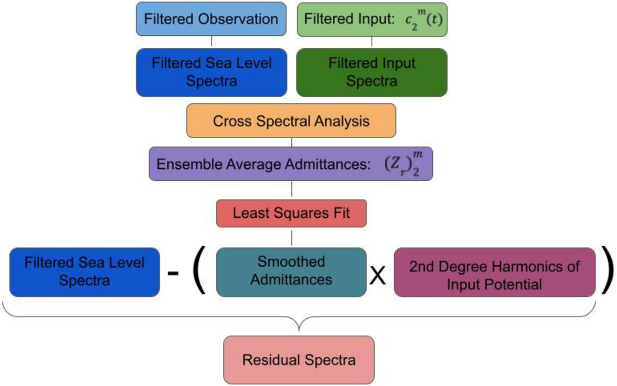
- The admittance estimate can therefore be rewritten as

$$Z_r = \frac{< G_r H^* >}{< G_r G_r^* >} \tag{8}$$

- Note that due to the random phase angle of the noise, the contribution to the ensemble average is negligible and thus our admittance estimate is not biased by noise.

Overview of Python Implementation

Least Squares Formalism



- We can also express Equation 8 as a discrete form of the impulse response relation such that

$$Z(f) = w_0 \sum_{s=1}^{S} w_s e^{-2\pi i f \tau_s} \tag{9}$$

- Note that at the discrete frequencies $f = r/355\frac{c}{d}$ we find $Z_r = Z(f)$.
- After computing the coherent and non-coherent energy we can use this relationship to compute smooth admittances through a least squares procedure.

- Using the connection to the impulse response relation we recognize

$$\hat{\zeta}(t) = \sum_{s=0}^{s} = c^*(t - \tau_s)w_s \tag{10}$$

- The weights w_s in Equations 8 and 9 are determined by

$$\sigma^2 = min(< [\zeta(t) - \hat{\zeta}(t)]^2 >) \tag{11}$$

- The weights for a lumped analysis are calculated by solving the covariance matrix

$$[M_{ij}][w_i] = [R_i] \tag{12}$$

where $[M_{ij}] =< c_i c_j >$ is the cross-covariance between input functions and $[R_i] = c_i \zeta$ is the cross-covariance between input functions and the observed series.

- Calculating weights for a sequential analysis is comparatively simple with

$$[M_i][w_i] = [R_i] \tag{13}$$

Advantages of Response

- Response Analysis is not a tool for tidal forecasting/practitioners, rather it is a tool for the experimentalist oceanographer.
- The principal advantage of the response method is the ability to make separate admittance estimates for distinct inputs.
- Allows for the systematic study of weather and other non-linear inputs.
- Only feasible method to deal with high complexity spectral inputs.
- Constitutes a slight advantage over traditional Harmonic Analysis

Disadvantages of Response

- Due to the credo of smoothness, it is incumbent on the user to generate realistic input functions.
- Response weights must be carefully selected for a given site. Incorrect selection / over-selection can lead to significantly inflated response weights.
- Descriptions of implementation are sparse and generally disorganized.
- Response accuracy decreases with higher complexity admittances.

Closing Remarks

"Evanishing amid the storm.—
Nae man can tether time or tide;"
-Robert Burns

References

W. H. Munk and D. E. Cartwright.
Tidal spectroscopy and prediction.
Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 259:533–581, 1966.

Vice-Amiral A. Dos Santos Franco.
The munk-cartwright method for tidal prediction and analysis.
1968.

A. Lambert.
Earth tide analysis and prediction by the response method.
Journal of Geophysical Research, 79(32), 1974.

P. L. Woodworth and J. M. Vassie.
Reanalyses of maskelyne's tidal data at st. helena in 1761.
Earth System Science Data, 14(9):4387–4396, 2022.