

## L07: Particle Filter and Monte-Carlo localization

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### 1 Particle filter (PF)

$$\begin{cases} \overline{bel}(x_t) = \int p(x_t|x_{t-1}, u_t) bel(x_{t-1}) dx_{t-1} \\ bel(x_t) = \eta p(z|x_t) \overline{bel}(x_t) \end{cases} \quad (1)$$

Gaussian filters (Unimodal distributions):

- Kalman Filter                      Linear system
- Extended KF                      Non-Linear system
- Unscented KF                      Non-Linear system (more at extra notes on L06)

Non-parametric : Particle Filter (PF)

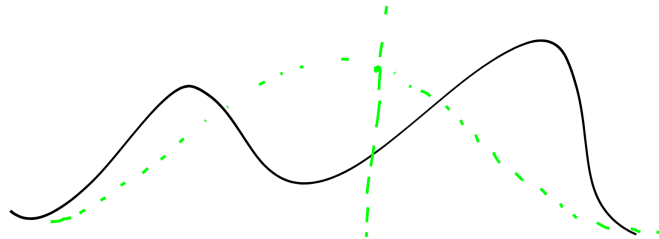


Figure 1: Non-parametric filters do not assume a unimodal distribution, such as KF which always approximates the solution to a Gaussian distribution (green line).

**Particle set:**  $X_t = \{\langle x_t^{[1]}, \omega_t^{[1]} \rangle, \dots, \langle x_t^{[M]}, \omega_t^{[M]} \rangle\}$ .

The particle set consists of  $M$  particles, each of them is a pair of a state  $x^{[m]}$  and a weight  $\omega^{[m]}$ .



Figure 2: Example of a particle set. On top, samples from a 1D PDF, on the bottom, a 2D Gaussian PDF with few samples drawn and the  $1\text{-}\sigma$  iso-contour plotted.

$$\begin{aligned} x^{[m]} &\sim p(x) && \text{Weighted samples. Particles become a good representation of PDFs} \\ \omega^{[m]} &= p_z(x^{[m]}) && (\text{if } M \text{ is large enough}) \end{aligned}$$

Q: What are the weights on sample mean and sample covariance?

1. Particle filter ( $X_{t-1}, u_t, z_t$ ):
2. for  $m = 1 : M$
3.  $x_t^{[m]} \sim p(x_t | u_t, x_{t-1}^{[m]})$
4.  $\omega_t^{[m]} = p(z_t | x_t^{[m]})$  propagation  $\overline{bel}(x_t)$
5.  $\overline{X}_t = \overline{X}_t \cup \langle x_t^{[m]}, \omega_t^{[m]} \rangle$  correction
6.  $X_t = \text{resampe}^*(\overline{X}_t)$  (better correction)  $X$  are "down" from  $bel(x_t)$  and not  $\overline{bel}$

[Gordon] reading introduces resampling as a requirement for the PF to work property

## 2 Bayes filter for full states

$$bel(x_{0:t}) = p(x_{0:t} | u_{1:t}, z_{1:t})$$

particles  $x_{0:t}^{[m]} = x_0^{[m]}, x_1^{[m]}, \dots, x_t^{[m]}$  Sequence of samples of states over time.

$$\begin{aligned} bel(x_{0:t}) &= \eta p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) p(x_{0:t} | z_{1:t-1}, u_{1:t}) \\ (\text{Markov} + \text{Product rule}) &= \eta p(z_t | x_t) p(x_t | x_{0:t-1}, z_{1:t-1}, u_{1:t}) p(x_{0:t-1} | z_{1:t-1}, u_{1:t}) = \\ &= \eta p(z_t | x_t) p(x_t | x_{t-1}, u_t) \underbrace{p(x_{0:t-1} | z_{1:t-1}, u_{1:t-1})}_{bel(x_{0:t-1})} \end{aligned}$$

We have obtained a new recursive form of the Bayes filter, but now considering the state variable to be a sequence of all estimates at all instants of time, i.e., time  $0 : t$ .

$$\begin{aligned} \overline{bel}(x_{0:t}) &= p(x_t | x_{t-1}, u_t) bel(x_{0:t-1}) \\ bel(x_{0:t}) &= \eta p(z_t | x_t) \overline{bel}(x_{0:t}) \end{aligned}$$

### 2.1 Prediction step

From this full state Bayes (no marginalization) given a particle  $x_{t-1}^{[m]} \sim bel(x_{0:t-1})$

$$\overline{bel}(x_{0:t}) \begin{cases} \overline{x}_t^{[m]} & \sim p(x_t | x_{t-1}^{[m]}, u_t) & \text{Sample drawn from the previous sample.} \\ \overline{\omega}_t^{[m]} & = 1 \cdot \omega_{t-1}^{[m]} & \text{importance factor from } bel. \end{cases} \quad (2)$$

In Fig. 3 is depicted an example of particles propagated (predicted)

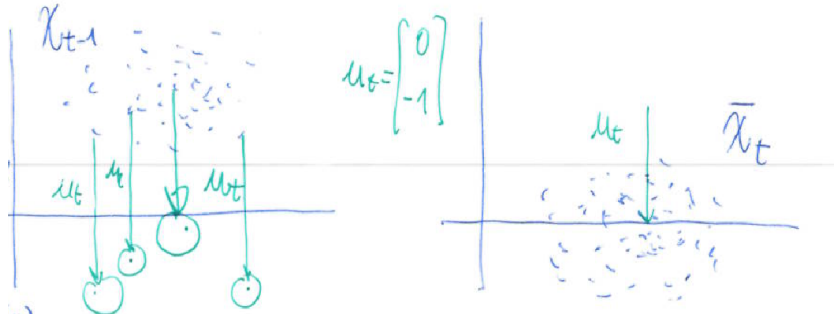


Figure 3: Example of the prediction step. For each particle, a propagation is sampled and this conforms the new particle set  $\bar{X}_t$ .

## Importance sampling

We will briefly introduce the concept of *importance sampling* before we continue with the PF derivation, since it plays an essential role on it.

$$\begin{aligned} \mathbb{E}_{x \sim p(x)} \{I(x \in A)\} &= \int I(x \in A) p(x) dx = \int I(x \in A) \frac{p(x)}{q(x)} \cdot q(x) dx \\ &= \mathbb{E}_{x \sim q} \{I(x \in A) \cdot \omega(x)\}, \end{aligned} \quad (3)$$

where  $\omega(x) = \frac{p(x)}{q(x)}$  is the Importance factor,  $p(x)$  is the target distribution, which we usually can't use directly and  $q(x)$  is the proposal distribution, more accessible and ready to use. The function  $I()$  in this context is the indicator function.

**Example:** Probability of sample a 1d r.v  $X$  in the interval  $[15, 17]$  if

$$p(x) = \mathcal{N}(0, 1) \quad A = \{x : 15 \leq x \leq 17\}$$

$$1) \quad p(x \in A) = \sum I(x^m \in A) p(x^m), \quad x^m \sim p(x^m)$$

$$2) \quad \text{Importance Sampling: } p(x \in A) = \sum I(x^m \in A) \underbrace{\frac{p(x^m)}{q(x^m)}}_{\omega^m} \cdot q(x^m), \quad x^m \sim q(x)$$

for instance  $q(x) = \mathcal{N}(16, 1)$  (proposal distribution)

With this proposal distribution we don't need trillions of samples but only hundreds.

$$\omega^m = \frac{\mathcal{N}(x^m; 0, 1)}{\mathcal{N}(x^m; 16, 1)} \quad (4)$$

## 2.2 Correction step

$$\underbrace{bel(x_{0:t})}_{\text{target distribution}} = \eta p(z_t | x_t) \cdot \underbrace{\bar{bel}(x_{0:t})}_{\text{proposal distribution}}, \quad (5)$$

where  $\bar{X}_t$  is the particle set representing the belief PDF  $\bar{bel}(x_{0:t})$ , our proposal distribution.

In order to correctly characterize the posterior  $bel(x_{0:t})$  we are going to weight the particles already drawn  $\bar{X}_t$  with proper weights  $\omega$  (importance factors).

$$x_t^{[m]} = \bar{x}_t^{[m]} \quad (\text{previously drawn}) \text{ from proposal distribution.}$$

$$\omega_t^{[m]} = \frac{\text{target distribution}}{\text{proposal distribution}} = \frac{\eta p(z_t | x_t^{[m]}) \cdot \overline{bel}(x_t^{[m]})}{\overline{bel}(x_t^{[m]})} = \eta p(z_t | x_t^{[m]}) \quad (6)$$

**Example:** Correction step applied to the particle set  $\bar{X}_t$

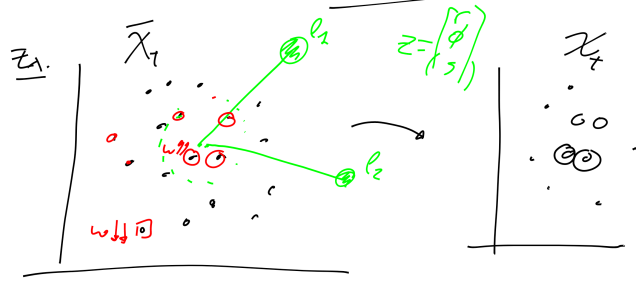


Figure 4: Example of correction step for a particle set.

Problem: creates an almost empty set of particles with weights non-zero and many particles with low weights  $\Rightarrow$  Degenerating over time.

### 3 PF Resampling

Resampling is the solution to the degeneracy occurring when propagating and correcting multiple times a particle set.

**Idea:** “survival of the fittest”. Only the most likely particles ( $\omega^m \uparrow$ ) ‘might’ survive

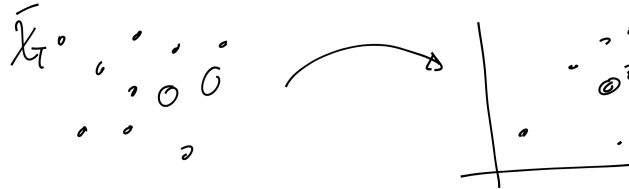


Figure 5: Resampling (the solution). Now, resampling guarantees that the highest values of  $\omega^m$  are more likely to *survive* but it also give chances to the particles with small values of importance factors to represent the next particle set. From  $M$  samples on  $\bar{X}_t$  we get  $M$  samples on  $X_t$  (closest to the  $bel(x_0)$ )

#### 3.1 Independent Resampling. First solution

We create a cumulative distribution function:

$$c_m = c_{m-1} + \omega^{[m]} \quad (\text{normalization should be considered}) \quad (7)$$

for  $m = 1 : M$

$$a \sim U[0; 1] \quad (\text{uniform distribution})$$

$$j = \text{find}(c_m, a)$$

$$X_t = X_t \cup \langle x_t^{[j]}, \omega_t^{[j]} \rangle$$

**Problem:** over time, independent resampling induces a loss of diversity in the particle population  $X_t$ .

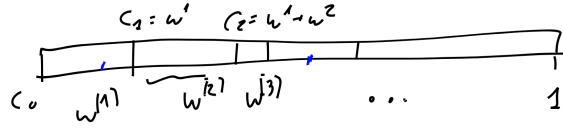


Figure 6: Independent Resampling.

### 3.2 Low-variance sampling

We create a similar distribution as in the independent sampling algorithm:

$$c_m = c_{m-1} + \omega^{[m]}.$$

The difference is that we no longer sample from this discrete distribution. Only an initial random configuration  $r$  is sampled, and then we add particles at intervals  $1/M$  over the full set.

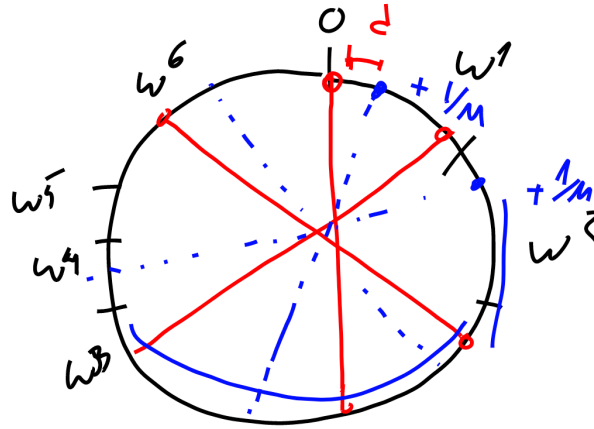


Figure 7: Low-variance sampling scheme. We select particles as equally spaced intervals.

Algorithm: low-variance sampling ( $\bar{X}_t$ ): (ProbRob 110)

$X_t = \phi$ ,  $c = \omega_t^{[1]}$ ,  $i = 1$ ;

$r \sim U(0, 1/M)$ ;

**for**  $m = 1:M$  **do**

$a = r + (m - 1) \cdot \frac{1}{M}$

**while**  $a > c$  **do**

$i++$ ;

$c = c + \bar{\omega}^{[i]}$ ;

**end**

$X_t = X_t \cup \langle \bar{x}_t^{[i]}, \frac{1}{M} \rangle$ ;

**end**

**return**  $X_t$

## 4 Monte-Carlo localization (MCL)

Reading: Dellaert'99.

We want to solve the Markov localization (L06) using PF.

$$bel(x_t) = p(x_t|U, Z, m) \rightarrow X_t, \quad (8)$$

where  $X_t$  is the particle set representing the posterior belief and  $m$  is the map of landmarks.

**Algorithm:** MCL  $(X_{t-1}, u_t, z_t, m)$  :

$\bar{X}_t = X_t = \phi$ ;

**for**  $m = 1:M$  **do**

$x_t^{[m]} = \text{sample\_motion\_model}(u_t, x_{t-1}^{[m]})$  (Section 2.1 and L05)  
     $\omega_t^{[m]} = \text{measurement\_model}(z_t, x_t^{[m]}, m)$  (Section 2.2 and L06)  
     $\bar{X}_t = \bar{X}_t \cup \langle x_t^{[i]}, \omega_t^{[m]} \rangle$

**end**

$X_t = \text{low\_variance\_sampling}(\bar{X}_t)$

## 5 Summary

- Particle filter is a version of the Bayes filter for full sequences  $x_{0:t}$ .
- Prediction step  $\bar{x}_t^{[m]} \sim p(x_t | x_{t-1}^{[m]}, u_t)$ .
- Correction step  $\omega_t^{[m]} = \eta p(z_t | x_t^{[m]})$ .
- Resampling regenerates the particle set.