Question 2

when $n = 100$

<u>data 1 $\sim Expo(\theta)$</u>

$\hat{\theta}_{MME} = 0,257$

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<u>data 2 $\sim N(\mu, \sigma^2)$</u>

$\mu_{MME} = \mu_{MLE} = 1,868$

$\sigma^2_{MME} = \sigma^2_{MLE} = 3,121$

<u>data 3 $\sim gamma(\alpha, \beta)$</u>

$\beta_{MME} \approx 0,899$

$\alpha_{MME} = 5,659$

There is no closed form for α's MLE estimator

a) **Data 1:**

→ MLE and MME estimators are equal, They have same variance, Their relative efficiency is 1.

<u>Data 2!</u>

→ The MLE is the sample mean and sample variance, which are unbiased and efficient. The MME is the same as the MLE so their efficiencies are equal.

<u>Data 3!</u>

→ MLE's are generally more efficient than MME's (MLE's have lower variance). However; since we do not have α_{MLE}, we can't compute without additional numerical analysis. Generally rel. efficiency would be less than 1, MLE's are more efficient.

Part b

data 1

$f(x;\theta) = \theta \cdot e^{-\theta x}$

$$CRLB = \frac{[\tau'(\theta)]^2}{I_n(\theta)} = \frac{\theta^2}{n} \qquad \hat{\theta} = \bar{x}$$

$n = 100$

$$E(\hat{\theta}) = \frac{\cancel{n}\theta}{\cancel{n}} = \bar{x} \approx 0.257$$

$$V(\hat{\theta}) = \frac{n \cdot \theta^2}{n^2} = \frac{\theta^2}{n} = \frac{\bar{x}^2}{n} \approx (0.0257)^2$$

UMVUE for $\theta \approx 3.892$ CRLB Var $= 0.151$

As sample size increase, θ is distributed Normally

$$\hat{\theta} \sim N(0.257, 0.0257^2)$$

data 2

$\sim N(\mu, \sigma^2)$ $\qquad \hat{\mu} = \bar{x} \approx 1.868$

$\hat{\mu} \sim N(1.868, \frac{\sigma^2}{n})$ $\qquad$ UMVU $\hat{\mu} = \bar{x} = 1.868$

$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$ $\qquad$ UMVU $\sigma^2 = 2.833$

Distributions of MLE's

$\hat{\mu}$ is normally distributed $\hat{\mu} \sim N(\mu, \sigma^2/n)$

$\hat{\sigma}$ follows a scaled chi-squared distribution χ^2

Data 3

$$\hat{\beta} = \frac{\sum \bar{x}_i - (\sum x_i)^2/n}{\sum x_i} = \hat{\beta}_{MME} = 0.899$$

$$\hat{\sigma} = \frac{\sum x_i}{n.\hat{\beta}} \qquad \hat{\sigma}_{MME} = 5.059$$

In real life, we use number methods to come close to MLE for (σ), and the approximation often come close to the UMVU estimator for larger samples ($n=100$ large enough) Since there is no closed-form solution, hence we can't provide the exact UMVU estimator in this case.

Part c

data 1 Sample mean ($\bar{x}$) is s.s for θ so;

Determine Pivotal Quantity; $Q = 2n\bar{x}\theta \sim \chi^2_{2n}$

$$P(\chi^2_{\frac{\alpha}{2},2n} \leq 2n\bar{x}\theta \leq \chi^2_{1-\frac{\alpha}{2},2n}) = 1-\alpha$$

$$P\left(\frac{\chi^2_{\frac{\alpha}{2},2n}}{2n\bar{x}} \leq \theta \leq \frac{\chi^2_{1-\frac{\alpha}{2},2n}}{2n\bar{x}}\right) = 1-\alpha$$

C.I for θ = (3.167, 4691) from python.

data 2

CI for μ ; use sample mean and sample std. dev.

$$\bar{x} \pm z_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$$

CI for μ (1.256 , 2.480)

CI for σ^2 ; use Chi-squared distribution.

$$\left(\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}, n-1}} , \frac{(n-1) \cdot s^2}{\chi^2_{\frac{\alpha}{2}, n-1}} \right)$$

CI for σ^2 (7.511 , 13.148)

data 3

the MLE's are approximately normally dist. for large sample, allowing us to construct confidence intervals

Use numerical optimization find $\hat{\alpha}$ and $\hat{\beta}$

$SE(\hat{\alpha}) = n\hat{\alpha}$, $SE(\hat{\beta}) = n\hat{\beta}$

$\hat{\alpha} \pm z_{\frac{\alpha}{2}} \cdot SE(\hat{\alpha})$ $\hat{\beta} \pm z_{\frac{\alpha}{2}} \times SE(\hat{\beta})$

CI for α (5.193 , 6.125) CI (0.713 , 1.085)