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LSSC

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Abstract

1 Overview

2 Theoritical Description

2.1 Notations

In order to keep coherence with [3], the notations used are the same. A picture of n pixels is seen as a column vector in $\mathbb{R}^n$. The noisy picture is noted $\mathbf{y}$ and the denoised one $\mathbf{x}$. The i-th pixel of x is noted $\mathbf{x}[i]$ and the patch centered in $\mathbf{x}[i]$ and of size m is noted $\mathbf{x}_i$.

2.2 Learned Sparse Coding

The idea behind this method is to assume that the denoised picture is a signal that can be approximated by a sparse linear combinations of elements from a basis set. The basis set is called a dictionary $\mathbf{D} \in \mathbb{R}^{m \times k}$ and is composed of k elements. The denoised patches are then computed from $\mathbf{D}$ with:

$$min_{\boldsymbol{\alpha} \in \mathbb{R}^k} ||\boldsymbol{\alpha}||_p \quad s.t. \quad ||\mathbf{y}_i - \mathbf{D}\boldsymbol{\alpha}||_2^2 \leqslant \epsilon \tag{1}$$

$\mathbf{D}\boldsymbol{\alpha}$ is the estimate of the denoised patch and $||\boldsymbol{\alpha}||_p$ is a regularization term that impose sparsity for $\boldsymbol{\alpha}$.
p is usually 0 or 1. Eq. 1 becomes NP-hard to solve when $p = 0$ and a greedy algorithm such as Orthogonal Matching Pursuit [5] can give an approximation. With $p = 1$, the problem is convex and solved efficiently with the LARS algorithm [6]. Experimental observations [7] have shown that the learning part is better with $p = 1$ and the recomposition part with $p = 0$.
ϵ can be chosen according to the value of the estimated standard deviation of the noise.

2.3 Simultaneous Sparse Coding

3 Algorithm Description

3.1 Algorithm Overview

The first part consists in initializing a dictionary that will denoise roughly the picture.
Once the picture is denoised a first time, a clustering is made in order to regroup similar patches for further treatment.
Then, iteratively for each clusters, the dictionary is updated using simultaneous sparse coding and the cluster is denoised.

3.2 Dictionnary Initialization

The initial dictionnary is first learned offline on the 10 000 images of the PASCAL VOC'07 database using the online dictionary learning procedure of [2]. This procedure is then used on the noisy picture in order to improve the dictionary efficiency.
In fact, only a fixed number T of patches in the picture are used to update the dictionary. However they are chosen so that they are independently and identically distributed in the picture.
The algorithm corresponds then to the minimization of eq.1 on the T patches with the l_1 norm using the LARS [6] algorithm.

Input : number of iterations $\mathbf{T}$, i.i.d. sampling of $\mathbf{T}$ patches of the noisy picture $\mathbf{Y}_T$, initial dictionary $\mathbf{D}^0 \in \mathbb{R}^{m \times k}$, regularization parameter λ
Output : learned dictionary $\mathbf{D}$
Initialization : $\mathbf{A}^0 \in \mathbb{R}^{k \times k} \leftarrow 0$, $\mathbf{B}^0 \in \mathbb{R}^{m \times k} \leftarrow 0$
for *t = 1..**T*** **do**
  $\mathbf{y}_t = \mathbf{Y}_T[\mathrm{t}]$
  Sparse coding: compute with LARS algorithm:

$$\boldsymbol{\alpha}^t = \mathrm{argmin}_{\boldsymbol{\alpha} \in \mathbb{R}^k} ||\boldsymbol{\alpha}||_1 \text{ s.t. } ||\mathbf{y}_t - \mathbf{D}^{t-1}\boldsymbol{\alpha}||_2^2 \leq \lambda$$

  $\mathbf{A}^t \leftarrow \mathbf{A}^{t-1} + \boldsymbol{\alpha}^t \boldsymbol{\alpha}^{tT}$
  $\mathbf{B}^t \leftarrow \mathbf{B}^{t-1} + \mathbf{y}_t^t \boldsymbol{\alpha}^{tT}$
  Update dictionary from $\mathbf{D}^{t-1}$ to $\mathbf{D}^t$ so that:

$$\mathbf{D}^t = \mathrm{argmin}_{\mathbf{D} \in \mathbb{R}^{m \times k}} \frac{1}{t} \sum_{i=1}^{t} (\frac{1}{2} |\mathbf{y}_t^i - \mathbf{D}\boldsymbol{\alpha}^i||_2^2 + \lambda ||\boldsymbol{\alpha}^i||_1)$$

end
return $\mathbf{D}^T$

Algorithm 1: Online Dictionary Learning

Input :input dictionary $\mathbf{D}$= [$\mathbf{d}^1$, ... , $\mathbf{d}^k$] $\in \mathbb{R}^{m \times k}$,
$\mathbf{A}$ = [$\mathbf{a}^1$, ... , $\mathbf{a}^k$] $\in \mathbb{R}^{k \times k}$, $\mathbf{B}$ = [$\mathbf{b}^1$, ... , $\mathbf{b}^k$]$\in \mathbb{R}^{m \times k}$
Output : updated dictionary $\mathbf{D}$
repeat
  for *j = 1..k* **do**
    update the j^{th} column:
    if *A(j,j) = 0* **then**

$$\mathbf{d}^j \leftarrow 0$$

    end
    else

$$\mathbf{u}^j \leftarrow \frac{1}{\mathbf{A}(j,j)} (\mathbf{b}^j - \mathbf{D}\mathbf{a}^j) + \mathbf{d}^j$$

$$\mathbf{d}^j \leftarrow \frac{1}{\max(||\mathbf{u}^j||_2, 1)} \mathbf{u}^j$$

    end
  end
until *convergence Marc: apparemment, dans le code on a une boucle sur params.updateIteration = 1. Est-ce qu'il ne vaudrait mieux pas calculer l'argmin à chaque boucle et s'arrêter lorsque c'est plus petit qu'une certaine valeur ?* ;
return **D**

Algorithm 2: Dictionary Update updateDictionary Marc: Algo validé

Input : Gram matrix of the dictionary $G = \mathbf{D}^T\mathbf{D} \in \mathbb{R}^{k\times k}$ where $\mathbf{D} = [\mathbf{d}^1, \dots, \mathbf{d}^k] \in \mathbb{R}^{m\times k}$, noisy patch $\mathbf{y}$, constraint λ
Output : sparse vector $\boldsymbol{\alpha} \in \mathbb{R}^k$
—INITIALIZATION—
Marc : $\mathcal{A}$ n'est pas définie, et $\boldsymbol{D}_{\mathcal{A}}$ non plus.
normX $\in \mathbb{R} \leftarrow ||\mathbf{y}||_2^2$
coeffs $\in \mathbb{R}^k \leftarrow \mathbf{0}$
Most correlated element
$\hat{\mathbf{c}} \in \mathbb{R}^k \leftarrow \mathbf{D}^T\mathbf{y}$
$C \in \mathbb{R}^+ \leftarrow \max_{j=1..m}(|\hat{\mathbf{c}}_j|)$
currentInd $\leftarrow j$ s.t. $\hat{\mathbf{c}}_j = C$
if *normX* $> \lambda$ *Marc : pourquoi > et pas < ?* **then**
  return 0
end
newAtom $\leftarrow$ **True**
for *i = 1..k* **do**
  —NEW ATOM—
  if *newAtom* **then**
    $\mathcal{A}$[i] $\leftarrow$ currentInd
    $G_A[i^{th}$ line$] \leftarrow G[\mathcal{A}[\text{i}]^{th}$ line$]$
    $G_S \leftarrow \boldsymbol{D}_{\mathcal{A}}^T\boldsymbol{D}_{\mathcal{A}}$
    UPDATE G_S^{-1}
  end
  —VARIABLES UPDATES—
  u $\leftarrow G_S^{-1}(\text{sgn}(\hat{\mathbf{c}}_j))_{j\in\mathcal{A}}$ Marc : qu'est-ce que ça veut dire ? de quelle taille est u ?
  ratio $\leftarrow \left(-\frac{\text{coeffs}[j]}{u_j}\right)_{j\in[1;i]}$
  stepMAX $\leftarrow \min^+$(ratio) ($\min^+$ means its the minimum between positive values)
  criticalInd $\leftarrow$ j s.t. ratio[j] = stepMAX Marc : criticalInd n'est jamais plus utilisé par la suite ??!!
  C $\leftarrow \hat{\mathbf{c}}$[0] Marc : ça ne devrait pas être C = max $\hat{\mathbf{c}}[j]$?
  $\gamma \leftarrow \min^+\left(\frac{C\pm\hat{\mathbf{c}}_j}{1\pm(G_A u)[j]}\right)_{j\notin\mathcal{A}}$ Marc : ça fait 2 possibilités ou bien 4 avec les ± ?
  —POLYNOMIAL RESOLUTION—
  $a \leftarrow \sum_{j\in\mathcal{A}} \text{sgn}(\hat{\mathbf{c}}[\mathcal{A}[j]])u[j]$
  $b \leftarrow \sum_{j\in\mathcal{A}} \hat{\mathbf{c}}[\mathcal{A}[j]]u[j]$
  $c \leftarrow$ normX - λ
  $\Delta \leftarrow b^2 - ac$
  stepMAX2 $\leftarrow \min(\frac{b-\sqrt{\Delta}}{a}, C)$
  —FINAL STEP & BREAK—
  $\gamma \leftarrow \min(\gamma$, stepMAX, stepMAX2)
  coeffs $\leftarrow$ coeffs + γu Marc : a priori coeffs et u ne sont pas de la même taille
  $\hat{\mathbf{c}} \leftarrow \hat{\mathbf{c}} - \gamma G_A u$
  normX $\leftarrow$ normX $+ a\gamma^2 - 2b\gamma$
  if $|\gamma| < 1e^{-15}$ || $\gamma = stepMAX2$ || $normX < 1e^{-15}$ || $normX - \lambda < 1e^{-15}$ **then**
    break Marc : ici on travaille en float, donc je pense qu'on peut mettre $1e^{-6}$
  end
  if $\gamma = stepMAX$ **then**
    DOWNDATE G_S^{-1}
    newAtom $\leftarrow$ **False**
    i $\leftarrow$ i-2 Marc c'est possible que i devienne négatif ? Ça risque de poser problème non ?
  end
  else
    newAtom $\leftarrow$ **True**
  end
end
return $\boldsymbol{\alpha} = sort(\text{coeffs}, A)$ (sorted coefficient with respect to the increasing order of A

Input : Gram matrix $G_S \in \mathbb{R}^{i \times i}$, and its former inverse to update $G_S^{-1} \in \mathbb{R}^{i-1 \times i-1}$
Output : updated $G_S^{-1} \in \mathbb{R}^{i \times i}$
if $i = 1$ **then**
  return $\frac{1}{G_S}$
end
$u \leftarrow G_S^{-1} G_S[i^{th} \text{ line}]$
$\sigma \leftarrow \frac{1}{G_s(i,i) - u.G_S[i^{th} \text{ line}]}$
$G_s^{-1}(i,i) \leftarrow \sigma$
$G_s^{-1}[i^{th} \text{ line}] \leftarrow$ -σu
return $G_s^{-1} \leftarrow G_s^{-1} + \sigma u u^T$

Algorithm 4: Update invert algorithm updateGram Marc: Algo correspondant, mais à valider.

Input : pseudo-Gram matrix $G_A \in \mathbb{R}^{i \times i}$ Gram matrix $G_S \in \mathbb{R}^{i \times i}$, and its inverse $G_S^{-1} \in \mathbb{R}^{i \times i}$, active indexes list $\mathcal{A} \in \mathbb{R}^i$, sparse coefficient list coeffs $\in \mathbb{R}^k$, criticalInd $\in [1; k]$, current iteration i
Output : downdated matrices $G_S, G_S^{-1}, G_A \in \mathbb{R}^{i-1 \times i-1}$, downdated lists $\mathcal{A} \in \mathbb{R}^{i-1}$, coeffs $\in \mathbb{R}^k$
$\sigma \leftarrow \frac{1}{G_S^{-1}(\text{criticalInd,criticalInd})}$
$u \leftarrow G_S^{-1}[\text{criticalInd}^{th} \text{ line}]$ without its criticalIndth coefficient
for j= *criticalInd:i* **do**
  $G_A[j^{th} \text{ line}] \leftarrow G_A[(j+1)^{th} \text{ line}]$
  for k= *1:criticalInd-1* **do**
    $G_S(j,k) \leftarrow G_S(j+1,k)$
    $G_S^{-1}(j,k) \leftarrow G_S^{-1}(j+1,k)$
  end
  for k= *criticalInd:i* **do**
    $G_S(j,k) \leftarrow G_S(j+1,k+1)$
    $G_S^{-1}(j,k) \leftarrow G_S^{-1}(j+1,k+1)$
  end
  $\mathcal{A}[j] \leftarrow \mathcal{A}[j+1]$
  coeffs[j] $\leftarrow$ coeffs[j+1]
end
coeffs[i] $\leftarrow 0$
$G_s^{-1} \leftarrow G_s^{-1} - \sigma u u^T$

Algorithm 5: Downdate invert algorithm downdateGram Marc: Algo ne correspondant pas, j'ai créé une nouvelle fonction downdateGramBis dont je pense est une meilleure version. Il faudra la tester et la valider.

Glossary

Image Credits

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4 References

References

[1] B. Efron, T. Hastie, I. Johnstone, and R. Tibshirani *Least angle regression*. Ann. Statist., 32(2):407499, 2004.

[2] J.Mairal, F. Bach, J. Ponce, and G. Sapiro *Online dictionary learning for sparse coding*. ICML, 2009.

[3] J. Mairal, F. Bach, J. Ponce, G. Sapiro and A. Zisserman *Non-local Sparse Models for Image Restoration*. International Conference on Computer Vision, 2009.

[4] J. Mairal *Reprsentations parcimonieuses en apprentissage statistique, traitement dimage et vision par ordinateur*. PhD thesis, 2010.

[5] S. Mallat and Z. Zhang *Matching pursuit in a timefrequency dictionary*. IEEE T. SP, 41(12):33973415, 1993.

[6] B. Efron, T. Hastie, I. Johnstone, and R. Tibshirani *Least angle regression*. Ann. Statist., 32(2):407499, 2004.

[7] M. Elad and M. Aharon *Image denoising via sparse and redundant representations over learned dictionaries*. IEEE T. IP, 54(12):37363745, 2006.