Derivation Document

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I. POTENTIAL ENERGY

Assuming there exists a potential energy function in the separable form:

$$V = V_x(x)V_y(y). \tag{1}$$

In that case, the forces in the horizontal and vertical directions can be expressed as:

$$\hat{F}_x(x,y) = -\frac{dV_x(x)}{dx}V_y(y), \tag{2}$$

$$\hat{F}_y(x,y) = -V_x(x)\frac{dV_y(y)}{dy}. \tag{3}$$

We assume the horizontal and vertical components of the forces to be in the form:

$$\hat{F}_x(x,y) = \frac{\pi}{x_{\text{to}}}\sin\left(\pi\frac{x}{x_{\text{to}}}\right)V_y(y), \tag{4}$$

$$\hat{F}_y(x,y) = V_x(x)k(l_0 - y). \tag{5}$$

According to (1), the following relation must be satisfied:

$$\frac{\partial \hat{F}_x(x,y)}{\partial y} = \frac{\partial \hat{F}_y(x,y)}{\partial x}. \tag{6}$$

Using this relation, we obtain

$$\frac{\pi}{x_{\text{to}}}\sin\left(\pi\frac{x}{x_{\text{to}}}\right)\frac{dV_y(y)}{dy} = \frac{dV_x(x)}{dx}k(l_0 - y). \tag{7}$$

Rearranging (7), we get:

$$\frac{\frac{dV_x(x)}{dx}}{\frac{\pi}{x_{\text{to}}}\sin\left(\pi\frac{x}{x_{\text{to}}}\right)} = \frac{\frac{dV_y(y)}{dy}}{k(l_0 - y)} = c_0, \tag{8}$$

where c_0 is an unknown constant.

Fist, we integrate

$$\frac{dV_x(x)}{dx} = c_0\frac{\pi}{x_{\text{to}}}\sin\left(\pi\frac{x}{x_{\text{to}}}\right) \tag{9}$$

to obtain

$$V_x(x) = -c_0\left(\cos\left(\pi\frac{x}{x_{\text{to}}}\right) + c_1\right). \tag{10}$$

Next, we integrate

$$\frac{dV_y(y)}{dy} = c_0 k(l_0 - y) \tag{11}$$

to obtain

$$V_y(y) = c_0\left(kl_0 y - \frac{k}{2}y^2 + c_2\right). \tag{12}$$

Now, we need to determine three constants c_0, c_1, and c_2.

First we determine c_0. Substituting (9) into (2) and equating the result with (4) we find:

$$\begin{aligned}\hat{F}_x(x,y) &= -\frac{\mathrm{d}V_x(x)}{\mathrm{d}x}V_y(y) \\ &= -c_0\frac{\pi}{x_{\text{to}}}\sin\left(\pi\frac{x}{x_{\text{to}}}\right)V_y(y) = \frac{\pi}{x_{\text{to}}}\sin\left(\pi\frac{x}{x_{\text{to}}}\right)V_y(y),\end{aligned} \tag{13}$$

and consequently,

$$c_0 = -1. \tag{14}$$

In order to check the consistency of this result, we also substitute (11) into (3) and equate the result with (5) to find:

$$\begin{aligned}\hat{F}_y(x,y) &= -V_x(x)\frac{\mathrm{d}V_y(y)}{\mathrm{d}y} \\ &= -V_x(x)c_0 k(l_0 - y) = V_x(x)k(l_0 - y),\end{aligned} \tag{15}$$

which also gives,

$$c_0 = -1. \tag{16}$$

Finally, we check the dimensional consistency of the approximation. Namely the following must hold:

$$[V] = [V_x(x)][V_y(y)] = Nm$$

and

$$[F_x] = [\frac{dV_x(x)}{dx}][V_y(y)] = N$$

and

$$[F_y] = [V_x(x)][\frac{dV_y(y)}{dy}] = N$$

All these conditions are satisfied if

$$[c_0] = [-]$$

Therefore $c_0 = -1$.

Note that to satisfy the above conditions, the following must be true

$$[F_x] = [\frac{\pi}{x_{\text{to}}}\sin\left(\pi\frac{x}{x_{\text{to}}}\right)][V_y(y)] = N$$

and

$$[F_y] = [V_x(x)][k(l_0 - y)] = N$$

From the force relations we obtain

$$[\frac{\pi}{x_{\text{to}}} \sin\left(\pi \frac{x}{x_{\text{to}}}\right)] = [\frac{dV_x(x)}{dx}]$$

$$[k(l_0 - y)] = [\frac{dV_y(y)}{dy}]$$

Multiplying the force relations we obtain

$$[\frac{\pi}{x_{\text{to}}} \sin\left(\pi \frac{x}{x_{\text{to}}}\right)][k(l_0 - y)][V_x(x)][V_y(y)] = N^2$$

and therefore

$$[\frac{\pi}{x_{\text{to}}} \sin\left(\pi \frac{x}{x_{\text{to}}}\right)][k(l_0 - y)] = \frac{N}{m}$$

All these relations are satisfied with the proposed approximation.

Next, we determine c_1 such that the spring is fully extended at toe-off:

$$\hat{F}_x(x_{\text{to}}, y_{\text{to}}) = F_x(x_{\text{to}}, y_{\text{to}}) = 0, \tag{17}$$

$$\hat{F}_y(x_{\text{to}}, y_{\text{to}}) = F_y(x_{\text{to}}, y_{\text{to}}) = 0, \tag{18}$$

$$\hat{V}(x_{\text{to}}, y_{\text{to}}) = V(x_{\text{to}}, y_{\text{to}}) = 0. \tag{19}$$

Conditions (18) and (19) are satisfied if

$$V_x(x_{\text{to}}) = -c_0\left(\cos(\pi) + c_1\right) = 0, \tag{20}$$

which relation implies

$$c_1 = 1. \tag{21}$$

Condition (17) is satisfied if

$$\frac{dV_x(x = x_{\text{to}})}{dx} = c_0 \frac{\pi}{x_{\text{to}}} \sin(\pi) = 0, \tag{22}$$

which is always true. Therefore $c_1 = 1$.

Finally, we determine c_2 such that,

$$\hat{V}(x_{\text{hs}}, y_{\text{hs}}) = V(x_{\text{hs}}, y_{\text{hs}}). \tag{23}$$

Based on (23), (10) and (12), we obtain,

$$-c_0^2\left(\cos\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) + c_1\right)\left(kl_0 y_{\text{hs}} - \frac{k}{2} y_{\text{hs}}^2 + c_2\right) = V(x_{\text{hs}}, y_{\text{hs}}). \tag{24}$$

Substituting (16) and (21), into the above relation, we obtain,

$$-\left(\cos\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) + 1\right)\left(kl_0 y_{\text{hs}} - \frac{k}{2} y_{\text{hs}}^2 + c_2\right) = V(x_{\text{hs}}, y_{\text{hs}}). \tag{25}$$

By solving the above relation for c_2, we obtain

$$c_2 = -\frac{V(x_{\text{hs}}, y_{\text{hs}})}{\cos\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) + 1} - kl_0 y_{\text{hs}} + \frac{k}{2} y_{\text{hs}}^2. \tag{26}$$

We are now in position to define the potential energy function. Using (10), (16) and (21), we find

$$V_x(x) = \cos\left(\pi\frac{x}{x_{\text{to}}}\right) + 1. \tag{27}$$

Similarly, using (12), (16) and (26), we find

$$V_y(y) = \frac{V(x_{\text{hs}}, y_{\text{hs}})}{\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + 1} - kl_0(y - y_{\text{hs}}) + \frac{k}{2}(y^2 - y_{\text{hs}}^2). \tag{28}$$

II. EQUATION OF MOTION

With the force components now identified, we can integrate to obtain the velocities under the assumption

$$y = y_{\text{hs}} \tag{29}$$

Let us start with the horizontal direction:

$$m\ddot{x} = \frac{\pi}{x_{\text{to}}}\sin\left(\pi\frac{x}{x_{\text{to}}}\right)V_y(y_{\text{hs}}), \tag{30}$$

$$m\frac{d\dot{x}}{dx}\dot{x} = \frac{\pi}{x_{\text{to}}}\sin\left(\pi\frac{x}{x_{\text{to}}}\right)V_y(y_{\text{hs}}) \tag{31}$$

$$\int_{\dot{x}_{\text{hs}}}^{\dot{x}}\dot{x}d\dot{x} = \frac{\pi}{x_{\text{to}}}\frac{V_y(y_{\text{hs}})}{m}\int_{x_{\text{hs}}}^{x}\sin\left(\pi\frac{x}{x_{\text{to}}}\right)dx. \tag{32}$$

$$\frac{1}{2}(\dot{x}^2 - \dot{x}_{\text{hs}}^2) = \frac{V_y(y_{\text{hs}})}{m}\left(\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \cos\left(\pi\frac{x}{x_{\text{to}}}\right)\right). \tag{33}$$

$$\dot{x}^2 = \dot{x}_{\text{hs}}^2 + \frac{2V_y(y_{\text{hs}})}{m}\left(\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \cos\left(\pi\frac{x}{x_{\text{to}}}\right)\right). \tag{34}$$

This gives us the kinetic energy relationship in the horizontal direction and thus the horizontal velocity as a function of horizontal position

$$\dot{x}(x)^2 = \dot{x}_{\text{hs}}^2 + \frac{2V(x_{\text{hs}}, y_{\text{hs}})}{m\left(\cos(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}) + 1\right)}\left(\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \cos\left(\pi\frac{x}{x_{\text{to}}}\right)\right). \tag{35}$$

Let us now consider the vertical velocity,

$$m\ddot{y} = V_x(x)k(l_0 - y_{\text{hs}}) - mg \tag{36}$$

or

$$m\frac{d\dot{y}}{dx}\dot{x} = V_x(x)k(l_0 - y_{\text{hs}}) - mg \tag{37}$$

or

$$\frac{d\dot{y}}{dx} = \frac{k}{m}(l_0 - y_{\text{hs}})\frac{V_x(x)}{\dot{x}(x)} - g\frac{1}{\dot{x}(x)} \tag{38}$$

or

$$d\dot{y} = \frac{k}{m}(l_0 - y_{\text{hs}})\frac{V_x(x)}{\dot{x}(x)}dx - g\frac{dx}{\dot{x}(x)} \tag{39}$$

or

$$\int_{\dot{y}_{hs}}^{\dot{y}} d\dot{y} = \frac{k}{m}(l_0 - y_{hs}) \int_{x_{hs}}^{x} \frac{V_x(x)}{\dot{x}(x)} dx - g \int_{x_{hs}}^{x} \frac{dx}{\dot{x}(x)} \tag{40}$$

or

$$\dot{y} - \dot{y}_{hs} = \frac{k}{m}(l_0 - y_{hs}) \int_{x_{hs}}^{x} \frac{V_x(x)}{\dot{x}(x)} dx - g \int_{x_{hs}}^{x} \frac{dx}{\dot{x}(x)} \tag{41}$$

To integrate the vertical velocity, we assume $\dot{x}(x) \approx \dot{x}_{hs}$ and obtain

$$\dot{y} - \dot{y}_{hs} = \frac{k}{m\dot{x}_{hs}}(l_0 - y_{hs}) \int_{x_{hs}}^{x} V_x(x) dx - \frac{g}{\dot{x}_{hs}} \int_{x_{hs}}^{x} dx. \tag{42}$$

or

$$\dot{y} - \dot{y}_{hs} = \frac{k}{m\dot{x}_{hs}}(l_0 - y_{hs}) \int_{x_{hs}}^{x} \left(\cos\left(\pi \frac{x}{x_{to}}\right) + 1 \right) dx \\ - \frac{g}{\dot{x}_{hs}}(x - x_{hs}). \tag{43}$$

or

$$\dot{y} - \dot{y}_{hs} = \frac{k}{m\dot{x}_{hs}}(l_0 - y_{hs})(x - x_{hs}) \\ + \frac{k}{m\dot{x}_{hs}}(l_0 - y_{hs}) \int_{x_{hs}}^{x} \cos\left(\pi \frac{x}{x_{to}}\right) dx \\ - \frac{g}{\dot{x}_{hs}}(x - x_{hs}). \tag{44}$$

or

$$\dot{y} - \dot{y}_{hs} = \frac{k}{m\dot{x}_{hs}}(l_0 - y_{hs})(x - x_{hs}) \\ + \frac{x_{to}}{\pi} \frac{k}{m\dot{x}_{hs}}(l_0 - y_{hs}) \left(\sin\left(\pi \frac{x}{x_{to}}\right) - \sin\left(\pi \frac{x_{hs}}{x_{to}}\right) \right) \\ - \frac{g}{\dot{x}_{hs}}(x - x_{hs}). \tag{45}$$

or

$$\dot{y} - \dot{y}_{hs} = \frac{x_{to}}{\pi} \frac{k}{m\dot{x}_{hs}}(l_0 - y_{hs}) \left(\sin\left(\pi \frac{x}{x_{to}}\right) - \sin\left(\pi \frac{x_{hs}}{x_{to}}\right) \right) \\ \left(\frac{k}{m\dot{x}_{hs}}(l_0 - y_{hs}) - \frac{g}{\dot{x}_{hs}} \right)(x - x_{hs}).$$

Finally, we obtain

$$\dot{y}(x) = \dot{y}_{hs} + c_{y1}\left(\sin\left(\pi \frac{x}{x_{to}}\right) - \sin\left(\pi \frac{x_{hs}}{x_{to}}\right) \right) + c_{y2}(x - x_{hs}) \tag{46}$$

where

$$c_{y1} = \frac{x_{to}}{\pi} \frac{k}{m\dot{x}_{hs}}(l_0 - y_{hs}), \tag{47}$$

and

$$c_{y2} = \frac{k}{m\dot{x}_{hs}}(l_0 - y_{hs}) - \frac{g}{\dot{x}_{hs}} = \frac{\pi}{x_{to}} c_{y1} - \frac{g}{\dot{x}_{hs}}. \tag{48}$$

We can also present this relation in the following way:

$$\dot{y}(x) = \dot{y}_{hs} + c_y(k)\left(\sin\left(\pi \frac{x}{x_{to}}\right) - \sin\left(\pi \frac{x_{hs}}{x_{to}}\right) + \frac{\pi}{x_{to}}(x - x_{hs}) \right) - \frac{g}{\dot{x}_{hs}}(x - x_{hs}) \tag{49}$$

where

$$c_y(k) = \frac{x_{to}}{\pi} \frac{k}{m\dot{x}_{hs}}(l_0 - y_{hs}). \tag{50}$$

III. STEADY STATE STIFFNESS

$$\dot{x}(x)^2 = \dot{x}_{\text{hs}}^2 + \underbrace{\frac{2V(x_{\text{hs}}, y_{\text{hs}})}{m\left(\cos(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}) + 1\right)}}_{c_x(k)}\left(\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \cos\left(\pi\frac{x}{x_{\text{to}}}\right)\right). \tag{51}$$

$$\dot{x}(x)^2 = \dot{x}_{\text{hs}}^2 + c_x(k)\left(\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \cos\left(\pi\frac{x}{x_{\text{to}}}\right)\right). \tag{52}$$

Note that the intuition one may use here is that the potential energy of the approximate system in the x direction (more precisely $2V/m$ in the x direction) is given by

$$c_x(k)\left(\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \cos\left(\pi\frac{x}{x_{\text{to}}}\right)\right) \tag{53}$$

where x_{hs} is the coordinate that defines the initial condition! This means that if we want to concatenate the velocity prediction because two different stiffness are used then it should be done by changing k (and nothing else) in $c_x(k)$ but chaining the initial condition defined by x_{hs} in the above formula.

Therefore

$$\dot{x}(0)^2 = \dot{x}_{\text{hs}}^2 + c_x(k_1)\left(\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - 1\right) \tag{54}$$

and

$$\dot{x}(x_{\text{to}})^2 = \dot{x}(0)^2 + c_x(k_2)\left(\cos\left(\pi\frac{0}{x_{\text{to}}}\right) + 1\right) \tag{55}$$

such that

$$\dot{x}(x_{\text{to}})^2 = \dot{x}_{\text{hs}}^2 + c_x(k_1)\left(\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - 1\right) + 2c_x(k_2). \tag{56}$$

Now, to find the steady state condition, we assume

$$\dot{x}(x_{\text{to}})^2 = \dot{x}_{\text{hs}}^2. \tag{57}$$

Consequently, we obtain

$$0 = c_x(k_1)\left(\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - 1\right) + 2c_x(k_2), \tag{58}$$

and

$$\frac{c_x(k_2)}{c_x(k_1)} = \frac{1}{2}\left(1 - \cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)\right), \tag{59}$$

and

$$\frac{k_2}{k_1} = \frac{1}{2}\left(1 - \cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)\right). \tag{60}$$

This relation will ensure that the toe off velocity at the end of the step in the x direction is the same as the heel strike velocity at the beginning of the step in the x direction. Assuming

$$\frac{k_2^*}{k_1} = \frac{1}{2}\left(1 - \cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)\right). \tag{61}$$

and

$$k_2 = k_2^* + \Delta k_2, \tag{62}$$

and

$$c_x(k_2) = c_x(k_2^*) + c_x(\Delta k_2), \tag{63}$$

and

$$c_x(k_2) = \frac{1}{2}c_x(k_1)\left(1 - \cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)\right) + c_x(\Delta k_2), \tag{64}$$

we find

$$\dot{x}(x_{\text{to}})^2 = \dot{x}_{\text{hs}}^2 + c_x(k_1)\left(\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - 1\right) + c_x(k_1)\left(1 - \cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)\right) + 2c_x(\Delta k_2). \tag{65}$$

or

$$\dot{x}(x_{\text{to}})^2 = \dot{x}_{\text{hs}}^2 + 2c_x(\Delta k_2) \tag{66}$$

where

$$c_x(k) = \frac{2V(x_{\text{hs}}, y_{\text{hs}})}{m\left(\cos(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}) + 1\right)} = \frac{k}{m}\frac{\left(l_0 - \sqrt{x_{\text{hs}}^2 + y_{\text{hs}}^2}\right)^2}{\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + 1} \tag{67}$$

Consequently,

$$\dot{x}(x_{\text{to}})^2 = \dot{x}_{\text{hs}}^2 + \frac{2\Delta k_2}{m}\frac{\left(l_0 - \sqrt{x_{\text{hs}}^2 + y_{\text{hs}}^2}\right)^2}{\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + 1} \tag{68}$$

Note that when $x_{\text{hs}} = -x_{\text{to}}$ then

$$\lim_{x_{\text{hs}} \to -x_{\text{to}}} \frac{\left(l_0 - \sqrt{x_{\text{hs}}^2 + y_{\text{hs}}^2}\right)^2}{\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + 1} = 0 \tag{69}$$

because

$$\lim_{x_{\text{hs}} \to -x_{\text{to}}} \frac{\left(l_0 - \sqrt{x_{\text{hs}}^2 + y_{\text{hs}}^2}\right)^2}{\cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + 1} = \lim_{x_{\text{hs}} \to -x_{\text{to}}} \frac{2\left(l_0 - \sqrt{x_{\text{hs}}^2 + y_{\text{hs}}^2}\right)\frac{-x_{\text{hs}}}{\sqrt{x_{\text{hs}}^2 + y_{\text{hs}}^2}}}{-\frac{\pi}{x_{\text{to}}}\sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)} = \lim_{x_{\text{hs}} \to -x_{\text{to}}} \frac{= 0}{\neq 0} = 0 \tag{70}$$

IV. TAKE OFF CONDITION

$$\dot{y}(x) = \dot{y}_{\text{hs}} + c_y(k)\left(\sin\left(\pi\frac{x}{x_{\text{to}}}\right) - \sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + \frac{\pi}{x_{\text{to}}}(x - x_{\text{hs}})\right) - \frac{g}{\dot{x}_{\text{hs}}}(x - x_{\text{hs}}) \tag{71}$$

where

$$c_y(k) = \frac{x_{\text{to}}}{\pi}\frac{k}{m\dot{x}_{\text{hs}}}(l_0 - y_{\text{hs}}). \tag{72}$$

Note that the intuition one may use here is that the vertical momentum of the approximate system (more precisely p_y/m) is given by

$$c_y(k)\left(\sin\left(\pi\frac{x}{x_{\text{to}}}\right) - \sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + \frac{\pi}{x_{\text{to}}}(x - x_{\text{hs}})\right) - \frac{g}{\dot{x}_{\text{hs}}}(x - x_{\text{hs}}) \tag{73}$$

where x_{hs} is the coordinate that defines the initial condition! This means that if we want to concatenate the velocity prediction because two different stiffness are used then it should be done by changing k (and nothing else) in $c_y(k)$ but chaining the initial condition defined by x_{hs} in the above formula.

Therefore

$$\dot{y}(0) = \dot{y}_{\text{hs}} - c_y(k_1)\left(\sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + \frac{\pi}{x_{\text{to}}}x_{\text{hs}}\right) + \frac{g}{\dot{x}_{\text{hs}}}x_{\text{hs}} \tag{74}$$

and

$$\dot{y}(x_{\text{to}}) = \dot{y}(0) + c_y(k_2)\left(\sin\left(\pi\frac{x_{\text{to}}}{x_{\text{to}}}\right) - \sin\left(\pi\frac{0}{x_{\text{to}}}\right) + \frac{\pi}{x_{\text{to}}}(x_{\text{to}} - 0)\right) - \frac{g}{\dot{x}_{\text{hs}}}(x_{\text{to}} - 0) \tag{75}$$

such that

$$\dot{y}(x_{\text{to}}) = \dot{y}(0) + c_y(k_2)\pi - \frac{g}{\dot{x}_{\text{hs}}}x_{\text{to}} \tag{76}$$

Consequently

$$\dot{y}(x_{\text{to}}) = \dot{y}_{\text{hs}} - c_y(k_1)\left(\sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + \frac{\pi}{x_{\text{to}}}x_{\text{hs}}\right) + \frac{g}{\dot{x}_{\text{hs}}}x_{\text{hs}} + c_y(k_2)\pi - \frac{g}{\dot{x}_{\text{hs}}}x_{\text{to}} \tag{77}$$

or

$$\dot{y}(x_{\text{to}}) = \dot{y}_{\text{hs}} - c_y(k_1)\left(\sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + \pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + c_y(k_2)\pi + \frac{g}{\dot{x}_{\text{hs}}}(x_{\text{hs}} - x_{\text{to}}) \tag{78}$$

Now, to find the take off condition, we set

$$\dot{y}(x_{\text{to}}) \leq 0. \tag{79}$$

$$\dot{y}_{\text{hs}} - c_y(k_1)\left(\sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + \pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + c_y(k_2)\pi - \frac{g}{\dot{x}_{\text{hs}}}(x_{\text{to}} - x_{\text{hs}}) \leq 0. \tag{80}$$

$$c_y(k_2)\pi - c_y(k_1)\left(\sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + \pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) \leq \frac{g}{\dot{x}_{\text{hs}}}(x_{\text{to}} - x_{\text{hs}}) - \dot{y}_{\text{hs}}. \tag{81}$$

Now let us introduce the following substitution

$$c_y(k) = Ck, \tag{82}$$

where

$$C = \frac{x_{\text{to}}}{\pi}\frac{1}{m\dot{x}_{\text{hs}}}(l_0 - y_{\text{hs}}). \tag{83}$$

Consequently,

$$Ck_2\pi - Ck_1\left(\sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + \pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) \leq \frac{g}{\dot{x}_{\text{hs}}}(x_{\text{to}} - x_{\text{hs}}) - \dot{y}_{\text{hs}}. \tag{84}$$

Now let us assume that we want to find the range of stiffness in early stance k_1, such that no take off accrues if we use the (previously derived) one step stiffness policy to reach steady state

$$\frac{k_2}{k_1} = \frac{1}{2}\left(1 - \cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)\right). \tag{85}$$

In this case, we find,

$$C\frac{1}{2}\left(1 - \cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)\right)k_1\pi - Ck_1\left(\sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) + \pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) \leq \frac{g}{\dot{x}_{\text{hs}}}(x_{\text{to}} - x_{\text{hs}}) - \dot{y}_{\text{hs}}. \tag{86}$$

such that

$$C\left(\frac{\pi}{2}\left(1 - \cos\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)\right) - \sin\left(\pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \pi\frac{x_{\text{hs}}}{x_{\text{to}}}\right)k_1 \leq \frac{g}{\dot{x}_{\text{hs}}}(x_{\text{to}} - x_{\text{hs}}) - \dot{y}_{\text{hs}}. \tag{87}$$

or

$$\frac{x_{\text{to}}}{\pi} \frac{1}{m} (l_0 - y_{\text{hs}}) \left(\frac{\pi}{2} \left(1 - \cos\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) \right) - \sin\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \pi \frac{x_{\text{hs}}}{x_{\text{to}}} \right) k_1 \leq \left(g(x_{\text{to}} - x_{\text{hs}}) - \dot{x}_{\text{hs}} \dot{y}_{\text{hs}} \right). \tag{88}$$

or

$$\left(\frac{\pi}{2} \left(1 - \cos\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) \right) - \sin\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \pi \frac{x_{\text{hs}}}{x_{\text{to}}} \right) k_1 \leq \frac{\pi}{x_{\text{to}}} \frac{m}{(l_0 - y_{\text{hs}})} \left(g(x_{\text{to}} - x_{\text{hs}}) - \dot{x}_{\text{hs}} \dot{y}_{\text{hs}} \right), \tag{89}$$

or

$$\left(\frac{\pi}{2} \left(1 - \cos\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) \right) - \sin\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \pi \frac{x_{\text{hs}}}{x_{\text{to}}} \right) \frac{k_1 (l_0 - y_{\text{hs}})}{mg} \leq \frac{\pi}{x_{\text{to}}} \left(x_{\text{to}} - x_{\text{hs}} - \frac{\dot{x}_{\text{hs}} \dot{y}_{\text{hs}}}{g} \right), \tag{90}$$

or

$$\left(\pi \sin\left(\frac{\pi}{2} \frac{x_{\text{hs}}}{x_{\text{to}}}\right)^2 - \sin\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \pi \frac{x_{\text{hs}}}{x_{\text{to}}} \right) \frac{k_1 (l_0 - y_{\text{hs}})}{mg} \leq \frac{\pi}{x_{\text{to}}} \left(x_{\text{to}} - x_{\text{hs}} - \frac{\dot{x}_{\text{hs}} \dot{y}_{\text{hs}}}{g} \right). \tag{91}$$

or

$$\frac{k_1 (l_0 - y_{\text{hs}})}{mg} \leq \frac{\pi}{x_{\text{to}}} \frac{(x_{\text{to}} - x_{\text{hs}}) - \frac{\dot{x}_{\text{hs}} \dot{y}_{\text{hs}}}{g}}{\pi \sin\left(\frac{\pi}{2} \frac{x_{\text{hs}}}{x_{\text{to}}}\right)^2 - \sin\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \pi \frac{x_{\text{hs}}}{x_{\text{to}}}}. \tag{92}$$

or

$$\frac{k_1 (l_0 - y_{\text{hs}})}{mg} \leq \frac{1 - \frac{x_{\text{hs}}}{x_{\text{to}}} - \frac{\dot{x}_{\text{hs}} \dot{y}_{\text{hs}}}{g x_{\text{to}}}}{\sin\left(\frac{\pi}{2} \frac{x_{\text{hs}}}{x_{\text{to}}}\right)^2 - \frac{1}{\pi} \sin\left(\pi \frac{x_{\text{hs}}}{x_{\text{to}}}\right) - \frac{x_{\text{hs}}}{x_{\text{to}}}}. \tag{93}$$