

## L05: Motion and sensor model

Gonzalo Ferrer

7 February 2023

### 1 Introduction to motion models

$$x_t = g(x_{t-1}, u_t, \varepsilon_t) \quad \text{in general Non-linear.} \quad (1)$$

As designers we should find the right Transition function  $g(\cdot)$  that better describes our system.

Ex:

$\left. \begin{array}{l} \text{free point 2D, 3D} \\ \text{wheeled robot} \end{array} \right\}$  Introduction for today.  
 car,  
 airplane,  
 quadrotor  
 serial manipulator, ...

### 2 Free point 2D: Kinematics

$$x_t = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{position} \quad u_t = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \quad \text{velocity} \quad (2)$$



Figure 1: Free point 2D: Kinematics

This corresponds to a free moving point, for now deterministic model (no noise). There are two approaches:

Discrete-time model	$x_t = g(x_{t-1}, u_t, \varepsilon_t) _{\varepsilon_t=0}$	
Continuous-time model	$\dot{x}_t = f(x, u)$	(Transition eq.)

$$\dot{x} = u = \begin{bmatrix} v_x(t) \\ v_y(t) \end{bmatrix} \quad \text{cont-time velocities} \quad (3)$$

$$x(t) = x(0) + \int_0^t f(x, u) dt' \quad \text{continuous-time trajectory} \quad (4)$$

## 2.1 Numerical method: Euler method, 1st order method

$$\dot{x} = \frac{dx}{dt} \simeq \frac{x(\Delta t) - x(0)}{\Delta t} \quad (5)$$

$$x(\Delta t) \simeq x(0) + \Delta t \cdot \dot{x} = x(0) + \Delta t \cdot f(x(0), u(0)) \quad (6)$$

An alternative to the Euler method are Runge-Kutta methods: which are higher order integration methods.

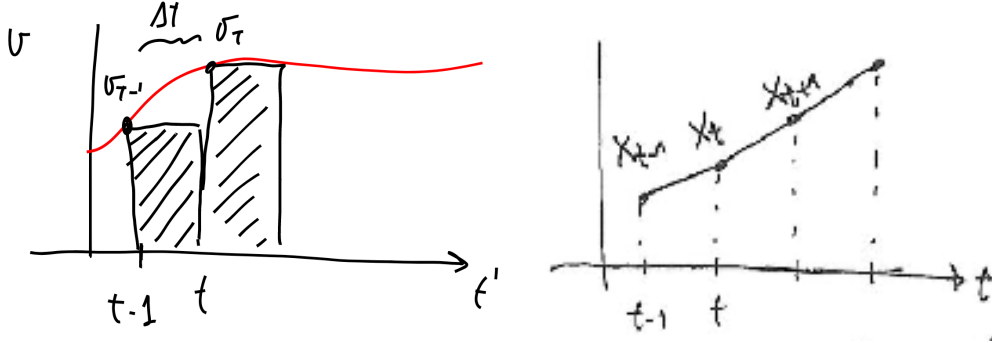


Figure 2: Euler method. On the left, piece-wise constant integration of velocity. On the right, piece-wise linear trajectory.

As a result, the 2D Kinematic point:

$$x_t = x_{t-1} + \Delta t \cdot f(x, u) = x_{t-1} + \Delta t \begin{bmatrix} v_x \\ v_y \end{bmatrix} \quad (\text{Transition function}) \quad (7)$$

$$A = I_{2 \times 2} \quad B = \Delta t \cdot I_{2 \times 2}$$

## 2.2 Probabilistic model 2D Kinematic point

$$x_t = x_{t-1} + \Delta t \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, R) \quad (8)$$

Noise is added to the state space (ProbRob p.127)

$$R = \begin{bmatrix} \alpha_1 v_x^2 + \alpha_2 v_y^2 & 0 \\ 0 & \alpha_3 v_x^2 + \alpha_4 v_y^2 \end{bmatrix} \quad (9)$$

Sometimes, the noise variable is more convenient to be added in the action space:

$$x_t = I \cdot x_{t-1} + \Delta t \cdot I \begin{bmatrix} v_x + \varepsilon'_x \\ v_y + \varepsilon'_y \end{bmatrix} = x_{t-1} + \Delta t \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \underbrace{\Delta t \begin{bmatrix} \varepsilon'_x \\ \varepsilon'_y \end{bmatrix}}_{\varepsilon} \quad (10)$$

where  $\varepsilon' \sim \mathcal{N}(0, M)$  can also be expressed in the action space if we transform the r.v. such as  $\varepsilon \sim \mathcal{N}(0, BMB^T)$ . It will be crucial to check in which space the noise is defined.

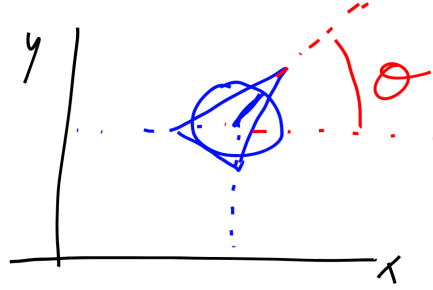


Figure 3: 2D pose. Position and orientation (or heading)

### 3 2D pose: Position and orientation (or heading)

$$x = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \quad \text{is the state variable.} \quad \dot{x} = \underbrace{\begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}}_u \quad \text{is the control variable.}$$

$$x_t = x_{t-1} + \Delta t \cdot u = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_{t-1} + \Delta t \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}_t \quad (11)$$

Q: are the heading and position variables decoupled?

There are 3 control variables for 3 state variables  $\Rightarrow \forall x \in X$  is reachable.

Linear approximation  $\rightarrow$  build probabilities model as before.

### Wheeled robots in 2D

$$x_t = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \quad \text{State is a pose: position and orientation}$$

Why not 3D? Wheeled robots (normally) stay on the ground, which we locally approximate as a plane (2D)

### 4 Kinematic Unicycle

We have already presented the c-t transition equation  $\dot{x} = f(x, u)$ . This is a deterministic function.

Wheels add constraints: heading and velocity are related, see Fig. 4.

$$\frac{dy}{dx} = \tan(\theta) \quad (12)$$

$$\frac{\dot{y}}{\dot{x}} = \frac{\sin \theta}{\cos \theta} \Rightarrow (-\dot{x} \sin \theta + \dot{y} \cos \theta) \cdot v = 0 \quad (13)$$

$$\Rightarrow \begin{matrix} \dot{x} = v \cdot \cos \theta \\ \dot{y} = v \cdot \sin \theta \end{matrix} \quad \text{proof: } -v \cos \theta \sin \theta + v \sin \theta \cos \theta = 0 \quad (14)$$

$$\dot{x} = f(x, u) = \begin{bmatrix} v \cdot \cos \theta \\ v \cdot \sin \theta \\ \omega \end{bmatrix} \quad u = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

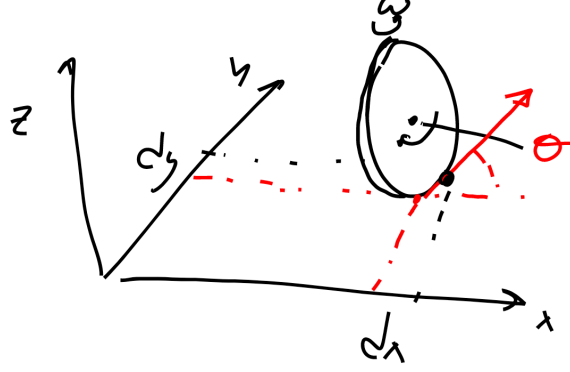


Figure 4: Kinematic Unicycle. Imagine a large disk, it can only move on the theta direction for infinitesimally small increments. Therefore, the constraints to the system follow from this fact.

Still  $\forall x \in X$  is reachable. (Proof in Lavalle book).

$$x_t = \underbrace{\begin{bmatrix} x_{t-1} + \Delta t \cdot v_t \cos \theta_{t-1} \\ y_{t-1} + \Delta t \cdot v_t \sin \theta_{t-1} \\ \theta_{t-1} + \Delta t \cdot \omega_t \end{bmatrix}}_{g(x_{t-1}, u_t, \varepsilon_t)|_{\varepsilon_t=0}} = I \cdot x_{t-1} + \underbrace{\begin{bmatrix} \Delta t \cdot \cos \theta_{t-1} & 0 \\ \Delta t \cdot \sin \theta_{t-1} & 0 \\ 0 & \Delta t \end{bmatrix}}_{B(x) - \text{Non-linear}} \begin{bmatrix} v \\ \omega \end{bmatrix} \quad (15)$$

## 4.1 Unicycle: probabilistic model

Let's first consider the Taylor expansion of the non-linear function  $g(\cdot)$ :

$$x_t = g(x_{t-1}, u_t, \varepsilon_t)|_{\varepsilon_t=0} \simeq g(\mu_{t-1}, u_t, \varepsilon_t)|_{\varepsilon_t=0} + G_t(x_{t-1} - \mu_{t-1}) + V_t(\underbrace{\tilde{u}_t - u_t}_{\varepsilon_t}). \quad (16)$$

The given action  $u_t$ , could be seen as the mean, and we have defined in the previous classes that this is a deterministic variable. The real action value after considering noise could also be defined as single random variable  $\tilde{u}_t = u_t + \epsilon_t$ , where the noise variable  $\epsilon \sim \mathcal{N}(0, M)$ . Note then that the Jacobian is necessary only for when the noise is in the action space.

$$G_t = \frac{\partial g(x_{t-1}, u_t)}{\partial x_{t-1}} \bigg|_{\mu_{t-1}} = \begin{bmatrix} 1 & 0 & -\sin \theta_{t-1} \cdot \Delta t v_t \\ 0 & 1 & \cos \theta_{t-1} \cdot \Delta t v_t \\ 0 & 0 & 1 \end{bmatrix} \quad (17)$$

$\partial/\partial x \qquad \qquad \partial/\partial y \qquad \qquad \partial/\partial \theta$

$$V_t = \frac{\partial g(x_{t-1}, u_t)}{\partial u_t} \bigg|_{u_t} = \begin{bmatrix} \cos \theta_{t-1} \cdot \Delta t & 0 \\ \sin \theta_{t-1} \cdot \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \quad (18)$$

Add noise in the state space:

$$x_t = g(x_{t-1}, u_t, \varepsilon_t) = g(x_{t-1}, u_t, 0) + \varepsilon_t \quad (19)$$

where we want to emphasize the fact that the transition function will produce a new state and the noise is added directly to this state  $x_t$ .

$$\begin{aligned} x_{t-1} &\sim \mathcal{N}(\mu_{t-1}, \Sigma_{t-1}) \quad , \quad \varepsilon_t \sim \mathcal{N}(0, R) \quad (\text{state space noise}) \\ \Rightarrow x_t &\sim N(g(\mu_{t-1}, u_t, 0), G_t \Sigma_{t-1}^T + R) \end{aligned} \quad (20)$$



Figure 5: Circular path. In the first order approximation we use linear segments, which has associated some error.

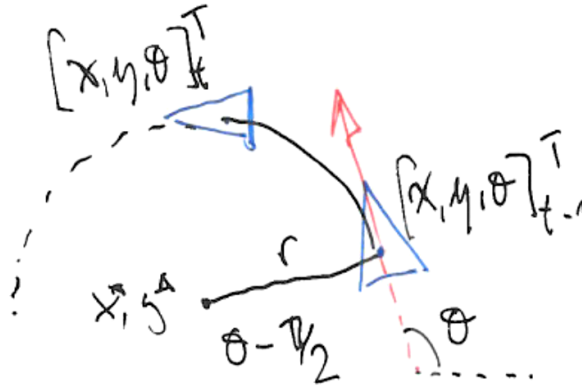


Figure 6: Approximation of the unicycle. In this approach there is a center of rotation  $(x^*, y^*)$ , from which the robot describes an arc.

Q: Show this at home.

**Example:** Circular path give the controls  $\theta = 1, \omega = 1$ .

Can we do better than Euler Integration? ProbRob Ch5.3 uses a sequence of arcs to approximate the unicycle:

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_t = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_{t-1} + \begin{bmatrix} -\frac{v}{\omega} \sin \theta + \frac{v}{\omega} \sin(\theta + \omega \Delta t) \\ \frac{v}{\omega} \cos \theta - \frac{v}{\omega} \cos(\theta + \omega \Delta t) \\ \omega \Delta t + \underbrace{\gamma \Delta t}_{\text{Extra term: final orientation}} \end{bmatrix}_t = g(x_{t-1}, u_t, 0) \quad (21)$$

## 4.2 Probabilistic Arc Kinematic motion model

$$\begin{bmatrix} \hat{v} \\ \hat{\omega} \end{bmatrix} = \begin{bmatrix} v + \varepsilon_v \\ \omega + \varepsilon_\omega \end{bmatrix}, \quad \varepsilon \sim \mathcal{N}(0, M) \quad (\text{Jacobians in ProbRob p.127})$$

$$x_t \sim N(g(\mu_{t-1}, u_t, 0), \quad G_t \Sigma_{t-1} G_t^T + V_t M V_t^T). \quad (22)$$

## 5 Odometry model

Counts of increments on spinning wheels (observation) more accurate than velocity models.

$$u = \begin{bmatrix} \delta_{rot1} \\ \delta_{trans} \\ \delta_{rot2} \end{bmatrix} = \begin{bmatrix} \frac{\text{atan2}(y_t - y_{t-1}, x_t - x_{t-1})}{\sqrt{(x_t - x_{t-1})^2 + (y_t - y_{t-1})^2}} \\ \theta_t - \theta_{t-1} - \delta_{rot1} \end{bmatrix} \quad (23)$$

5

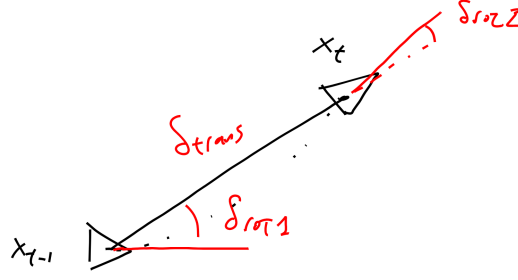


Figure 7: Odometry model. The robot first rotates, then translates in the heading direction and translates a second time to reach any possible end configuration.

Discrete-time transition function:

$$x_t = g(x_{t-1}, u_t, \varepsilon_t)|_{\varepsilon_t=0} = \begin{bmatrix} x_{t-1} + \delta_{trans} \cos(\theta + \delta_{rot1}) \\ y_{t-1} + \delta_{trans} \sin(\theta + \delta_{rot1}) \\ \theta_{t-1} + \delta_{rot1} + \delta_{rot2} \end{bmatrix} \quad (24)$$

## 5.1 Probabilistic odometry model

$$x_t = g(x_{t-1}, u_t, \varepsilon_t) \quad \begin{matrix} \text{(noise in action space)} \\ \varepsilon_t \sim \mathcal{N}(0, M) \end{matrix} \quad \text{(ProbRob p.139)}$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{\delta_{rot1}} \\ \varepsilon_{\delta_{trans}} \\ \varepsilon_{\delta_{rot2}} \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} \alpha_1 \delta_{rot1}^2 + \alpha_2 \delta_{trans}^2 & 0 & 0 \\ 0 & \alpha_3 \delta_{trans}^2 + \alpha_4 (\delta_{rot1}^2 + \delta_{rot2}^2) & 0 \\ 0 & 0 & \alpha_1 \delta_{rot2}^2 + \alpha_2 \delta_{trans}^2 \end{bmatrix} \right) \quad (25)$$

## 5.2 Jacobian for odometry model

$$G_t = \frac{\partial g(x_{t-1}, u_t, \varepsilon_t)}{\partial x_{t-1}} \bigg|_{\mu_{t-1}, \varepsilon_t=0} = \begin{bmatrix} 1 & 0 & -\delta_{trans} \sin(\theta + \delta_{rot1}) \\ 0 & 1 & \delta_{trans} \cos(\theta + \delta_{rot1}) \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \partial/\partial x & \partial/\partial y & \partial/\partial \theta \end{matrix} \quad (26)$$

$$V_t = \frac{\partial g(x_{t-1}, u_t, \varepsilon_t)}{\partial u_t} \bigg|_{\mu_{t-1}, \varepsilon_t=0} = \begin{bmatrix} -\delta_{trans} \cdot \sin(\theta + \delta_{rot1}) & \cos(\theta + \delta_{rot1}) & 0 \\ \delta_{trans} \cdot \cos(\theta + \delta_{rot1}) & \sin(\theta + \delta_{rot1}) & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \partial/\partial \delta_{rot1} & \partial/\partial \delta_{trans} & \partial/\partial \delta_{rot2} \end{matrix} \quad (27)$$

## 6 2D-Rigid Body Transformations

2D poses can be interpreted as a transformation between coordinate frames  $\rightarrow$  XYT parameterization. In the example below, between the global frame  $g$  and the robot frame  $r$ .

$${}^g x_r = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

We need to project point  ${}^l p$  in the local frame (or robot frame) to the global frame  $g$ .

$${}^g p = \begin{bmatrix} x_{local} \cdot \cos \theta_g - y_{local} \sin \theta_g + t_x \\ x_{local} \cdot \sin \theta_g + y_{local} \cos \theta_g + t_y \end{bmatrix} = \begin{bmatrix} \cos \theta_g & -\sin \theta_g & t_x \\ \sin \theta_g & \cos \theta_g & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{local} \\ y_{local} \\ 1 \end{bmatrix} \quad (28)$$

(homogeneous coordinates)

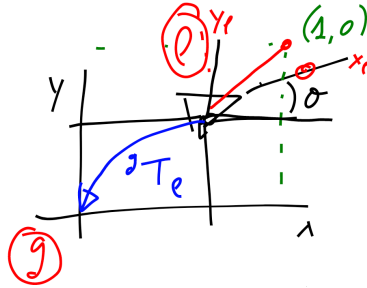


Figure 8: 2D-Rigid Body Transformation. From the local frame to the world frame. This transformation is expressed as a 2D pose.

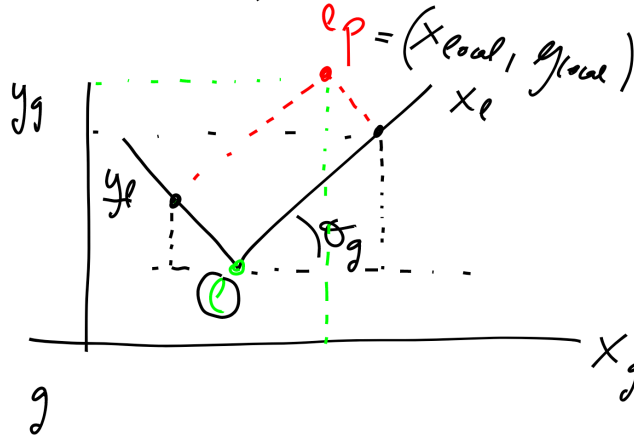


Figure 9: 2D-Rigid Body Transformations 2

We can express this transformation more compactly as a linear transformation of homogeneous coordinates:

$${}^g p = \begin{bmatrix} {}^g R_l(\theta_g) & t \\ 0_{1 \times 2} & 1 \end{bmatrix} \cdot {}^l p = {}^g T_l \cdot {}^l p = {}^g p. \quad (29)$$

Note that the superscript with the point coordinates, has to match the subscript of the transformation, and as a result the new frame of the point is the global frame.

## 7 RBT composition

Two solutions to transform the same point, one directly from  $b \rightarrow g$  and the second  $b \rightarrow a \rightarrow g$ :

$$\begin{aligned} {}^g p &= {}^g T_b \cdot {}^b p \\ {}^a p &= {}^a T_b \cdot {}^b p \\ {}^g p &= {}^g T_a \cdot {}^a p = {}^g T_a \cdot {}^a T_b \cdot {}^b p \end{aligned}$$

## 8 Sensor models

$$\begin{aligned} p(z_t | x_t) & \quad \text{Observation distribution (Lecture 4)} \\ z_t = h(x_t, \delta_t) & \quad \text{Observation function} \end{aligned}$$

**Example:** Accelerometer, sensor, radar, Lidar, camera, ...

We will propose a sensor model, but any function relating state and observation is a valid sensor model.

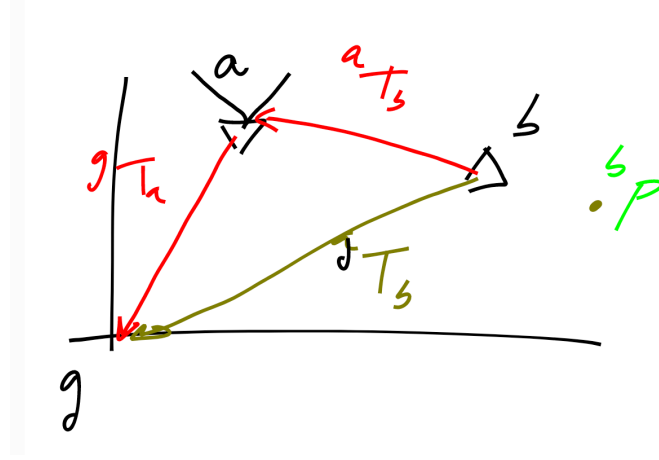


Figure 10: RBT composition. Two solution to transform the point  ${}^b p$ .

## 8.1 Feature-based measurements model

The idea is to extract features  $f$  from observations  $f(z_t) = \{f_1, f_2, \dots, f_n\}$ .

**Example:** lines, corners, point description, objects, etc...

**Landmark definition:** feature which corresponds to physical objects.

$$(2D) \quad f_i = \begin{bmatrix} \text{'range'} \\ \text{'bearing'} \\ \text{'signature'} \end{bmatrix} = \begin{bmatrix} r \\ \phi \\ s \end{bmatrix} \quad \begin{array}{l} s \text{ is optional to alleviate data association problem.} \\ \text{(color, size, } \mathbb{R}^{60} \text{ embedding, etc ...)} \end{array}$$

each feature corresponds to a location  $[m_{i,x}, m_{i,y}]^T$

$$\begin{bmatrix} r_t^j \\ \phi_t^j \\ s_t^j \end{bmatrix} = \begin{bmatrix} \sqrt{(m_{i,x} - x)^2 + (m_{i,y} - y)^2} \\ \text{atan2}(m_{i,y} - y, m_{i,x} - x) - \theta \\ s_i \end{bmatrix} + \begin{bmatrix} \delta_{\sigma_r^2} \\ \delta_{\sigma_\phi^2} \\ \delta_{\sigma_s^2} \end{bmatrix}, \quad \delta \sim \mathcal{N}(0, \Sigma_\delta) \quad (30)$$

The data association problem: the  $i^{th}$  map location  $m_i$  corresponds to the  $j^{th}$  feature.

$$f^j = h(x_t, m_i) + \delta_t \quad (31)$$

$$f \sim \mathcal{N}(f(z_t); h(\mu_t, m_i), \Sigma_\delta) \quad \text{Probabilistic model} \quad (32)$$

Sampling  $x_t$  from  $m_i$  in ProbRob 180.

## 9 Summary

- Transition function:  $x_t = g(x_{t-1}, u_t)$ , discrete-time model obtained by integrating  $\dot{x} = f(x, u)$  (NL)
- Probabilistic model:
  - Add noise to the state/action space
  - Linearize  $g(\cdot)$
  - Covariance transformation:  $x_t \sim N(g(\mu_{t-1}, u_t), G_t \Sigma G_t^T + R)$
- RBT 2D:  $p = {}^g T_l \cdot {}^l p = {}^g T_a \cdot {}^a T_l \cdot {}^l p$  – chain of transformations



- Observation function  $\rightarrow$  Landmarks:

$$m_i = [m_{i,x} \quad m_{i,y}]^T$$

$$\text{(features from sensor data:)} \quad z = h(x, m_i) = [r \quad \phi \quad s]^T$$

– range, bearing, appearance

$$p(z|x) \sim N(z; h(\mu_x, m), \Sigma_z)$$