1PF1 Complex Analysis

1P1 Series | Michaelmas Term 1995
4 Lectures | Dr D W Murray
1 Tutorial Sheet | Notes revised 1/12/95

Textbooks

The basics are in many books, for example: G Stephenson, "Mathematical Methods for Science Students" (Longman) or G James, "Modern Engineering Mathematics" (Addison-Wesley, 1992). The more advanced material for the later lectures can be found in E Kreysig "Advanced Engineering Mathmetics", 5th ed, Chapters 12-13, and Sokolnikoff and Redheffer, "Mathematics and Physics of Modern Engineering" (McGraw-Hill).

Overview

Complex analysis proves a useful tool for solving a wide variety of problems in engineering science — the analysis of ac electrical circuits, the solution of linear differential equations with constant coefficients, the representation of wave forms, and so on.

Lectures 1 and 2, given in 1st Week, cover algebraic preliminaries and elementary functions of complex variables. Much of the material appears in A-level courses in pure mathematics, though some material on functions of complex numbers will be new to you.
Lectures 3 and 4 given in 2nd Week cover more applied material, looking at phasors and complex representations of waves. These topics will be new to almost all.

Chapter 1

Algebraic Preliminaries

1.1 The form of the complex number

A **complex number**, z, has the form

$$z = x + iy \tag{1.1}$$

where x and y are **real numbers** and i is the **imaginary unit** whose existence is postulated such that

$$i^2 = -1 \ . \tag{1.2}$$

Quantity x is the **real part** of z and y is the **imaginary part**

$$x = \text{Re}(z) \qquad y = \text{Im}(z) \ . \tag{1.3}$$

Be careful to note that Im(z) is a *real* quantity.
A real number is thus a complex number with zero imaginary part. A complex number with zero real part is said to be **pure imaginary**. There is one complex number that is real and pure imaginary — it is of course, zero. There is no particular need therefore to write zero as $(0 + i0)$.

1.2 Operations on complex numbers

1.2.1 Addition, subtraction

Addition and subtraction of two complex numbers $z_1 = (a + ib)$ and $z_2 = (c + id)$ involves the separate addition and subtraction of the real and imaginary parts:

$$z = z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d) \tag{1.4}$$
$$z = z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i(b - d) \ . \tag{1.5}$$

Thus

$$\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2) \tag{1.6}$$

and so on for imaginary parts and subtraction.

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are identical iff $z_1 - z_2 = 0$, which requires the real and imaginary parts of both numbers to be identical — ie:

$$a = c \quad \text{and} \quad b = d \ . \tag{1.7}$$

Thus when we write an equation involving complex numbers, we are effectively writing **two** real equations

$$\begin{aligned} \text{Re}(\text{lefthandside}) &= \text{Re}(\text{righthandside}) & (1.8) \\ \text{Im}(\text{lefthandside}) &= \text{Im}(\text{righthandside}) \ . & (1.9) \end{aligned}$$

1.2.2 Multiplication, division

Multiplication again follows rules familiar from real quantities, though wherever i^2 appears we replace it by -1:

$$\begin{aligned} z = (a+ib)(c+id) &= ac + iad + ibc + i^2 bd \\ &= ac + iad + ibc - bd \\ &= (ac - bd) + i(bc + ad) \ . \end{aligned} \tag{1.10}$$

Division is defined as the inverse of multiplication. Suppose

$$z = x + iy = \frac{a+ib}{c+id} \tag{1.11}$$

then to find x and y:

$$(x+iy)(c+id) = (xc - yd) + i(yc + dx) = a + ib \tag{1.12}$$

giving two linear simultaneous equations in the unknowns x and y

$$\begin{aligned} xc - yd &= a & (1.13) \\ yc + dx &= b & (1.14) \end{aligned}$$

which are straightforward enough to solve.

However it is much more convenient to turn the denominator into a real number by multiplying top and bottom by $(c - id)$:

$$\begin{aligned} z = x + iy &= \frac{(a+ib)(c-id)}{(c+id)(c-id)} & (1.15) \\ &= \frac{(ac+bd) + i(bc-ad)}{c^2+d^2} & (1.16) \end{aligned}$$

Equating real and imaginary parts

$$\begin{aligned} x &= \frac{ac+bd}{c^2+d^2} & (1.17) \\ y &= \frac{bc-ad}{c^2+d^2} \ . & (1.18) \end{aligned}$$

1.3 Complex conjugate

We just saw the important result that multiplying $(c+id)$ by $(c-id)$ gives a real and positive quantity (c^2+d^2). For any complex number $z = x+iy$, the **complex conjugate** is defined as

$$\overline{z} = (x - iy) \tag{1.19}$$

so that

$$z\overline{z} = x^2 + y^2 \quad . \tag{1.20}$$

Note the following results:

$$\overline{(\overline{z})} = z \tag{1.21}$$

and

$$\frac{1}{2}(z+\overline{z}) \quad = \quad \mathrm{Re}(z) \tag{1.22}$$

$$\frac{1}{2i}(z-\overline{z}) \quad = \quad \mathrm{Im}(z) \tag{1.23}$$

Note too that

$$(\overline{z_1} + \overline{z_2} + \overline{z_3} + \ldots) = \overline{(z_1 + z_2 + z_3 + \ldots)} \quad . \tag{1.24}$$

Now consider the product $(\overline{z_1})(\overline{z_2})$. If $z_1 = (a+ib)$ and $z_2 = (c+id)$ this product is $(a-ib)(c-id) = (ac-bd) - i(bc+ad)$. But this is $\overline{(z_1 z_2)}$. This can be extended to any number of products, so that we obtain the result:

$$(\overline{z_1}\,\overline{z_2}\,\overline{z_3}\ \ldots) = \overline{(z_1 z_2 z_3 \ldots)} \quad . \tag{1.25}$$

An equivalent result holds for division

$$\frac{\overline{z_1}}{\overline{z_2}} = \overline{\left(\frac{z_1}{z_2}\right)} \quad . \tag{1.26}$$

(The derivation is left to you!)

1.4 The modulus

The quantity

$$A = \sqrt{z\overline{z}} = \sqrt{x^2+y^2} = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} \tag{1.27}$$

is called the **modulus** of the complex number z. We will meet this again later.

The following results are of occasional use:

$$|z_1||z_2||z_3|\ldots \quad = \quad |z_1 z_2 z_3 \ldots| \tag{1.28}$$

$$\frac{|z_1|}{|z_2|} \quad = \quad \left|\frac{z_1}{z_2}\right| \quad , \tag{1.29}$$

proofs of which are left as an exercise.

♣ Examples

1. Evaluate (a) $(1+2i)+(3+2i)-4/i^3$; (b) $\frac{(i-2)(1-2i)}{(i-3)(1-3i)}$; and (c) $(1+2i)^{-2}$.

   Ans:

   (a) $(1+2i)+(3+2i)-4/i^3 = 4+4i+4/i = 4+4i-4i = 4$.

   (b) $\frac{(i-2)(1-2i)}{(i-3)(1-3i)} = \frac{5i}{10i} = \frac{1}{2}$.

   (c) $(1+2i)^{-2} = \frac{(1-2i)^2}{[(1+2i)(1-2i)]^2} = \frac{1-4i+4i^2}{5^2} = \frac{-3-4i}{25}$

2. If both the sum and product of two complex numbers are real, show that either the numbers are real or one is the complex conjugate of the other.

   Ans: Let the numbers be $(a+ib)$ and $(c+id)$. The conditions are $\mathrm{Im}((a+ib)+(c+id))=0$ and $\mathrm{Im}((a+ib)(c+id))=0$. The first gives $b=-d$ and the second $ad+bc=0$. Combining we find $b(c-a)=0$ — so *either* $b=0$ and $d=0$ (both real); *or* $a=c$ and $b=-d$ (complex conjugates).

3. Show that $z=(\pm 1\pm i)/\sqrt{2}$ satisfies the equation $z^4+1=0$ for all combinations of the signs.

   Ans: Taking positive signs first

$$
\begin{aligned}
[\frac{1}{\sqrt{2}}(1+i)]^4 &= \frac{1}{4}(1+4i+6i^2+4i^3+i^4) \\
&= \frac{1}{4}(1+4i-6-4i+1) = -1 \quad ,
\end{aligned}
$$

   so this satisfies the equation. We could grind out the value for the complex conjugate $[\frac{1}{\sqrt{2}}(1-i)]^4$, but the previous example indicated that its value will be the complex conjugate of -1, that is -1 again. Taking the negative of the original number $[\frac{1}{\sqrt{2}}(-1-i)]^4$ just gives $(-1)^4[\frac{1}{\sqrt{2}}(1+i)]^4$ which again is -1, and again we can use the result of the previous example to get a fourth value of -1. Thus the proposition is verified.

An important point from the last example is that there are FOUR fourth roots of -1. Indeed later we see that there are n complex $n-$th roots of any complex number, when n is an integer. That is, there there are n solutions to $z^n = a+ib$.

1.5 Geometric representation of complex numbers and operations

Just as the single part of a real number can be represented by a point on the real line, so the two parts of a complex number can be represented by a point on the **complex plane**, also referred to as the **Argand diagram** or **z-plane**. Convention dictates that the abcissa is the real axis and the ordinate the imaginary axis.

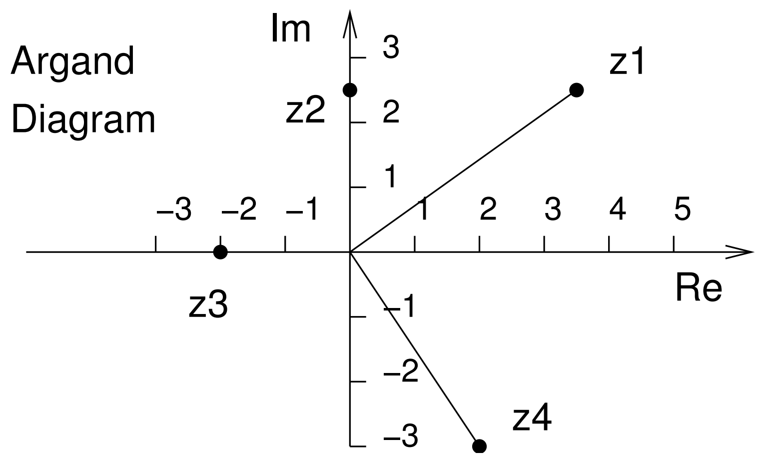

Figure 1.1: Numbers $z_1 = 3.5 + i2.5$, $z_2 = i2.5$ and $z_3 = -2$ and $z_4 = 2 - 3i$ on the Argand diagram.

1.5.1 Addition, subtraction

Now consider the representation of an addition of two complex numbers in the Argand diagram. Because we add the real and imaginary parts separately, the addition is like the addition of two vectors using the parallelogram construction. Ie

$$(a + ib) + (c + id) = (a + c) + i(b + d) \tag{1.30}$$

works in a very similar way to

$$(a\hat{\mathbf{x}} + b\hat{\mathbf{y}}) + (c\hat{\mathbf{x}} + d\hat{\mathbf{y}}) = (a + c)\hat{\mathbf{x}} + (b + d)\hat{\mathbf{y}} \ \ , \tag{1.31}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are vectors. Subtraction is the equivalent of adding the negative, so the construction is similarly straightforward.

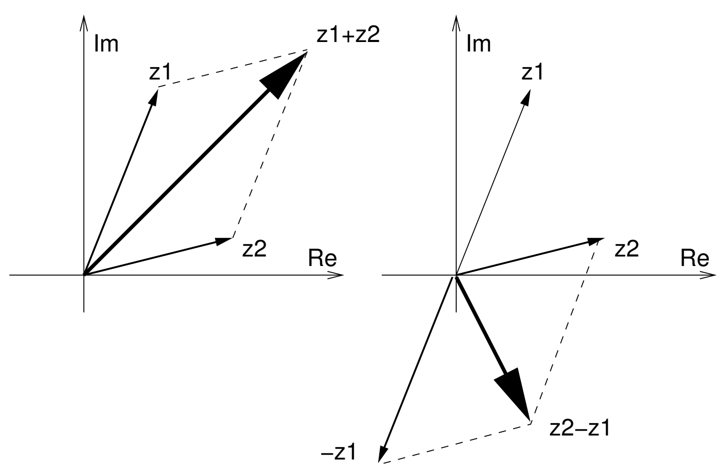

Figure 1.2: Addition and subtraction of two complex numbers.

1.5.2 Multiplication, division

What about multiplication? Here the vector analogy breaks down — it is not like taking scalar or vector products. The example in Figure 1.3 however hints at a simple geometrical interpretation. It appears that the modulus of the complex product is the product of the individual moduli, and the angles from the real axis are added.

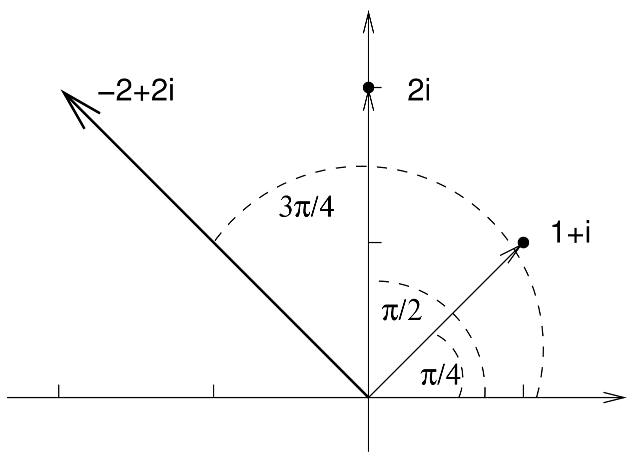

Figure 1.3: Multiplication of two complex numbers, $(1 + i)$ by $2i$

1.6 The Modulus-argument or Polar representation

For the above and several other reasons it is useful to introduce the modulus-argument representation for complex numbers, using in effect the (r, θ) of polar coordinates. From Figure 1.4 it is immediately

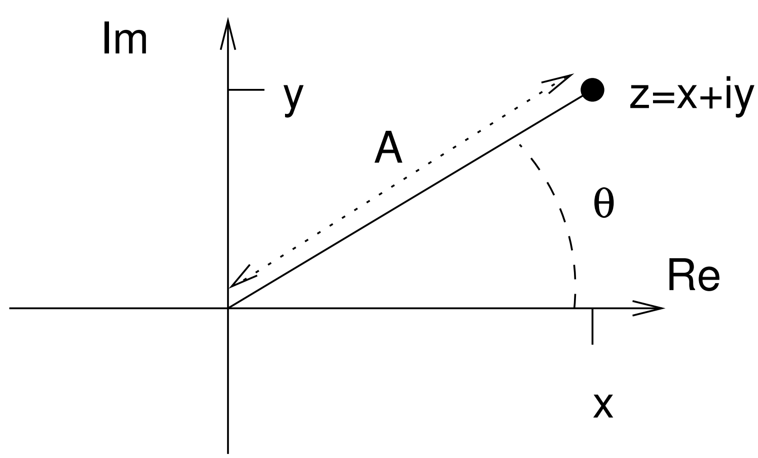

Figure 1.4: Modulus-argument representation using A and θ.

obvious that

$$z = x + iy = A(\cos\theta + i\sin\theta) \tag{1.32}$$

where $A = \mathrm{mod}(z) = \sqrt{x^2 + y^2}$ is the modulus and $\theta = \arg(z) = \tan^{-1}(y/x)$ is the **argument** of the complex number. Note that given x and y there is an ambiguity in the quadrants — between 1 and 3, and between 2 and 4 — in y/x. You need to consider the sign of x or y to resolve this. (Eg if $y/x > 0$ and $y < 0$ the number is in the third quadrant.)

Now try multiplying two numbers $z_1 = A_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = A_2(\cos\theta_2 + i\sin\theta_2)$.

$$\begin{aligned} z_1 z_2 &= A_1(\cos\theta_1 + i\sin\theta_1)A_2(\cos\theta_2 + i\sin\theta_2) \qquad (1.33) \\ &= A_1A_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)] \end{aligned}$$

But use of the well known trigonometric relationships

$$(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) = \cos(\theta_1 + \theta_2) \qquad (1.34)$$

$$(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) = \sin(\theta_1 + \theta_2) \qquad (1.35)$$

gives

$$z_1 z_2 = A_1A_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \ , \qquad (1.36)$$

which confirms what we guessed at earlier — the moduli multiply, the arguments add.
Note the obvious corollary that when we multiply a complex number by $\cos\theta + i\sin\theta$, the resulting number is rotated by angle θ in the complex plane.

1.6.1 De Moivre's Theorem

What we have just seen is a generalization of de Moivre's Theorem, which states

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \ . \qquad (1.37)$$

As we shall see now there is a representation of complex numbers which embodies all these properties and makes manipulation of multiplication, division, powers and roots of complex numbers particularly straightforward.

1.7 The Exponential Representation

We just saw that to multiply two complex numbers we multiply the moduli A and add the arguments θ. Where else do we see addition arising from multiplication? ... Of course, in logarithms and exponentials.

In fact we can write a complex number as

$$z = Ae^{i\theta} \qquad (1.38)$$

where we define

$$e^{i\theta} = \cos\theta + i\sin\theta \ . \qquad (1.39)$$

One justification for this definition is using the series expansions for $\exp$, $\cos$ and $\sin$. Using McLaurin's series

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \dots\right) \\ &= \cos\theta + i\sin\theta \end{aligned} \tag{1.40}$$

(but to get this we have to assume that we can differentiate $e^{i\theta}$ wrt θ to get $ie^{i\theta}$ and so on).

Once we accept the definition, many results follow straightforwardly. For example,

$$z_1 z_2 z_3 \dots = (A_1 A_2 A_3 \dots) e^{i(\theta_1 + \theta_2 + \theta_3 + \dots)} \tag{1.41}$$

$$\frac{1}{z_2} = \frac{1}{A_2} e^{-i\theta_2} \tag{1.42}$$

$$\frac{z_1}{z_2} = \frac{A_1}{A_2} e^{i(\theta_1 - \theta_2)} \tag{1.43}$$

$$\overline{z} = A e^{-i\theta} \tag{1.44}$$

A few points:

- Many, perhaps because they can't "visualize" $e^{i\theta}$, get the urge to convert any complex number to the $(x + iy)$ real and imaginary parts representation. Resist! Accept the mod-arg representation and use it fully!
- Multiplication and division of several complex numbers is usually easier using the $Ae^{i\theta}$ representation. It saves expanding brackets (ie $(a + ib)(c + id)(e + if)\dots$) which requires more work and (yes, even for you!) is very prone to error.
- When taking powers and roots of complex numbers (see below) using the $Ae^{i\theta}$ representation is a must!
- The modulus-argument representation does not buy you anything when adding and subtracting complex numbers.
- You can express arguments in radians (the default) or degrees (if you must).

♣ Examples

1. Express $4 + 3i$ in modulus-argument form.

   Ans: The modulus is $\sqrt{4^2 + 3^2} = 5$ and the argument is $\theta = \tan^{-1}\frac{3}{4}$ in the first quadrant. Thus $\theta = 0.644$. So $4 + 3i = 5e^{i0.644}$.

2. Show using the modulus-argument representation that $z\overline{z}$ is real.

   Ans: Suppose $z = Ae^{i\theta}$. Then $\overline{z} = Ae^{-i\theta}$ and their product is $z\overline{z} = A^2 e^{i(\theta-\theta)} = A^2$, ie entirely real.

1.8 Winding number and the argument's principal value

It is clear that the modulus-argument description of any complex number is not unique. We can wind any number of 2π radians, both positive and negative, onto the argument and the number remains the same. That is

$$e^{i\theta} \quad \text{is identical to} \quad e^{i(\theta \pm k2\pi)}, k = 0, 1, 2, 3, \ldots \quad . \tag{1.45}$$

There is no absolute requirement to "unwind" arguments, but there is a **Principal Value** of the argument such that

$$\pi < \theta \geq \pi \; . \tag{1.46}$$

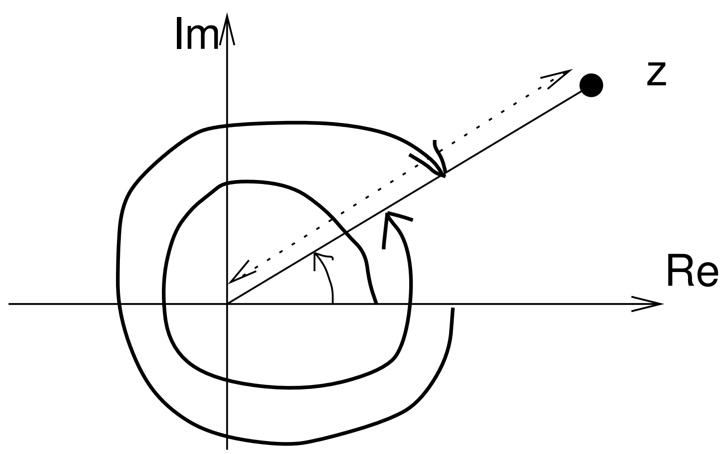

Figure 1.5: Adding or subtracting any number of 2π (radians) onto the argument leaves the complex number unchanged.

1.9 Summary

The important things we have seen are

- The definition of operations on complex numbers
- The geometric representation of numbers and operations on the Argand diagram
- The exponential representation (or modulus-argument representation) and how it is useful for multiplication, division, powers and roots.

In the next lecture we shall see how the exponential representation can be exploited to find powers and roots of complex numbers, and a further raft of elementary functions of complex numbers.

Chapter 2

Elementary functions of Complex Variables

2.1 Integer Powers of complex numbers

Consider taking the integer power of a complex number. Using the modulus argument representation:

$$z^n = (Ae^{i\theta})^n = A^n e^{in\theta} \quad . \tag{2.1}$$

So the argument is multiplied, and the modulus raised to the power.

♣ Examples

1. Evaluate (a) $z_a = (1+i)^2$ and (b)$z_b = (1+i)^{10}$.

   Ans:

   (a) It is quick enough to multiply out the brackets. $z_a = (1 + 2i + i^2) = 2i$. But to use the mod-arg way $(1+i) = \sqrt{2}e^{i\pi/4}$, so that $z_a = 2e^{i\pi/2} = 2i$.

   (b) To multiply out the brackets would be absolute madness! Using $(1+i) = \sqrt{2}e^{i\pi/4}$ we have $z_b = \sqrt{2}^{10} e^{i10\pi/4} = 32e^{i\pi/2} = 32i$, where we have "unwound" 2π from the argument. (Yes, of course we could have found $z_a^5 = (2i)^5 = 32i$ ...)

2. If $z = (1+i)$ find $[(z^4 + 2z^5)/\overline{z}]^2$.

   Ans: Converting to the mod-arg representation:

$$\begin{aligned} z &= \sqrt{2}\exp(i\pi/4) \\ \overline{z} &= \sqrt{2}\exp(-i\pi/4) \\ z^4 &= 4\exp(i\pi) = -4 \quad \text{and} \\ 1 + 2z &= 3 + 2i = \sqrt{13}\exp(i0.381) \end{aligned}$$

   Thus

$$[(z^4 + 2z^5)/\overline{z}]^2 = [z^4(1+2z)/\overline{z}]^2 \quad = \quad \left(\frac{-4\sqrt{13}\exp(i0.381)}{\sqrt{2}\exp(-i\pi/4)}\right)^2$$

$$
\begin{aligned}
&= \frac{(16)(13)}{2}\left(\exp[i(0.381+\pi/4)]\right)^2 \\
&= 104\exp(i2.33)\ .
\end{aligned}
$$

2.2 Integer Roots of complex numbers

Using the arguments just developed for powers, it is tempting when thinking about roots to divide the argument and root the modulus, so that the n-th root would be

$$
(Ae^{i\theta})^{\frac{1}{n}} = A^{\frac{1}{n}} e^{i\frac{\theta}{n}} \ . \tag{2.2}
$$

This solution is flawed, or rather, **incomplete**, because there exist a total of n solutions for the n-th root, when n is an integer. How can we express the other roots?
Recall from the section on winding numbers that

$$
e^{i\theta} = e^{i(\theta \pm k2\pi)}\ , \quad k = 0, 1, 2, 3, \ldots \ . \tag{2.3}
$$

We are trying to solve the equation

$$
z^n = Ae^{i\theta} \tag{2.4}
$$

for z, the n-th root of $Ae^{i\theta}$. Strictly however we should be solving

$$
z^n = Ae^{i(\theta \pm k2\pi)}\ , \quad k = 0, 1, 2, 3, \ldots \ . \tag{2.5}
$$

Our solution should be

$$
z = A^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} \pm \frac{k2\pi}{n}\right)}\ , \quad k = 0, 1, 2, 3, \ldots \ . \tag{2.6}
$$

The important point is that these are NOT now all the same number. The solution arguments start at θ/n, and are separated by units of $2\pi/n$, not 2π.
So why aren't there are an infinite number of roots? Simply because not *all* the different values of k give different solutions. Once we have counted up to $k = n$ the number is

$$
z = A^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + 2\pi\right)} \tag{2.7}
$$

which is identical with the $k = 0$ number.
So, for integer n, there are n distinct n-th roots of a complex number $Ae^{i\theta}$,

$$
z = A^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{k2\pi}{n}\right)}\ , \quad k = 0, 1, 2, 3, \ldots, (n-1) \ . \tag{2.8}
$$

All solutions have argument $A^{\frac{1}{n}}$, so they lie on a circle in the Argand diagram. The first solution has argument θ/n and the remainder are separated in argument by $2\pi/n$.

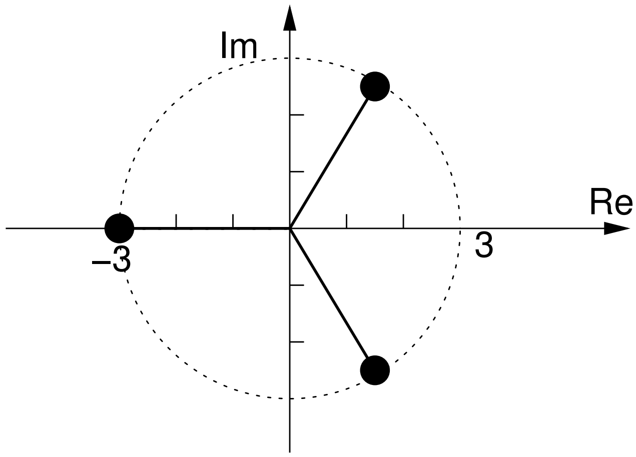

Figure 2.1: The three cube roots of (-27)

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♣ Examples

1. Find the cube root(s) of -27, and sketch them on the Argand diagram.

   Ans: We immediately know that we should find three solutions, and they will be separated in argument by $2\pi/3$. We need to solve for z where $z^3 = -27 = 27e^{i\pi}$. Written out:

$$\begin{aligned} z &= 3e^{i\left(\frac{\pi}{3}+\frac{k2\pi}{3}\right)}\ , k = 0, 1, 2 \\ &= \frac{3}{2}(1 + i\sqrt{3}), \quad -3, \ \ \frac{3}{2}(1 - i\sqrt{3})\ . \end{aligned}$$

   The solutions are shown on the Argand diagram in Figure 2.1. (Fortunately one of them *is* -3!)

2. Find all the values of $(-2)^{\frac{1}{4}}e^{\frac{-i\pi}{4}}$ and sketch them on an Argand diagram.

   Ans: There are 4 fourth roots of (-2).

$$z^4 = -2 = 2e^{i\pi}$$

   so

$$z = 2^{\frac{1}{4}}e^{i(\frac{\pi}{4}+k\frac{\pi}{2})} \quad k = 0, 1, 2, 3$$

   We need to multiply by $e^{\frac{-i\pi}{4}}$ (which rotates the solution clockwise by $\pi/4$) so the required values are

$$2^{\frac{1}{4}}e^{ik\frac{\pi}{2}}, \quad k = 0, 1, 2, 3$$

   as shown in Figure 2.2.

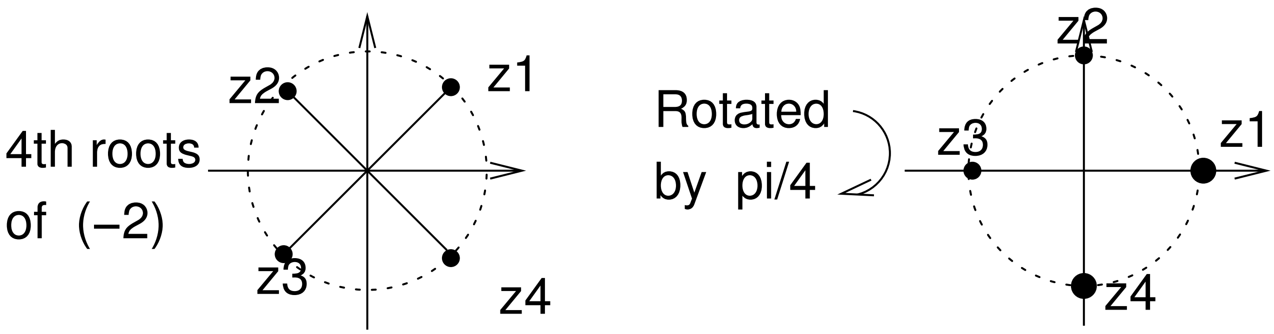

Figure 2.2: Example 2. Note that multiplying by $e^{-i\pi/4}$ rotates anti-clockwise by $-\pi/4$ — ie clockwise by $\pi/4$.

3. Solve $(z+1)^3 = (z-1)^3$.

   Ans: It is very tempting to expand both sides and cancel, creating a quadratic in z. This is hard work. A better way is to write

   $$\left(\frac{z+1}{z-1}\right)^3 \;=\; 1$$

   which requires $z \neq 1$ (it doesn't — see later). Thus

   $$\left(\frac{z+1}{z-1}\right) \;=\; 1^{\frac{1}{3}} e^{i\frac{m2\pi}{3}} \quad m = 0, 1, 2 \;.$$

   or writing $\phi = m2\pi/3$

   $$\left(\frac{z+1}{z-1}\right) \;=\; e^{i\phi} \;.$$

   Rearranging we have the solution for z as

   $$z \;=\; -\frac{1+e^{i\phi}}{1-e^{i\phi}} \;.$$

   We could stop here, but if we multiply top and bottom by $e^{-i\phi/2}$ we get

   $$z \;=\; +\frac{e^{-i\phi/2}+e^{i\phi/2}}{-e^{-i\phi/2}+e^{i\phi/2}} \;.$$

   or

   $$z \;=\; -i\cot\frac{\phi}{2} \;.$$

So the three solutions are

$$z = -i\cot 0\,, \quad -\,i\cot\frac{\pi}{3}\,, \quad -\,i\cot\frac{2\pi}{3} \ .$$

(Note that $-i\cot 0$ is an infinite distance along the negative imaginary axis. Is this a sensible solution to the original equation?)

2.3 Euler formulae

In this last example, the step

$$\frac{e^{-i\phi/2}+e^{i\phi/2}}{-e^{-i\phi/2}+e^{i\phi/2}} = -i\cot\frac{\phi}{2} \ .$$

may have surprised you, but such relationships follow straightforwardly from the exponential representation. If

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.9}$$

then

$$e^{-i\theta} = \cos\theta - i\sin\theta \ . \tag{2.10}$$

Thus

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \quad \text{and} \quad e^{i\theta} - e^{-i\theta} = 2i\sin\theta \tag{2.11}$$

So we obtain the Euler relationships:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \ . \tag{2.12}$$

Dividing the two we have

$$\tan\theta = -i\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}} \quad \text{and} \quad \cot\theta = +i\frac{e^{i\theta}+e^{-i\theta}}{e^{i\theta}-e^{-i\theta}} \ . \tag{2.13}$$

2.4 More advanced functions

So far we have defined operations on complex numbers sufficient to evaluate any algebraic function of the form

$$\frac{a_0z^m + a_1z^{m-1} + \ldots + a_m}{b_0z^n + b_1z^{n-1} + \ldots + b_n}$$

where m and n are integers or rational fractions.
Now we derive some rather more advanced elementary functions of complex numbers, though at this stage we will not explore their properties in depth.

2.5 Trigonometrical functions of complex numbers

Cosine. From the Euler relationships,

$$\cos\theta = (e^{i\theta} + e^{-i\theta})/2 \quad \text{and} \quad \sin\theta = (e^{i\theta} - e^{-i\theta})/2i \ , \tag{2.14}$$

it follows that

$$\cos z = (e^{iz} + e^{-iz})/2 \quad \text{and} \quad \sin z = (e^{i\theta} - e^{-iz})/2i \ . \tag{2.15}$$

But if $e^{i\theta} = \cos\theta + i\sin\theta$, it follows that

$$e^z = e^{(x+iy)} = e^x e^{iy} = e^x(\cos y + i\sin y) \tag{2.16}$$

and further that

$$e^{iz} = e^{(ix-y)} = e^{-y}(\cos x + i\sin x) \ \text{ and} \tag{2.17}$$
$$e^{-iz} = e^{(-ix+y)} = e^{y}(\cos x - i\sin x) \ . \tag{2.18}$$

Hence, putting equations (2.4) and (2.5) into (2.2):-

$$\cos z = \frac{1}{2}\Big(\cos x(e^y + e^{-y}) + i\sin x(e^{-y} - e^y)\Big) \tag{2.19}$$
$$= \cos x\cosh y - i\sin x\sinh y \ , \tag{2.20}$$

using the standard definitions of hyperbolic cosine and sin of a real variable $\cosh y = (e^y{+}e^{-y})/2$, $\sinh y = (e^y - e^{-y})/2$.
A different way to this formula for the cosine of a complex number is as follows.

$$\cos z = \cos(x+iy) = \cos x\cos iy - \sin x\sin iy \ . \tag{2.21}$$

But putting $\theta = iy$ into the Euler formulae (eq 1.56) gives:

$$\cos iy = (e^{-y} + e^y)/2 = \cosh y \tag{2.22}$$
$$\sin iy = (e^{-y} - e^y)/2i = i\sinh y \ . \tag{2.23}$$

So that

$$\cos z = \cos x\cosh y - i\sin x\sinh y \ , \tag{2.24}$$

Sine. The derivation of $\sin z$ follows similarly (see the examples below).

Tangent etc. Other trigonometric functions are defined by extension as

$$\tan z = \frac{\sin z}{\cos z} \ , \qquad \text{cosec}\ z = 1/\sin z \tag{2.25}$$

and so on.
You might suppose these relationships to be unsurprising. But actually it is remarkable that *all* the relationships of analytic trigonometry with real numbers remain valid for complex numbers. One begins to respect the complex plane as a “natural” extension of the real line.

♣ Examples

1. Evaluate $\sin^2 z + \cos^2 z$.

   Ans: From the above discussion, we expect the answer unity ...

   We have the definition of $\cos z$ above, but we need to find $\sin z$.

$$\begin{aligned} \sin z &= \sin(x+iy) = \sin x \cos iy + \cos x \sin iy \\ &= \sin x \cosh y + i \cos x \sinh y \quad . \end{aligned}$$

   Now

$$\cos z = \cos x \cosh y - i \sin x \sinh y \quad ,$$

   so that

$$\begin{aligned} \sin^2 z + \cos^2 z &= \sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y + \\ &\qquad 2i \sin x \cosh y \cos x \sinh y + \\ &\qquad\quad \cos^2 x \cosh^2 y - \sin^2 x \sinh^2 y - \\ &\qquad\qquad 2i \cos x \cosh y \sin x \sinh y \\ &= (\sin^2 x + \cos^2 x)(\cosh^2 y - \sinh^2 y) \\ &= 1 \qquad\qquad QED. \end{aligned}$$

2.6 Hyperbolic functions of complex numbers

By analogy with their definitions in the real domain:

$$\sinh z = \frac{1}{2}(e^z - e^{-z}) \tag{2.26}$$

$$\cosh z = \frac{1}{2}(e^z + e^{-z}) \quad . \tag{2.27}$$

You should check that relationships familiar from real analysis hold here too.

2.7 Logarithm of complex numbers

We want to find the natural log of a complex number:

$$w = u + iv = \ln z \tag{2.28}$$

where $\ln$ denotes a natural logarithm. Writing $z = A \exp i(\theta \pm k2\pi), k = 0, 1, 2, ...$ we get

$$\ln z = u + iv = \ln A + i(\theta \pm k2\pi) \tag{2.29}$$

Note that $\ln A$ can be written

$$\ln A = \ln|z| = \frac{1}{2}\ln(x^2+y^2) \quad . \tag{2.30}$$

Much more important is to note carefully that $\ln$ is a *multi-valued* function, but the $k = 0$ branch (ie $v = \theta$) is called the principal value.

2.8 Curves and regions in the complex plane

When we solve an equation $f(x) = 0$ in the real domain we recover a set of solutions for x, solutions which can be represented as points on the one-dimensional real line. There may be zero solutions (eg $x^2 + 1 = 0$), a single solution (eg $x - 2 = 0$), 2 solutions (eg $x^2 - 4 = 0$), 3 solutions (eg $x^3 - x^2 - 4x + 4 = 0$), 4 solutions and so on, or an infinite set of solutions (eg $\cos x - 0.5 = 0$).
When we solve an equation $f(z) = 0$ we obtain solutions which are represented as points in the two-dimensional complex plane. Again we can have equations which yield no, single or multiple solutions. In general though, equations in the complex domain create more interesting solution sets than those in the real domain, jumping up a dimension so that often real point solutions become complex curve solutions, and real line solutions become complex region solutions. A couple of examples will clarify this. Suppose we solved $|x| - 1.5$ we would arrive at 2 real solutions $x = \pm 1.5$. But in the complex domain there are an *infinite* number of solutions to $|z| = 1.5$. Why? Because it traces out the circle, centred at the origin, with radius 1.5, in the Argand plane. (Notice that in this case, the two real solutions occur where the complex circle intersects the real axis.)
Inequalities which yield a continuum of solutions on the real line might result from regions in the complex plane. Consider, for example, $|x| \leq 1.5$. Solutions on the real line lie in the interval $[-1.5, 1.5]$. Solutions to the complex inequality $|z| \leq 1.5$ however lie anywhere on a disc (centre at origin, radius 1.5) in the complex plane.

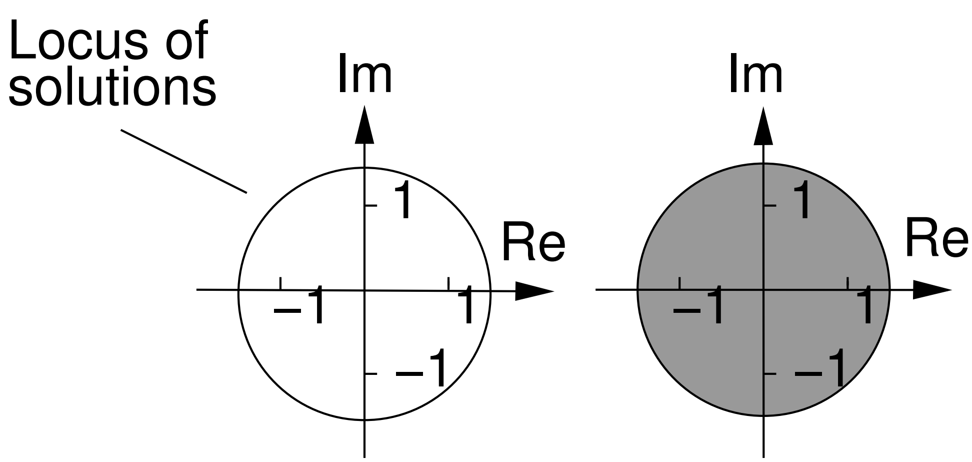

Figure 2.3: Solutions to $|z| = 1.5$ and $|z| \leq 1.5$.

2.9 How to find loci

Interpretation and definition of loci on the complex plane are achieved by various means, and we give a few examples here. In some cases, the geometric interpretation is obvious (as in the circles above) but for the less obvious your aim should be to get equations, possibly parametric, relating y and x.

2.9.1 Lines

We mention two ways of defining a straight line in the complex plane.

1. Suppose we know that the complex point a lies on the line, and it makes an angle θ with the real axis, so its slope is $\tan\theta$. Then

$$\arg(z-a) = \begin{cases} \theta \\ \pi - \theta \end{cases} \tag{2.31}$$

defines the locus of points z on the line.

2. Suppose the required line is the perpendicular bisector of the line joining two points a and b. Then the locus of z is given by

$$|z-a| = |z-b| \ . \tag{2.32}$$

2.9.2 Circles

There are several ways of defining circles. For example

1. The locus of point on a circle of radius R, centre at a is of course

$$|z-a| = R \ . \tag{2.33}$$

2. This method uses Apollonius' Theorem, which say that if a point P moves such that the ratio of its distances from two fixed points A and B is a constant other than unity, then P moves on a circle. If the complex fixed points are a and b then

$$\left|\frac{z-a}{z-b}\right| = k \quad (k \neq 1) \tag{2.34}$$

is the locus of a circle. Notice that in this case the geometrical interpretation in terms of centre and radius not so obvious — see the exercises below for an example.

2.9.3 Other loci

It is of course possible to describe more exotic curves. In general given an relationship involving a complex $z = x + iy$, you should try to obtain a relationship relating x and y which conforms to a well known template — eg $(x/a)^2 + (y/b)^2 = 1$ for an ellipse, and so on.

♣ Examples

1. Trace the locus defined by $|z - 2i| = 2$.

Ans: It may be obvious that this is a circle with centre at $2i$ and of radius 2. To show this formally, let $z = (x + iy)$, then

$$|x + i(y-2)| = 2$$

or

$$x^2 + (y-2)^2 = 4$$

which indeed we recognize as the equation of a circle, radius 2, center (0,2). See Figure 2.4(a).

2. What is the locus of z if

$$\left|\frac{z+i}{z-1}\right| = \sqrt{2}\,? \tag{2.35}$$

Ans: Recall that $|z_1/z_2| = |z_1|/|z_2|$, so

$$\frac{|x+i(y+1)|}{|x+i(y-1)|} = \sqrt{2}$$

Square both sides to obtain

$$\frac{x^2+(y+1)^2}{x^2+(y-1)^2} = 2.$$

Hence

$$(x-2)^2 + (y-1)^2 = 4$$

which is a circle centre $(2,1)$ and radius 2.

3. Trace the locus define by $\text{Im}(z) = [\text{Re}(z)]^2$.

Ans: Writing $z = (x+iy)$, we have

$$y = x^2$$

which is a parabola as shown in the Figure 2.4(b).

4. Trace the locus define by $arg(z+3) = \tan^{-1}2$.

Ans: Writing $z = (x+iy)$,

$$\tan^{-1}\frac{y}{x+3} = \tan^{-1}2$$

so

$$y = 2x + 6$$

giving the locus sketched in Figure 2.4.

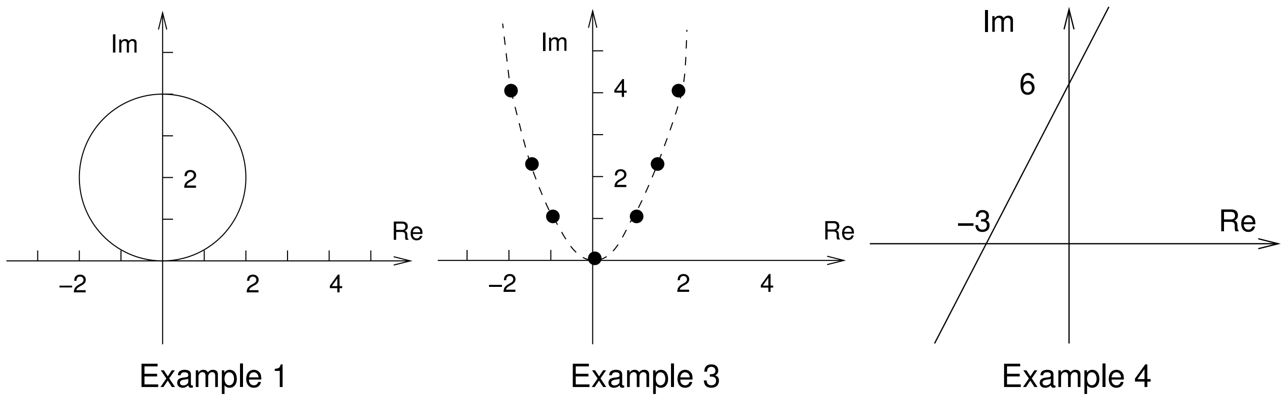

Figure 2.4: Locus for Examples 1, 3 and 4

2.10 Mappings between complex planes: introduction

Finally Suppose the complex numbers $w = u + iv$ and $z = x + iy$ are related by some function f of z:

$$w = f(z) \ . \tag{2.36}$$

When we let the function recipe get to work on z we will, in general, obtain a real part which depends on both x and y and and imaginary part which depends on both x and y. We can then equate these with u and v respectively. So u and v are in general functions of both x and y; ie

$$w = f(z) = u(x, y) + iv(x, y) \ . \tag{2.37}$$

As we consider different values of (x, y), that is, different points on the z-plane, we arrive at different results (u, v) — different points on the w-plane. The function f is said to define a *mapping* from one plane to the other. It turns out that such mappings prove valuable in advanced complex analysis (see lectures 3 and 4) though as yet they may appear somewhat trivial.

♣ Examples

1. How is the point $2 + i$ in the z-plane transformed under the mapping

$$w = z^2 = x^2 - y^2 + i2xy \ . \tag{2.38}$$

Ans: Equating real and imaginary part gives

$$u(x, y) = x^2 - y^2 \tag{2.39}$$
$$v(x, y) = 2xy \ . \tag{2.40}$$

So the point is mapped onto $(2^2 - 1^2) + i(2.2.1) = (3 + 4i)$

The mapping can be sketched using two Argand diagrams. This particular function is single-valued and the mapping is one-to-one because for each z there is unique w. The function $w = \sqrt{z}$ is many-valued and the mapping is one-to-many.

Note that this particular transformation is non-linear, because u and v are non-linear functions of x and y.

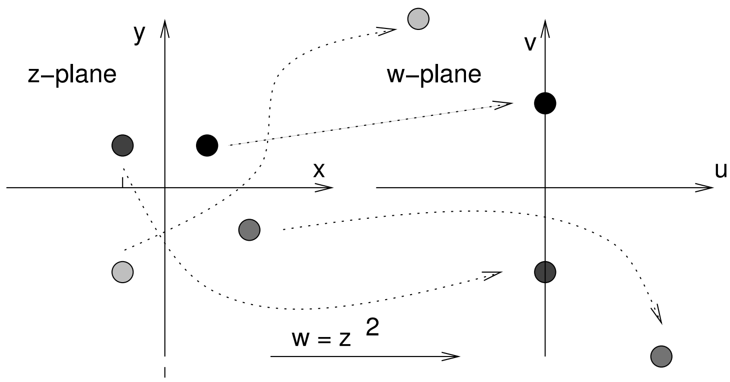

Figure 2.5: A few points mapped from the z to the w plane under $w = z^2$.

2.11 Linear transformations of complex numbers

In multivariable real analysis, a linear transformation from the xy plane to the uv plane has the general form

$$u(x, y) = ax + by + c \quad \text{DEFINITION (A)} \tag{2.41}$$
$$v(x, y) = dx + ey + f \ , \tag{2.42}$$

where a,b,c etc are real constants. Such transformations preserve straight lines — that is a straight line in the z-plane will map to a straight line in w. This is easy to show. Consider the mapping of the line $y = Mx + C$.

$$u(x, y) = ax + b(Mx + C) + c = Px + Q \tag{2.43}$$
$$v(x, y) = dx + e(Mx + C) + f \ = Rx + S, \tag{2.44}$$

so that

$$v = \frac{R}{P}u - \frac{R}{P}Q + S \tag{2.45}$$
$$= Nu + D \tag{2.46}$$

a straight line in w.
Now it is not immediately obvious that we can make such an arbitrary linear transformation as defined at (A) using complex numbers. A straightforward transferal of the notion of a linear relationship from the real domain into the complex domain would suggest that a linear transform be described by

$$w = Z_1 z + Z_2 \quad \text{DEFINITION (B)} \tag{2.47}$$

where Z_1 and Z_2 are complex constants.
Now w needs a real part $u = ax+by+c$. We can fix this up by making $Z_1 = (a-ib)$ and $\text{Re}(Z_2) = c$. Suppose in fact we make $Z_2 = (c + id)$. Then we find

$$w = ax + by + c + i(-bx + ay + d) \ . \tag{2.48}$$

So the real part is fine, but the imaginary part depends on a and b. Can we do better by another choice of Z_1 and Z_2? The short answer is NO! The fact is that according to definition (A) of a linear transformation we need 6 independent constants, but using Z_1 and Z_2 the most we can get is 4.

Definition (B) has a simple interpretation. Any Z_1 can be representated as $A_1 e^{i\theta_1}$, which corresponds to a rotation by θ_1 about the origin, followed by a scaling by factor A_1. Adding Z_2 then performs a translation. So definition (B) permits

- Rotation in the plane about the origin
- Isotropic Scaling (Expansion or contraction)
- Translation in the plane

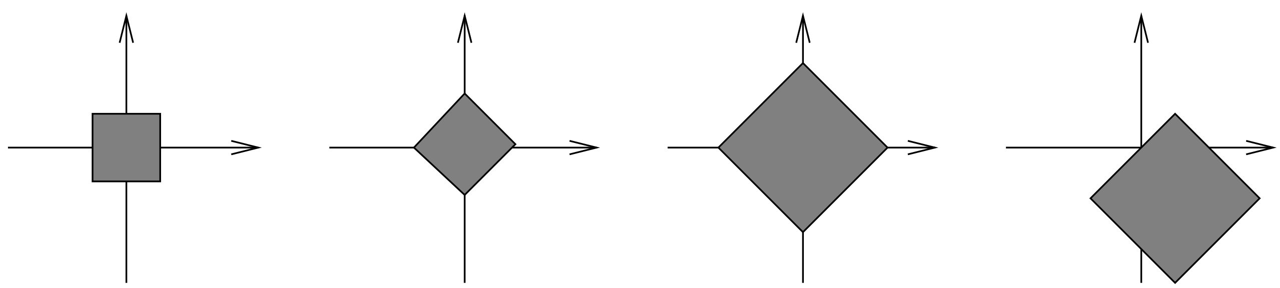

Figure 2.6: Mapping with $w = Z_1 z + Z_2$ allows rotation of z about the origin, isotropic scaling and translation.

What is missing from (B) that is in (A)? The answer lies in Figure 2.7. Using the above three operations could we go from the left-handed mug to the right-handed one? No, we are short of a reflection operation, which is a preserver of straight lines, and so must be a linear operation.

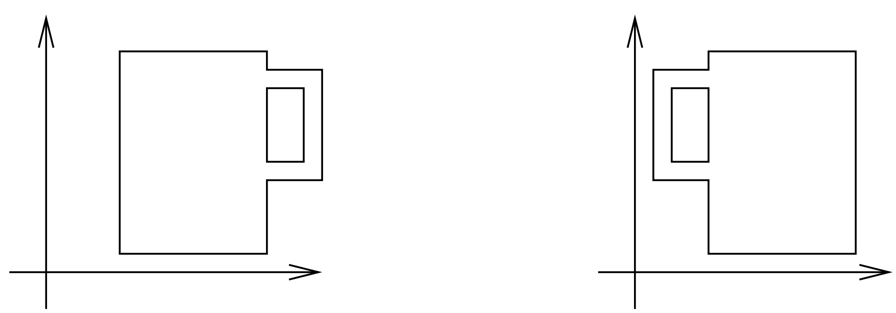

Figure 2.7: Reflection. This operation cannot be achieved using definition (B). We need definition (C).

There is a very simple way of reflecting the complex plane about the real axis, and that is *to take the complex conjugate*. Thus rather than definition (B) we use

$$w = Z_1 z + Z_2 + Z_3 \overline{z} \quad \text{DEFINITION (C)} \tag{2.49}$$

for a linear transformation. Notice that by bringing in Z_3 we have brought the number of independent constants up to 6, and so we can describe the general linear transformation using this recipe.
So definition (C) permits

- Rotation in the plane

- Non-isotropic Scaling
- Translation in the plane
- Reflection

2.11.1 Building a transformation incrementally

A linear transformation followed by a linear transformation creates overall just another linear transformation, so complicated transformations can be built up out of a succession of elementary ones. One can evisage passing through several w frames until the final one is reached:

$$z \to w = z \to w_1 \to w_2 \to \ldots \to w \tag{2.50}$$

The elementary transformations are:

Rotation anticlockwise through an angle ϕ about the origin is effected by multiplication by $e^{i\phi}$.

$$w = e^{i\phi} z \ . \tag{2.51}$$

Expansion and contraction about the origin is achieved by multiplication by a non-negative *real* constant. Obviously numbers less that 1 contract.

$$w = \alpha z \quad (\text{Im}(\alpha) = 0 \ , \text{Re}(\alpha) \geq 0) \ . \tag{2.52}$$

(Note that if α were negative, it is equivalent to a rotation of π followed by a $|\alpha|$ expansion.)

Translation is effected simply by adding a constant to z.

$$w = z + Z_2 \tag{2.53}$$

Reflection about the *real* axis is achieved by taking the complex conjugate.

$$w = \overline{z} \ . \tag{2.54}$$

As we have stated, a general transformation can be constructed from combinations of the above. But a particular transformation could (apparently) be built in different ways — say a rotation followed a translation followed by a reflection instead of a translation followed by a reflection followed by a rotation, or whatever. When you write down the two transformations in "elemental form" they might appear quite different, but if they achieve the same overall transformation they MUST be the same. If you boil the two expressions down to either description (A) or (C), you will find that they are.
For example, the transformation

$$w = [2e^{i\pi}z + (2 + 4i)] \ \ (\text{rotate, thenscale, then translate}) \tag{2.55}$$

is identical to

$$w = 2[e^{i\pi}z + (1 + 2i)] \ \ (\text{rotate, translate, thenscale}). \tag{2.56}$$

Try finding the equivalent descriptions in form (A) and (C). (The following may help you.)
Given a general transformation using description (A) it is easy to find the equivalent form (C). To find Z_1, Z_2 and Z_3, go back to the description (A) and rewrite with $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2$:

$$\begin{aligned} u &= ax + by + c \\ &= a(z + \overline{z})/2 + b(z - \overline{z})/2 + c \qquad (2.57) \\ v &= dx + ey + f \\ &= d(z + \overline{z})/2 + e(z - \overline{z})/2 + f \qquad (2.58) \end{aligned}$$

Thus

$$\begin{aligned} w &= u + iv \qquad (2.59) \\ &= [(a + b)/2 + i(d + e)/2]z + [(a - b)/2 + i(d - e)/2]\overline{z} + [c + if] \ . \qquad (2.60) \end{aligned}$$

So

$$\begin{aligned} Z_1 &= [(a + b)/2 + i(d + e)/2] \qquad (2.61) \\ Z_2 &= [c + if] \qquad (2.62) \\ Z_3 &= [(a - b)/2 + i(d - e)/2] \ . \qquad (2.63) \end{aligned}$$

Note too that you require at least three points to recover a transformation uniquely.

♣ Example

Develop a transformation that can achieve a rotation of $\pi/3$ and an expansion of 2, *both* about the point $z_1 = (1 + i)$.
Ans: The effect we want is sketched in Figure 2.8. But we know only how to rotate and expand about the origin, so first translate the coordinates so that z_1 is at the origin of the w_a-plane.

$$w_a = z - z_1 \ .$$

Now rotate and expand into w_b

$$w_b = 2e^{i\frac{\pi}{3}} w_a \ .$$

Finally translate back into the w-plane

$$w = w_c + z_1 \ .$$

Thus in total:

$$\begin{aligned} w &= 2e^{i\frac{\pi}{3}}(z - z_1) + z_1 \\ &= 2e^{i\frac{\pi}{3}} z + (1 - 2e^{i\frac{\pi}{3}})(1 + i) \\ &= 2e^{i\frac{\pi}{3}} z + [(1 - \sqrt{3}) - i] \ . \end{aligned}$$

So notice again how the sequence of translation, rotation, scaling, then translation is converted into rotation, scaling, then translation.

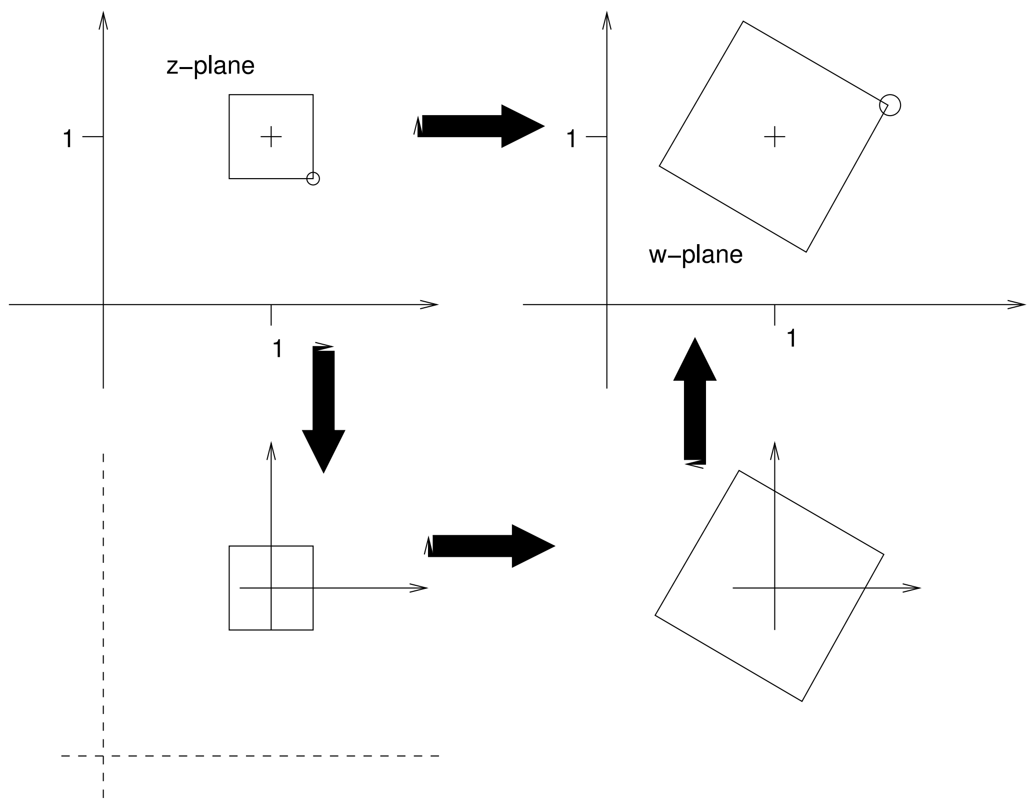

Figure 2.8: The effects of the transformation on a square.

2.12 Non-linear mappings

Earlier we discussed how transformations mapped points, lines and regions from the complex plane z to another complex plane w.
So far we have concentrated on the linear transformation. Now we look further at non-linear transformations.
A few concrete examples of how to sketch the transformations will suggest lines of attack.

The transformation $w = 1/z$.

$$\begin{aligned} w &= u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \\ \Rightarrow u &= \frac{x}{x^2+y^2} \\ v &= \frac{-y}{x^2+y^2} \end{aligned}$$

Now consider have various geometric features map under the transformation

Rectangular mesh in $x - y$.

How do lines of $x = c$ (constant) and $y = d$ (constant) map onto the w-plane?
If $x = c$,

$$\begin{aligned} u &= c/(c^2 + y^2) \\ \text{and } v &= -y/(c^2 + y^2) \\ \Rightarrow u/v &= -c/y \\ \Rightarrow 1 &= \frac{u}{c(u^2 + v^2)} . \\ \text{Thus } u^2 + v^2 &= \frac{1}{c}u \\ \text{so that } \left(u - \frac{1}{2c}\right)^2 + v^2 &= \left(\frac{1}{2c}\right)^2 . \end{aligned}$$

Which is a circle of radius $1/2c$, centred at $u = 1/2c$, $v = 0$. As c gets bigger the $x = c$ line moves further from the imaginary axis in the z-plane, but the circle get smaller with its centre moving along the real u-axis nearer to the origin, so that all circles pass through the origin. Notice too that a line with $c > 0$ gives a circle in the $u > 0$ half plane.
Similar equations hold for the $y = d$ lines. They give rise to circles above and below the real axis, again all going through the origin. Now however, the circle arising from $d > 0$ lies in the $v < 0$ half plane.
Figure 2.9 show the transformation in the w-plane of a rectangular grid in the z-plane. Notice that the mapping appears to preserve angles of intersection: lines meeting a right angles in z lead to circles meeting at right angles.

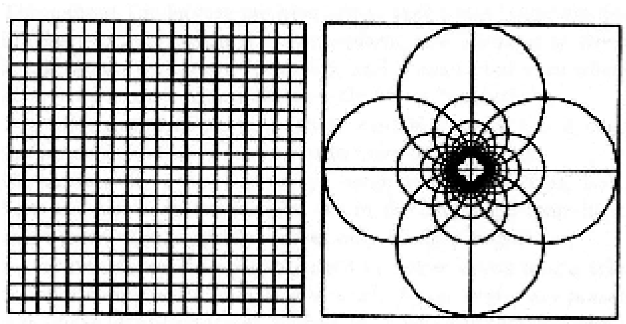

Figure 2.9: Mapping of x constant and y constant grid under $w = 1/z$.

Mapping of radial circles and lines.

One should not only think about the transformation of the x-constant, y-constant mesh. What about circles centred at the origin and radial lines? To look at this geometry it is useful to represent the number in mod-arg form as in

$$w = Be^{i\phi} = f(Ae^{i\theta}) = 1/z \tag{2.64}$$

$$= \frac{1}{A}e^{-i\theta} \tag{2.65}$$

Thus a circle of radius A in the z-plane. This will get mapped to a circle of radius $B = 1/A$ in w-plane. Also a straight line through the origin at angle θ in z gets mapped into a straight line through the origin at angle $-\theta$ in w, as sketched in figure 2.10. However if we pass anticlockwise around a circle in the z-plane, the we move clockwise around the mapping in the w-plane. Note again that angles are preserved.

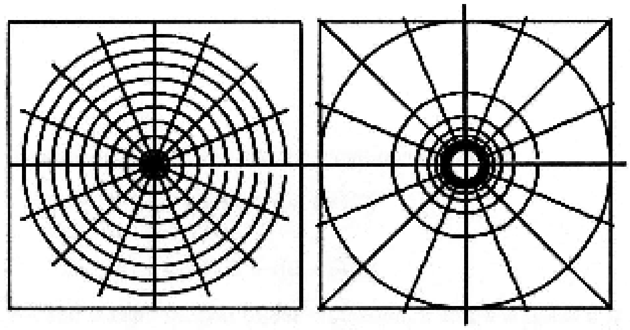

Figure 2.10: Mapping of circles and radial lines under $w = 1/z$.

The transformation $w = z^2$.

$$
\begin{aligned}
w &= u + iv = (x+iy)^2 \\
\Rightarrow u &= x^2 - y^2 \\
v &= 2xy
\end{aligned}
$$

Now consider how lines of $x = c$ (constant) and $y = d$ (constant) map onto the w-plane. If $x = c$,

$$
\begin{aligned}
u &= c^2 - y^2 \\
\text{and } v &= 2cy \\
\Rightarrow u &= c^2 - v^2/4c^2
\end{aligned}
$$

which is a parabola, symmetric about the $u-$axis.
Similarly for $y = d$:

$$
u = v^2/4d^2 - d^2
$$

which are reflections of the $x = c$ parabola about the imaginary v-axis. Figure 2.11 show the transformation in the w-plane of a rectangular grid in the z-plane. Again, it looks as though angles of intersection under this mapping are preserved.

2.13 Angle preserving properties

Throughout the lecture we have noted that some mappings preserve angles, so that lines that intersect at some angle in in the z-plane, also intersect at the same angle in the w-plane. This is called the “conformal” property, and is connected with whether or not the complex function involved possesses a derivative — that is is “analytic”.

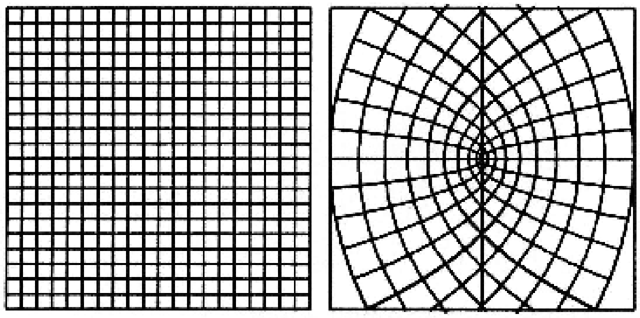

Figure 2.11: Mapping of x constant and y constant grid under $w = z^2$.

You will meet this idea properly in the 2MA series, but it is worth examining briefly which of the functions we have seen seem to have this property.
Certainly $w = z^2$ and $w = 1/z$ seem to be conformal, but so far we have looked only at lines and circles intersecting at $90°$ in the conformal mappings. You may be thinking that the angle-preserving property applies only to right angles.
So for something different, Figure 2.12 below shows how a triangle with angles of $60°$ and $30°$ transforms under the mapping $w = z^2$. Again angles are preserved!

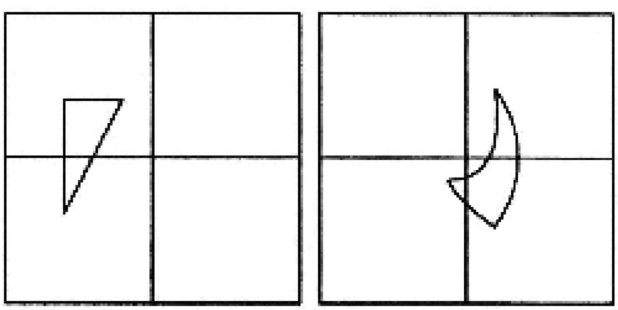

Figure 2.12: Mapping a triangle under $w = z^2$ — not just right angles are preserved!

Are linear transformations conformal? Certainly rotation and translation and isotropic expansion all preserve angles, and so a linear transformation of the form $w = Z_1 z + Z_2$ *appears* conformal (and in fact it is).
What about the general form of the linear transformation? Although perhaps not immediately obvious, the $Z_3\bar{z}$ term together with z term in description (C) of a linear transformation seen earlier allows different "non-isotropic" expansions in the x and y directions. For example, suppose we wish to make

$$u = 2x \tag{2.66}$$
$$v = 3y \; . \tag{2.67}$$

To find Z_1, Z_2 and Z_3, use the equations in Lecture 2:

$$Z_1 = [(a+b)/2 + i(d+e)/2] \tag{2.68}$$
$$Z_2 = [c + if] \tag{2.69}$$
$$Z_3 = [(a-b)/2 + i(d-e)/2] \; . \tag{2.70}$$

For the transformation under discussion, $a = 2, b = 0, c = 0, d = 0, e = 3, f = 0$, an we find

$$Z_1 = [1 + i3/2] \quad (2.71)$$
$$Z_2 = [0] \quad (2.72)$$
$$Z_3 = [1 - i3/2] \ . \quad (2.73)$$

Figure 2.13 suggests that a linear transformation not involving $\overline{z}$ preserves the angle between lines (it is possible to prove this!), and the figure shows that a linear transformation involving $\overline{z}$ is not conformal.

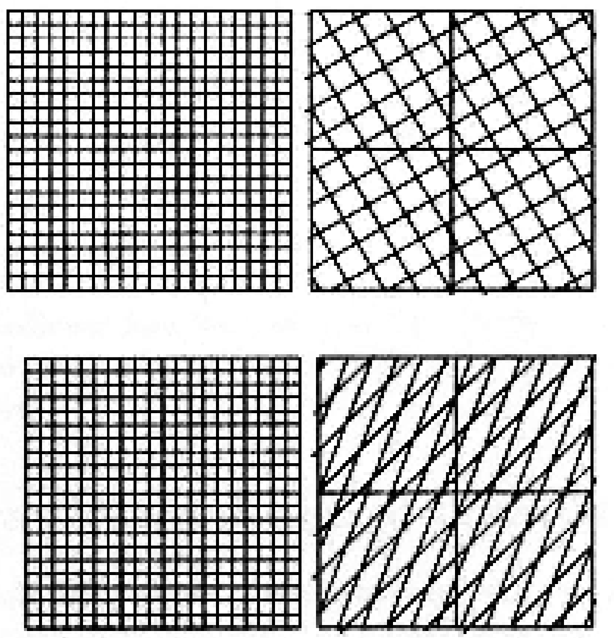

Figure 2.13: The linear transformation $w = Z_1 z + Z_3$ is conformal but the general linear transformation involving complex conjugates $w = Z_1 z + Z_3 \overline{z} + Z_2$ (eg $(1 + 2i)z + (2 + i)\overline{z} + 3 + i$) is not.

2.14 Summary

This lecture has covered the derivation of a number of functions of complex variables. Throughout you will have seen how the complex definitions are such as to preserve many of the properties of the real functions, reinforcing the idea that the complex domain is a very natural extension of the real domain.
Finally we discussed how curves and regions in the complex can be defined as solutions to complex equations.

Chapter 3

Time varying complex numbers and Phasors

Complex algebra is widely used in the analysis of linear systems. This lecture introduces the ideas.

The algebra is simple, but the engineering systems used as an example — ac circuits — might not be wholly familiar to you. You should not be too concerned if it all doesn't quite fall into place at this stage as you will see all the material again.

This lecture aims merely to link the complex algebra you now know with that which later engineering lectures might assume.

3.1 Oscillatory complex numbers

We have seen that the exponential representation allows any complex number to be represented as $Ae^{i\theta}$.

Consider now a complex number z_1 which is a function of time t

$$z_1(t) = Ae^{i\omega t} \quad , \tag{3.1}$$

where ω is some positive real number.

At $t = 0$, $z_1(t)$ is pure real, but as t increases, $z(t)$ traces out a circle about the origin with radius A in the Argand plane. When $t = 2\pi/\omega$, $z(t)$ is as it was at $t = 0$. The angular frequency of rotation is ω, which is equivalent to frequency $f = \omega/2\pi$ and period $T = 2\pi/\omega$, as shown in Figure 3.1.

If we plot the real part of $z(t)$ as a function of t we have

$$\text{Re}(z_1(t)) = A\cos\omega t \quad , \tag{3.2}$$

a quantity which varies cosinusoidally with time.

Now consider a second time varying complex number

$$z_2(t) = Be^{i(\omega t+\phi)} \quad , \tag{3.3}$$

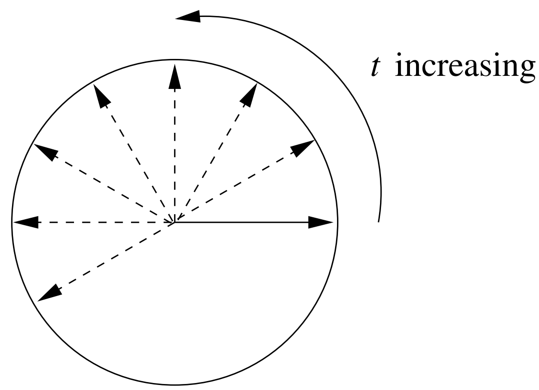

Figure 3.1: $Ae^{i\omega t}$ rotates with angular frequency ω in the complex plane.

where ϕ is a real number which is constant. We have already seen that $z_1(t) = Ae^{i\omega t}$ rotates around as ωt increases, but what does $z_2(t) = Be^{i(\omega t+\phi)}$ look like? Writing it as

$$\begin{aligned} z_2(t) &= Be^{i(\omega t+\phi)} \\ &= Be^{i\omega t}e^{i\phi} \end{aligned}$$

shows clearly that z_2 also rotates around at the angular frequency ω, but is always *rotated* by the fixed angle ϕ forward of z_1.

The real part of this number is

$$\mathrm{Re}(z_2(t)) = B\cos(\omega t + \phi) \quad , \tag{3.4}$$

again a cosinusoidal variation with angular frequency ω. Now however the amplitude is B, but more interestingly leading the first by a **phase angle** ϕ. A sketch of the real parts of the two numbers is given in Figure 3.2 When the phase angle is positive, the second "leads" the first, when negative the second "lags" — alternative descriptions "advanced" and "retarded" in phase.

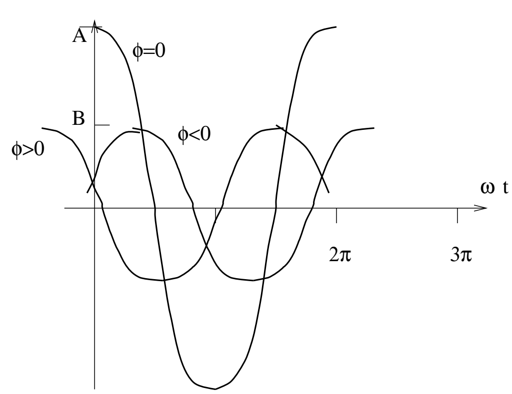

Figure 3.2: A sketch of $A\cos\omega t$ and $B\cos(\omega t + \phi)$ for ϕ positive and negative.

3.2 Phasor diagrams

As you shall see, such time-varying complex numbers occur very frequently as the solutions to the differential equations describing physical systems, and it is important therefore to have a good way of representing them diagrammatically.

Returning to z_1 and z_2, given that they both rotate with time at the *same* angular frequency, the important *differences* in magnitude and phase angle ϕ can be captured on a static diagram by choosing a particular value of time at which to "freeze" the quantities. Such frozen quantities are call **phasors**, and the diagram called a **phasor diagram**.

The exact choice of time doesn't matter, but it is often convenient to choose a time which makes one of the frozen phasors lie along the real axis — this defines the **reference phase**.

For example, if we choose $t = 0$, then z_1 becomes the reference phase, as in Figure 4.3a. But the choice is up to you: choosing another time (what is it?) will make z_2 the reference phase, as in Figure 3.3. The two diagrams are entirely equivalent: the important thing is the phases differences and relative amplitudes.

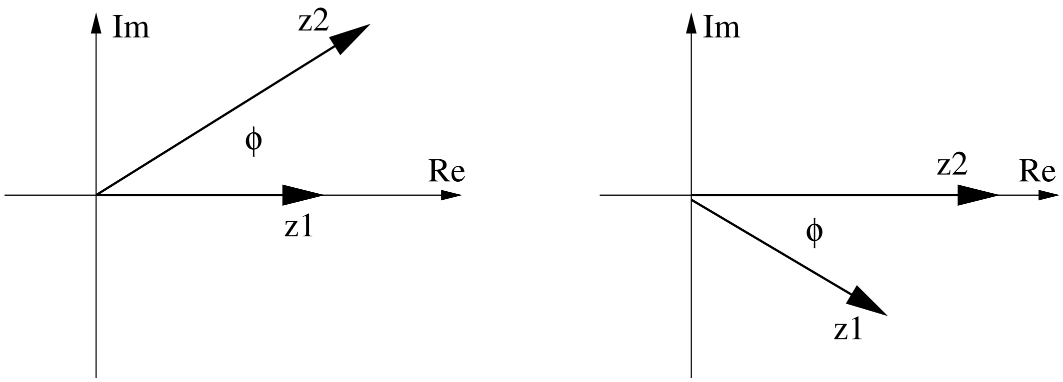

Figure 3.3: Phasor Diagrams using different reference phases

3.3 Linear response

Using complex numbers to represent both magnitude and phase simultaneously provides a convenient and powerful shorthand when describing linear systems.

One of the properties of a linear system is that if you excite the system by shaking it at a particular frequency, the response of the system has the SAME frequency, but may, and probably will, have a DIFFERENT amplitude and DIFFERENT phase. The amplitude and phase are likely to be functions of frequency.

By encoding the amplitude and phases of excitation and response at a particular frequency ω as the modulus and argument of a complex number, we can describe the effect of the system by a complex number called the **transfer function**. In the example in the figure it is obvious that $B = GA$ and

$\phi = \gamma + \theta$. The transfer function is $Ge^{i\gamma}$. Typically, both G and γ will be functions of the frequency ω, and determining how they vary with frequency is an important part of analyzing a system. For example, if $G \approx 1$ when ω is small but $G \approx 0$ when ω is high we would describe the system as a low pass filter — it might be filtering out tape hiss, or high frequency road vibrations.

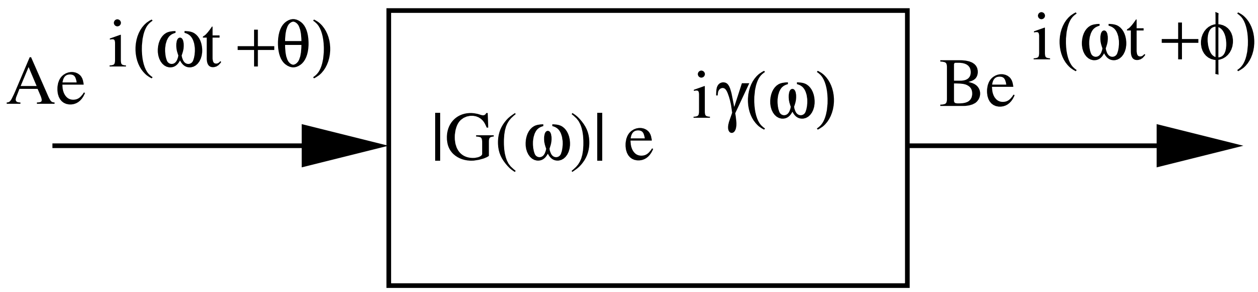

Figure 3.4: The response of a system to an excitation is determined by the transfer function.

3.3.1 Example of a linear system

If you aren't convinced yet, let's grind through the equations. Let's pick a simple second order linear system, such as the spring/damper system you coded up in the C3 lab session. There you will remember that we arrived at an equation

$$m\ddot{y} + \nu\dot{y} + ky = kd(t).$$

or

$$\ddot{y} + A\dot{y} + By = Bd(t)$$

where $d(t)$ was the commanded or desired position of the disk-drive head, and $y(t)$ was the actual position. In the laboratory we applied a 0-1 step as $d(t)$, but here we will applied a co-sinusoidal exciation, using the complex form. That is

$$d(t) = e^{i\omega t}$$

Now assume that the response y has the same frequency, but different amplitude and phase:

$$y(t) = ae^{i(\omega t+\phi)} = ae^{i\phi}e^{i\omega t} ,$$

so

$$\dot{y} = i\omega ae^{i\phi}e^{i\omega t} ,$$

and

$$\ddot{y} = -\omega^2 a e^{i\phi} e^{i\omega t} \ .$$

Putting these into the differential equation we have

$$-\omega^2 a e^{i\phi} e^{i\omega t} + Ai\omega a e^{i\phi} e^{i\omega t} + Ba e^{i\phi} e^{i\omega t} = Be^{i\omega t} \ .$$

Notice that our guess for the form $y(t)$ was valid because the temporal function $e^{i\omega t}$ is the same both sides of the equation. Dividing through by $e^{i\omega t}$ and taking out factors we have

$$ae^{i\phi}(-\omega^2 + B + iA\omega) = B$$

so that

$$ae^{i\phi} = \frac{B}{(B - \omega^2 + iA\omega)} \ .$$

Converting the RHS to mod-arg form we have

$$ae^{i\phi} = \frac{B}{\sqrt{(B-\omega^2)^2 + A^2\omega^2}} e^{i\tan^{-1}\left(\frac{-A\omega}{B-\omega^2}\right)} \ .$$

We can then equate LH modulus with RH modulus, and LH argument with RH argument.

As the C3 lab notes mention, and as your upcoming lectures on differential equations will show, one can write $B = \omega_0^2$ and $A = 2\zeta\omega_0$, where ω_0 is the natural frequency and ζ the damping parameter, so that

$$a = \frac{\omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2}$$

and

$$\tan\phi = -\frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega^2} \ .$$

The interpretation of these results is covered in the differential equations course. The important point to grasp here is that using the $e^{i\omega t}$ solution has allowed you to handle magnitude and phase together.

Is there an alternative? Yes. You write the excitation $d(t) = \cos\omega t$ and the response as $y(t) = \alpha\cos\omega t + \beta\sin\omega t$. In the tutorial sheet you are asked to derive a and ϕ from α and β.

3.4 AC circuit analysis

As a further example, let us evaluate the transfer function of a linear a.c. circuit. Because of the possible confusion of i with current, it is usual in engineering applications to use j instead of i for the imaginary unit.

3.4.1 The complex impedance of an inductor

First, consider an alternating voltage source connected across an inductor.

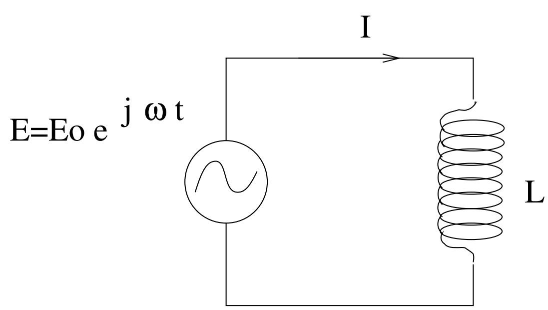

Figure 3.5: A generator E supplying a pure inductance L

First we will write the EMF of the source in complex form as:

$$E = E_0 e^{j\omega t} \quad . \tag{3.5}$$

Now let us evaluate the current I flowing in the inductor:

Method 1 — usings A-level facts

If you have "done" a.c. circuit at A-level, you will probably recall that:

- in an inductor the current "lags" the voltage by $90°$
- the inductor's impedance is ωL, so that the current magnitude is $I_0 = E_0/\omega L$.

So we expect the current in the inductor to be

$$\begin{aligned} I &= I_0 e^{j(\omega t - \pi/2)} && (3.6) \\ &= \frac{E_0}{\omega L} e^{j(\omega t - \pi/2)} && (3.7) \\ &= \left[\frac{e^{-j\pi/2}}{\omega L}\right] \left[E_0 e^{i\omega t}\right] && (3.8) \end{aligned}$$

$$= \frac{1}{j\omega L}\left[E_0 e^{i\omega t}\right] \qquad (3.9)$$

$$= \frac{1}{j\omega L} E \ . \qquad (3.10)$$

Method 2 — from first principles

First, write down the differential equation describing the system:

$$E - L\frac{dI}{dt} = 0 \ .$$

so that

$$I = \frac{\int E dt}{L}$$

But $E = E_0 e^{j\omega t}$ so that, noting that j and ω are constants, on integrating

$$I = \frac{1}{j\omega L} E_0 e^{j\omega t} = \frac{1}{j\omega L} E$$

as before.

The complex impedance of a the inductor is found by dividing the voltage by the current:

$$Z_L = \frac{E}{I} \qquad (3.11)$$

which from above is

$$Z_L = j\omega L \ . \qquad (3.12)$$

Thus the impedance is pure imaginary, independent of time, but dependent on frequency.

To summarize so far then, if we write the voltage across an inductor as a complex quantity, to get the complex current through the inductor we simply divide the complex voltage by $j\omega L$. We could plot this as a phasor diagram.

3.4.2 The complex impedance of a resistor

The relationship between complex voltage and current in a resistor is always given by $V = IR$. There is no phase difference between current and voltage, and the complex impedance is just $Z_R = R$ and purely real.

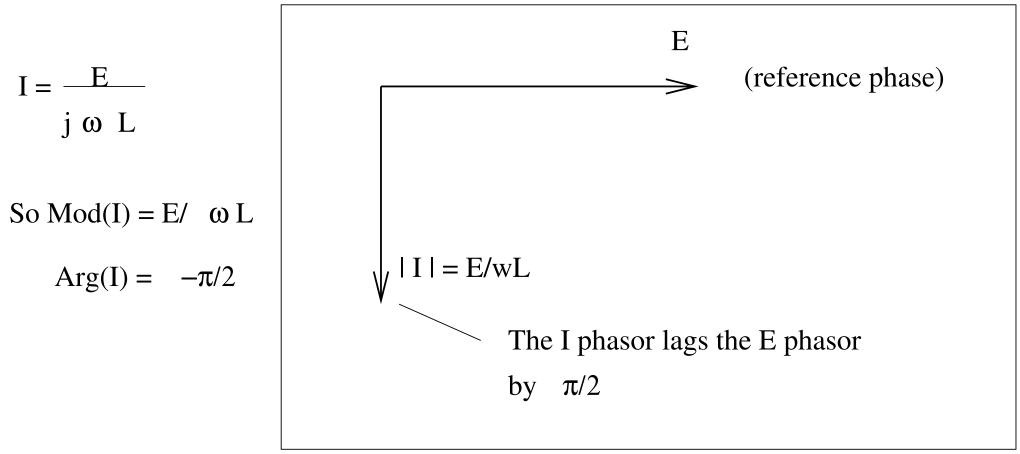

Figure 3.6: Phasor diagram

3.4.3 The complex impedance of a capacitor

If now we stick a capacitor across the source with complex EMF $E = E_0 e^{i\omega t}$, the charge in the capacitor is

$$Q = CE = CE_0 e^{j\omega t}$$

The current is just

$$\begin{aligned} I &= \frac{dQ}{dt} \\ &= Cj\omega E_0 e^{j\omega t} \\ &= j\omega CE \ . \end{aligned}$$

Evidently, the complex impedance of a capacitor is just

$$Z_C = 1/j\omega C \ .$$

Recall from A-level that the current in a capacitor leads the voltage by $\pi/2$. As $j\omega C = \omega C e^{j\pi/2}$, the complex solution will provide this phase shift.

Again we could plot this as a phasor diagram:

3.4.4 Voltage and current in more complicated circuits

Handling the voltages and currents in a single capacitor or single inductor was never difficult, but dealing with more complicated circuits using real solutions to differential equations is usually prohibitively messy. The messiness arises because to keep track of magnitude, phase and time, you have to carry around solutions of the form $A \cos \omega t + B \sin \omega t$.

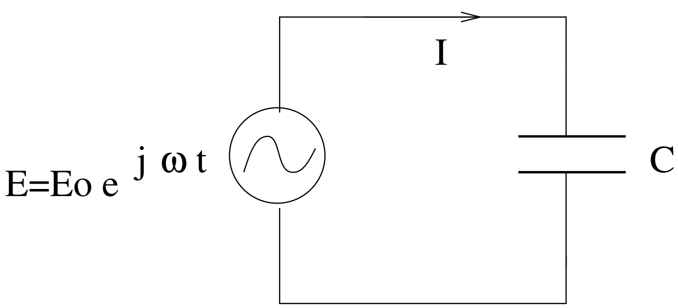

Figure 3.7: A generator E supplying a pure capacitance C

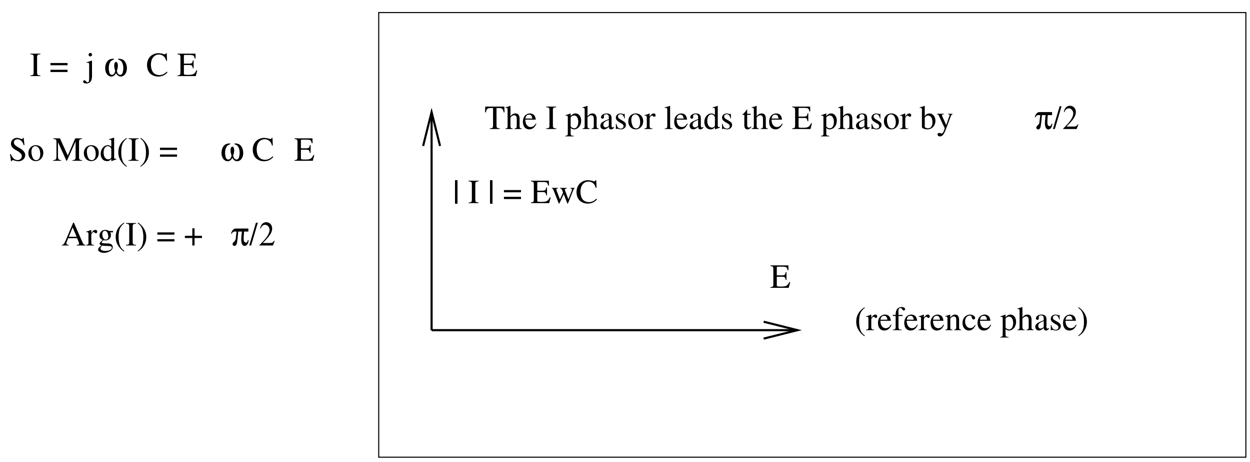

Figure 3.8: Phasor diagram

By introducing complex solutions, two advantages are apparent:

1. The temporal (time) part of the solution is always $e^{j\omega t}$ and can be forgotten about during calculation of the magnitude and phase.
2. The magnitude and phase are both described by a single complex number, and so Ohm's law can be re-introduced in complex form as $V = IZ$, where all the quantities are (potentially) complex.

The latter point is highly significant, because it allows us to work out the complex impedance of a circuit using the formulae for resistors in serial and parallel.

3.4.5 Example 1

In this example we look for the transfer function V_C/V of the following circuit:

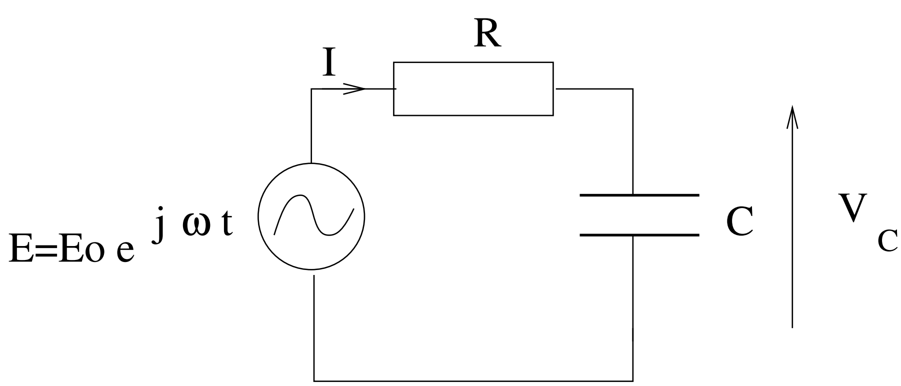

Figure 3.9: A simple R-C circuit

The total impedance of the series circuit is

$$
\begin{aligned}
Z_T &= Z_R + Z_C \\
&= R + 1/j\omega C
\end{aligned}
$$

The complex current flowing is therefore:

$$
\begin{aligned}
I &= \frac{E}{Z_T} \\
&= \frac{E}{R + 1/j\omega C}
\end{aligned}
$$

The voltage across the capacitor is

$$\begin{aligned}
V_C &= IZ_C \\
&= \frac{E}{R + 1/j\omega C}\frac{1}{j\omega C} \\
&= \frac{E}{j\omega CR + 1} \\
&= \frac{E}{\omega^2 C^2 R^2 + 1}(-j\omega CR + 1) \\
&= \frac{E}{\omega^2 C^2 R^2 + 1}\sqrt{(\omega^2 C^2 R^2 + 1)}e^{j\tan^{-1}(-\omega CR)} \\
&= \frac{E}{\sqrt{\omega^2 C^2 R^2 + 1}}e^{j\tan^{-1}(-\omega CR)} \\
&= \frac{E_0}{\sqrt{\omega^2 C^2 R^2 + 1}}e^{j\tan^{-1}(-\omega CR)}e^{j\omega t}
\end{aligned}$$

This says that the voltage across the capacitor has a magnitude of

$$|V_C| = \frac{E_0}{\sqrt{\omega^2 C^2 R^2 + 1}}$$

and is out of phase with the applied voltage by

$$\phi = \tan^{-1}(-\omega CR)$$

Note that both these quantities depend on the frequency ω. When ω is very small (ie, nearly d.c.) the magnitude is E_0, and the phase difference is 0. When ω is very large, the magnitude is zero, and the phase shift $-\pi/2$. Notice that the magnitude is reduced to 1/sqrt2 of its $\omega = 0$ value when $\omega = 1/RC$. This system acts as a *low pass filter* — the simplest version of the type of system that would eliminate crackle from a telephone line, or be used to drive bass speakers in an audio system. For example, if you wished to attenuate above $f \approx 4$kHz (or $w \approx 2.5 \times 10^4 \mathrm{rad}.s^{-1}$ you might choose $R = 1\mathrm{k}\Omega$ and $C = .04\mu$F.

In your electricity lectures you will learn about methods of plotting these quantities in both polar plots and Bode plots.

3.4.6 Example 2

We run through this more complicated example to convince you that, despite the mess, the basic idea of obtaining magnitude and phase remains the same. It might even persuade you that using complex impedances makes AC circuit analysis straightforward.

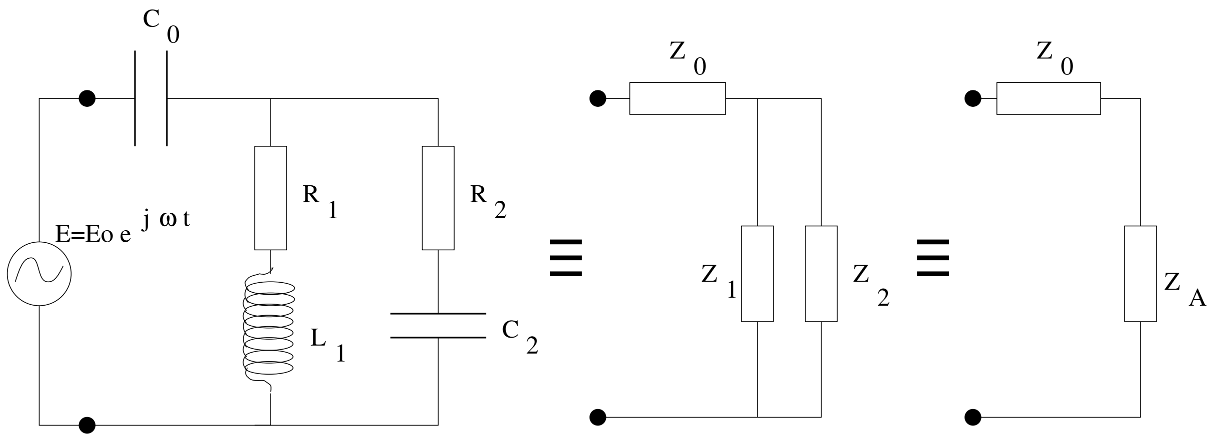

Figure 3.10: An L-C-R circuit

Suppose we impress a voltage $V = V_0 e^{j\omega t}$ on the circuit shown. What is the current I?

Writing down the total impedance

Because we can use the usual series and parallel formulae, it is possible to write:

$$\begin{aligned} Z_{Total} &= Z_0 + Z_A \\ &= Z_0 + \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)^{-1} \\ &= Z_0 + \frac{Z_1 Z_2}{Z_1 + Z_2} \end{aligned}$$

Now we need to work out what $Z_{0,1,2}$ are. From the expressions for impedance:

$$\begin{aligned} Z_0 &= \frac{1}{j\omega C_0} \\ Z_1 &= R_1 + j\omega L_1 \\ Z_2 &= R_2 + \frac{1}{j\omega C_2} \end{aligned}$$

Now simply slot in the expressions and grind ...

$$\begin{aligned} Z_{Total} &= \frac{1}{j\omega C_0} + \frac{(R_1 + j\omega L_1)(R_2 + 1/j\omega C_2)}{R_1 + j\omega L_1 + R_2 + 1/j\omega C_2} \\ &= \frac{-j}{\omega C_0} + \frac{R_1 R_2 + L_1/C_2 + j\left(\omega L_1 R_2 - R - 1/\omega C_2\right)}{(R_1 + R_2) + j(\omega L_1 - 1/\omega C_2)} \end{aligned}$$

Now multiply top and bottom by the complex conjugate of the bottom:

$$\begin{aligned} Z_{Total} &= \frac{-j}{\omega C_0} + \frac{[R_1R_2 + \frac{L_1}{C_2} + j\left(\omega L_1 R_2 - \frac{R_1}{\omega C_2}\right)][(R_1+R_2) - j(\omega L_1 - \frac{1}{\omega C_2})]}{(R_1+R_2)^2 + (\omega L_1 - \frac{1}{\omega C_2})^2} \\ &= a + jb \end{aligned}$$

where the real part is

$$a = \frac{\left(R_1R_2 + \frac{L_1}{C_2}\right)(R_1+R_2) + \left(\omega L_1 R_2 - \frac{R_1}{\omega C_2}\right)\left(\omega L_1 - \frac{1}{\omega C_2}\right)}{(R_1+R_2)^2 + \left(\omega L_1 - \frac{1}{\omega C_2}\right)^2}$$

and the imaginary part is

$$b = \frac{-1}{\omega C_0} + \frac{\left(\omega L_1 R_2 - \frac{R_1}{\omega C_2}\right)(R_1+R_2) - \left(R_1R_2 + \frac{L_1}{C_2}\right)\left(\omega L_1 - \frac{1}{\omega C_2}\right)}{(R_1+R_2)^2 + \left(\omega L_1 - \frac{1}{\omega C_2}\right)^2}$$

Now the current is $I = I_0 e^{j\omega t}$ where

$$\begin{aligned} I_0 &= V_0 \frac{1}{a+jb} \\ &= \frac{V_0}{a^2+b^2}(a - jb) \end{aligned}$$

Now turn this into modulus argument form:

$$\begin{aligned} I_0 &= \frac{V_0}{a^2+b^2}(a - jb) \\ &= \frac{V_0}{a^2+b^2}\sqrt{a^2+b^2}e^{j\tan^{-1}(-b/a)} \\ &= \frac{V_0}{\sqrt{a^2+b^2}}e^{j\tan^{-1}(-b/a)} \end{aligned}$$

This looks a bit messy, but all it is saying is that the magnitude of the current is

$$|I| = \frac{V_0}{\sqrt{a^2+b^2}},$$

and it out of phase with the reference phase, ie the source voltage in our case, by an angle

$$\phi = \tan^{-1}(-b/a) \ .$$

NB! that both the modulus and the phase difference are functions of frequency ω.

3.5 Summary

This lecture has explored the definition and use of phasors to describe complex numbers whose magnitude is constant but which rotate in the complex plane over time. You will meet them again soon in the analysis of ac circuits. We also introduced the idea of the complex transfer function of a system, and you will meet this again soon in the lectures on differential equations.

It is all too easy when studying systems from the engineering standpoint, for example ac circuits, to miss the link to the mathematics course, and to regard the techniques used, for example Bode plots, as witchcraft. The aim of this lecture has been to urge you to forge those links.

Chapter 4

Time varying complex numbers and Waves

Sound waves, ocean waves, Mexican waves, radio waves ... The concept of a wave is commonly used, but how well do we understand what comprises a wave?

In all our cases, we seem to have a travelling disturbance, although later we shall see standing waves. Usually we have medium — air, water, a crowd — but radio waves propagate in free space. Perhaps, as physicists just over 100 years ago wondered, there is an intangible “luminiferous ether”? A famous experiment carried out by Michelson and Morley in 1887 showed there was not. The Mexican wave shows that the medium does not have to move with the wave — spectators move up and down, but the wave moves around the stadium. Indeed the Mexican wave is an example of a transverse wave, where the wave motion is perpendicular to the disturbance. But we can equally well have longitudinal waves, where the disturbance is along the direction of the wave motion.

4.1 Mathematical description of wave motion

We will consider transverse waves because they are easier to visualize, but the mathematics of longitudinal waves is identical.

Suppose at time $t = 0$ we took a snapsot of our Mexican wave in the crowd. We would some height profile $y = g(x)$, with a peak at $x = x_p$, as shown in Figure 4.1. Travelling at velocity c, a time t later the disturbance would have reach $x_p + ct$. If we asked at any particular x what the height y was, we would answer that it was identical with $g(x)$ shifted back by ct. That is, the description of the travelling disturbance is just

$$y = g(x - ct) \ .$$

Notice that we have not had to say anything specific about the shape of g. For example in the Mexican wave, the crowd might rise quickly and fall slowly giving the shape in Figure 4.2(i) or vice versa as in 4.2(ii).

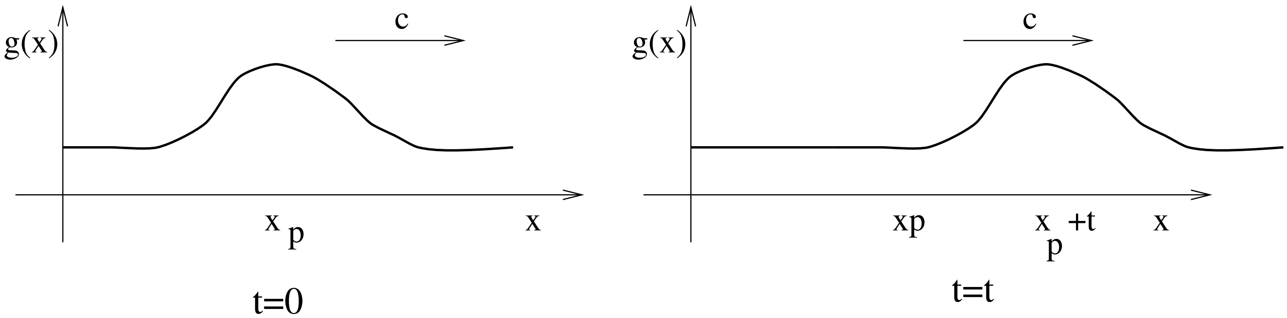

Figure 4.1: The profile at time 0 and time t. The disturbance has moved by ct where c is the wave velocity.

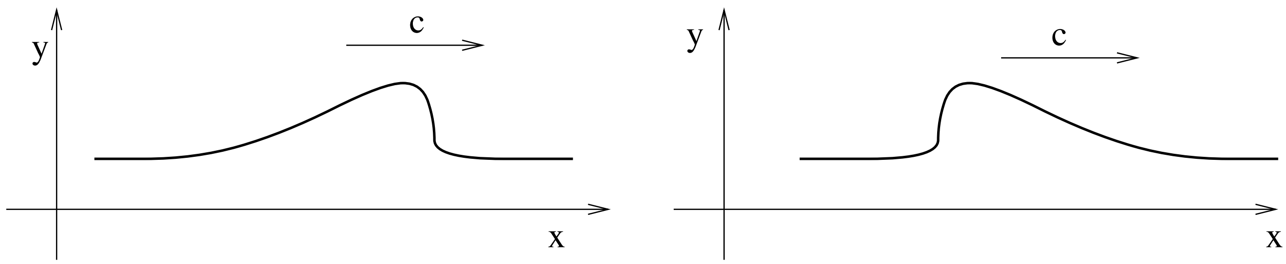

Figure 4.2: There is nothing special about a wave profile

The speed at which the wave moves through the crowd is a function of the interaction between individuals. Suppose this was a symmetric interaction — and in linear systems it is by definition — there is there no reason why a second wave could not propagate in the opposite direction at the same speed. It need not be the same shape as the first. Its description is

$$y = h(x + ct) \ .$$

The process is linear, so both waves can propagate at the same time. We come to the conclusion that the general description of a wave disturbance must be

$$y = g(x - ct) + h(x + ct) \ .$$

Notice again that we have not needed to say anything specific about the shape of either part of the wave.

4.2 * The wave equation

*** This is NOT on the syllabus, but is here for interest and completeness.**

We have arrived at a general solution for the wave equation by intuitive argument. The wave equation itself is a 2nd-order partial differential equation, and so beyond the syllabus.

However, when analysing some physical system, it is more usually to that the system gives rise to the wave equation and hence must allow waves to propagate, so it is useful to recognize the equation!

To find what the wave equation is, we need to differentiate y twice w.r.t. t and x.

Write $u = x - ct$ and $v = x + ct$, so that $y = g(u) + h(v)$. Then note that y is the sum of two functions, each of a single variable, so that

$$\frac{\partial y}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} + \frac{dh}{dv}\frac{\partial v}{\partial t} \tag{4.1}$$

$$= \frac{dg}{du}(-c) + \frac{dh}{dv}(c) \ . \tag{4.2}$$

But because dg/du is another function of u, $dg/du = \xi(u)$ say, and because dg/dv a function of v, $dh/dv = \eta(u)$ say, we can play the same game again and obtain

$$\frac{\partial^2 y}{\partial t^2} = -c\frac{d^2 g}{du^2}\frac{\partial u}{\partial t} + c\frac{d^2 h}{dv^2}\frac{\partial v}{\partial t} \tag{4.3}$$

$$= c^2\left(\frac{d^2 g}{du^2} + \frac{d^2 h}{dv^2}\right) \ . \tag{4.4}$$

Similarly

$$\frac{\partial y}{\partial x} = \frac{dg}{du}\frac{\partial u}{\partial x} + \frac{dh}{dv}\frac{\partial v}{\partial x} \tag{4.5}$$

$$= \frac{dg}{du} + \frac{dh}{dv} \tag{4.6}$$

and so

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 g}{du^2}\frac{\partial u}{\partial x} + \frac{d^2 h}{dv^2}\frac{\partial v}{\partial x} \tag{4.7}$$

$$= \left(\frac{d^2 g}{du^2} + \frac{d^2 h}{dv^2}\right) \tag{4.8}$$

$$= \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}. \tag{4.9}$$

So all we need recognize is

The wave equation is $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$.

and we know immediately that waves will propagate in the system with velocity c.

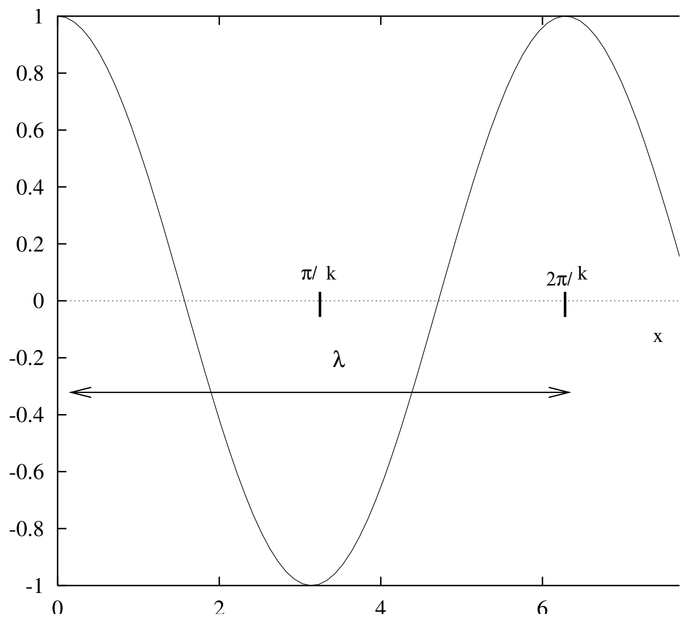

Figure 4.3:

4.3 Harmonic Waves

So far, we have left the description of the wave profile in terms of arbitrary functions g and h. Because arbitary profiles can be built out of a summation of cosine waves, the most important form to consider is that of the cosine (or sine) wave. These are harmonic waves, which are periodic — ie the profile repeats.

Considering only the wave travelling in the forward $+x$ direction, we shall write $y = g(x - ct)$ as

$$y = A\cos[-k(x - ct)] = A\cos[k(ct - x)] \;\; ,$$

where k and A are constants.

4.3.1 Amplitude, wavelength, wavenumber, frequency etc

A is the **amplitude** of the wave, but what is k? Consider the shape of the wave at any instant. We can conveniently choose $t = 0$. Then the profile is

$$y = A\cos[-kx] = A\cos[kx]$$

and the value at $x = x_1$ is $y_1 = A\cos[kx_1]$. This value will next appear at x_2 where $kx_2 = kx_1 + 2\pi$, or

$$x_2 = x_1 + \frac{2\pi}{k} \;\; .$$

But $x_2 - x_1$ is the **wavelength** λ, so that

$$\lambda = \frac{2\pi}{k} \ .$$

The quantity k is called the **wavenumber**.

Now consider the motion at a particular position x. Again it is convenient to choose $x = 0$. Then

$$y = A\cos[kct] = \ .$$

Comparing with

$$y = A\cos[\omega t]$$

we see that a particular point undergoes shm of amplitude A with an angular frequency of

$$\omega = kc = \frac{2\pi}{\lambda}c.$$

But $\omega = 2\pi f$ where f is the frequency in Hz, so we reach the important result that

$$\boxed{c = f\lambda = \frac{\omega}{k}}$$

Finally we could return to our expression for the wave and write it in the "conventional" form

$$y = A\cos[\omega t - kx]$$

and we could also consider the possibility of a phase shift ϕ:

Thus **a Harmonic wave travelling along** $+x$ **is**

$$y = A\cos[\omega t - kx + \phi]$$

and a **Harmonic wave travelling along** $-x$ **is**

$$y = A\cos[\omega t + kx + \phi]$$

each with a wave velocity $c = w/k$.

4.4 Mod-arg representation of an harmonic wave

We can (of course) represent an harmonic wave using the exponential

An harmonic wave travelling along $+x$ is $y = Ae^{i[\omega t - kx + \phi]}$
and
An harmonic wave travelling along $-x$ is $y = Ae^{i[\omega t + kx + \phi]}$

You should check that these expressions satisfy the wave equation.

4.4.1 Phase shifts and complex amplitudes

So far we have assumed that A is a real number. What would happen if A were complex — $\tilde{A}$ say. By writing

$$y = Ae^{i[\omega t - kx + \phi]} \quad (4.10)$$

$$= Ae^{i\phi}e^{i[\omega t - kx} \quad (4.11)$$

$$= \tilde{A}e^{i[\omega t - kx} \quad (4.12)$$

$$(4.13)$$

we see that a **complex amplitude simply encodes a phase shift**.

4.4.2 Absorption and complex wavenumbers

In an absorbing medium, the amplitude of a wave decays exponentially with distance. If the amplitude is A at $x = 0$ and the wave is travelling along $+x$, then at some distance x, the amplitude is

$$Ae^{-\alpha x}$$

where α is an absorption coefficient.

An absorption coefficient can be represented by using a complex wavenumber $\tilde{k}$. Let

$$\tilde{k} = k' - ik''$$

where both k' and k'' are real. Then

$$y = Ae^{i[\omega t - \tilde{k}x]} \quad (4.14)$$

$$= Ae^{i[\omega t - (k' - ik'')x]} \quad (4.15)$$

$$= Ae^{-k''x}e^{i[\omega t - k'x]} \quad (4.16)$$

$$(4.17)$$

so that k' is the usual wavenumber, and k'' is the absorption coefficient.

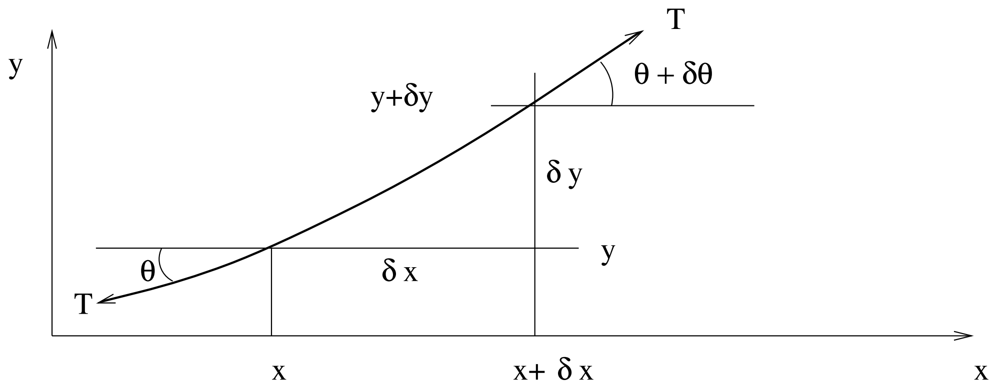

Figure 4.4: Transverse travelling waves on an infinite string

4.5 * Examples of systems which support waves

*** The derivations are NOT on the syllabus, but are here for information.**

Earlier we noted that the analysis of a system (mechanical, electrical etc) might result in our showing that the system could be described by the wave equation, and so we would then know that waves propagate in the system. Here we look at two examples, transverse waves in a string, and longitudinal waves in a fluid.

4.5.1 Example I: transverse travelling waves in an infinite string

The angle θ is always small, so that $\cos\theta = 1$ and $\sin\theta = \theta$. The tension T is uniform throughout the string, and so the there is no resultant force in the x direction on the element of original length δx, and mass $\rho\delta x$.

However, in the y-direction:

$$T\sin(\theta+\delta\theta) - T\sin(\theta) = \rho\delta x\frac{\partial^2 y}{\partial t^2}$$

or

$$T\delta\theta = \rho\delta x\frac{\partial^2 y}{\partial t^2} \ .$$

But $\partial y/\partial x = \tan\theta \approx \theta$ and so

$$\delta\theta = \frac{\partial^2 y}{\partial x^2}\delta x \ .$$

Thus

$$T\frac{\partial^2 y}{\partial x^2}\delta x \ \ = \rho\delta x\frac{\partial^2 y}{\partial t^2} \ .$$

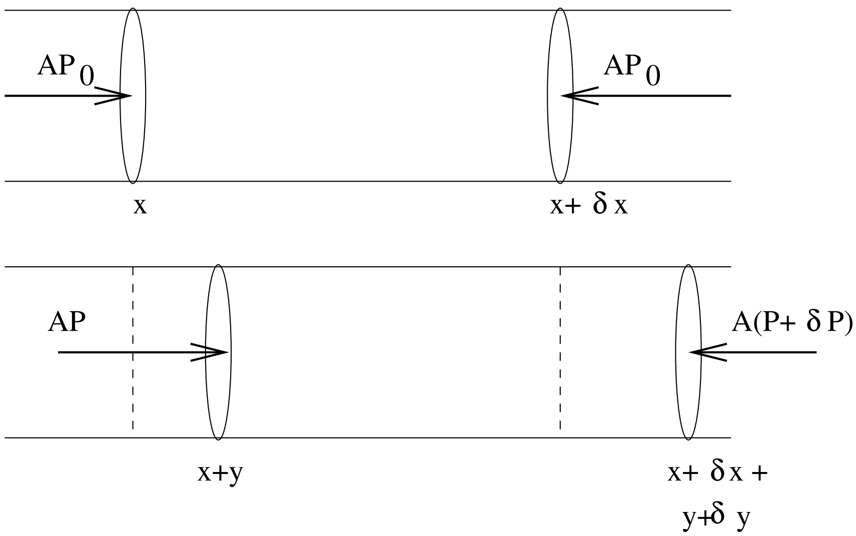

Figure 4.5: Longitudinal waves in a fluid.

or

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \quad .$$

This is the wave equation, and transverse waves propagate with velocity $c = \sqrt{T/\rho}$, where T is the tension and ρ the mass per unit length of the string.

4.5.2 Example II: longitudinal waves in fluids

The equilibrium position shown on top is disturbed as shown. The increase in pressure

$$p = P - P_0 = -K\frac{\partial y}{\partial x}$$

where K is the bulk modulus of the fluid. Thus

$$\frac{\partial p}{\partial x} = -K\frac{\partial^2 y}{\partial x^2}.$$

Now consider Newton's 2nd Law:

$$AP - A(P + \delta P) = A\rho\delta x\frac{\partial^2 y}{\partial t^2} \tag{4.18}$$

$$\delta P = \rho\delta x\frac{\partial^2 y}{\partial t^2} \tag{4.19}$$

Thus

$$-\frac{\partial P}{\partial x} = \rho\frac{\partial^2 y}{\partial t^2} \tag{4.20}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{K}\frac{\partial^2 y}{\partial t^2} \tag{4.21}$$

So waves propagate with velocity $\sqrt{K/\rho}$.

4.6 Standing waves

Travelling waves transfer energy in the direction of travel.

Suppose now that we had two travelling waves of equal amplitude travelling in opposite directions. There would be no nett energy transfer. Such a wave is called a **standing wave**.

Using the exponential representation:

$$y = Ae^{i[kx-\omega t]} + Ae^{i[kx+\omega t]} \tag{4.22}$$

$$= Ae^{ikx}\left(e^{-i\omega t} + e^{i\omega t}\right) \tag{4.23}$$

$$= Ae^{ikx}(2\cos\omega t)\ . \tag{4.24}$$

Taking just the real part

$$y = 2A\cos kx \cos\omega t\ .$$

The effective amplitude is now $2A\cos kx$, so that $y = 0$ at all times when

$$kx = \frac{\pi}{2} \pm n\pi.$$

These are called NODES. The wave has a maximum amplitude at ANTI-NODES when

$$kx = \pm n\pi\ .$$

Note that no energy can be transported through the nodes.

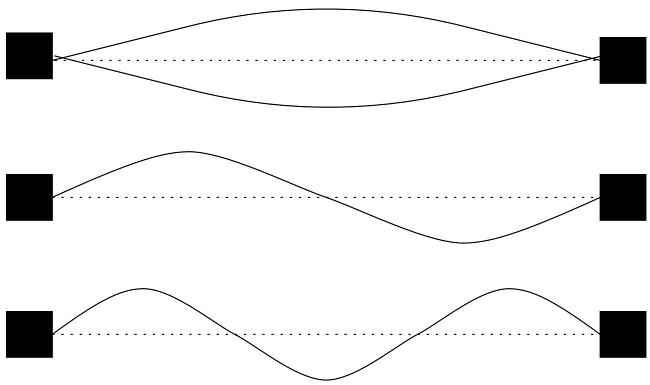

Figure 4.6: Harmonic waves on a string.

4.7 Example I: standing waves in a plucked string

Transverse standing waves occur on a string which is clamped at both ends and plucked. The longest wavelength which can persist as a standing wave is when the clamps at either end are nodes. If the distance between the clamps is L then

$$\frac{\lambda_0}{2} = L \ .$$

The wave velocity is still $c = \sqrt{T/\rho}$ so that the lowest frequency, the fundamental frequency, is

$$f_0 = \frac{c}{\lambda_0} = \frac{1}{2L}\sqrt{\frac{T}{\rho}} \ .$$

Higher frequencies or harmonics can also persist. What are these frequencies?

4.7.1 Finding the wave form

Suppose we clamp a string of length L and stretch it to a shape $f(x)$ before releasing it. How would the standing wave appear? We could of course (in 2nd year) solve the wave equation for $y(x,t)$ with the boundary conditions that $y(0,t) = 0$ and $y(L,t) = 0$ for all time and that at $y(x,0) = f(x)$ and that $\dot{y}(x,0) = 0$.

But there is more intuitive way. We know that a standing wave is made up of two equal amplitude waves travelling in opposite direction. This must be true then at $t = 0$. So both waves have the profile $\frac{1}{2}f(x)$. We make the profile periodic over a range of $2L$, and then set the waves travelling in opposite directions, we should reproduce the required wave form.

An example will make this clearer.

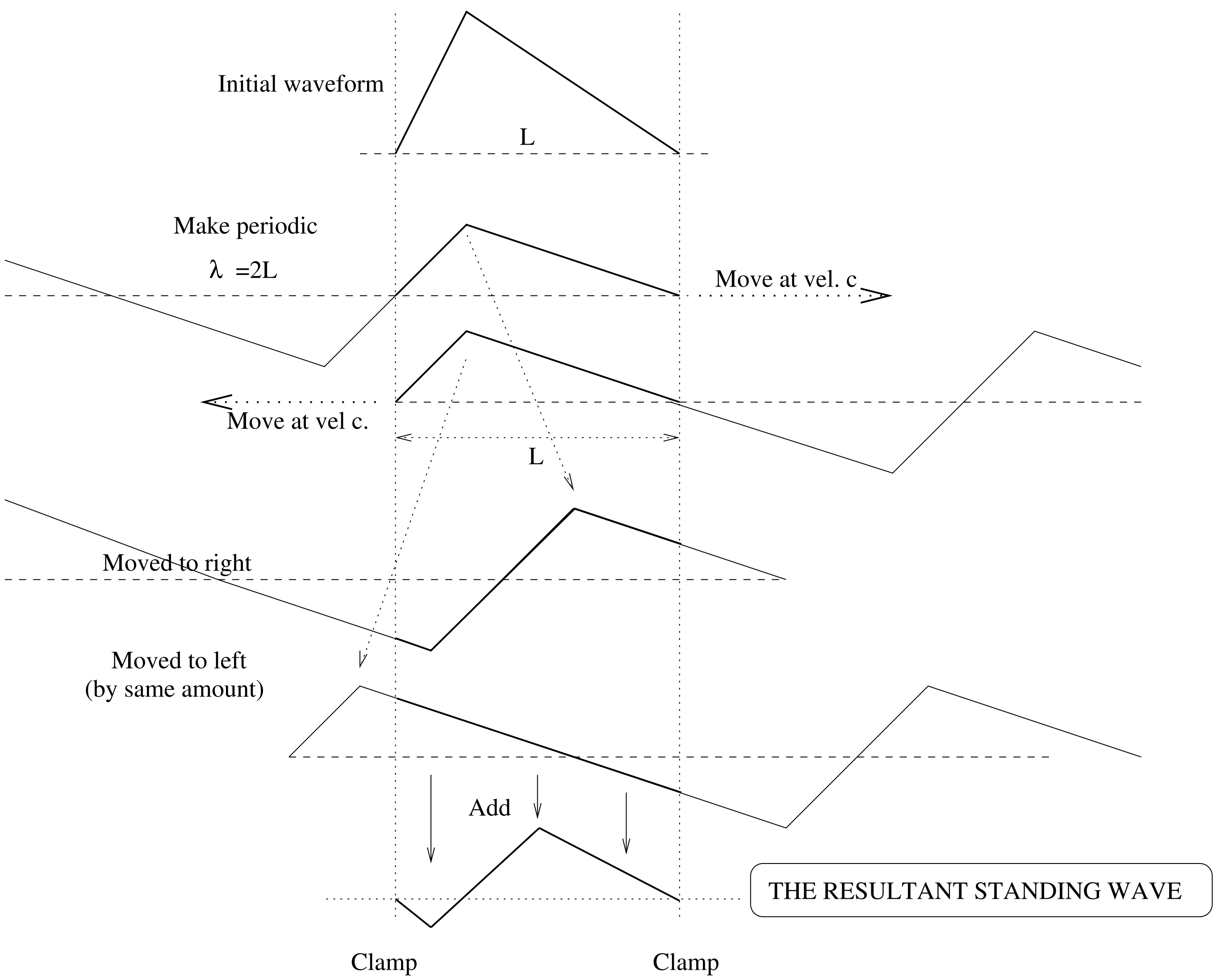
Initial waveform
L
Make periodic
λ =2L
Move at vel. c
Move at vel c.
L
Moved to right
Moved to left
(by same amount)
Add
THE RESULTANT STANDING WAVE
Clamp
Clamp

Figure 4.7:

4.8 Example II: standing waves in open and closed pipes

Finally, we briefly explore standing waves in open and closed pipes. In a open pipe, there are antinodes at the lip and the open end, so the if the length of the pipe is L, the fundamental has wavelength $\lambda_0 = 2L$. If however the pipe is stopped, there must be an antinode at the stopper. Thus $\lambda_0 = 4L$. Thus a stopped pipe will have a fundamental frequency half that of an open pipe (and hence it sounds an octave lower).

NB! the diagrams might suggest that the waves are transverse. They are certainly still longitudinal.

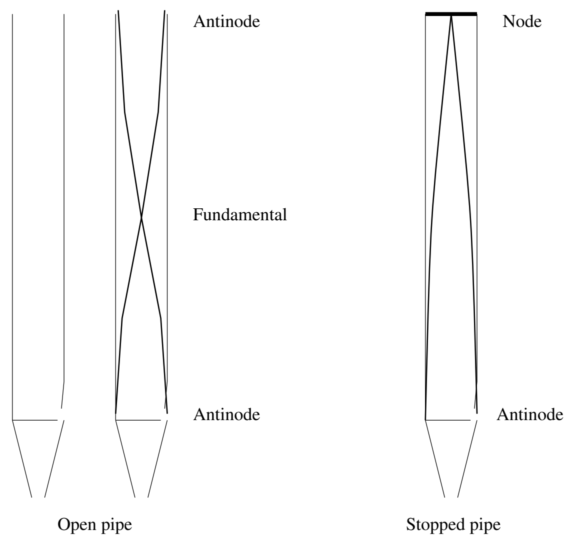

Figure 4.8: Open and stopped pipes

Chapter 5

Analytic functions of a complex variable

In Lectures 2 we explored some elementary functions of the complex number z. In lecture 3 we saw that some mappings possess an angle preserving or "conformal" property.

These properties arise from the the notions of the limit, continuity and differentiability of complex functions. To get at these notions we need to use partial differentiation in the Cauchy-Riemann relationships.

Throughout the following, we assume that the function is

$$w = f(z) = u(x, y) + iv(x, y)$$

where u and v are real functions of x and y.

5.1 Neighbourhoods, Limits and Continuity

Neighbourhoods. It is possible to define a neighbourhood around a point z_0 in the z-plane, such that any point z in the neighbourhood satisfies

$$|z - z_0| < \delta \qquad (\delta > 0) \ . \tag{5.1}$$

A neighbourhood thus defined will be referred to as the δ-neighbourhood of z_0.

Limits. The definition of a limit is similar to that for a real function. If $f(z)$ is a single-valued function of z, and w_0 is a complex constant and if for every $\epsilon > 0$, no matter how small, there exists a positive number $\delta(\epsilon)$ such that $|f(z) - w_0| < \epsilon$ for all z within the δ neighbourhood of z_0, then w_0 is the limit of $f(z)$ as z approaches z_0. In brief

$$\lim_{z \to z_0} f(z) = w_0 \ . \tag{5.2}$$

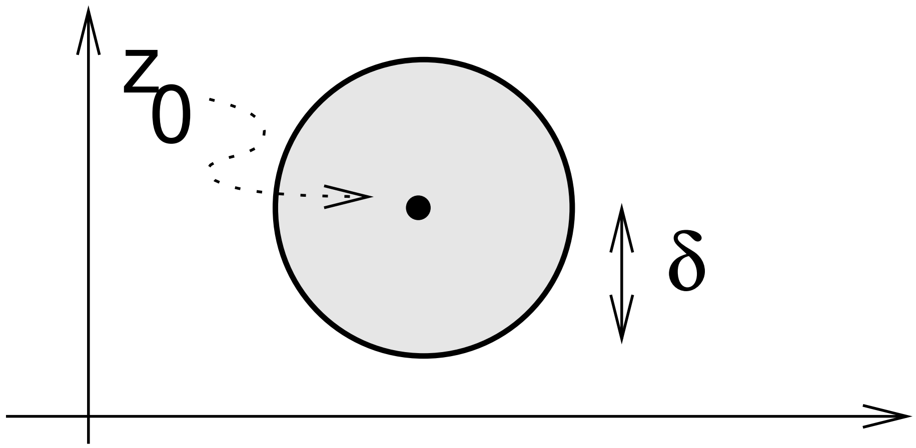

Figure 5.1: The δ neighbourhood around z_0.

Continuity. This is closely related to the limit. The function $f(z)$ is continuous at a point z_0 provided

$$\lim_{z \to z_0} f(z) = f(z_0) \tag{5.3}$$

In other words, for a function $f(z)$ to be continuous at a point, z_0, it must have a value at that point, and that value must equal the limit as z approaches z_0.

There are various points arising:

- Sums, differences, products and quotients of continuous functions are continuous functions, though in the case of quotients, the divisor must be non-zero at the point.
- A continuous function of a continuous function is a continuous function.
- A necessary and sufficient condition for $f(z) = u(x, y) + iv(x, y)$ to be continuous is that u and v be continuous.

5.2 Derivatives of complex function

Discussions of limits and continuity inevitably lead to derivatives!

The derivative of a function $w = f(z)$ of a complex variable is defined by

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z} \right] \ . \tag{5.4}$$

Note that this definition is formally identical to that for functions of single real variables, so that the well known formulae such as

$$\frac{d}{dz}(w_1 \pm w_2) = w_1' \pm w_2' \tag{5.5}$$

$$\frac{d}{dz}(w_1 w_2) = w_1' w_2 + w_1 w_2' \tag{5.6}$$

$$\frac{d}{dz}(w_1/w_2) = \frac{w_2 w_1' - w_1 w_2'}{w_2^2} \tag{5.7}$$

$$\frac{d}{dz} w^n = n w^{n-1} w' \tag{5.8}$$

should hold good.

But the definition of the derivative may have surprised you. You may have expected something of the problem seen in partial differentiation where the direction in which we approach the point at which we want to find the derivative is significant. (As you will recollect, for a function $h(x, y)$ if we approach along the x-axis we obtain $\partial h/\partial x$ and if along y, $\partial h/\partial y$.) Surely the same problem exists here?

Indeed it does, as Δz is given by

$$\Delta z = \Delta x + i\Delta y \ , \tag{5.9}$$

and we could approach a point z_0 parallel to the real axis ($\Delta y = 0$) or parallel to the imaginary axis ($\Delta x = 0$), or anything in between.

But our desire is to keep complex analysis as near real analysis as possible ...

To side-step this difficulty, derivatives are only defined for functions $f(z)$ where Δf is the same *irrespective* of the direction of approach to z_0. This restriction is quite substantial — for example a straightforward function like

$$w = f(z) = \overline{z} = x - iy \tag{5.10}$$

does not have a derivative — df/dz is simple *not defined*. The good news however is that many functions *do* possess this quality.

The next question is, of course, how can we tell whether or not a function of a complex variable has a derivative?

5.3 The Cauchy-Riemann equations

The function $f(z)$ is

$$w = f(z) = u(x, y) + iv(x, y) \tag{5.11}$$

The definition of the derivative

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} \right] \tag{5.12}$$

can then be rewritten as

$$\frac{dw}{dz} = \tag{5.13}$$
$$\lim_{\Delta x, \Delta y \to 0} \left[\frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y} \right] .$$

Now suppose we approach parallel to the real axis, so that $\Delta y = 0$:

$$\begin{aligned}
\frac{dw}{dz} &= \lim_{\Delta x \to 0} \left[\frac{[u(x+\Delta x, y) + iv(x+\Delta x, y)] - [u(x,y) + iv(x,y)]}{\Delta x} \right] && (5.14)\\
&= \lim_{\Delta x \to 0} \left[\frac{[u(x+\Delta x, y) - u(x,y)] + i[v(x+\Delta x, y) - v(x,y)]}{\Delta x} \right] && (5.15)\\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} && (5.16)
\end{aligned}$$

Now suppose we approach parallel to the imaginary axis, so that $\Delta x = 0$:

$$\begin{aligned}
\frac{dw}{dz} &= \lim_{\Delta y \to 0} \left[\frac{[u(x, y+\Delta y) + iv(x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{i\Delta y} \right] && (5.17)\\
&= \lim_{\Delta y \to 0} \left[\frac{-i[u(x, y+\Delta y) - u(x,y)] + [v(x, y+\Delta y) - v(x,y)]}{\Delta y} \right] && (5.18)\\
&= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} && (5.19)
\end{aligned}$$

If these two expressions for df/dz are equal, then

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \ . \tag{5.20}$$

and, equating real and imaginary parts, we arrive at the celebrated Cauchy-Riemann equations

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} && (5.21)\\
\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \ . && (5.22)
\end{aligned}$$

Satisfying the Cauchy-Riemann equalities at a point is the necessary and sufficient condition for a function $w = f(z) = u(x,y) + iv(x,y)$ to have a derivative at a particular point.

5.4 The derivative itself

Note the fact if the function $f(z)$ satisfies Cauchy-Riemann at a point then its derivative at that point may be derived in two ways: either

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{or} \frac{dw}{dz} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad . \tag{5.23}$$

But, if the function is analytic, we know that the formal definition of the derivative is the same as that for a real variable. Thus one can write the derivative as though z were a real variable. For example (we show this below) $f(z) = \cos(z^2)$ is analytic. So its derivative is $df/dz = -2z\sin(z^2)$.

5.5 Analytic functions

An analytic function is a slightly higher life-form that a function which has a derivative.

For a function of a complex variable to be analytic at a point, it must satisfy the Cauchy-Riemann equations not only at that point but also in a *neighbourhood* of that point.

♣ Examples

1. Verify the earlier claim that $f(z) = \overline{z}$ is non-analytic.

   Ans:

   $$w = f(z) = u(x,y) + iv(x,y) = x - iy$$

   so that

   $$u(x,y) = x \quad \text{and} \quad v(x,y) = -y \ .$$

   Thus

   $$\begin{aligned} \frac{\partial u}{\partial x} &= 1 \quad ; \quad \frac{\partial v}{\partial y} = -1 \quad \text{violated} \\ -\frac{\partial u}{\partial y} &= 0 \quad ; \quad \frac{\partial v}{\partial x} = 0 \ . \end{aligned}$$

   Thus $\overline{z}$ is non-analytic.

2. Determine whether $f(z) = z\overline{z}$ is analytic.

   Ans: For this function

$$u(x,y) = x^2 + y^2; \qquad v(x,y) = 0$$

   so

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x \quad ; \qquad \frac{\partial v}{\partial y} = 0 \\ -\frac{\partial u}{\partial y} &= -2y \quad ; \qquad \frac{\partial v}{\partial x} = 0 \ . \end{aligned}$$

   These are equal only at $x = 0, y = 0$. Thus the derivative exists at this point, but not in a neighbourhood around $x = 0, y = 0$. Thus the function is nowhere analytic.

3. Determine whether $f(z) = z^2$ is analytic. If it is, find its derivative.

   Ans: For this function

$$u(x,y) = x^2 - y^2; \qquad v(x,y) = 2xy$$

   so

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x \quad ; \qquad \frac{\partial v}{\partial y} = 2x \\ -\frac{\partial u}{\partial y} &= 2y \quad ; \qquad \frac{\partial v}{\partial x} = 2y \ . \end{aligned}$$

   and the Cauchy-Riemann equations are satisfied everywhere. Thus the the function is everywhere analytic.

   According to our earlier assertion we would expect the derivative to be

$$f'(z) = 2z \ .$$

   We can check this using

$$\begin{aligned} \frac{df}{dz} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= 2x + i2y \\ &= 2z \ . \end{aligned}$$

5.6 Properties of analytic functions

We now derive briefly certain properties of analytic functions. We shall leave further exploration of these, in particular those of conformal mappings and harmonic functions, until Lecture 4.

5.6.1 Properties of analytic functions: I

Recalling the properties of continuous functions we can state that

- Sum, differences, products and quotients of analytic functions are analytic functions, though in the case of quotients, the divisor must be non-zero at the point.
- An analytic function of an analytic function is an analytic function.
- An analytic function of an non-analytic function is non-analytic.

♣ Example

1. Show that $\cos(z^2)$ is analytic and find its derivative.

   Ans: We know that z^2 is analytic, but is $\cos z$ an analytic function? Recall that $\cos z = \cos x \cosh y - i \sin x \sinh y$. Hence

   $$u(x, y) = \cos x \cosh y; \qquad v(x, y) = -\sin x \sinh y$$

   so the Cauchy-Riemann equations are built from

   $$\begin{aligned} \frac{\partial u}{\partial x} &= -\sin x \cosh y \quad ; \qquad \frac{\partial v}{\partial y} = -\sin x \cosh y \\ -\frac{\partial u}{\partial y} &= -\cos x \sinh y \quad ; \qquad \frac{\partial v}{\partial x} = -\cos x \sinh y \ . \end{aligned}$$

   and are satisfied everywhere. Hence $\cos z$ is analytic, and as z^2 is analytic, $\cos(z^2)$ is analytic.

   For the doubters who like to do everything the long way ... In full

   $$\cos(z^2) = \cos(x^2 - y^2) \cosh(2xy) - i \sin(x^2 - y^2) \sinh(2xy)$$

   so

   $$u = \cos(x^2 - y^2) \cosh(2xy); \qquad v = -\sin(x^2 - y^2) \sinh(2xy)$$

so the Cauchy-Riemann equations are built from

$$\begin{aligned}
\frac{\partial u}{\partial x} &= -2x.\sin(x^2-y^2).\cosh(2xy)+\cos(x^2-y^2).2y.\sinh(2xy) \\
\frac{\partial v}{\partial y} &= 2y.\cos(x^2-y^2).\sinh(2xy)-\sin(x^2-y^2).2x.\cosh(2xy) \\
-\frac{\partial u}{\partial y} &= -2y.\sin(x^2-y^2).\cosh(2xy)-\cos(x^2-y^2).2x.\sinh(2xy) \\
\frac{\partial v}{\partial x} &= -2x.\cos(x^2-y^2).\sinh(2xy)-\sin(x^2-y^2).2y.\cosh(2xy) \ .
\end{aligned}$$

and they are indeed satisfied.

We *expect* the derivative to be

$$df/dz = -2z\sin(z^2)$$

To check this first expand this out

$$df/dz = -2[x+iy][\sin(x^2-y^2)\cosh(2xy)+i\cos(x^2-y^2)\sinh(2xy)]$$

where we give use the definition of $\sin z$ from Lecture 2. But using the partial derivatives:

$$\begin{aligned}
df/dz &= \partial u/\partial x + i\partial v/\partial x \\
&= \left[-2x.\sin(x^2-y^2).\cosh(2xy)+\cos(x^2-y^2).2y.\sinh(2xy)\right] \\
&\quad +i\left[-2x.\cos(x^2-y^2).\sinh(2xy)-\sin(x^2-y^2).2y.\cosh(2xy)\right] \\
&= -2[x+iy][\sin(x^2-y^2)\cosh(2xy)+i\cos(x^2-y^2)\sinh(2xy)]
\end{aligned}$$

Thus our expectation is fulfilled.

5.7 Properties II: harmonic functions

Laplace's equation is a second order partial differential equation which arises in the analysis of several physical systems. If $\phi=\phi(x,y)$ Laplace's equation is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \ . \tag{5.24}$$

We now show that if $w=f(z)$ is an analytic function and its real and imaginary parts have continuous second partial derivatives, then these parts separately satisfy Laplace's equation. Any function which satisfies Laplace's equation is called an *harmonic function*, but two functions $u(x,y)$ and $v(x,y)$

which satisfy Laplace and make $w = u(x, y) + iv(x, y)$ analytic are said to be *conjugate harmonic functions*.

Given the C-R equalities, the proof follows simply. If

$$w = f(z) = u(x, y) + iv(x, y) \tag{5.25}$$

is analytic, then the Cauchy-Riemann equalities hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{5.26}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \ . \tag{5.27}$$

Differentiating the first wrt x and the second wrt y:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{5.28}$$

$$\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \ . \tag{5.29}$$

Using $v_{xy} = v_{yx}$,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \ . \tag{5.30}$$

Ie

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ , \tag{5.31}$$

and u satisfies Laplace's equation.

Doing things the other way round and differentiating the first C-R relation wrt y and the second wrt x:

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \tag{5.32}$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \ . \tag{5.33}$$

Using $u_{xy} = u_{yx}$,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \ , \tag{5.34}$$

and v satisfies Laplace's equation.

♣ Example

1. Show that $f(z) = e^z$ is analytic and verify that its real and imaginary parts are conjugate harmonic functions.

   Ans: Recall that $e^z = e^x(\cos y + i\sin y)$. Thus $u = e^x \cos y$ and $v = e^x \sin y$. The partial derivatives

   $$\begin{aligned} u_x = e^x \cos y \quad &; \quad v_y = e^x \cos y \\ -u_y = e^x \sin y \quad &; \quad v_x = e^x \sin y \end{aligned}$$

   confirm that the C-R equalities hold everywhere, and so the function is analytic. Laplace's equation

   $$\begin{aligned} u_{xx} + u_{yy} &= e^x \cos y - e^x \cos y = 0 \\ v_{xx} + v_{yy} &= e^x \sin y - e^x \sin y = 0 \end{aligned}$$

   is evidently satisfied by both.

5.8 Properties III: conformal mappings

We noted in Lecture 2 that the relationship $w = f(z)$ sets up a mapping from the z-plane to the w-plane.

If $w = f(z)$ is analytic in some region R of the z-plane, it turns out that there is a remarkable connection between curves C in the region R and their maps C' in the equivalent region R' in the w-plane.

Consider nearby points P at z and Q at $z + \Delta z$ on the curve C in the $z-$plane and let the length of curve between them be Δs. Then consider the mappings of P and Q and the curve in between them into points P' at w, Q' at $w + \Delta w$ and curve C' in the w-plane. Let the arc length be $\Delta s'$. First notice how that the ratio of arc-lengths is

$$\lim_{\Delta z \to 0} \frac{\Delta s'}{\Delta s} = \lim_{\Delta z \to 0} \frac{|\Delta w|}{|\Delta z|} \tag{5.35}$$

$$= \lim_{\Delta z \to 0} \left| \frac{\Delta w}{\Delta z} \right| \quad \text{recall §1.4} \tag{5.36}$$

$$= \left| \frac{dw}{dz} \right| \tag{5.37}$$

$$= |f'(z)| \quad . \tag{5.38}$$

Thus under an analytic function, each curve passing through a particular point on the $z-$plane (where $f'(z) \neq 0$), suffers the same stretching as it is mapped onto the $w-$plane, no matter what the curve's orientation.

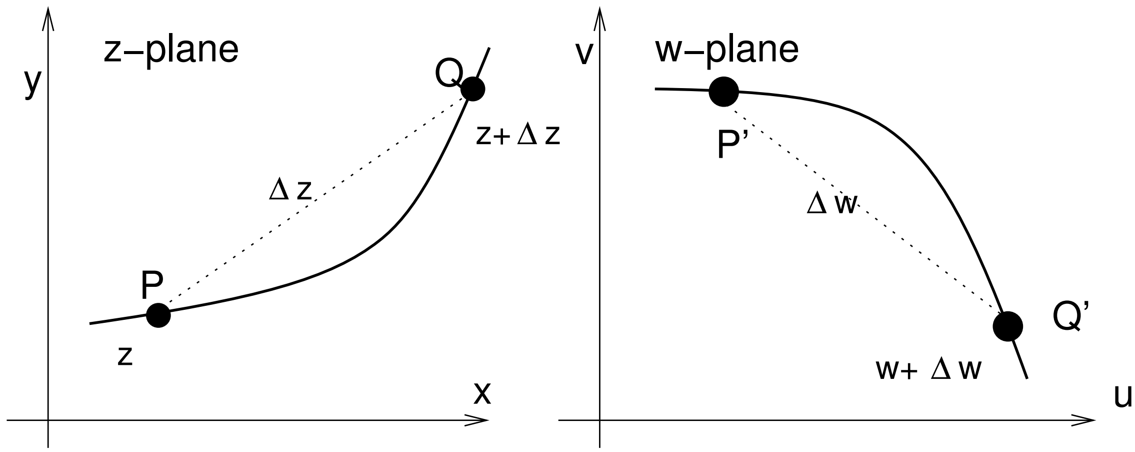

Figure 5.2: Mapping of a curve to a curve under an analytic function.

Now consider the orientation of the tangents to the curves in the z- and w-planes. Before taking the limit, the angle between the chord PQ from $z \to z + \Delta z$ and the real x-axis is simply[1] $\arg(\Delta z)$. Similarly the angle between the mapped chord $w \to w + \Delta w$ and the real u-axis is $\arg(\Delta w)$. Thus in the limit the *difference* in angle is

$$\lim_{\Delta z \to 0} (\arg(\Delta w) - \arg(\Delta z)) = \lim_{\Delta z \to 0} \arg\left(\frac{\Delta w}{\Delta z}\right) = \arg\left(\frac{dw}{dz}\right) \quad . \tag{5.39}$$

But again dw/dz depends only on a point, so:

- the angle between the tangents of any two curves intersecting at a point in the z-plane is the same as the angle between the mappings of those curves at their intersection in the $w-$plane, provided the function $w = f(z)$ is analytic at the point and provided df/dz is non-zero at the point.

Such an angle preserving mapping is called *conformal*.

5.9 Some Engineering with analytic functions: fluid flow

Again, the following analysis is beyond the 1MA syllabus! But it should convince the skeptic that mappings of analytic functions are not merely pretty, but have engineering applications ...

Consider the steady flow of an incompressible fluid of unit depth over the xy-plane as sketched in Figure 5.3. The lines of flow of the fluid particles are arrowed in the Figure and they cross a curve C in the xy-plane.

[1]Remember, Δz is a complex number just like any other!

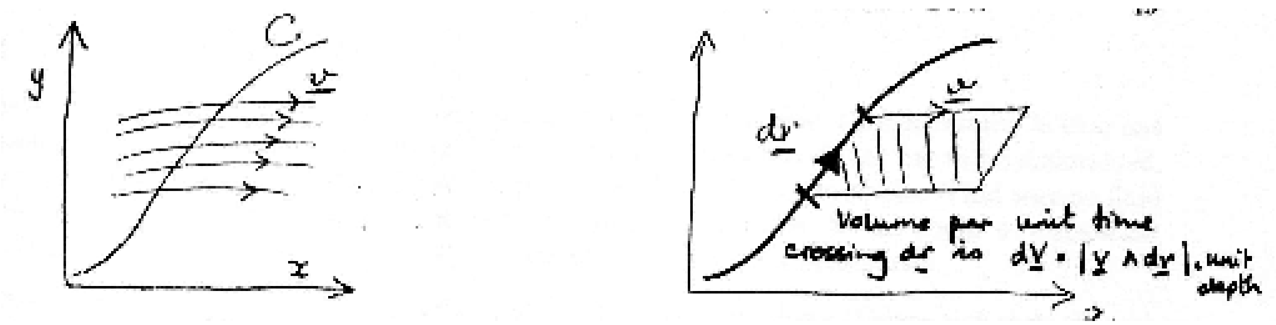

Figure 5.3: Fluid flowing past a curve C in the xy-plane.

The first thing we want to do is to derive the volume of fluid that crosses the curve per unit time. Let the velocity of flow at some point (x, y) be the vector quantity $\mathbf{v}$, which broken into components,

$$\mathbf{v} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} \tag{5.40}$$

where v_x is the velocity component in the x-direction, $\hat{\mathbf{x}}$ is a unit vector in that direction, and similarly for v_y and $\hat{\mathbf{y}}$. The element of volume crossing a short section $d\mathbf{r}$ of the curve C in an unit time is the volume, dV, of the parallelopiped shaded in the figure. This is given by the scalar triple product

$$dV = \hat{\mathbf{z}} \cdot (\mathbf{v} \times d\mathbf{r}) = \begin{vmatrix} 0 & 0 & 1 \\ v_x & -v_y & 0 \\ dx & dy & 0 \end{vmatrix} \tag{5.41}$$

where we have used $d\mathbf{r} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy$. Hence

$$dV = v_x dy - v_y dx \ \ . \tag{5.42}$$

Hence the total volume of fluid crossing in unit time the curve is just the line integral:

$$V = \int_C (v_x dy - v_y dx) \ \ . \tag{5.43}$$

Introducing the stream function

If C is a closed curve, and the region contains no sources or sinks of fluid, we must have

$$\int_{C\ Closed} (v_x dy - v_y dx) = 0 \ \ . \tag{5.44}$$

But this implies that $(v_x dy - v_y dx)$ is the total differential of some function $\Psi(x, y)$, so

$$d\Psi = (v_x dy - v_y dx) \ \ . \tag{5.45}$$

because then the integral around a loop is $\int_{(x_0,y_0)}^{(x_0,y_0)} d\Psi$ is simply $\Psi(x_0, y_0) - \Psi(x_0, y_0)$, which is indeed zero.

But

$$d\Psi = \frac{\partial\Psi}{\partial x}dx + \frac{\partial\Psi}{\partial y}dy \tag{5.46}$$

so that

$$v_y = -\frac{\partial\Psi}{\partial x} \qquad v_x = \frac{\partial\Psi}{\partial y} \quad . \tag{5.47}$$

(Remember v_x here is the velocity component in the x direction, NOT $\partial v/\partial x$.)

Also, using $\partial^2\Psi/\partial x\partial y = \partial^2\Psi/\partial y\partial x$,

$$-\frac{\partial v_y}{\partial y} = \frac{\partial v_x}{\partial x} \quad , \tag{5.48}$$

or

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{5.49}$$

(The quantity on the lefthandside is called the *divergence* of $\mathbf{v}$ and any vector field with zero divergence is is *solenoidal*, as you will discover in 2MA.)

The streamlines.

The lines of $\Psi = constant$ are the tracks of flow of fluid particles or *streamlines*. How do we know this? The volume crossing C is the integral $\int_C d\Psi$ which is always zero around a loop, but also zero when taken between any two points on a line of $\Psi = constant$. The only way for no fluid to cross the curve is for it to travel along the curve. Thus $\Psi = constant$ lines are the streamlines.

Introducing the vector potential.

Now we add another condition, and that is that the flow is "irrotational" ($\mathrm{curl}\ \mathbf{v} = 0$). Then "it can be shown that" there exists another scalar function $\Phi(x, y)$, called the velocity potential, such that

$$\mathbf{v} = \hat{\mathbf{x}}\frac{\partial\Phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Phi}{\partial y} \quad . \tag{5.50}$$

Thus $v_x = \partial\Phi/\partial x$ and $v_y = \partial\Phi/\partial y$.

The stream function and vector potential satisfy Cauchy-Riemann.

But earlier we found that $v_x = \partial\Psi/\partial y$ and $v_y = -\partial\Psi/\partial x$, so that

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y} \tag{5.51}$$

$$-\frac{\partial\Phi}{\partial y} = \frac{\partial\Psi}{\partial x} \tag{5.52}$$

So the stream function, Ψ, and the velocity potential, Φ, are related by the Cauchy-Riemann equalities! Now we could write

$$w = F(z) = \Phi(x,y) + i\Psi(x,y) \tag{5.53}$$

and we know that $F(z)$ is analytic. We also know that both the stream function and the velocity potential are conjugate harmonic functions, both satisfying Laplace's equation. We can also write

$$\frac{dF}{dz} = \frac{\partial\Phi}{\partial x} + i\frac{\partial\Psi}{\partial x} \tag{5.54}$$

$$= v_x - iv_y \ . \tag{5.55}$$

Flow around obstacles.

Now think about what happens if you stick an obstacle in the flow. One possibility is that out conditions for solenoidal or irrotational flow get broken. However, if they are to be maintained, we must ensure that the obstacle's boundary is coincident with a streamline. That way no fluid tries to enter or exit the obstacle and thus the flow is undisturbed. So we would $\Psi = constant$ curves when mapped into the z-plane to trace out the obstacle's boundaries.

This suggests an informal method of solution to fluid flow problems. Given and obstacle with a boundary defined in the z-plane, we try out functions $w = \Phi + i\Psi = f(z)$ which we know to be analytic, then look to see how the lines of constant Ψ in the w-plane map onto the z-plane. If one of the mapped lines follows the obstacle boundary, we have found the correct analytic function describing the flow, and the other lines describe the streamlines, as sketched in Figure 5.4

♣ Examples

A couple of examples will illustrate this idea.

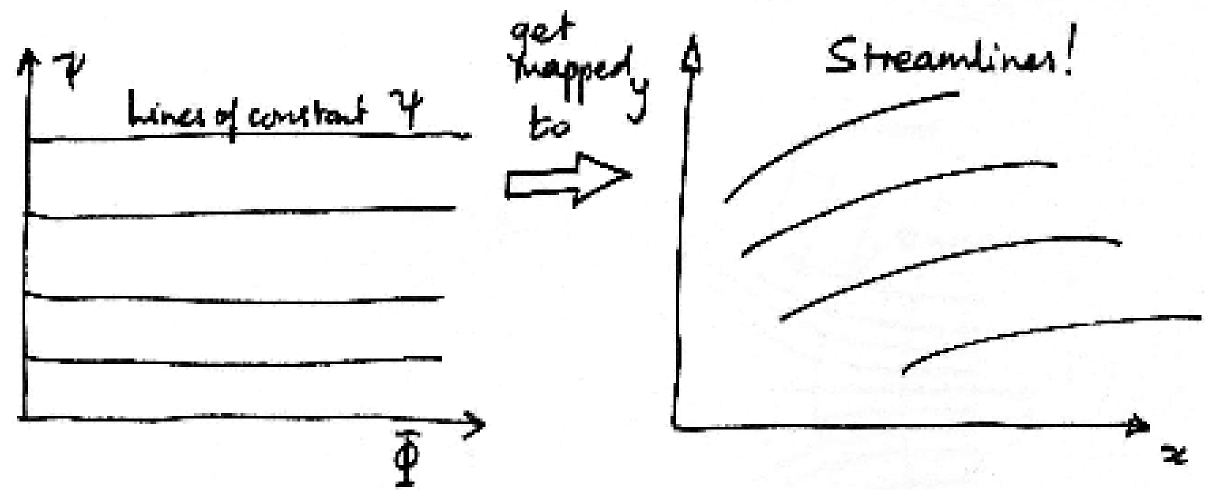

Figure 5.4: Mapping lines of constant Ψ into the z-plane.

Example 1: 45° corner

To determine the flow pattern around a 45° corner we consider the function

$$w = \Phi + i\Psi = az^4 \quad a > 0. \tag{5.56}$$

Equating real and imaginary parts, we have

$$\Phi = a(x^4 + y^4 - 6x^2y^2) \tag{5.57}$$
$$\Psi = 4axy(x^2 - y^2) \tag{5.58}$$

so that the streamlines are

$$\Psi = 4axy(x^2 - y^2) = constant. \tag{5.59}$$

We see that $x = y$ and $x = 0$ are the streamlines corresponding to the obstacle boundary, and Figure 5.5 shows a plot of other streamlines. The velocity components are

$$v_x = \partial\Phi/\partial x = 4ax(x^2 - 3y^2) \tag{5.60}$$
$$v_y = \partial\Phi/\partial y = 4ay(y^2 - 3x^2) \ . \tag{5.61}$$

Example 2: a circular column

Consider the analytic function

$$F(z) = c\left(z + \frac{a^2}{z}\right) = \Phi + i\Psi \ \ c > 0. \tag{5.62}$$

To proceed, it is most convenient to write $z = re^{i\theta}$ and hence

$$\Phi = c\left(r + \frac{a^2}{r}\right)\cos\theta \tag{5.63}$$

$$\Psi = c\left(r - \frac{a^2}{r}\right)\sin\theta \tag{5.64}$$

from which it is readily seen that $r = a$ is a streamline. The velocity components are found from

$$\frac{dF}{dz} = v_x - iv_y = c\left(1 - \frac{a}{z^2}\right) \ , \tag{5.65}$$

and it is clear that $\mathbf{v} = 0$ when $x = \pm a$ and $y = 0$. These points are called stagnation points. So the flow is what we would obtain if the obstruction is a circular cylinder, and the streamlines are sketched in the figure.

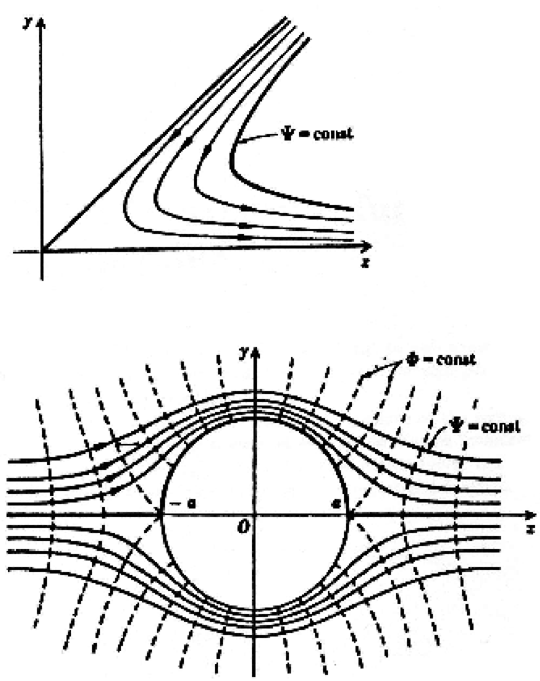

Figure 5.5: Mapping lines of constant Ψ into the z-plane for: top — the function of example 1, the $45°$ bend; and bottom — example 2, the circular column.