Meccanica **34:** 281–285, 1999.

BRIEF NOTES

Non-piecewise Representation of Discontinuous Functions and its Application to the Clebsch Method for Beam Deflections

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(Received: 23 April 1998; accepted in revised form: 14 May 1999)

Key words: Non-piecewise beam equations, Non-piecewise integration, Discontinuity functions, Clebsch method, Mechanics of materials.

1. Introduction

The Clebsch (or pointed brackets) method to determine beam deflections is very useful in the solution of problems with concentrated or otherwise discontinuous loads [1, 2]. However, its use has not been as widespread as one would expect. Some mechanics of materials books and many mechanical design books don't even mention it. Perhaps the reason is that it makes use of a rather special representation of the Macauley and the singularity functions which are the basis of the method [1]. The most fundamental of these functions is the Heaviside unit step. Most symbolic mathematics software include the unit step as a preprogrammed function and thus it may be called from the beam equations and used for this purpose [3–6]. This will, very likely, help popularize this approach. This preprogrammed unit step is represented in either of the two following forms: (1) the customary, (2) its Laplace transform.

The customary, non-unified, piecewise, representation requires at least two value assignments and two 'zoning', conditional statements, one in connection with each of the values. The Laplace transform of these functions is very simple but it forces the user to shift back and forth from the x domain to the s domain and this is a rather cumbersome process even when using the sophisticated symbolic mathematics software. A third alternative is presented in this article.

2. Proposed Representation of Basic Functions

The following are the proposed representations of the three well-known functions:

Absolute value or Vee

$$V_a = |x - a| = +\sqrt{(x - a)^2} \tag{1}$$

Relay or Jump

$$J_a = \frac{x-a}{V_a} = \frac{V_a}{x-a} \tag{2}$$

Heaviside unit step

$$H_a = \tfrac{1}{2}(1 + J_a) \tag{3}$$

It is worth noticing that the proposed representation, Equations (1)– (3) requires, in each case, only one equality valid from $-\infty$ to $+\infty$ and, therefore, it requires no inequalities at all. Thus this proposed representation may be incorporated, as is, directly into the beam equations, if so desired.

3. Basic Operations

Taking care to maintain $\sqrt{x^2}$ or $\sqrt{(x-a)^2}$ as such and using relations (1)–(3) as well as the following formula from an ordinary table of integrals [7]:

$$\begin{aligned}\int x^m (a+bx^q)^p \,\mathrm{d}x \;&=\; \frac{x^{m-q+1}(a+bx^q)^{p+1}}{b(qp+m+1)} \\ &\quad - \frac{a(m-q+1)}{b(qp+m+1)} \int x^{m-q}(a+bx^q)^p \,\mathrm{d}x,\end{aligned} \tag{4}$$

the following integration formula is established:

$$\int H_a (x-a)^N \,\mathrm{d}(x-a) = H_a \frac{(x-a)^{N+1}}{N+1}, \quad N \neq -1 \tag{5}$$

It is interesting to verify that symbolic integration in the *Mathematica* program, for a specified value of N, yields results in accordance with relation (5).

A unified representation of discontinuous functions may be built up using the Heaviside unit step as a ‘switch-on’ function and its negative as a ‘switch off’ function. Thus the function shown in Figure 1 is represented thus:

$$g(x) = f_1(x) + H_a[-f_1(x) + f_2(x)]. \tag{6}$$

Referring to the product of unit step functions, it is fairly obvious that

$$\text{If } b \geqslant a \qquad H_a H_b = H_b. \tag{7}$$

4. Example

A beam built-in at both ends is loaded as shown in Figure 2. At 5 m from the left end there is an abrupt change in the rigidity so that its value on each side is: $\mathrm{EI_L} = 1000\,\mathrm{kN\,m^2}$ and $\mathrm{EI_R} = 4000\,\mathrm{kN\,m^2}$. A single equation will be established to describe each of the following functions along the full length of the beam: the bending moment, the slope of the elastic curve and the deflection.

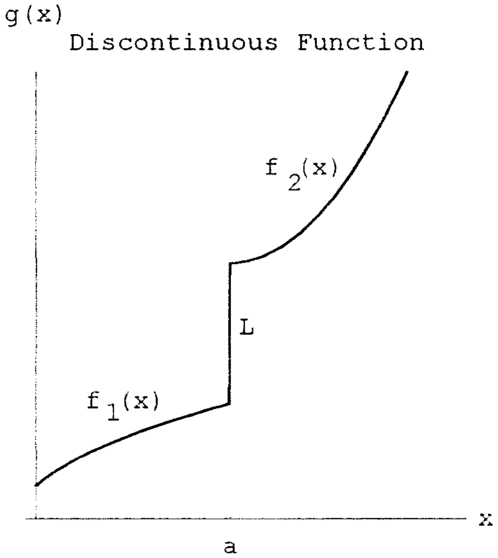

Figure 1. Curve represented by Equation 6. If L is not zero, $g(x)$ is a discontinuous function. If L is zero, $g(x)$ is a broken line function.

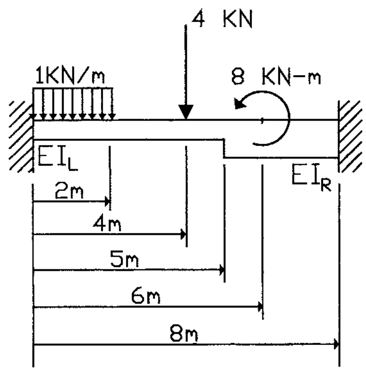

Figure 2. Statically indeterminate beam with discontinuous loads and discontinuous rigidity.

5. Solution

Figure 3 shows the free body diagram of the whole beam. Taking the origin at the left end, and using the scheme of Equation (6), the moment, at any point of the beam, may be expressed as

$$M = -M_L + R_L x - \tfrac{1}{2}x^2 + H_2[(+\tfrac{1}{2}x^2) + (-2x + 2)] - 4H_4(x - 4) - 8H_6. \tag{8}$$

Proceeding in this fashion:

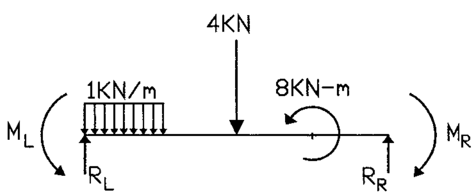

Figure 3. Free body diagram.

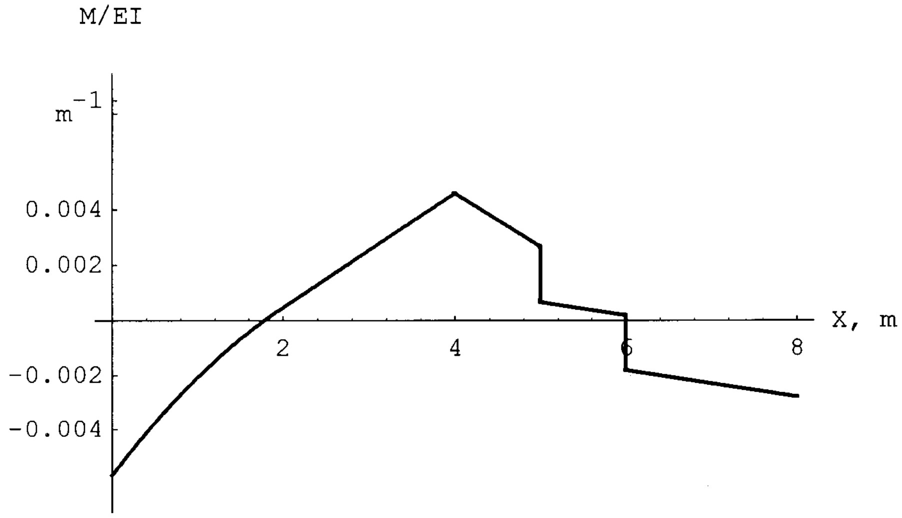

Figure 4. M/EI diagram plotted directly from Equation (10).

$$\frac{1}{EI} = 0.001\left(1 - \frac{3}{4}H_5\right). \tag{9}$$

Multiplying the last two equations and applying Equation (7) yields

$$\begin{aligned}\frac{M}{EI} = y'' \;=\; & 0.001[-M_L + R_L x - \tfrac{1}{2}x^2 + \tfrac{1}{2}H_2(x-2)^2 - 4H_4(x-4) - \\ & - \tfrac{3}{4}H_5\{(-M_L + 5R_L - 12) + (R_L - 6)(x-5)\} - 2H_6]. \end{aligned}\tag{10}$$

Integrating in accordance with relation (5) yields the slope of the elastic curve:

$$\begin{aligned} y' = \theta \;=\; & 0.001[-M_L x + \tfrac{1}{2}R_L x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{6}H_2(x-2)^3 - 2H_4(x-4)^2 - \\ & - \tfrac{3}{4}H_5\{(-M_L + 5R_L - 12)(x-5) + \\ & + \tfrac{1}{2}(R_L - 6)(x-5)^2\} - 2H_6(x-6) + C_1]. \end{aligned}\tag{11}$$

Integrating Equation (11) yields the deflection curve:

$$\begin{aligned} y \;=\; & 0.001[-\tfrac{1}{2}M_L x^2 + \tfrac{1}{6}R_L x^3 - \tfrac{1}{24}x^4 + \tfrac{1}{24}H_2(x-2)^4 - \tfrac{2}{3}H_4(x-4)^3 - \\ & - \tfrac{3}{4}H_5\{\tfrac{1}{2}(-M_L + 5R_L - 12)(x-5)^2 + \\ & + \tfrac{1}{6}(R_L - 6)(x-5)^3\} - H_6(x-6)^2 + C_1 + C_2 x]. \end{aligned}\tag{12}$$

Analytic continuation is established automatically by use of relations (5) and (6) thus only one constant is required per integration. Substituting the boundary conditions in Equations (11) and (12) yields the values:

$$C_1 = 0, \quad C_2 = 0, \quad R_L = 4.0630\,\text{kN}, \quad M_L = 5.6541\,\text{kN}m \tag{13}$$

Thus, upon substitution of values (13) all the desired functions, i.e. the moment, the slope of the elastic curve and the deflection have been established, each represented by a single equation valid along the full length of the beam. Figures 4 and 5 are direct plots of Equations (10) and (12), respectively.

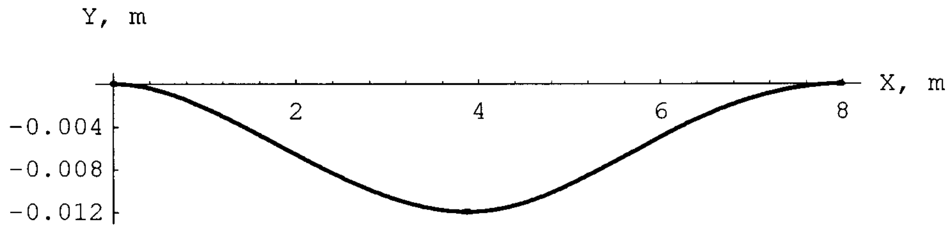

Figure 5. Beam deflection plotted directly from Equation (12).

6. Conclusions

An alternative representation for the discontinuity functions used in the Clebsch method has been proposed in terms of elementary, widely used, algebraic operators, familiar to the student long before he reaches university level. He does not even need to have ever heard of the Laplace transform. And, in any case, no symbolic mathematics software is required.

Although the proposed representation is different, the properties of these functions remain the same. Their integration is carried out in an elementary, familiar fashion but lead to the same, long established results. The representation permits these functions to be incorporated into the beam equations so that they may be handled even in a non-programmable calculator. The beam equations may be plotted by simply keying them directly into a graphics calculator. When used in a computer the proposed representation simplifies programming. All of the advantages of the Clebsch method are preserved, of course.

Acknowledgements

The author wishes to express his appreciation to the reviewers for their valuable criticisms, and to Heriberto Vega-Sámano for the diagrams of Figures 2 and 3.

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