MA030282: Mathematics for Engineers
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Project 2: Inverted Pendulum

The inverted pendulum plant is a nonlinear system, and it is widely used for testing different control algorithms. There is a wide variety of this system's applications, such as crane stabilization, vehicle development with pendulum system "Segway," and modeling for seismic control of building structures. In the aerospace field, it is used for the active control of a rocket to keep it upright at the time of takeo˙, the modeling of biped robots, satellite positioning, balance stabilization of ships and aircraft. Let's first consider a single inverted pendulum with mass m_1 and length l (see Fig. 1), mounted on a cart of mass m_0 with wheels.

1. System of Pendulum Inverted on a Cart (PIC)

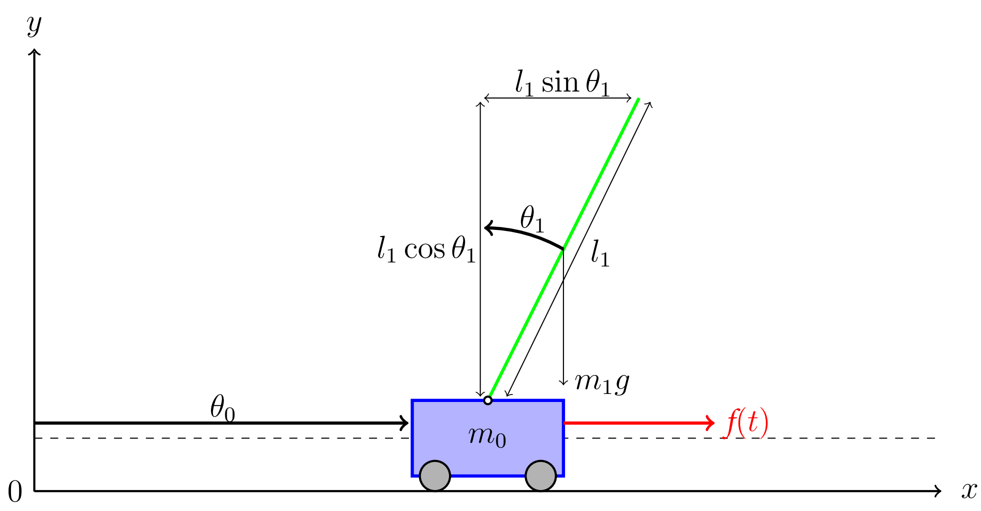


Figure 1: Pendulum Inverted on a Cart (PIC)

In order to write down the equation of motion for PIC system we used the second law of Newton:

$$m_1 g \frac{l_1}{2} \sin\theta_1 = \left(\frac{m_1 l^2{}_1}{12} + m_1 \left(\frac{l_1}{2}\right)^2 \right) \ddot{\theta}_1$$

$$\sin\theta_1 \to \theta_1$$

$$\frac{ml_1}{12}\theta_1 = \frac{ml_1{}^2}{3}\ddot{\theta}_1$$

$$\theta_1 = \frac{2l_1}{3g}\ddot{\theta}_1$$

$$f + \frac{\left(\frac{m_1 l_1^2}{12} + m_1 \left(\frac{l_1}{2}\right)^2\right)\ddot{\theta}_1}{{}^{l_1}/_2} = (m_0 + m_1)a$$

$$f + m_1 g \frac{l_1}{2} \theta_1 \frac{l_1}{2} = (m_0 + m_1)a$$

$$f + m_1 g \frac{\theta_0}{l_1} = (m_0 + m_1)\ddot{\theta}_0$$

$$\ddot{\theta}_0 - \frac{m_1}{m_0 + m_1} \frac{g}{l_1} \theta_0 = f$$

Another approach to get a motion equation consists of using Lagrange equations for Lagrangian. Theoretical part is presented in 2.1, the solution for part 1 is presented at the end of the document.

2. System of Double Pendulum Inverted on a Cart (DPIC)

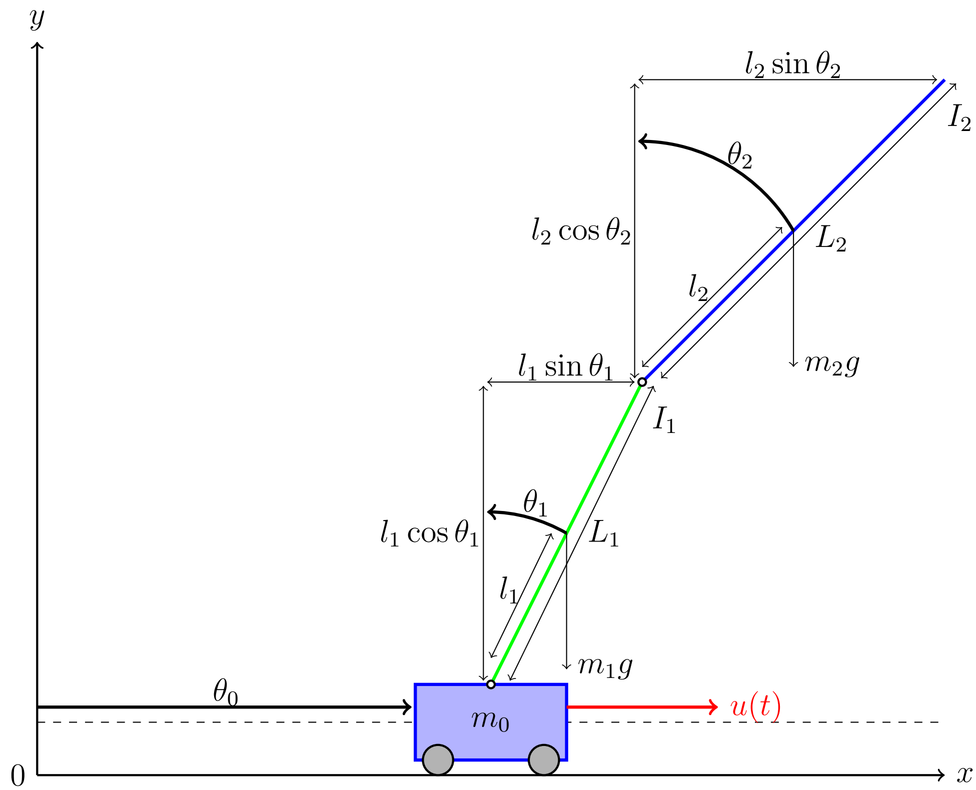


Figure 2: Double Pendulum Inverted on a Cart (DPIC)

2.1 Derivation via Lagrange

It is necessary to calculate the equations of motion for the DPIC system. There are multiple methods to derive the system of equations modeling the DPIC system. For example, a Newtonian approach regarding forces would eventually calculate the system equations. However, it is most elegant and efficient to use the Lagrange equations for a particular Lagrangian.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}}\right) - \frac{\partial L}{\partial \boldsymbol{\theta}} = \mathbf{q} \tag{2.1}$$

L is the Lagrangian, $\mathbf{q}$ is a vector of generalized forces which act in the direction of each component in θ, these forces are not included in the kinetic and potential energies of the cart and each pendulum link. The control force $u(t)$ is one of these forces. This method of derivation is similar to that of [9] but explains it in full. Let

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{pmatrix}$$

Recall that θ_0 is the horizontal position of the cart and θ_1 and θ_2 are the angles of the first and second pendulum links to the vertical. Because of these components, the control force $u(t)$, which is the force acting on the cart, only acts horizontally and does not affect the angles directly. Negating other external forces, $\mathbf{q}$ is simply

$$\mathbf{q} = \begin{pmatrix} u(t) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(t) = \mathbf{H}u(t)$$

Thus, looking at the Lagrange equation (1.1) for each component of θ the following system of equations is obtained.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_0}\right) - \frac{\partial L}{\partial \theta_0} = u(t)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

The Lagrangian, L is the difference between the total kinetic energy, E_{kin}, and the total gravitational potential energy, E_{pot} of the system.

$$L = E_{kin} - E_{pot} \tag{2.2}$$

These energies can be broken down into energies of the specific components. The cart, the first pendulum link, and the second pendulum link.

$$E_{kin} = E_{kin}^{(0)} + E_{kin}^{(1)} + E_{kin}^{(2)}$$
$$E_{pot} = E_{pot}^{(0)} + E_{pot}^{(1)} + E_{pot}^{(2)}$$

Here the subscript notation denotes either kinetic or potential energy and the superscript indicates which component of the system is being referred to. $E^{(0)}_{kin}$ is the kinetic energy of the cart and $E^{(2)}_{pot}$ is the gravitational potential energy of the second, or top, pendulum link. Using Figure 2, the specific coordinates of each component in the DPIC system can be calculated, this will help calculate the Lagrangian (2.2). For the position of the cart one has

$$x_0 = \theta_0$$
$$y_0 = 0$$

The position of the midpoint of the first pendulum link

$$x_1 = \theta_0 + l_1 \sin\theta_1$$
$$y_1 = l_1 \cos\theta_1$$

The position of the midpoint of the second pendulum link

$$x_2 = \theta_0 + L_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$y_2 = L_1 \cos\theta_1 + l_2 \cos\theta_2$$

Using these coordinates the energy components can be calculated. First, using standard results from Newtonian mechanics, calculate the kinetic and potential energy, $E^{(0)}_{kin}$ and $E^{(0)}_{pot}$, for the cart

$$E^{(0)}_{kin} = \frac{1}{2} m_0 \dot{\theta}_0^2$$
$$E^{(0)}_{pot} = 0$$

Next look at the bottom pendulum link. Due to modeling each pendulum link with the center of mass at the midpoint of that link, the kinetic energy has two components. These are translational kinetic energy and rotational kinetic energy.

$$
\begin{aligned}
E_{kin}^{(1)} &= E_{kin}^{(1)}(trans) + E_{kin}^{(1)}(rot) \\
E_{kin}^{(1)} &= \frac{1}{2}m_1\big(\dot{x}_1^2 + \dot{y}_1^2\big) + \frac{1}{2}I_1\dot{\theta}_1^2 \\
E_{kin}^{(1)} &= \frac{1}{2}m_1\left\{\Big(\frac{d}{dt}\big[\theta_0 + l_1\sin\theta_1\big]\Big)^2 + \Big(\frac{d}{dt}\big[l_1\cos\theta_1\big]\Big)^2\right\} + \frac{1}{2}I_1\dot{\theta}_1^2 \\
E_{kin}^{(1)} &= \frac{1}{2}m_1\Big[\big(\dot{\theta}_0 + l_1\dot{\theta}_1\cos\theta_1\big)^2 + \big(-l_1\dot{\theta}_1\sin\theta_1\big)^2\Big] + \frac{1}{2}I_1\dot{\theta}_1^2 \\
E_{kin}^{(1)} &= \frac{1}{2}m_1\big[\dot{\theta}_0^2 + 2l_1\dot{\theta}_0\dot{\theta}_1\cos\theta_1 + l_1^2\dot{\theta}_1^2\cos^2\theta_1 + l_1^2\dot{\theta}_1^2\sin^2\theta_1\big] + \frac{1}{2}I_1\dot{\theta}_1^2 \\
E_{kin}^{(1)} &= \frac{1}{2}m_1\big[\dot{\theta}_0^2 + 2l_1\dot{\theta}_0\dot{\theta}_1\cos\theta_1 + l_1^2\dot{\theta}_1^2\big] + \frac{1}{2}I_1\dot{\theta}_1^2 \\
E_{kin}^{(1)} &= \frac{1}{2}m_1\dot{\theta}_0^2 + \frac{1}{2}\big(m_1l_1^2 + I_1\big)\dot{\theta}_1^2 + m_1l_1\dot{\theta}_0\dot{\theta}_1\cos\theta_1
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
E_{pot}^{(1)} &= m_1gy_1 \\
E_{pot}^{(1)} &= m_1gl_1\cos\theta_1
\end{aligned}
$$

Finally, calculate the energies for the top pendulum link. As before, for the same reasons, the kinetic energy has two components, translational and rotational.

$$
\begin{aligned}
E_{kin}^{(2)} &= \frac{1}{2}m_2\big(\dot{x}_2^2 + \dot{y}_2^2\big) + \frac{1}{2}I_2\dot{\theta}_2^2 \\
&= \frac{1}{2}m_2\left\{\left(\frac{d}{dt}\big[\theta_0 + L_1\sin\theta_1 + l_2\sin\theta_2\big]\right)^2 + \left(\frac{d}{dt}\big[L_1\cos\theta_1 + l_2\cos\theta_2\big]\right)^2\right\} \\
&\quad + \frac{1}{2}I_2\dot{\theta}_2^2 \\
&= \frac{1}{2}m_2\Big[\big(\dot{\theta}_0 + L_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2\big)^2 + \big(-L_1\dot{\theta}_1\sin\theta_1 - l_2\dot{\theta}_2\sin\theta_2\big)^2\Big] \\
&\quad + \frac{1}{2}I_2\dot{\theta}_2^2 \\
&= \frac{1}{2}m_2\Big[\dot{\theta}_0^2 + 2L_1\dot{\theta}_0\dot{\theta}_1\cos\theta_1 + 2l_2\dot{\theta}_0\dot{\theta}_2\cos\theta_2 + L_1^2\dot{\theta}_1^2\cos^2\theta_1 \\
&\quad + 2L_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2 + l_2^2\dot{\theta}_2^2\cos^2\theta_2 + L_1^2\dot{\theta}_1^2\sin^2\theta_1 \\
&\quad + 2L_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2 + l_2^2\dot{\theta}_2^2\cos^2\theta_2\Big] + \frac{1}{2}I_2\dot{\theta}_2^2 \\
&= \frac{1}{2}m_2\big[\dot{\theta}_0^2 + 2L_1\dot{\theta}_0\dot{\theta}_1\cos\theta_1 + 2l_2\dot{\theta}_0\dot{\theta}_2\cos\theta_2 + L_1^2\dot{\theta}_1^2\big(\cos^2\theta_1 + \sin^2\theta_1\big) \\
&\quad + 2L_1l_2\dot{\theta}_1\dot{\theta}_2\big(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\big) + l_2^2\dot{\theta}_2^2\big(\cos^2\theta_2 + \sin^2\theta_2\big)\big] \\
&\quad + \frac{1}{2}I_2\dot{\theta}_2^2 \\
&= \frac{1}{2}m_2\big[\dot{\theta}_0^2 + 2L_1\dot{\theta}_0\dot{\theta}_1\cos\theta_1 + 2l_2\dot{\theta}_0\dot{\theta}_2\cos\theta_2 + L_1^2\dot{\theta}_1^2 \\
&\quad + 2L_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + l_2^2\dot{\theta}_2^2\big] + \frac{1}{2}I_2\dot{\theta}_2^2 \\
&= \frac{1}{2}m_2\big[\dot{\theta}_0^2 + 2L_1\dot{\theta}_0\dot{\theta}_1\cos\theta_1 + 2l_2\dot{\theta}_0\dot{\theta}_2\cos\theta_2 + L_1^2\dot{\theta}_1^2 \\
&\quad + 2L_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + l_2^2\dot{\theta}_2^2\big] + \frac{1}{2}I_2\dot{\theta}_2^2 \\
&= \frac{1}{2}m_2\dot{\theta}_0^2 + \frac{1}{2}m_2L_1^2\dot{\theta}_1^2 + \frac{1}{2}\left(m_2l_2^2 + I_2\right)\dot{\theta}_2^2 + m_2L_1\dot{\theta}_0\dot{\theta}_1\cos\theta_1 \\
&\quad + m_2l_2\dot{\theta}_0\dot{\theta}_2\cos\theta_2 + m_2L_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)
\end{aligned}
$$

Also

$$
\begin{aligned}
E_{pot}^{(2)} &= m_1gy_2 \\
&= m_1g(L_1\cos\theta_1 + l_2\cos\theta_2)
\end{aligned}
$$

Adding all of these kinetic and potential energies together gives the overall energies for the system.

$$\begin{aligned}
E_{kin} &= E_{kin}^{(0)} + E_{kin}^{(1)} + E_{kin}^{(2)} \\
&= \frac{1}{2}m_0\dot{\theta}_0^2 + \frac{1}{2}m_1\dot{\theta}_0^2 + \frac{1}{2}(m_1 l_1^2 + I_1)\dot{\theta}_1^2 + m_1 l_1\dot{\theta}_0\dot{\theta}_1\cos\theta_1 \\
&\quad + \frac{1}{2}m_2\dot{\theta}_0^2 + \frac{1}{2}m_2 L_1^2\dot{\theta}_1^2 + \frac{1}{2}(m_2 l_2^2 + I_2)\dot{\theta}_2^2 + m_2 L_1\dot{\theta}_0\dot{\theta}_1\cos\theta_1 \\
&\quad + m_2 l_2\dot{\theta}_0\dot{\theta}_2\cos\theta_2 + m_2 L_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) \\
&= \frac{1}{2}(m_0+m_1+m_2)\dot{\theta}_0^2 + \frac{1}{2}(m_1 l_1^2 + m_2 L_1^2 + I_1)\dot{\theta}_1^2 + \frac{1}{2}(m_2 l_2^2 + I_2)\dot{\theta}_1^2 \\
&\quad + (m_1 l_1 + m_2 L_1)\dot{\theta}_0\dot{\theta}_1\cos\theta_1 + m_2 l_2\dot{\theta}_0\dot{\theta}_2\cos\theta_2 + m_2 L_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)
\end{aligned}$$

Also

$$\begin{aligned}
E_{pot} &= E_{pot}^{(0)} + E_{pot}^{(1)} + E_{pot}^{(2)} \\
&= 0 + m_1 g l_1\cos\theta_1 + m_1 g(L_1\cos\theta_1 + l_2\cos\theta_2) \\
&= g(m_1 l_1 + m_2 L_1)\cos\theta_1 + m_2 g l_2\cos\theta_2
\end{aligned}$$

The Lagrangian (2.2) is

$$\begin{aligned}
L &= E_{kin} - E_{pot} \\
&= \frac{1}{2}(m_0+m_1+m_2)\dot{\theta}_0^2 + \frac{1}{2}(m_1 l_1^2 + m_2 L_1^2 + I_1)\dot{\theta}_1^2 + \frac{1}{2}(m_2 l_2^2 + I_2)\dot{\theta}_2^2 \\
&\quad + (m_1 l_1 + m_2 L_1)\cos\theta_1\dot{\theta}_0\dot{\theta}_1 + m_2 l_2\cos\theta_2\dot{\theta}_0\dot{\theta}_2 + m_2 L_1 l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 \\
&\quad - g(m_1 l_1 + m_2 L_1)\cos\theta_1 - m_2 g l_2\cos\theta_2
\end{aligned}$$

Now that the Lagrangian is known, explicitly calculate the partial, and full deriva-tives for the system of equations (2.1).

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_0}\right) - \frac{\partial L}{\partial \theta_0} &= u(t) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} &= 0 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} &= 0
\end{aligned}
$$

Now, calculating the derivatives

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_0} &= (m_0 + m_1 + m_2)\dot{\theta}_0 + (m_1 l_1 + m_2 L_1)\cos\theta_1 \dot{\theta}_1 + m_2 l_2 \cos\theta_2 \dot{\theta}_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_0}\right) &= (m_0 + m_1 + m_2)\ddot{\theta}_0 + (m_1 l_1 + m_2 L_1)\cos\theta_1 \ddot{\theta}_1 + m_2 l_2 \cos\theta_2 \ddot{\theta}_2 \\
&\quad - (m_1 l_1 + m_2 L_1)\sin\theta_1 \dot{\theta}_1^2 - m_2 l_2 \sin\theta_2 \dot{\theta}_2^2 \\
\frac{\partial L}{\partial \theta_0} &= 0 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_0}\right) - \frac{\partial L}{\partial \theta_0} &= (m_0 + m_1 + m_2)\ddot{\theta}_0 + (m_1 l_1 + m_2 L_1)\cos\theta_1 \ddot{\theta}_1 + m_2 l_2 \cos\theta_2 \ddot{\theta}_2 \\
&\quad - (m_1 l_1 + m_2 L_1)\sin\theta_1 \dot{\theta}_1^2 - m_2 l_2 \sin\theta_2 \dot{\theta}_2^2 \\
&= u(t)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_1} &= (m_1 l_1^2 + m_2 L_1^2 + I_1)\dot{\theta}_1 + (m_1 l_1 + m_2 L_1)\cos\theta_1 \dot{\theta}_0 \\
&\quad + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) &= (m_1 l_1^2 + m_2 L_1^2 + I_1)\ddot{\theta}_1 + (m_1 l_1 + m_2 L_1)\cos\theta_1 \ddot{\theta}_0 \\
&\quad + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 - (m_1 l_1 + m_2 L_1)\sin\theta_1 \dot{\theta}_0 \dot{\theta}_1 \\
&\quad - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_2 \\
\frac{\partial L}{\partial \theta_1} &= -(m_1 l_1 + m_2 L_1)\sin\theta_1 \dot{\theta}_0 \dot{\theta}_1 - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 \\
&\quad + g(m_1 l_1 + m_2 L_1)\sin\theta_1 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} &= (m_1 l_1^2 + m_2 L_1^2 + I_1)\ddot{\theta}_1 + (m_1 l_1 + m_2 L_1)\cos\theta_1 \ddot{\theta}_0 \\
&\quad + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 - (m_1 l_1 + m_2 L_1)\sin\theta_1 \dot{\theta}_0 \dot{\theta}_1 \\
&\quad - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_2 \\
&\quad + (m_1 l_1 + m_2 L_1)\sin\theta_1 \dot{\theta}_0 \dot{\theta}_1 + m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 \\
&\quad - g(m_1 l_1 + m_2 L_1)\sin\theta_1 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} &= (m_1 l_1^2 + m_2 L_1^2 + I_1)\ddot{\theta}_1 + (m_1 l_1 + m_2 L_1)\cos\theta_1 \ddot{\theta}_0 \\
&\quad + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 \\
&\quad - g(m_1 l_1 + m_2 L_1)\sin\theta_1 \\
&= 0
\end{aligned}
$$

finally,

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_2} &= m_2 l_2 \cos\theta_2 \dot{\theta}_0 + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 + (m_2 l_2^2 + I_2)\dot{\theta}_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) &= m_2 l_2 \cos\theta_2 \ddot{\theta}_0 + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + (m_2 l_2^2 + I_2)\ddot{\theta}_2 \\
&\quad - m_2 l_2 \sin\theta_2 \dot{\theta}_0 \dot{\theta}_2 - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_1 \\
\frac{\partial L}{\partial \theta_2} &= -m_2 l_2 \sin\theta_2 \dot{\theta}_0 \dot{\theta}_2 + m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 \\
&\quad + m_2 g l_2 \sin\theta_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} &= m_2 l_2 \cos\theta_2 \ddot{\theta}_0 + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + (m_2 l_2^2 + I_2)\ddot{\theta}_2 \\
&\quad - m_2 l_2 \sin\theta_2 \dot{\theta}_0 \dot{\theta}_2 - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_1 \\
&\quad + m_2 l_2 \sin\theta_2 \dot{\theta}_0 \dot{\theta}_2 - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 \\
&\quad - m_2 g l_2 \sin\theta_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} &= m_2 l_2 \cos\theta_2 \ddot{\theta}_0 + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + (m_2 l_2^2 + I_2)\ddot{\theta}_2 \\
&\quad - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 - m_2 g l_2 \sin\theta_2 \\
&= 0
\end{aligned}
$$

This gives the calculated system, in full

$$
\begin{aligned}
(m_0 + m_1 + m_2)\ddot{\theta}_0 + (m_1 l_1 + m_2 L_1)\cos\theta_1 \ddot{\theta}_1 + m_2 l_2 \cos\theta_2 \ddot{\theta}_2 \\
-(m_1 l_1 + m_2 L_1)\sin\theta_1 \dot{\theta}_1^2 - m_2 l_2 \sin\theta_2 \dot{\theta}_2^2 = u(t)
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
(m_1 l_1^2 + m_2 L_1^2 + I_1)\ddot{\theta}_1 + (m_1 l_1 + m_2 L_1)\cos\theta_1 \ddot{\theta}_0 \\
+ m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 \\
-g(m_1 l_1 + m_2 L_1)\sin\theta_1 = 0
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
m_2 l_2 \cos\theta_2 \ddot{\theta}_0 + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + (m_2 l_2^2 + I_2)\ddot{\theta}_2 \\
-m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 - m_2 g l_2 \sin\theta_2 = 0
\end{aligned}
\tag{2.5}
$$

2nd-Order System

The system (2.3-2.5) is a nonlinear second-order system of the form

$$\mathbf{D}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \mathbf{G}(\boldsymbol{\theta}) = \mathbf{H}u \tag{2.6}$$

where

$$\mathbf{D}(\boldsymbol{\theta}) = \begin{bmatrix} m_0 + m_1 + m_2 & (m_1 l_1 + m_2 L_1)\cos\theta_1 & m_2 l_2 \cos\theta_2 \\ (m_1 l_1 + m_2 L_1)\cos\theta_1 & m_1 l_1^2 + m_2 L_1^2 + I_1 & m_2 L_1 l_2 \cos(\theta_1 - \theta_2) \\ m_2 l_2 \cos\theta_2 & m_2 L_1 l_2 \cos(\theta_1 - \theta_2) & m_2 l_2^2 + I_2 \end{bmatrix}$$

$$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} 0 & -(m_1 l_1 + m_2 L_1)\sin\theta_1 \dot{\theta}_1 & -m_2 l_2 \sin\theta_2 \dot{\theta}_2 \\ 0 & 0 & m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2 \\ 0 & -m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 & 0 \end{bmatrix}$$

$$\mathbf{G}(\boldsymbol{\theta}) = \begin{bmatrix} 0 \\ -(m_1 l_1 + m_2 L_1) g \sin\theta_1 \\ -m_2 g l_2 \sin\theta_2 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

To simplify the system, we adopt the choices from [9]. Here

$$\implies l_1 = \frac{1}{2}L_1 \qquad l_2 = \frac{1}{2}L_2$$

$$I_1 = \frac{1}{12}m_1L_1^2 \qquad I_2 = \frac{1}{12}m_2L_2^2$$

this updates the system with $\mathbf{D}(\boldsymbol{\theta})$

$$= \begin{bmatrix} m_0 + m_1 + m_2 & (\frac{1}{2}m_1 + m_2)L_1\cos\theta_1 & \frac{1}{2}m_2L_2\cos\theta_2 \\ (\frac{1}{2}m_1 + m_2)L_1\cos\theta_1 & (\frac{1}{3}m_1 + m_2)L_1^2 & \frac{1}{2}m_2L_1L_2\cos(\theta_1 - \theta_2) \\ \frac{1}{2}m_2L_2\cos\theta_2 & \frac{1}{2}m_2L_1L_2\cos(\theta_1 - \theta_2) & \frac{1}{3}m_2L_2^2 \end{bmatrix}$$

and

$$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} 0 & -(\frac{1}{2}m_1 + m_2)L_1\sin\theta_1\dot{\theta}_1 & -\frac{1}{2}m_2L_2\sin\theta_2\dot{\theta}_2 \\ 0 & 0 & \frac{1}{2}m_2L_1L_2\sin(\theta_1 - \theta_2)\dot{\theta}_2 \\ 0 & -\frac{1}{2}m_2L_1L_2\sin(\theta_1 - \theta_2)\dot{\theta}_1 & 0 \end{bmatrix}$$

$$\mathbf{G}(\boldsymbol{\theta}) = \begin{bmatrix} 0 \\ -\frac{1}{2}(m_1 + m_2)L_1g\sin\theta_1 \\ -\frac{1}{2}m_2gL_2\sin\theta_2 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Note. $\mathbf{D}(\boldsymbol{\theta})$ is symmetric and nonsingular, $\implies \mathbf{D}^{-1}(\boldsymbol{\theta})$ exists and is also symmetric

1st-Order System

In order to use control theory on this system, it makes most sense to convert the system to first-order by manipulating the system and employing a few ‘tricks’. Recall the system

$$\mathbf{D}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \mathbf{G}(\boldsymbol{\theta}) = \mathbf{H}u$$

disregarding the arguments for now in the notation

$$\begin{aligned}
\mathbf{D}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} + \mathbf{G} = \mathbf{H}u \\
\mathbf{D}\ddot{\boldsymbol{\theta}} &= -\mathbf{C}\dot{\boldsymbol{\theta}} - \mathbf{G} + \mathbf{H}u \\
\ddot{\boldsymbol{\theta}} &= -\mathbf{D}^{-1}\mathbf{C}\dot{\boldsymbol{\theta}} - \mathbf{D}^{-1}\mathbf{G} + \mathbf{D}^{-1}\mathbf{H}u
\end{aligned}$$

which is

$$\begin{aligned}
\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & -\mathbf{D}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathbf{D}^{-1}\mathbf{G} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{D}^{-1}\mathbf{H} \end{bmatrix} u \\
\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & -\mathbf{D}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathbf{D}^{-1}\mathbf{G} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{D}^{-1}\mathbf{H} \end{bmatrix} u \\
\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} - \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & -\mathbf{D}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathbf{D}^{-1}\mathbf{G} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{D}^{-1}\mathbf{H} \end{bmatrix} u \\
\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix} &= \begin{bmatrix} 0 & I \\ 0 & -\mathbf{D}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathbf{D}^{-1}\mathbf{G} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{D}^{-1}\mathbf{H} \end{bmatrix} u
\end{aligned}$$

Let $\mathbf{x} = \begin{bmatrix} \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \ddot{\boldsymbol{\theta}} \end{bmatrix}$

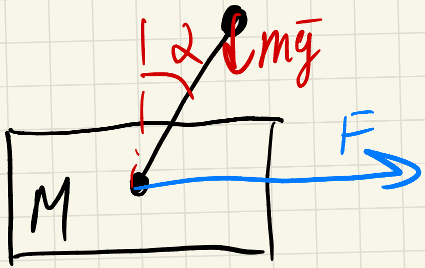
$$\begin{aligned}
\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & I \\ 0 & -\mathbf{D}^{-1}\mathbf{C} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -\mathbf{D}^{-1}\mathbf{G} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{D}^{-1}\mathbf{H} \end{bmatrix} u \\
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & I \\ 0 & -\mathbf{D}^{-1}\mathbf{C} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -\mathbf{D}^{-1}\mathbf{G} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{D}^{-1}\mathbf{H} \end{bmatrix} u
\end{aligned}$$

gives the first-order system

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})u + \mathbf{L}(\mathbf{x}) \tag{2.7}$$

where

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} 0 & I \\ 0 & -\mathbf{D}^{-1}\mathbf{C} \end{bmatrix}, \quad \mathbf{B}(\mathbf{x}) = \begin{bmatrix} 0 \\ \mathbf{D}^{-1}\mathbf{H} \end{bmatrix}, \quad \mathbf{L}(\mathbf{x}) = \begin{bmatrix} 0 \\ -\mathbf{D}^{-1}\mathbf{G} \end{bmatrix}.$$



$$T = \frac{M v_1^2}{2} + \frac{m v_2^2}{2}$$

$$\begin{cases} x_1 = x \\ v_1 = \dot{x} \\ x_2 = x - l\sin\alpha \\ y_2 = l\cos\alpha \end{cases}$$

$$v_2^2 = \left((x - l\sin\alpha)'\right)^2 + \left((l\cos\alpha)'\right)^2 = (\dot{x} - l\cos\alpha\cdot\dot{\alpha})^2 + l^2\sin^2\alpha\cdot\dot{\alpha}^2 =$$

$$= \dot{x}^2 - 2\dot{x}l\cos\alpha\cdot\dot{\alpha} + l^2\cos^2\alpha\cdot\dot{\alpha}^2 + l^2\sin^2\alpha\cdot\dot{\alpha}^2 = \dot{x}^2 - 2\dot{x}\dot{\alpha}\cos\alpha\cdot l + l^2\dot{\alpha}^2$$

$$L = \frac{M\dot{x}^2}{2} + \frac{m}{2}\cdot\left(\dot{x}^2 - 2\dot{x}\dot{\alpha}\cos\alpha\, l + l^2\cdot\dot{\alpha}^2\right) - mgl\cos\alpha$$

$$L = \frac{M\dot{x}^2}{2} + \frac{m}{2}\left(\dot{x}^2 - 2\dot{x}\dot{\alpha}\cos\alpha\, l + l^2\cdot\dot{\alpha}^2\right) - mgl\cos\alpha$$

$$\frac{\partial L}{\partial \dot{\alpha}} = \left(-m\dot{x}\dot{\alpha}\cos\alpha\, l + \frac{m}{2} l^2\cdot\dot{\alpha}^2\right)' = -m\dot{x}\cos\alpha\, l + m l^2\dot{\alpha}$$

$$\frac{\partial L}{\partial \dot{\alpha}}\cdot\frac{d}{dt} = -m\ddot{x}\cos\alpha\, l + m\dot{x}\sin\alpha\cdot\dot{\alpha} l + m l^2\ddot{\alpha}$$

$$\frac{\partial L}{\partial \alpha} = m\dot{x}\dot{\alpha}\sin\alpha + mgl\sin\alpha$$

$$-\cancel{m}\ddot{x}\cos\alpha\,\cancel{l} + \cancel{m\dot{x}\sin\alpha\cdot\dot{\alpha}\, l} + \cancel{m}\,l^{\cancel{2}}\ddot{\alpha} - \cancel{m\dot{x}\dot{\alpha}\sin\alpha} - \cancel{m}g\cancel{l}\sin\alpha = 0$$

$$-\ddot{x}\cos\alpha + l\ddot{\alpha} - g\sin\alpha = 0 \Rightarrow l\ddot{\alpha} = g\sin\alpha + \ddot{x}\cos\alpha$$

$$L = \frac{M\dot{x}^2}{2} + \frac{m}{2}\left(\dot{x}^2 - 2\dot{x}\dot{\alpha}\cos\alpha\, l + l^2 \cdot \dot{\alpha}^2\right) - mg\, l\cos\alpha$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m\dot{x} - m\dot{\alpha}\cos\alpha\, l$$

$$\frac{\partial L}{\partial \dot{x}} \cdot \frac{d}{dt} = M\ddot{x} + m\ddot{x} - m\ddot{\alpha}\cos\alpha\, l + m\dot{\alpha}\sin\alpha \cdot \dot{\alpha}\, l$$

$$\frac{\partial L}{\partial x} = 0$$

$$\boxed{\begin{cases} (M+m)\ddot{x} - m\ddot{\alpha}\cos\alpha\, l + m\dot{\alpha}^2\sin\alpha\, l = F \\ l\ddot{\alpha} = g\sin\alpha + \ddot{x}\cos\alpha \end{cases}}$$