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Computational Intelligence

Subject8: Fuzzy Logic and Inference

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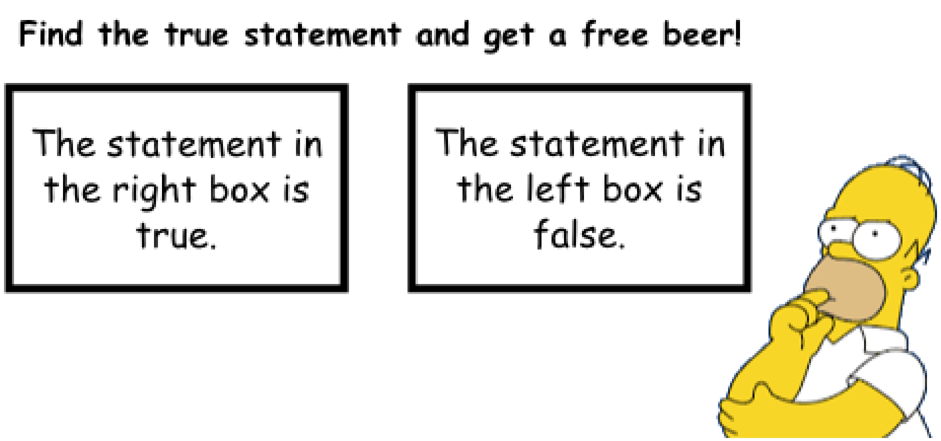
Agenda

- Classical logic
- Multi-valued logic
- Fuzzy logic
- Linguistic hedges
- Fuzzy inference

Classical Logic

- "**Liar Paradox**" in philosophy and logic
  - David: "I said in my alarm, Every man is a liar!"
    - If true, then David also is lying because he is a man!
    - If he too is lying, the statement consequently is not true!



Classical Logic

- Important concepts

Title	Description
Logic	Concluding based on some statements/propositions
Proposition	A statement than can be true or false
Propositional Logic	The logic based on a combination of several propositions
Logic Variable	The variable introducing a proposition
Logic Function	The function operating on one/several LV(s)

Classical Logic

Primitive

- Performs a logical operation
- Samples: OR, NOT, AND, Implication, etc.
- **Functional completeness**: a set of operators which can be used to express all possible truth tables by combining its members

Propositional Formula

- A type of syntactic formula which is well formed and has a truth value

$$(A \ and \ B) \ or \ (C \ and \ D) \ \rightarrow E$$

Classical Logic

Tautology

- A formula or assertion that is <u>true</u> in every possible interpretation

Contradiction

- A formula or assertion that is <u>false</u> in every possible interpretation

a	b	$(a \to b)$	$(a \wedge (a \to b))$	$(a \wedge (a \to b)) \to b$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Tautology

$(a \to b) \equiv (-a \vee b)$

Classical Logic

Boolean Algebra

Law/Theorem	Law of Addition	Law of Multiplication
Identity Law	$x + 0 = x$	$x \cdot 1 = x$
Complement Law	$x + x' = 1$	$x \cdot x' = 0$
Idempotent Law	$x + x = x$	$x \cdot x = x$
Dominant Law	$x + 1 = 1$	$x \cdot 0 = 0$
Involution Law	$(x')' = x$	
Commutative Law	$x + y = y + x$	$x \cdot y = y \cdot x$
Associative Law	$x+(y+z) = (x+y)+z$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Distributive Law	$x \cdot (y+z) = x \cdot y+x \cdot z$	$x+y\cdot z = (x+y) \cdot (x+z)$
Demorgan's Law	$(x+y)' = x' \cdot y'$	$(x \cdot y)' = x' + y'$
Absorption Law	$x + (x \cdot y) = x$	$x \cdot (x + y) = x$

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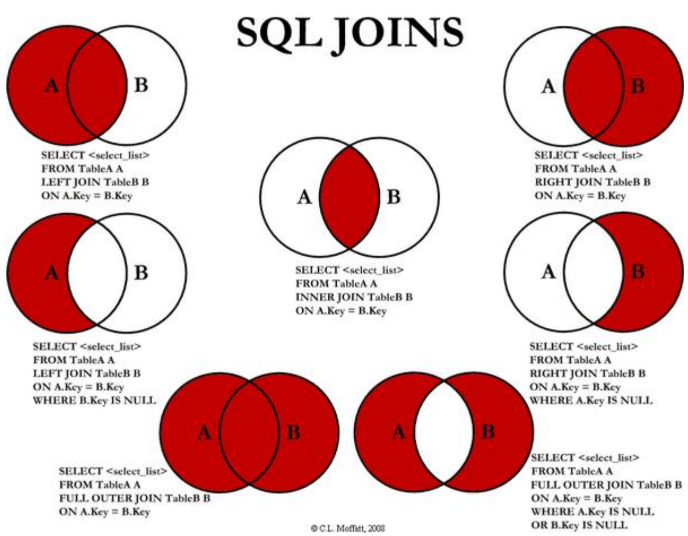
Classical Logic

Set Theory

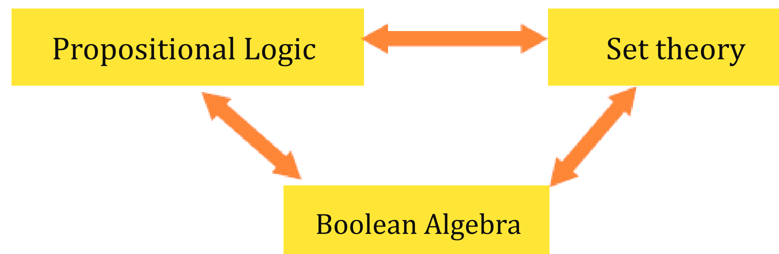


Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Classical Logic

Isomorphism

- If there are equivalent concepts in different systems



Operator	Propositional Logic	Boolean Algebra	Set Theory
Union	$\cup$	$+$	$\vee$
Intersection	$\cap$	$\cdot$	$\wedge$
Empty set	$\emptyset$	$\circ$	$\circ$
Subsequence	$\subseteq$	$\leq$	➔

Classical Logic

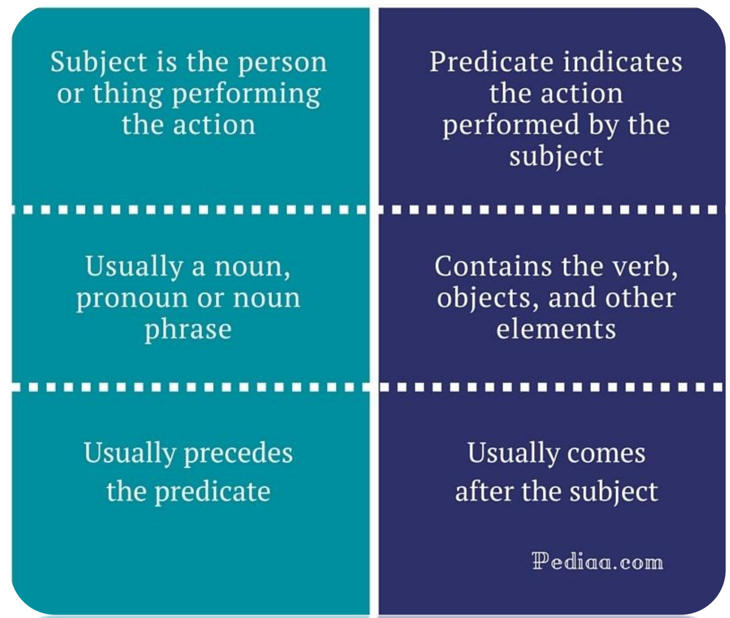
- Propositions, Subjects, and Predicates

Iran is in the north-west of Asia

Subject Predicate

The predicate

- Must contain a verb
- Applies on the subject
- Is a characteristic/set indicator function



Classical Logic

- Predicate - other properties
  - Extendable for more than one variable

X is a student of the class.

X = { Mohammadi, Rezaee, Alizadeh}

  - Extendable using existential and universal quantifiers

$$\exists x\ P(x) = \bigvee P(x)$$ *"there exists", "there is at least one", or "for some"*

$$\forall x\ P(x) = \bigwedge P(x)$$ *"given any" or "for all"*

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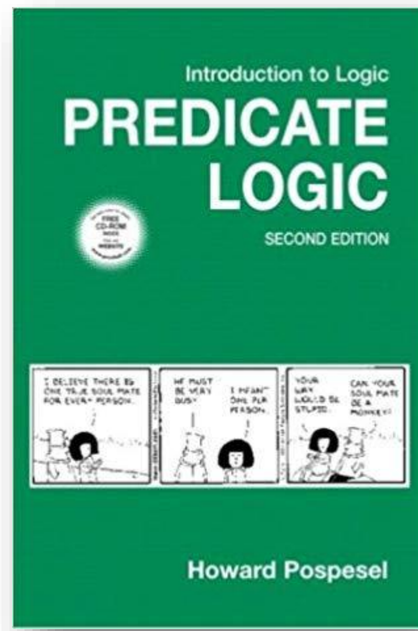
Classical Logic

Predicate Logic

Covers more concepts, including:

- Constants, like “Ali” or 100
- Variables, like x and y
- Predicates
- Universal and existential quantifiers
- Functions to describe objects



Classical Logic

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Inference

$(a \to b) \equiv (-a \vee b)$

Modus Ponens

$$\frac{\begin{array}{c} P \Rightarrow Q \\ P \end{array}}{Q}$$

If it is Tuesday, I will go to the gym
Today is Tuesday
So, I will go to the gym

Modus Tollens

$$\frac{\begin{array}{c} P \Rightarrow Q \\ -Q \end{array}}{-P}$$

If it is Tuesday, I will go to the gym
Today I didn't go to the gym
So, it is not Tuesday

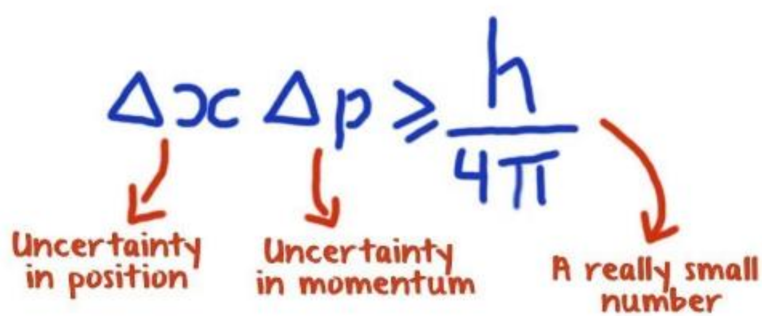
Chain Rule

$$\frac{\begin{array}{c} P \Rightarrow Q \\ Q \Rightarrow R \end{array}}{P \Rightarrow R}$$

If it is Tuesday, I will go to the gym
If I go to the gym, I will see Joe
So, If it is Tuesday, I will see Joe

Multi-valued Logic

- Classical Logic has two outcomes: true or false
- Based on the Heisenberg's uncertainty principle
- Three-valued logic:
  - Including: True, False, and Unsure



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Multi-valued Logic

Three-valued logic

(F, false; U, unknown; T, true)

NOT(A)

A	¬A
F	T
U	U
T	F

AND(A, B)

A ∧ B		B		
		F	U	T
A	F	F	F	F
	U	F	U	U
	T	F	U	T

OR(A, B)

A ∨ B		B		
		F	U	T
A	F	F	U	T
	U	U	U	T
	T	T	T	T

Multi-valued Logic

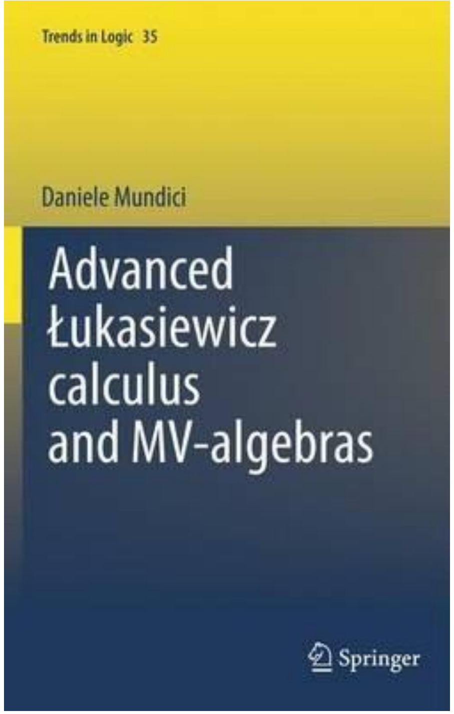
Multi-valued logic by Lukasiewicz

- Members have a degree of truth

$$T_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$$

Lukasiewicz Isomorphism

- Multi-valued logic ➔ Fuzzy set
- Binary-valued element ➔ Crisp set

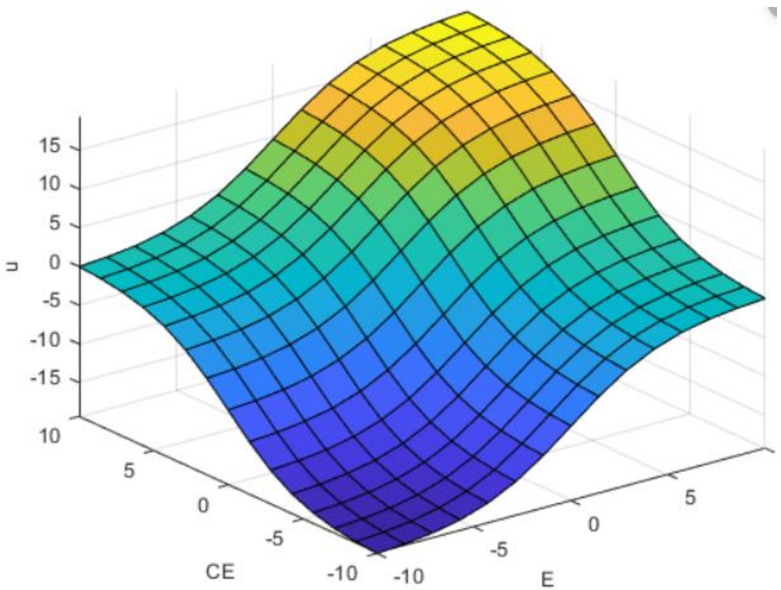


Fuzzy Logic

- Here, instead of true/false, the membership degrees are used

Different types of propositions

- Unconditional and unqualified
- Unconditional and qualified
- Conditional and unqualified
- Conditional and qualified



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Unconditional and unqualified

The temperature is very high today

$$P: \mu \; is \; F$$

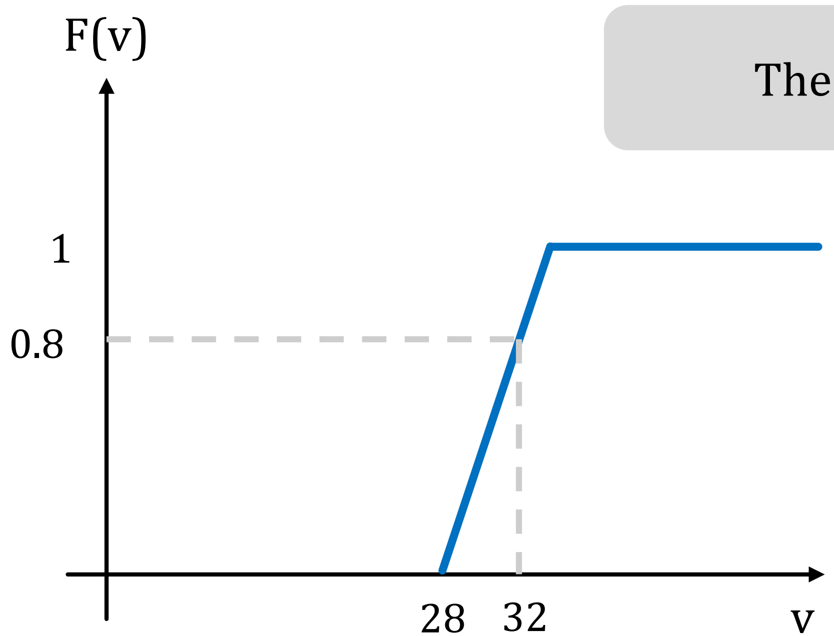
P $= a \; fuzzy \; preposition$
μ $= a \; variable \; of \; a \; universal \; set \; V$
F $= a \; fuzzy \; set \; on \; V$

$$T(P) = F(v)$$

$T(P) = the \; degree \; of \; truth \; of \; P$
$F(v) = the \; membership \; grade \; of \; F$

Fuzzy Logic

Unconditional and unqualified



The temperature is very high today

$$v = 32$$

$$F(32) = 0.8$$

$$T(P) = 0.8$$

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Unconditional and qualified

The possibility of rain is 20%, which is ignorable!

$$P\colon \mu \text{ is } F \text{ is } S$$

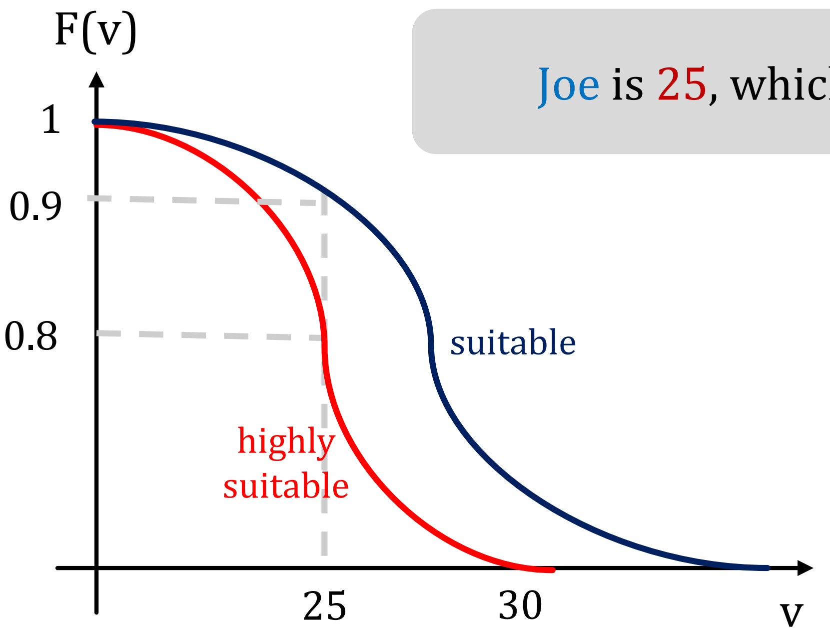
P = a fuzzy preposition
μ = a variable of a universal set V
F = a fuzzy set on V
S = a fuzzy truth qualifier

$$T(P) = S(F(v))$$

$T(P)$ = the degree of truth of P
$F(v)$ = the membership grade of F

Fuzzy Logic

Unconditional and qualified



Joe is 25, which makes him suitable for the job!

$$v = 25$$

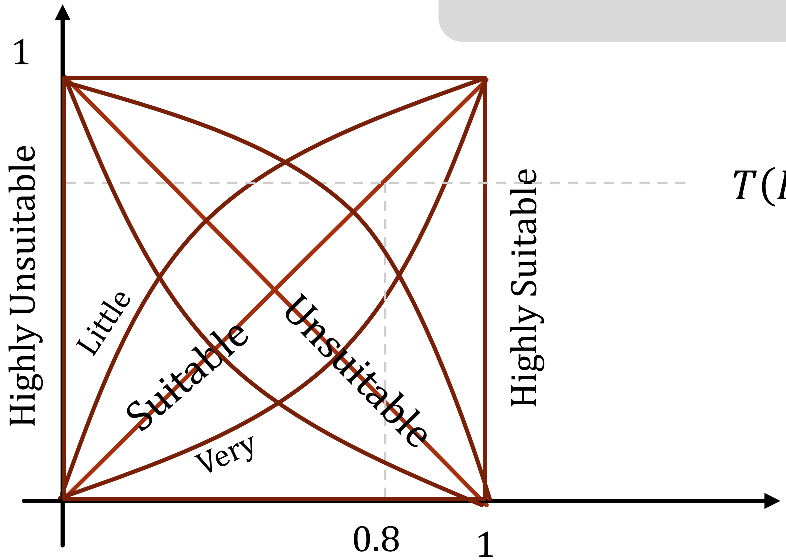
$$F(25) = 0.8 \; in \; highly \; suitable$$

$$F(25) = 0.9 \; in \; suitable$$

Fuzzy Logic

Unconditional and qualified

Joe is 20, which makes him suitable for the job!

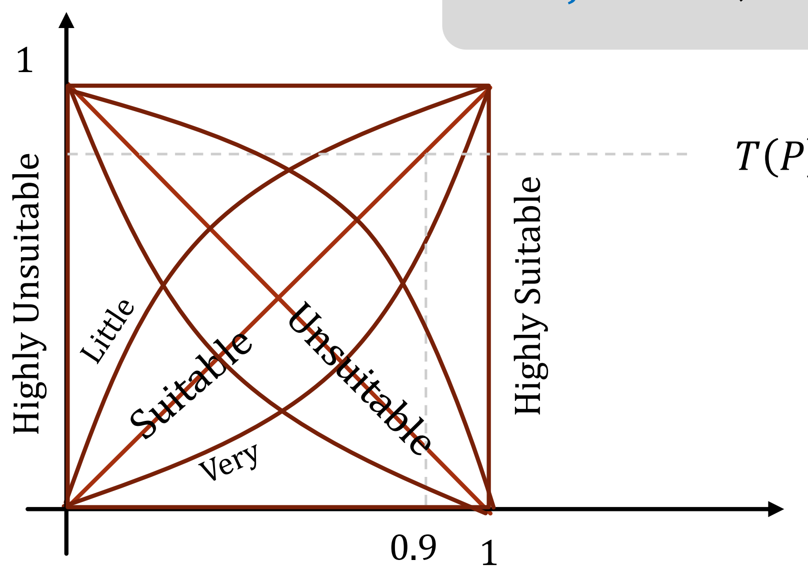


$$T(P) = S(0.8)\ for\ highly\ suitable$$

Fuzzy Logic

Unconditional and qualified

Joe is 20, which makes him suitable for the job!



$T(P) = S(0.9)\ for\ suitable$

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Conditional and unqualified

If the temperature is low, then the speed of the fan is high!

$$P\colon if\ X\ is\ A, then\ Y\ is\ B$$

$P\ = a\ fuzzy\ preposition$

$X\&Y\ = variables\ of\ universal\ sets$

$A\&B\ = fuzzy\ sets\ on\ X\ and\ Y$

$$\mathrm{R}(x, y) = \gamma[A(x), B(y)]$$

$R(x, y) = a\ fuzzy\ set\ on\ X * Y$

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Conditional and unqualified

- Consider these two Fuzzy numbers with a relation as γ:

$$A = \frac{0.1}{x_1} + \frac{0.8}{x_2} + \frac{1}{x_3} \qquad B = \frac{0.5}{y_1} + \frac{1}{y_2} \qquad \gamma[A(x), B(y)] = \min(1, 1 - a + b)$$

- Thus:

$$\mathrm{R}(x, y) = \gamma[A(x), B(y)] = \frac{1}{x_1 y_1} + \frac{1}{x_1 y_2} + \frac{0.7}{x_2 y_1} + \frac{1}{x_2 y_2} + \frac{0.5}{x_3 y_1} + \frac{1}{x_3 y_2}$$

min(1, 1-(0.1)+0.5) for x1 and y1 = 1	min(1, 1-(0.8)+0.5) for x1 and y1 = 1

Fuzzy Logic

Conditional and qualified

- A combination of the previous methods …

If the temperature is low, then “the fan speed becomes 25rpm” is a good choice!

If Joe is young, then “his age is 20” is a true sentence!

Linguistic Hedges

How to extend the propositions?

- Using some simple tools, called LH
  - Allows speakers and writers to signal caution, or probability
  - Samples: Very, more, less, often, rather, etc.

Joe likes to do the job himself

Joe often likes to do the job himself

Joe often likes to do most of the jobs himself

Linguistic Hedges

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Important notes:

- LHs can be applied on Fuzzy sets, not Crisp sets
  - Sample: The line is too straight !?!?
- Use "power" for some words like "often" or "very"
- Use "Square root" for some words like "sometimes" or "little"

The temperature is very high today

$$v = 32 \;\rightarrow F(32) = 0.8 \;\rightarrow T(P) = 0.8$$

$$T(P) = 0.8^2 = 0.64$$ Very high

$$T(P) = \sqrt{0.8} = 0.89$$ A little high

Fuzzy Inference

- Based on the Classical sets inference rules

$(a \rightarrow b) \equiv (-a \vee b)$

Modus Ponens	Modus Tollens	Chain Rule
$P \rightarrow Q$	$P \rightarrow Q$	$P \rightarrow Q$
P	$-Q$	$Q \rightarrow R$
Q	$-P$	$P \rightarrow R$
If it is Tuesday, I will go to the gym Today is Tuesday So, I will go to the gym	If it is Tuesday, I will go to the gym Today I didn't go to the gym So, it is not Tuesday	If it is Tuesday, I will go to the gym If I go to the gym, I will see Joe So, If it is Tuesday, I will see Joe

Fuzzy Inference

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Modus Ponens (Classical)

- **Rule:** if P is true, then Q is concluded
- **Phases:** observation and conclusion
- No new knowledge is produced (exact inference)

$$\frac{\begin{array}{c} P \rightarrow Q \\ P \end{array}}{Q}$$

If Joe passes the exam, he will get the job
Observation: Joe passed the exam
Conclusion: He will get the job

Fuzzy Inference

Modus Ponens (Fuzzy)

- **Rule:** if P is true, then Q is concluded
- **Phases:** observation and conclusion
- New knowledge is produced
  - The outcome is not definite (approximate inference)

if X is A then Y is B
X is A'

Y is B'

If Joe study hard, he will get a good score
Observation: Joe studies very hard
Conclusion: He will get a great score

Fuzzy Inference

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Modus Tollens (Classical)

- **Rule:** if *P* is true, then *Q* is concluded
- **Phases:** observation and conclusion
- No new knowledge is produced (exact inference)

$$\frac{P \rightarrow Q \quad -Q}{-P}$$

If Joe passes the exam, he will get the job
Observation: Joe didn't get a good job
Conclusion: He didn't pass the exam

Fuzzy Inference

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Modus Tollens (Fuzzy)

- **Rule:** if P is true, then Q is concluded
- **Phases:** observation and conclusion
- New knowledge is produced
  - The outcome is not definite (approximate inference)
- To calculate A' we need:
  - The relation between X and Y
  - The value of B'

if X is A then Y is B
Y is not B'

X is not A'

$$A'(y) = \sup \min[B'(y), R(x,y)]$$

$$R(x,y):\ \gamma(A(x), B(y))$$

Fuzzy Inference

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Modus Tollens (Fuzzy)

- For instance, if: $A = \frac{0.5}{x_1} + \frac{1}{x_2} + \frac{0.6}{x_3}$ $\quad B = \frac{1}{y_1} + \frac{0.4}{y_2}$
- And R is defined as: $R = \frac{1}{x_1 y_1} + \frac{0.9}{x_1 y_2} + \frac{1}{x_2 y_1} + \frac{0.4}{x_2 y_2} + \frac{1}{x_3 y_2} + \frac{0.8}{x_3 y_2}$
- Let's assume: $B' = \frac{0.9}{y_1} + \frac{0.7}{y_2}$

$A'(y) = \sup \min[B'(y), R(x, y)]:$

$$A'(x_1) = \sup \min[B'(y), R(x_1, y)] = \max[\min(0.9, 1), \min(0.7, 0.9)] = 0.9$$
$$A'(x_2) = \sup \min[B'(y), R(x_2, y)] = \max[\min(0.9, 1), \min(0.7, 0.4)] = 0.9$$
$$A'(x_3) = \sup \min[B'(y), R(x_3, y)] = \max[\min(0.9, 1), \min(0.7, 0.8)] = 0.9$$

$$\Rightarrow A'(y) = \frac{0.9}{x_1} + \frac{0.9}{x_2} + \frac{0.9}{x_3}$$

Fuzzy Inference

Chain Rule (Classical)

- Several rules connected together
- Conclusion can be met using other statements

$$\frac{P \rightarrow Q \quad Q \rightarrow R}{P \rightarrow R}$$

If Joe passes the exam, he will get the job
If Joe has a good job, he can buy a car
Conclusion: If Joe passes the exam, he can buy a car

Fuzzy Inference

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Chain Rule (Fuzzy)

- Several rules connected together
- There are three relations for all Fuzzy sets

$$\frac{\begin{array}{l}\text{if } X \text{ is } A \text{ then } Y \text{ is } B\\ \text{if } Y \text{ is } B \text{ then } Z \text{ is } C\end{array}}{\text{if } X \text{ is } A \text{ then } Z \text{ is } C}$$

$$R1(x,y):\ \gamma(A(x),B(y))$$

$$R2(y,z):\ \gamma(B(y),C(z))$$

$$R3(x,z):\ \gamma\big(A(x),C(z)\big) = \sup\min[R1(x,y),R2(x,y)]$$

Fuzzy Inference

Chain Rule (Fuzzy)

- For instance, if: $A = \frac{0.5}{x_1} + \frac{1}{x_2} + \frac{0.6}{x_3}$ $\quad B = \frac{1}{y_1} + \frac{0.4}{y_2}$ $\quad C = \frac{0.2}{z_1} + \frac{1}{z_2}$
- And J defines the relations: $J(a,b) = \begin{cases} 1 & if\ a \leq b \\ b & if\ a > b \end{cases}$
- We will have:

$$R1(x,y) = \begin{array}{ccc} y1 & y2 & \\ 1 & 0.4 & x1 \\ 1 & 0.4 & x2 \\ 1 & 0.4 & x3 \end{array} \qquad R2(y,z) = \begin{array}{ccc} y1 & y2 & \\ 0.2 & 1 & z1 \\ 0.2 & 1 & z2 \end{array} \Rightarrow R3(x,z) = \begin{array}{ccc} z1 & z2 & \\ 0.2 & 1 & x1 \\ 0.2 & 1 & x2 \\ 0.2 & 1 & x3 \end{array}$$

What's Next?

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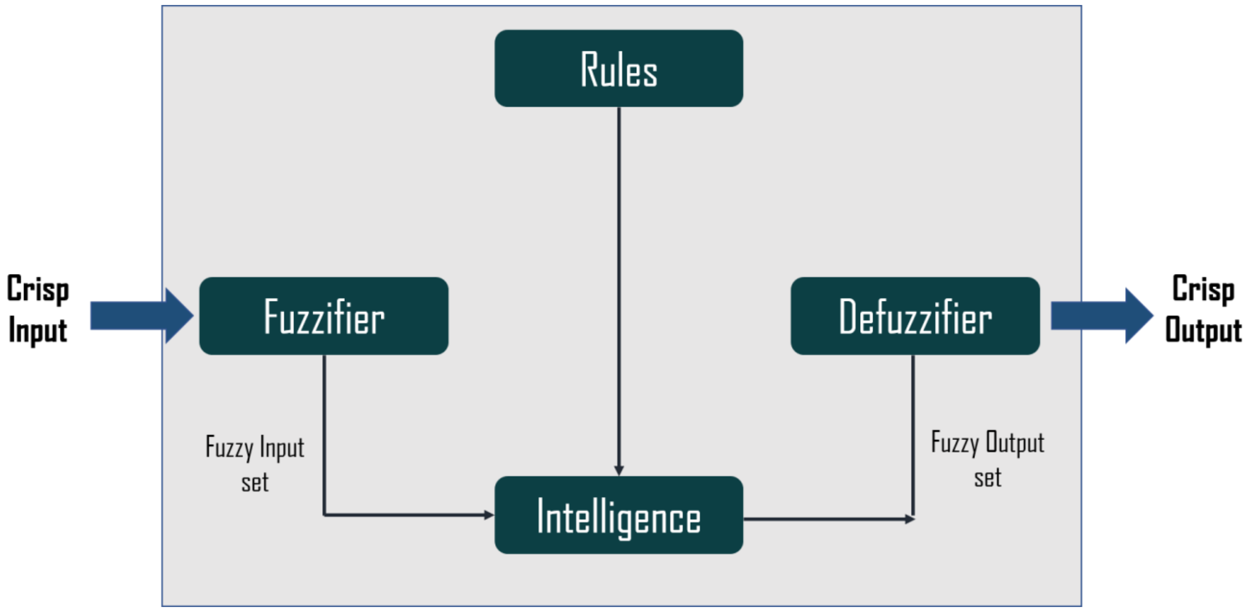
- Fuzzy Controllers



Questions?