Part I

n-dimension Spherical coordinates and the volumes of the n-ball in $\mathbb{R}^n$ $(by\ Wen\ Shih)$

1 Introduction

We know that ω_n, the surface area of the unit ball in $\mathbb{R}^n$, gets involved in the fundamental solution for the Laplace operator. We are also very familiar with the case in $\mathbb{R}^2$ and $\mathbb{R}^3$. However, for high dimension case, $n \geq 4$, it is not easy to imagine. The purpose of this note is to discuss how to calculate ω_n for $n \geq 4$.

Let $V_n(R) = \int_{B_R(0)} 1dx = \int_0^R (\int_{|x|=r} 1dS)dr$ denote the volume of the ball with radius R in $\mathbb{R}^n$ and $S_n(R)$ denote the surface area of the ball with radius R in $\mathbb{R}^n$. Then it is clear that $S_n(R) = \frac{d}{dR}V_n(R) = \int_{|x|=R} 1dS = R^{n-1}\int_{|x|=1} 1dS$. Since $V_n(R) = \int_{B_R(0)} 1dx = R^n \int_{B_1(0)} 1dx$ as well, $S_n(R) = \frac{d}{dR}V_n(R) = nR^{n-1}\int_{B_1(0)} 1dx$, and hence, the surface area of the unit ball in $\mathbb{R}^n$ is $\omega_n = S_n(1) = \int_{|x|=1} 1dS = n\int_{B_1(0)} 1dx = nV_n(1)$. So, once we know $V_n(R)$, we can obtain ω_n immediately. However, to calculate $V_n(R) = \int_{B_R(0)} 1dx$, we not only need n-dimension Spherical coordinates, but also $\int_0^\pi \sin^n \theta d\theta$, for $n \in \mathbb{N}$.

2 Observation for n-dimension Spherical coordinates

2.1 2D & 3D case:

In 2D case, we usually let $x = r\cos\theta_1$, $y = r\sin\theta_1$, where $r \geq 0$ and $0 \leq \theta_1 < 2\pi$, then $dxdy = |\frac{\partial(x,y)}{\partial(r,\theta)}|drd\theta_1$ and

$$\frac{\partial(x,y)}{\partial(r,\theta_1)} = \det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 \\ \sin\theta_1 & r\cos\theta_1 \end{bmatrix} = r\cos^2\theta_1 + r\sin^2\theta_1 = r.$$

So, $dxdy = rdrd\theta_1$.

In 3D case, we usually let $x = r\sin\theta_1\cos\theta_2$, $y = r\sin\theta_1\sin\theta_2$, $z = r\cos\theta_1$, where $r \geq 0$, $0 \leq \theta_1 \leq \pi$, and $0 \leq \theta_2 < 2\pi$, then $dxdydz = |\frac{\partial(x,y,z)}{\partial(r,\theta_1,\theta_2)}|drd\theta_1 d\theta_2$ and

$$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \det\begin{bmatrix} \sin\theta_1\cos\theta_2 & r\cos\theta_1\cos\theta_2 & -r\sin\theta_1\sin\theta_2 \\ \sin\theta_1\sin\theta_2 & r\cos\theta_1\sin\theta_2 & r\sin\theta_1\cos\theta_2 \\ \cos\theta_1 & -r\sin\theta_1 & 0 \end{bmatrix} = r^2\sin\theta_1.$$

So, $dxdydz = r^2\sin\theta_1 drd\theta_1 d\theta_2$.

2.2 Generalization:

In 2D case, if we rewrite $x \to x_1$ and $y \to x_2$, then

$$\begin{cases} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1 \end{cases},$$

where $r \geq 0$ and $0 \leq \theta_1 < 2\pi$. Since $dx_1dx_2 = |\frac{\partial(x_1,x_2)}{\partial(r,\theta)}|drd\theta_1$ and

$$\frac{\partial(x_1,x_2)}{\partial(r,\theta_1)} = \det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 \\ \sin\theta_1 & r\cos\theta_1 \end{bmatrix} = r\cos^2\theta_1 + r\sin^2\theta_1 = r, \tag{1}$$

$dx_1dx_2 = rdrd\theta_1$.

In 3D case, if we rewrite $z \to x_1$, $x \to x_2$, $y \to x_3$ (Note: the order satisfies right-hand rule), then let

$$\begin{cases} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1\cos\theta_2 \\ x_3 = r\sin\theta_1\sin\theta_2 \end{cases}, \quad \begin{array}{l} \text{Note: } r\cos\theta_1 \in spen(x_1) \text{ and } r\sin\theta_1 \in spen(x_2,x_3) \\ \text{Then in } x_2x_3\text{-space, it is a 2D case, but } r \to r\sin\theta_1 \\ \text{instead. So, } x_2 = r\sin\theta_1\cos\theta_2 \;\&\; x_3 = r\sin\theta_1\sin\theta_2. \end{array}$$

where $r \geq 0$, $0 \leq \theta_1 \leq \pi$, and $0 \leq \theta_2 < 2\pi$. Since $dx_1dx_2dx_3 = |\frac{\partial(x_1,x_2,x_3)}{\partial(r,\theta_1,\theta_2)}|drd\theta_1d\theta_2$ and

$$\frac{\partial(x_1,x_2,x_3)}{\partial(r,\theta_1,\theta_2)} = \det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 & 0 \\ \sin\theta_1\cos\theta_2 & r\cos\theta_1\cos\theta_2 & -r\sin\theta_1\sin\theta_2 \\ \sin\theta_1\sin\theta_2 & r\cos\theta_1\sin\theta_2 & r\sin\theta_1\cos\theta_2 \end{bmatrix} \tag{2}$$

$$\begin{aligned}
&= r\sin\theta_1\sin\theta_2\det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 \\ \sin\theta_1\sin\theta_2 & r\cos\theta_1\sin\theta_2 \end{bmatrix} + r\sin\theta_1\cos\theta_2\det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 \\ \sin\theta_1\cos\theta_2 & r\cos\theta_1\cos\theta_2 \end{bmatrix} \\
&= r\sin\theta_1\sin^2\theta_2\det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 \\ \sin\theta_1 & r\cos\theta_1 \end{bmatrix} + r\sin\theta_1\cos^2\theta_2\det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 \\ \sin\theta_1 & r\cos\theta_1 \end{bmatrix} \\
&= r\sin\theta_1\sin^2\theta_2\cdot r + r\sin\theta_1\cos^2\theta_2\cdot r \qquad \text{(by (1))} \\
&= r^2\sin\theta_1.
\end{aligned} \tag{3}$$

Similarly, in 4D case, if we let

$$\begin{cases} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1\cos\theta_2 \\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3 \\ x_4 = r\sin\theta_1\sin\theta_2\sin\theta_3 \end{cases}, \quad \begin{array}{l} \text{Note: } r\cos\theta_1 \in \mathit{spen}(x_1) \text{ and } r\sin\theta_1 \in \mathit{spen}(x_2,x_3,x_4) \\ \text{Then in } x_2x_3x_4\text{-space, it is a 3D case, but } r \to r\sin\theta_1 \\ \text{instead. So, } x_2 = r\sin\theta_1\cos\theta_2,\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3 \\ \&\ x_4 = r\sin\theta_1\sin\theta_2\sin\theta_3. \end{array}$$

where $r \geq 0$, $0 \leq \theta_1 \leq \pi$, $0 \leq \theta_2 \leq \pi$, and $0 \leq \theta_3 < 2\pi$. Since $dx_1dx_2dx_3dx_4 = |\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(r,\theta_1,\theta_2,\theta_3)}|drd\theta_1d\theta_2d\theta_3$ and

$$\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(r,\theta_1,\theta_2,\theta_3)} \tag{4}$$

$$= \det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 & 0 & 0 \\ \sin\theta_1\cos\theta_2 & r\cos\theta_1\cos\theta_2 & -r\sin\theta_1\sin\theta_2 & 0 \\ \sin\theta_1\sin\theta_2\cos\theta_3 & r\cos\theta_1\sin\theta_2\cos\theta_3 & r\sin\theta_1\cos\theta_2\cos\theta_3 & -r\sin\theta_1\sin\theta_2\sin\theta_3 \\ \sin\theta_1\sin\theta_2\sin\theta_3 & r\cos\theta_1\sin\theta_2\sin\theta_3 & r\sin\theta_1\cos\theta_2\sin\theta_3 & r\sin\theta_1\sin\theta_2\cos\theta_3 \end{bmatrix} \tag{5}$$

$$\begin{aligned}
&= r\sin\theta_1\sin\theta_2\sin\theta_3\det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 & 0 \\ \sin\theta_1\cos\theta_2 & r\cos\theta_1\cos\theta_2 & -r\sin\theta_1\sin\theta_2 \\ \sin\theta_1\sin\theta_2\sin\theta_3 & r\cos\theta_1\sin\theta_2\sin\theta_3 & r\sin\theta_1\cos\theta_2\sin\theta_3 \end{bmatrix} \\
&+ r\sin\theta_1\sin\theta_2\cos\theta_3\det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 & 0 \\ \sin\theta_1\cos\theta_2 & r\cos\theta_1\cos\theta_2 & -r\sin\theta_1\sin\theta_2 \\ \sin\theta_1\sin\theta_2\cos\theta_3 & r\cos\theta_1\sin\theta_2\cos\theta_3 & r\sin\theta_1\cos\theta_2\cos\theta_3 \end{bmatrix} \\
&= r\sin\theta_1\sin\theta_2\sin^2\theta_3\det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 & 0 \\ \sin\theta_1\cos\theta_2 & r\cos\theta_1\cos\theta_2 & -r\sin\theta_1\sin\theta_2 \\ \sin\theta_1\sin\theta_2 & r\cos\theta_1\sin\theta_2 & r\sin\theta_1\cos\theta_2 \end{bmatrix} \\
&+ r\sin\theta_1\sin\theta_2\cos^2\theta_3\det\begin{bmatrix} \cos\theta_1 & -r\sin\theta_1 & 0 \\ \sin\theta_1\cos\theta_2 & r\cos\theta_1\cos\theta_2 & -r\sin\theta_1\sin\theta_2 \\ \sin\theta_1\sin\theta_2 & r\cos\theta_1\sin\theta_2 & r\sin\theta_1\cos\theta_2 \end{bmatrix} \\
&= r\sin\theta_1\sin\theta_2\sin^2\theta_3\cdot r^2\sin\theta_1 + r\sin\theta_1\sin\theta_2\cos^2\theta_3\cdot r^2\sin\theta_1 \qquad \text{(by (2) and (3))} \\
&= r^3\sin^2\theta_1\sin\theta_2.
\end{aligned} \tag{6}$$

In 5D case, if we let

$$\begin{cases} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1\cos\theta_2 \\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3 \\ x_4 = r\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 \\ x_5 = r\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4 \end{cases}, \quad \begin{array}{l} \text{Note: } r\cos\theta_1 \in \mathit{spen}(x_1) \text{ and } r\sin\theta_1 \in \mathit{spen}(x_2,x_3,x_4,x_5) \\ \text{Then in } x_2x_3x_4x_5\text{-space, it is a 4D case, but } r \to r\sin\theta_1 \\ \text{instead. So, } x_2 = r\sin\theta_1\cos\theta_2,\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3, \\ x_4 = r\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4,\ x_5 = r\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4 \end{array}$$

where $r \geq 0$, $0 \leq \theta_1 \leq \pi$, $0 \leq \theta_2 \leq \pi$, $0 \leq \theta_3 \leq \pi$, and $0 \leq \theta_4 < 2\pi$. By (4), (5), (6), and the same process as above, we obtain

$$\frac{\partial(x_1,x_2,x_3,x_4,x_5)}{\partial(r,\theta_1,\theta_2,\theta_3,\theta_4)} = r^4\sin^3\theta_1\sin^2\theta_2\sin\theta_3. \tag{7}$$

From the above observation, we can guess the n-dimension Spherical coordinates and the Jacobian should be (8) and (9) in Theorem 1, respectively.

3 Observation for $V_n(R)$

$$
\begin{aligned}
V_1(R) &= \int_{B_R(0)} 1dx_1 = 2R \\
V_2(R) &= \int_{B_R(0)} 1dx_1dx_2 = \int_0^{2\pi}\int_0^R rdrd\theta_1 = \int_0^R rdr \cdot \int_0^{2\pi} 1d\theta_1 (= \pi R^2) \\
V_3(R) &= \int_{B_R(0)} 1dx_1dx_2dx_3 = \int_0^{2\pi}\int_0^{\pi}\int_0^R r^2 \sin\theta_1 drd\theta_1d\theta_2 \qquad \text{(by(3))} \\
&= \int_0^R r^2\sin\theta_1 dr \cdot \int_0^{\pi} \sin\theta_1 d\theta_1 \cdot \int_0^{2\pi} 1d\theta_2 (= \frac{4}{3}\pi R^3) \\
V_4(R) &= \int_{B_R(0)} 1dx_1dx_2dx_3dx_4 = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\int_0^R r^3 \sin^2\theta_1 \sin\theta_2 drd\theta_1d\theta_2d\theta_3 \qquad \text{(by (6))} \\
&= \int_0^R r^3 dr \cdot \int_0^{\pi} \sin^2\theta_1 d\theta_1 \cdot \int_0^{\pi} \sin\theta_2 d\theta_2 \cdot \int_0^{2\pi} 1d\theta_3 (= \frac{1}{2}\pi^2 R^4) \\
V_5(R) &= \int_{B_R(0)} 1dx_1dx_2dx_3dx_4dx_5 = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\int_0^R r^4 \sin^3\theta_1 \sin^2\theta_2 \sin\theta_3 drd\theta_1d\theta_2d\theta_3d\theta_4 \qquad \text{(by (7))} \\
&= \int_0^R r^4 dr \cdot \int_0^{\pi} \sin^3\theta_1 d\theta_1 \cdot \int_0^{\pi} \sin^2\theta_2 d\theta_2 \cdot \int_0^{\pi} \sin\theta_3 d\theta_3 \cdot \int_0^{2\pi} 1d\theta_4 (= \frac{8}{15}\pi^2 R^5)
\end{aligned}
$$

From this observation, we can expect that we need $\int_0^{\pi} \sin^n\theta d\theta$, for $n \in \mathbb{N}$ to calculate $V_n(R)$.

4 Main Results

By the above observations, we can obtain the following results.

Theorem 1 (n-dimension Spherical coordinates) *Let*

$$
\begin{cases}
x_1 = r\cos\theta_1 \\
x_2 = r\sin\theta_1\cos\theta_2 \\
x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3 \\
x_4 = r\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 \\
\vdots \\
x_{n-2} = r\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-3}\cos\theta_{n-2} \\
x_{n-1} = r\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-3}\sin\theta_{n-2}\cos\theta_{n-1} \\
x_n = r\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-3}\sin\theta_{n-2}\sin\theta_{n-1}
\end{cases}, \tag{8}
$$

where $r \geq 0,\ 0 \leq \theta_i \leq \pi,\ i = 1, 2, ..., n-2,$ *and* $0 \leq \theta_{n-1} < 2\pi,$ *then*

$$
\frac{\partial(x_1, x_2, \cdots, x_{n-1}, x_n)}{\partial(r, \theta_1, \theta_2, \cdots, \theta_{n-2}, \theta_{n-1})} = r^{n-1}\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin^2\theta_{n-3}\sin\theta_{n-2}. \tag{9}
$$

Lemma 2 ($\int_0^{\pi}\sin^n\theta d\theta$, for $n \in \mathbb{N}$) *(1) If n is even, say, $n = 2m$ for some $m \in \mathbb{N}$, then*

$$
\int_0^{\pi} \sin^{2m}\theta d\theta == \frac{2m-1}{2m}\frac{2m-3}{2m-2}\cdots\frac{1}{2}\cdot\pi
$$

or

$$
\int_0^{\pi} \sin^{n}\theta d\theta = \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\pi.
$$

(2) If n is odd, say, $n = 2m-1$ for some $m \in \mathbb{N}$, then

$$
\int_0^{\pi} \sin^{2m-1}\theta d\theta == \begin{cases} \frac{2m-2}{2m-1}\frac{2m-4}{2m-3}\cdots\frac{4}{5}\frac{2}{3}\cdot 2 & , \ m \geq 2 \\ 2 & , \ m = 1 \end{cases}
$$

or

$$\int_0^{\pi} \sin^n \theta d\theta = \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{4}{5}\frac{2}{3}\cdot 2 & , n \geq 3 \\ 2 & , n = 1 \end{cases}.$$

Once we have Theorem 1, then

$$\begin{aligned} V_n(R) &= \int_{B_R(0)} 1 dx_1 dx_2 \cdots dx_n \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{R} r^{n-1}\sin^{n-2}\theta_1 \sin^{n-3}\theta_2\cdots\sin^2\theta_{n-3}\sin\theta_{n-2} dr d\theta_1 d\theta_2\cdots d\theta_{n-2}d\theta_{n-1} \\ &= \int_0^R r^{n-1}dr\cdot\int_0^{\pi}\sin^{n-2}\theta_1 d\theta_1\cdot\int_0^{\pi}\sin^{n-3}\theta_2 d\theta_2\cdot\cdots\cdot\int_0^{\pi}\sin\theta_{n-2}d\theta_{n-2}\cdot\int_0^{2\pi}1 d\theta_{n-1} \\ &= \frac{2\pi R^n}{n}\cdot\int_0^{\pi}\sin^{n-2}\theta_1 d\theta_1\cdot\int_0^{\pi}\sin^{n-3}\theta_2 d\theta_2\cdot\cdots\cdot\int_0^{\pi}\sin\theta_{n-2}d\theta_{n-2} \end{aligned} \tag{10}$$

and it follows from Lemma 2 that we have the following Theorem.

Theorem 3 (The volumes of the n-ball in $\mathbb{R}^n$) *(1) If n is even, say, $n = 2m$ for some $m \in \mathbb{N}$, then*

$$V_{2m}(R) = \frac{R^{2m}\pi^m}{m!}.$$

(2) If n is odd, say, $n = 2m-1$ for some $m \in \mathbb{N}$, then

$$V_{2m-1}(R) = \frac{2^m \pi^{m-1} R^{2m-1}}{(2m-1)(2m-3)\cdots 3\cdot 1}.$$

Remark 4 (Recall Gamma function) *Gamma function is dedined as*

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1}e^{-y}dy \text{ for } \alpha > 0,$$

and it has the following properties:

$$\begin{array}{l} (1)\ \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \text{ for } \alpha > 0 \\ (2)\ \Gamma(n) = (n-1)! \text{ for } n \in \mathbb{N} \\ (3)\ \Gamma(\frac{1}{2}) = \sqrt{\pi} \end{array}.$$

Theorem 5 ($V_n(R)$ in terms of Gamma function) *For $n \in \mathbb{N}$,*

$$V_n(R) = \frac{2R^n\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}.$$

5 Conclusion

Now we have known $V_n(R)$ from Theorem 5, then we are able to obtain $\omega_n = nV_n(1)$ immediately. See the following table.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$\cdots$	n
$V_n(R)$	$2R$	πR^2	$\frac{4}{3}\pi R^3$	$\frac{1}{2}\pi^2 R^4$	$\frac{8}{15}\pi^2 R^5$	$\cdots$	$2R^n\pi^{\frac{n}{2}}/n\Gamma(\frac{n}{2})$
$V_n(1)$	2	π	$\frac{4}{3}\pi$	$\frac{1}{2}\pi^2$	$\frac{8}{15}\pi^2$	$\cdots$	$2\pi^{\frac{n}{2}}/n\Gamma(\frac{n}{2})$
ω_n	2	2π	4π	$2\pi^2$	$\frac{8}{3}\pi^2$	$\cdots$	$2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$

6 Proofs

6.1 Proof of Theorem 1

It follows the same process in Section 2.2 and by indution.

6.2 Proof of Lemma 2

First, we compute $\int \sin^n \theta d\theta$ by integration by parts,

$$\begin{aligned}\int \sin^n \theta d\theta &= \int \sin^{n-1} \theta d(-\cos\theta) = -\sin^{n-1}\theta\cos\theta + (n-1)\int \cos^2\theta \sin^{n-2}\theta d\theta \\ &= -\sin^{n-1}\theta\cos\theta + (n-1)\int \sin^{n-2}\theta d\theta - (n-1)\int \sin^n \theta d\theta,\end{aligned}$$

and hence,

$$n\int \sin^n \theta d\theta = -\sin^{n-1}\theta\cos\theta + (n-1)\int \sin^{n-2}\theta d\theta$$

or

$$\int \sin^n \theta d\theta = -\frac{1}{n}\sin^{n-1}\theta\cos\theta + \frac{n-1}{n}\int \sin^{n-2}\theta d\theta. \tag{11}$$

Second, if n is even, say, $n = 2m$, for some $m \in \mathbb{N}$, then by (11)

$$\begin{aligned}\int_0^\pi \sin^{2m}\theta d\theta &= \frac{2m-1}{2m}\int_0^\pi \sin^{2m-2}\theta d\theta \\ &= \frac{2m-1}{2m}\Big(-\frac{1}{2m-2}\sin^{2m-3}\theta\cos\theta|_0^\pi + \frac{2m-3}{2m-2}\int_0^\pi \sin^{2m-4}\theta d\theta\Big) \\ &= \frac{2m-1}{2m}\frac{2m-3}{2m-2}\int_0^\pi \sin^{2m-4}\theta d\theta \\ &= \cdots \\ &= \frac{2m-1}{2m}\frac{2m-3}{2m-2}\cdots\frac{1}{2}\int_0^\pi 1 d\theta \\ &= \frac{2m-1}{2m}\frac{2m-3}{2m-2}\cdots\frac{1}{2}\cdot\pi\end{aligned}$$

or

$$\int_0^\pi \sin^n\theta d\theta = \frac{2m-1}{2m}\frac{2m-3}{2m-2}\cdots\frac{1}{2}\cdot\pi = \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\pi.$$

If n is odd, say, $n = 2m-1$, for some $m \in \mathbb{N}$, then by (11) for $m \geq 2$,

$$\begin{aligned}\int_0^\pi \sin^{2m-1}\theta d\theta &= \frac{2m-2}{2m-1}\int_0^\pi \sin^{2m-3}\theta d\theta \\ &= \frac{2m-2}{2m-1}\Big(-\frac{1}{2m-3}\sin^{2m-4}\theta\cos\theta|_0^\pi + \frac{2m-4}{2m-3}\int_0^\pi \sin^{2m-5}\theta d\theta\Big) \\ &= \frac{2m-2}{2m-1}\frac{2m-4}{2m-3}\int_0^\pi \sin^{2m-5}\theta d\theta \\ &= \cdots \\ &= \frac{2m-2}{2m-1}\frac{2m-4}{2m-3}\cdots\frac{4}{5}\int_0^\pi \sin^3\theta d\theta \\ &= \frac{2m-2}{2m-1}\frac{2m-4}{2m-3}\cdots\frac{4}{5}\Big(-\frac{1}{3}\sin^2\theta\cos\theta|_0^\pi + \frac{2}{3}\int_0^\pi \sin\theta d\theta\Big) \\ &= \frac{2m-2}{2m-1}\frac{2m-4}{2m-3}\cdots\frac{4}{5}\frac{2}{3}\int_0^\pi \sin\theta d\theta \\ &= \frac{2m-2}{2m-1}\frac{2m-4}{2m-3}\cdots\frac{4}{5}\frac{2}{3}\cdot 2,\end{aligned}$$

for $m = 1$,

$$\int_0^\pi \sin\theta d\theta = 2.$$

Or

$$\int_0^\pi \sin^n \theta d\theta = \begin{cases} \frac{2m-2}{2m-1}\frac{2m-4}{2m-3}\cdots\frac{4}{5}\frac{2}{3}\frac{2}{1} & , \ m \geq 2 \\ 2 & , \ m = 1 \end{cases}$$
$$= \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{4}{5}\frac{2}{3}\cdot 2 & , \ n \geq 3 \\ 2 & , \ n = 1 \end{cases} .$$

6.3 Proof of Theorem 3 & Theorem 5

By (10), we have $V_n(R) = \frac{2\pi R^n}{n} \cdot \int_0^\pi \sin^{n-2}\theta_1 d\theta_1 \cdot \int_0^\pi \sin^{n-3}\theta_2 d\theta_2 \cdot \cdots \cdot \int_0^\pi \sin\theta_{n-2} d\theta_{n-2}$. For n is even, say, $n = 2m$ for some $m \in \mathbb{N}$, then by Lemma 2,

$$\begin{aligned} V_{2m}(R) &= \frac{2\pi R^{2m}}{2m} \cdot \int_0^\pi \sin^{2m-2}\theta_1 d\theta_1 \cdot \int_0^\pi \sin^{2m-3}\theta_2 d\theta_2 \cdots \int_0^\pi \sin\theta_{n-2} d\theta_{n-2} \\ &= \frac{2\pi R^{2m}}{2m}(\frac{2m-3}{2m-2}\frac{2m-5}{2m-4}\cdots\frac{3}{4}\frac{1}{2}\cdot\pi)(\frac{2m-4}{2m-3}\frac{2m-6}{2m-5}\cdots\frac{4}{5}\frac{2}{3}\cdot 2)(\frac{2m-5}{2m-4}\cdots\frac{3}{4}\frac{1}{2}\cdot\pi)(\frac{2m-6}{2m-5}\cdots\frac{4}{5}\frac{2}{3}\cdot 2)\cdots(\frac{1}{2}\pi)\cdot 2 \\ &= R^{2m}\frac{\pi}{m}\frac{\pi}{m-1}\frac{\pi}{m-2}\cdots\frac{\pi}{1} \\ &= \frac{R^{2m}\pi^m}{m!} \\ &= \frac{R^{2m}\pi^m}{m\cdot(m-1)!} \\ &= \frac{R^{2m}\pi^m}{m\Gamma(m)} \end{aligned}$$

or

$$V_n(R) = \frac{R^n \pi^{\frac{n}{2}}}{(\frac{n}{2})!} = \frac{2R^n\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}.$$

For n is odd, say, $n = 2m-1$ for some $m \in \mathbb{N}$, then

$$\begin{aligned} V_{2m-1}(R) &= \frac{2\pi R^{2m-1}}{2m-1} \cdot \int_0^\pi \sin^{2m-3}\theta_1 d\theta_1 \cdot \int_0^\pi \sin^{2m-5}\theta_2 d\theta_2 \cdots \int_0^\pi \sin\theta_{n-2} d\theta_{n-2} \\ &= \frac{2\pi R^{2m-1}}{2m-1}(\frac{2m-4}{2m-3}\frac{2m-6}{2m-5}\cdots\frac{4}{5}\frac{2}{3}2)(\frac{2m-5}{2m-4}\cdots\frac{3}{4}\frac{1}{2}\pi)(\frac{2m-6}{2m-5}\cdots\frac{4}{5}\frac{2}{3}2)(\frac{2m-7}{2m-6}\cdots\frac{3}{4}\frac{1}{2}\pi)\cdots(\frac{2}{3}2)(\frac{1}{2}\pi)\cdot 2 \\ &= \frac{2\pi R^{2m-1}}{2m-1}\frac{2\pi}{2m-3}\frac{2\pi}{2m-5}\cdots\frac{2\pi}{3}\cdot\frac{2}{1} \\ &= \frac{2^m\pi^{m-1}R^{2m-1}}{(2m-1)(2m-3)\cdots 3\cdot 1} \end{aligned}$$

or

$$\begin{aligned} V_n(R) &= \frac{2^m\pi^{m-1}R^{2m-1}}{(2m-1)(2m-3)\cdots 3\cdot 1} \\ &= \frac{2^{\frac{n+1}{2}}\pi^{\frac{n-1}{2}}R^n}{n(n-2)\cdots 3\cdot 1} \\ &= \frac{2\pi^{\frac{n}{2}}R^n}{n}\frac{1}{(\frac{n}{2}-1)(\frac{n}{2}-2)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}} \\ &= \frac{2\pi^{\frac{n}{2}}R^n}{n}\frac{1}{(\frac{n}{2}-1)(\frac{n}{2}-2)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(\frac{1}{2})} \\ &= \frac{2\pi^{\frac{n}{2}}R^n}{n\Gamma(\frac{n}{2})}. \end{aligned}$$