CAIT
Mobile Robotics Lab
Perception in Robotics Course

L06: EKF and Localization

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1 Extended Kalman filter

Kalman filter: Linear system plus Gaussian prior

$$\left.\begin{aligned} \bar{\mu}_t &= A_t\mu_t + B_t u_t \\ \bar{\Sigma}_t &= A_t \Sigma_{t-1} A_t^T + R_t \end{aligned}\right\} \text{prediction (marginalize)}$$

$$\left.\begin{aligned} K_t &= \bar{\Sigma}_t C_t^T \left(C_t \bar{\Sigma}_t C_t^T + Q\right)^{-1} \\ \mu_t &= \bar{\mu}_t + K_t\left(z_t - C_t\bar{\mu}_t\right) \\ \Sigma_t &= \left(I - K_t C_t\right)\bar{\Sigma}_t \end{aligned}\right\} \text{correction (conditioning)}$$

Motion model: first order Taylor expansion

$$x_t = g_n(x_{t-1}, u_t, \epsilon_t) \approx g(\mu_{t-1}, u_t) + \frac{\partial g}{\partial x_{t-1}}\Big|_{\mu_{t-1}} (x_{t-1} - \mu_{t-1}) + \epsilon_t, \qquad \epsilon_t \sim \mathcal{N}(0, R)$$

In L05 discussed on how to model $g(\cdot)$ for different systems and how to obtain the probabilistic model.

Sensor model

We observe features of landmarks (L05):

$$z_t = h_n(x_t, \delta_t) \approx h(\mu_t) + \frac{\partial h}{\partial x_t}\Big|_{\mu_t} (x_t - \mu_t) + \delta_t, \qquad \delta_t \sim \mathcal{N}(0, Q)$$

Intuition on linearization

Linearizing assumes **errors**!

Equations of the Extended Kalman Filter

Inputs: $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$

1. $\bar{\mu}_t = g(\mu_{t-1}, u_t)$
2. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$
3. $K_t = \bar{\Sigma}_t H_t^T \left(H_t \bar{\Sigma}_t H_t^T + Q_t\right)^{-1}$
4. $\mu_t = \bar{\mu}_t + K_t \underbrace{\left(z_t - h(\bar{\mu}_t)\right)}_{\Delta z}$ where Δz is the innovation vector.
5. $\Sigma_t = (I - K_t H_t)\,\bar{\Sigma}_t$
   return μ_t, Σ_t (actually $\mathcal{N}(\mu_t, \Sigma_t)$)

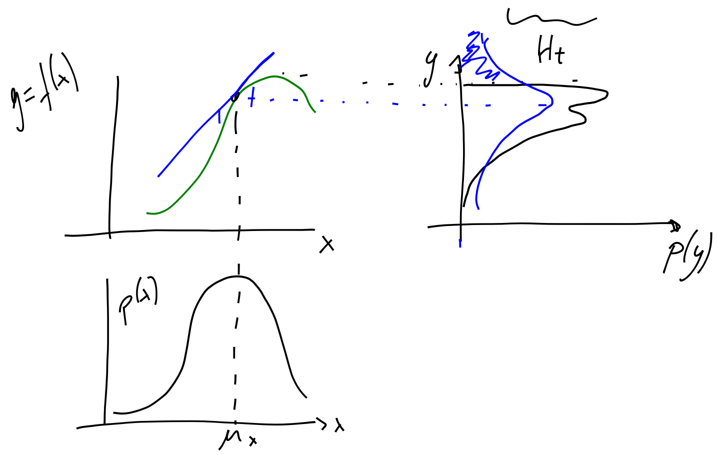

Figure 1: Linearization intuition. The random variable $x \sim p(x)$ is transformed by the function $y = f(x)$. The true distribution of $y \sim p(y)$ would not be Gaussian (black line in the top right), but after linearizing, the PDF is approximated as Gaussian.

Properties:

- EKF is very efficient $O(k^{2.4} + n^2)$
- Not optimal, but in practice works well (depends on the non-linearities, some are more problematic)

Compact initial distribution reduces the error because we are "near" the linearization point $(O(||\Delta x||))$

2 Other alternatives than linearizing

2.1 Unscented Transformation

2.2 Sampling (Particle filter in L07)

3 What is Localization?

"Estimate pose from the observed data"

Markov localization directly uses Bayes filter (see Figure 2):

$$\bar{bel}(x_t) = \int p(x_t|u_t, x_{t-1}, m)\ bel(x_{t-1})\ dx_{t-1}$$
$$bel(x_t) = \eta\ p(z_t|x_t, m)\ \bar{bel}(x_t)$$

4 EKF localization with known associations

First, we need to solve the data association problem landmark-observation.

We will assume known correspondences: $c^i = j$ $(z^i \to m_j)$ (from landmark m_j). For the case of 3 observations of each landmark:

$$p(z|x, m, c) = \prod_{i=1}^{3} p(z^i|x, m, c^i)$$

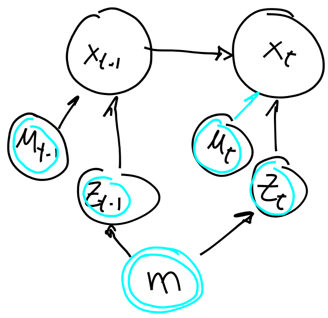

Figure 2: Bayes network corresponding to the localization problem. Observable variables are colored in cyan and estimated variables x's in black. Edges connecting z and x should be the other direction.

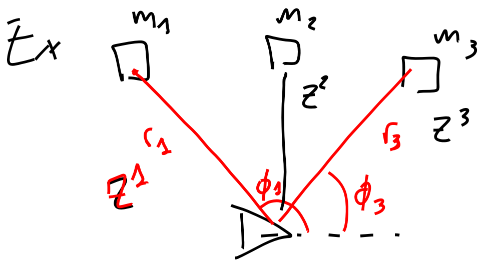

Figure 3: Data association. Now each landmark has a singature s and we can distinguish them.

Algorithm: EKF localization with known correspondences

Inputs: $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, m$ (ProbRob 204)

1. $G_t = \frac{\partial g(x_{t-1}, u_t)}{\partial x_{t-1}}$, $V_t = \frac{\partial g(x_{t-1}, u_t)}{\partial u_t}$,
2. $\bar{\mu}_t = g(\mu_{t-1}, u_t)$
3. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t M_t V_t^T = G_t \Sigma_{t-1} G_t^T + R_t$
4. $Q_t = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\phi^2 \end{bmatrix}$. Range and bearing observation noise. Known correspondences $\implies$ $\sigma_s^2 = 0$ (eliminate)
5. for $\{i : z_t^i = [r_t^i, \phi_t^i]^T\}$:
6. $\quad \hat{z}_t^i = \begin{bmatrix} \sqrt{(m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2} \\ atan2(m_{j,y} - \bar{\mu}_{t,y}, m_{j,x} - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{bmatrix}$
7. $\quad H_t^i = \frac{\partial h(x_t)}{\partial x_t}\Big|_{\bar{\mu}_t} = \begin{bmatrix} \frac{-(m_{j,x} - \bar{\mu}_{t,x})}{\sqrt{(m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2}} & \frac{-(m_{j,y} - \bar{\mu}_{t,y})}{\sqrt{(m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2}} & 0 \\ \frac{m_{j,y} - \bar{\mu}_{t,y}}{(m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2} & \frac{-(m_{j,x} - \bar{\mu}_{t,x})}{(m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2} & -1 \end{bmatrix}$
8. $\quad S_t^i = H_t^i \bar{\Sigma}_t (H_t^i)^T + Q_t$
9. $\quad K_t^i = \bar{\Sigma}_t (H_t^i)^T (S_t^i)^{-1}$
10. $\quad \bar{\mu}_t := \bar{\mu}_t + K_t^i \left(z_t^i - \hat{z}_t^i\right)$ (innovation vector for z_t^i)
11. $\quad \bar{\Sigma}_t := \left(I - K_t^i H_t^i\right) \bar{\Sigma}_t$
12. endfor

13. $\mu_t = \bar{\mu}_t \quad (\mu_t = \bar{\mu}_t + \sum_i K_t^i \left(z_t^i - \hat{z}_t^i\right))$
14. $\Sigma_t = \bar{\Sigma}_t$
15. return μ_t, Σ_t

Iterated EKF

Why are we updating the prediction belief $\bar{bel}(x_t)$ I times?
If we assume that $\{z_t^i\}_{i=1}^I$ are independent:

$$
\begin{aligned}
bel(x_t) =& \eta \cdot p(z|x, m, c) \cdot \bar{bel}(x_t) \\
=& \prod^I p(z^i|x, m, c) \cdot \bar{bel}(x_t) \\
=& p(z^1|x, m, c) \cdot \\
& p(z^2|x, m, c) \cdot \\
& \dots \cdot \\
& \underbrace{p(z^I|x, m, c) \cdot \bar{bel}(x_t)}_{\text{conditioning a joint Gaussian}}
\end{aligned}
$$

where we are iteratively conditioning a distribution, starting from $p(z^I|x, m, c) \cdot \bar{bel}(x_t)$ up to the first observation. Note that the order matters, since the linearization point at each term in the summation gets updated. Intuitively, we want them to be as close to the real solution in order to minimize the implicit linearization error.

Localization problems (taxonomy)

- Local (position tracking) [x_0 given] vs Global [x_0 unkown, kidnapped problem]
- Static vs Dynamic [moving furniture, doors, snow...]
- Passive vs Active [exploration, belief planning]
- Single-robot vs Multi-robot

5 Summary

- Extended Kalman Filter and linearization errors.
- Localization $\rightarrow$ EKF localization as a uni-modal solution.
- Map of landmarks:

$$m = \left\{ \begin{bmatrix} m_{1,x} \\ m_{1,y} \end{bmatrix} \begin{bmatrix} m_{2,x} \\ m_{2,y} \end{bmatrix} \dots \begin{bmatrix} m_{i,x} \\ m_{j,y} \end{bmatrix} \dots \right\} \text{ map of known landmark} \tag{1}$$

The localization problem becomes a state estimation problem $\Rightarrow$ EKF,UKF (in additional notes)

$$bel(x_t) = p(x_t|U, Z, m) \tag{2}$$

Assume (for now) $c_t^i = j$ $(z^i \rightarrow m_j)$ known correspondences