Lecture 4: Piecewise Cubic Interpolation

(*Compiled 15 September 2012*)

In this lecture we consider piecewise cubic interpolation in which a cubic polynomial approximation is assumed over each subinterval. In order to obtain sufficient information to determine these coefficients, we require continuity of the interpolating polynomials in neighboring intervals as well as the continuity of a number of derivatives. Depending on the number of derivatives that we require to be continuous at the "stitching points" we obtain different approximation schemes in which additional data about the function $f(x)$ being approximated might have to be furnished in order that there is sufficient data to determine the coefficients of the cubic polynomials.

Key Concepts: Piecewise cubic interpolation, Cubic Splines, Cubic Hermite Interpolation.

4 Piecewise Cubic Interpolation

4.1 Degree of freedom analysis of piecewise cubic interpolants

Consider the domain $[a, b]$ that is partitioned into N intervals having $N+1$ nodes and $N-1$ internal nodes. In each of these subintervals assume that different a cubic polynomial is to be constructed. Let $p_{3,N}(x)$ be the combination of all these cubic polynomials. We now discuss the alternative approximation schemes that one obtains depending on the different schemes of constraint.

Schemes of constraint:

(1) **Piecewise Cubic Hermite polynomials:** $p_{3,N}(x)$ and $p'_{3,N}(x)$ are continuous at interior nodes.

There are four unknown coefficients for each of the N intervals and $2(N-1)$ constraints due to the continuity requirements on $p_{3,N}(x)$ and $p'_{3,N}(x)$.

$$\Rightarrow \text{ DOF } : 4N - 2(N-1) = 2(N+1)$$
$$\Rightarrow \text{ specify function value and its derivative at all } N+1 \text{ nodes}$$

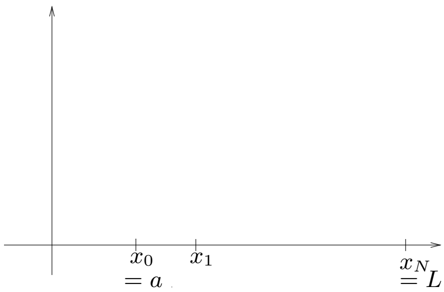
- Have to specify f & f' at all $N+1$ nodes!

(2) **Cubic Spline:** $p_{3,N}(x)$, $p'_{3,N}(x)$, and $p''_{3,N}(x)$ are all continuous at interior points.

$$\Rightarrow \text{ DOF } : 4N - 3(N-1) = N+3 = \underbrace{N+1}_{} + 2$$
$$\Rightarrow \text{ specify } f \text{ at all } N+1 \text{ nodes}$$
$$\text{and impose 2 EXTRA CONDITIONS}$$

The extra conditions are up to the user to prescribe depending on the application, e.g. $p''_3(x_0) = 0 = p''_3(x_N)$ which is called the natural spline.

4.2 Piecewise Cubic Hermite polynomials



DOF: $4N - 2(N-1) = 2(N+1) \Rightarrow$ prescribe $f(x_i)$ and $f'(x_i)$ at $i = 0, \ldots, N$.

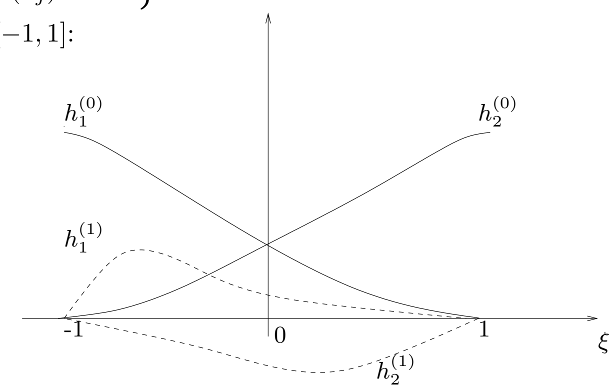
Representation of f in terms of Hermite basis functions $h_i^{(0)}(x)$ and $h_i^{(1)}(x)$:

$$f(x) \approx h(x) = \sum_{i=0}^{N} f(x_i) h_i^{(0)}(x) + \sum_{i=0}^{N} f'(x_i) h_i^{(1)}(x)$$

where

$$\left.\begin{array}{lcl} h_i^{(0)}(x_j) = \delta_{ij} & \text{and} & \frac{d}{dx} h_i^{(1)}(x_j) = \delta_{ij} \\ \frac{d}{dx} h_i^{(0)}(x_j) = 0 & & h_i^{(1)}(x_j) = 0 \end{array}\right\} \quad (1)$$

Constructing basis functions on the canonical interval $[-1, 1]$:



We use the linear Lagrange basis functions

$N_a^1(\xi) = \dfrac{1}{2}(1+\xi_a \xi) : N_a^1(\xi_b) = \delta_{ab}, \qquad \xi_{a,b} = \pm 1$

Let

$h_1^{(0)}(\xi) = (\alpha_1 \xi + \beta_1)[N_1^1(\xi)]^2; h_2^{(0)}(\xi) = (\alpha_2 \xi + \beta)[N_2^1(\xi)]^2$

$h_1^{(1)}(\xi) = (\gamma_1 \xi + \delta_1)[N_1^1(\xi)]^2; h_2^{(1)}(\xi) = (\gamma_2 \xi + \delta_2)[N_2^1(\xi)]^2$

To find α_i, β_i, γ_i and δ_i we impose the conditions (1):

$$\begin{array}{rl} 1 = & h_1^{(0)}(-1) = \beta_1 - \alpha_1 \quad 0 = h_1^{(0)'}(-1) = \alpha_1 1 + (\beta_1 - \alpha_1)\dfrac{2}{2}(1 - (-1))\dfrac{(-1)}{2} = 2\alpha_1 - \beta_1 \\ \therefore & \alpha_1 = 1 \text{ and } \beta_1 = 2 \\ \Rightarrow & \boxed{h_1^{(0)}(\xi) = \frac{1}{4}(2+\xi)(1-\xi)^2} \\ \text{Similarly} & \boxed{h_2^{(0)}(\xi) = \frac{1}{4}(2-\xi)(1+\xi)^2} \end{array}$$

For the derivative basis functions

$$\left.\begin{array}{l} 0 = h_1^{(1)}(-1) = (\delta_1 - \gamma_1) \\ 1 = h_1^{(1)}(-1) = 2\gamma_1 - \delta_1 \end{array}\right\} \Rightarrow \delta_1 = \gamma_1 = 1 \Rightarrow \boxed{h_1^{(1)}(\xi) = \frac{1}{4}(1+\xi)(1-\xi)^2}$$

$$\text{Similarly } \boxed{h_2^{(1)}(\xi) = \frac{1}{4}(\xi - 1)(1+\xi)^2}$$

Expressions for basis functions on an arbitrary interval $[x_i, x_{i+1}]$.

$$\begin{aligned}\text{Use the linear transformation } x(\xi) &= x_i N_1^{(1)}(\xi) + x_{i+1} N_2^{(1)}(\xi) \\ &= x_i \frac{1}{2}(1-\xi) + x_{i+1}\frac{1}{2}(1+\xi) \\ &= \left(\frac{x_i + x_{i+1}}{2}\right) + \frac{(x_{i+1} - x_i)}{2}\xi \\ \text{The inverse transformation is: } \xi(x) &= \frac{2x - (x_i + x_{i+1})}{(x_{i+1} - x_i)} = \frac{2x - (x_i + x_{i+1})}{\Delta x_i}\end{aligned}$$

$$\begin{aligned}1 + \xi &= \frac{x_{i+1} - x_i + 2x - x_i - x_{i+1}}{x_{i+1} - x_i} = \frac{2(x - x_i)}{(x_{i+1} - x_i)} \Rightarrow 2 + \xi = \frac{\Delta x_i + 2(x - x_i)}{\Delta x_i} \\ 1 - \xi &= \frac{2(x_{i+1} - x)}{x_{i+1} - x_i} \qquad \Rightarrow (2 - \xi) = \frac{\Delta x_i + 2(x_{i+1} - x)}{\Delta x_i}\end{aligned}$$

$$\therefore h_i^{(0)}(x) = \frac{[\Delta x_i + 2(x - x_i)](x_{i+1} - x)^2}{(\Delta x_i)^3} \tag{4.1}$$

$$h_{i+1}^{(0)}(x) = \frac{[\Delta x_i + 2(x_{i+1} - x)](x - x_i)^2}{(\Delta x_i)^3} \tag{4.2}$$

$$h_i^{(1)}(x) = \frac{(x - x_i)(x_{i+1} - x)^2}{(\Delta x_i)^2} \tag{4.3}$$

$$h_{i+1}^{(1)}(x) = -\frac{(x_{i+1} - x)(x - x_i)^2}{(\Delta x_i)^2} \tag{4.4}$$

Here $\Delta x_i = x_{i+1} - x_i$ and observe that

$$\begin{aligned}\frac{d\,h_i}{d\xi}(\xi) &= \frac{d\,h_i}{dx}(\xi(x)) \cdot \frac{dx}{d\xi} \\ &= \frac{d\,h_i}{dx}(\xi(x)) \cdot \frac{\Delta x_i}{2}\end{aligned}$$

Error involved:

For function values the error is given by:

$$|f(x) - h(x)| \leq ||f^{(4)}||_\infty \tfrac{\Delta x^4}{384}$$

while for derivatives the error is :

$$|f'(x) - h'(x)| \leq ||f^{(4)}||_\infty \tfrac{\sqrt{3}\Delta x^3}{216}$$

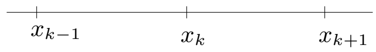
4.3 Piecewise Cubic Spline interpolation

NDOF: $4N - 3(N-1) = N + 1 + 2 \Rightarrow$ specify $f(x_i)$ at $x_0, \ldots, x_N$.
+ 2 extra conditions

4.3.1 *Derivation using Cubic Hermie interpolation*

Since we have similar piecewise cubic polynomials to the Piecewise Cubic Hermite polynomials on each subinterval but with additional continuity required at the $N-1$ interior nodes, our starting point is the Hermite cubic basis expansion. We then impose additional conditions to make up for the derivatives $f'(x_i)$ which are not known (or required) in the case of splines.

On $[x_k, x_{k+1}]$



$$\begin{aligned} s(x) = {} & f_k \frac{[\Delta x_k + 2(x - x_k)](x_{k+1} - x)^2}{(\Delta x_k)^3} + f_{k+1} \frac{[\Delta x_k + 2(x_{k+1} - x)](x - x_k)^2}{(\Delta x_k)^3} \\ & + s'_k \frac{(x - x_k)(x_{k+1} - x)^2}{(\Delta x_k)^2} + s'_{k+1} \frac{(x - x_{k+1})(x - x_k)^2}{(\Delta x_k)^2} \end{aligned}$$

where the f'_k and f'_{k+1} from the Hermite expansion have been replaced by the unknown quantities s'_k and s'_{k+1} which are to be determined by a system of equations which ensure that $s''(x)$ is continuous at internal nodes.

The first and second derivatives of $s(x)$ must be continuous at the interieor points $x_j,\ j = 1 \ldots N-1$. Continuity of the first derivative is already obtained by our choice of basis functions. So we apply continuity of $s''(x)$ to get equations for the coefficients s'_k. That is, we calculate $s''(x)$ on the two intervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ and require continuity at x_k. After some algebra.

$$\Delta x_k s'_{k-1} + 2(\Delta x_k + \Delta x_{k-1}) s'_k + \Delta x_{k-1} s'_{k+1} = 3(f[x_k, x_{k+1}]\Delta x_{k-1} + f[x_{k-1}, x_k]\Delta x_k)$$

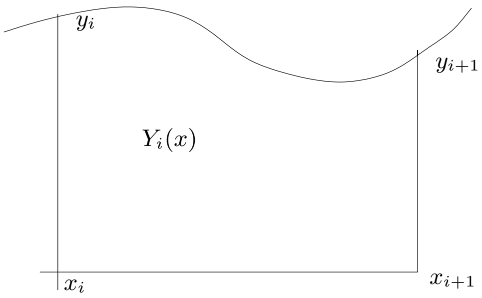
We have $N-1$ equations and $(N+1)$ unknowns $s'_0, s'_1, \ldots, s'_N$ so we need 2 more conditions. Say we specify

(I.) $f'_0 = s'_0$ and $s'_N = f'_N$ then on a uniform mesh $\Delta x_k = \Delta x$:

$$\begin{bmatrix} 4 & 1 & 0 & & \ldots & & 0 \\ 1 & 4 & 1 & 0 & \ldots & & 0 \\ 0 & 1 & 4 & 1 & & & \\ & 0 & & & \ddots & & \vdots \\ \vdots & & & \ddots & & & \\ & & & & & & 0 \\ & & & & & & 1 \\ 0 & & & \ldots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} s'_1 \\ s'_2 \\ \\ \vdots \\ \\ \vdots \\ \\ s'_{N-1} \end{bmatrix} = 3 \begin{bmatrix} f[x_1, x_2] + f[x_0, x_1] \\ f[x_2, x_3] + f[x_1, x_2] \\ \\ \vdots \\ \\ \vdots \\ \\ f[x_{N-1}, x_N] + f[x_{N-2}, x_{N-1}] \end{bmatrix} - \begin{bmatrix} s'_0 \\ \\ \vdots \\ \\ \\ \vdots \\ \\ s'_N \end{bmatrix}$$

A tridiagonal system of equations – easy to solve!

4.3.2 *Alternative cubic spline derivation for derivative primary variables*



$$
\begin{aligned}
\text{Let } Y_i(x) &= a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \\
Y_i'(x) &= 3a_i(x-x_i)^2 + 2b_i(x-x_i) + c_i \\
Y_i''(x) &= 6a_i(x-x_i) + 2b_i \\
y_i &= Y_i(x_i) = d_i \qquad y_{i+1} = Y_i(x_{i+1}) = a_i h_i^3 + b_i h_i^2 + c_i h_i + y_i \text{ where } h_i = x_{i+1} - x_i.
\end{aligned}
$$

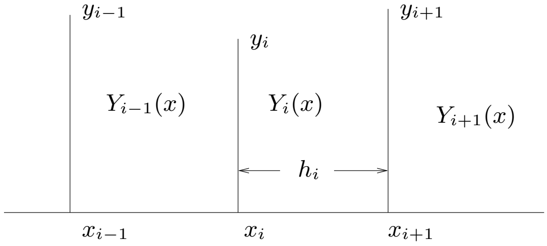
$$
\begin{aligned}
\text{Let } s_i' &= Y_i'(x_i) = c_i \qquad s_{i+1}' = Y_i'(x_{i+1}) = 3a_i h_i^2 + 2b_i h_i + s_i' \\
Y_i''(x_i) &= 2b_i = Y_{i-1}''(x_i) = 6a_{i-1}h_{i-1} + 2b_{i-1} \\
&\begin{bmatrix} h_i^3 & h_i^2 \\ 3h_i^2 & 2h_i \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} y_{i+1} - y_i - s_i' h_i \\ s_{i+1}' - s_i' \end{bmatrix} \\
a_i &= 2h_i(y_{i+1} - y_i - s_i' h_i) \Big/ (-h_i^4) - (s_{i+1}' - s_i')h_i^2 \Big/ (-h_i^4) \\
b_i &= \left\{ h_i^3(s_{i+1}' - s_i') - 3h_i^2(y_{i+1} - y_i - s_i' h_i) \right\} \Big/ (-h_i^4) \\
\therefore a_i &= -2\left(\frac{y_{i+1} - y_i}{h_i^3}\right) + \frac{s_{i+1}' + s_i'}{h_i^2} \\
b_i &= 3\left(\frac{y_{i+1} - y_i}{h_i^2}\right) - \left(\frac{s_{i+1}' + 2s_i'}{h_i}\right)
\end{aligned}
$$

$$
\therefore \frac{6}{h_i}\frac{\Delta y_i}{h_i} - \frac{2}{h_i}\left(s_{i+1}' + 2s_i'\right) = \frac{-12}{h_{i-1}}\frac{\Delta y_{i-1}}{h_{i-1}} + 6\left(\frac{s_i' + s_{i-1}'}{h_{i-1}}\right) + \frac{6}{h_{i-1}}\frac{\Delta y_{i-1}}{h_{i-1}} - 2\frac{(s_i' + 2s_{i-1}')}{h_{i-1}}
$$

$$
\therefore 6\left(\frac{\Delta y_i}{h_i}h_{i-1} + \frac{\Delta y_{i-1}}{h_{i-1}}h_i\right) = 2h_{i-1}s_{i+1}' + 4h_{i-1}s_i' + 4h_i s_i' + 2h_i s_{i-1}'
$$

$$
\therefore \quad \boxed{h_{i-1}s_{i+1}' + 2(h_{i-1} + h_i)s_i' + h_i s_{i-1}' = 3(f[x_{i-1}, x_i]h_i + f[x_i, x_{i+1}]h_{i-1}).}
$$

4.3.3 *Another spline derivation with s_i'' as primary variables*



$$
\begin{aligned}
\text{Let } Y_i(x) &= a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \\
Y_i'(x) &= 3a_i(x-x_i)^2 + 2b_i(x-x_i) + c_i \\
Y_i''(x) &= 6a_i(x-x_i) + 2b_i
\end{aligned}
$$

$$\textbf{at } x_i: \qquad y_i = Y_i(x_i) = d_i \qquad \leftarrow \text{ interpolates the function at } x_i \tag{4.5}$$

$$y_{i+1} = Y_i(x_{i+1}) = a_i h_i^3 + b_i h_i^2 + c_i h_i + y_i \leftarrow \text{ enforce continuity} \tag{4.6}$$

$$\text{Introduce primary variable}: s_i'' = Y_i''(x_i) = 2b_i \Rightarrow \boxed{b_i = s_i''/2} \tag{4.7}$$

$$\text{Build in continuity of } Y'' \Rightarrow s_{i+1}'' = Y_i''(x_{i+1}) = 6a_i h_i + s_i'' \Rightarrow \boxed{a_i = \tfrac{s_{i+1}''-s_i''}{6h_i}} \tag{4.8}$$

$$
\begin{aligned}
\therefore Y_i(x) &= \left(\frac{s_{i+1}''-s_i''}{6h_i}\right)(x-x_i)^3 + \frac{s_i''}{2}(x-x_i)^2 + c_i(x-x_i) + y_i \\
(2) \Rightarrow \quad y_{i+1} &= \left(\frac{s_{i+1}''-s_i''}{6h_i}\right)h_i^3 + \frac{s_i''}{2}h_i^2 + c_i h_i + y_i \\
c_i &= \left(\frac{y_{i+1}-y_i}{h_i}\right) - \left(\frac{s_{i+1}''-s_i''}{6}\right)h_i - \frac{s_i''}{2}hi \\
&= \frac{\Delta y_i}{h_i} - \left(\frac{s_{i+1}''+2s_i''}{6}\right)h_i
\end{aligned}
$$

Impose continuity of first derivatives:

$$
\begin{aligned}
Y_i'(x_i) &= 3a_i(x_i-x_i)^2 + 2b_i(x_i-x_i) + c_i = c_i \\
Y_{i-1}'(x_i) &= 3a_{i-1}h_{i-1}^2 + 2b_{i-1}h_{i-1} + c_{i-1}
\end{aligned}
$$

$$\therefore c_i = \frac{\Delta y_i}{h_i} - \left(\frac{s_{i+1}''+2s_i''}{6}\right)h_i = 3\left(\frac{s_i''-s_{i-1}''}{6h_{i-1}}\right)h_{i-1}^2 + s_{i-1}''h_{i-1} + \left\{\frac{\Delta y_{i-1}}{h_{i-1}} - \left(\frac{s_i''+2s_{i-1}''}{6}\right)h_{i-1}\right\}$$

$$\boxed{h_{i-1}s_{i-1}''(-3+6-2) + (2h_i+2h_{i-1})s_i'' + h_i s_{i+1}'' \qquad = 6\left(\tfrac{\Delta y_i}{h_i} - \tfrac{\Delta y_{i-1}}{h_{i-1}}\right)}$$

or

$$\boxed{h_{i-1}s_{i-1}'' + (2h_i+2h_{i-1})s_i'' + h_i s_{i+1}'' \qquad = 6\left(\tfrac{\Delta y_i}{h_i} - \tfrac{\Delta y_{i-1}}{h_{i-1}}\right)}$$

$$i = 1, \ldots N-1 \qquad\qquad N-1 \text{ EQ}$$

2 EXTRA CONDITIONS

4.3.4 *Extra Conditions*

(1) **Natural Spline:** $s_0'' = 0 = s_N''$ two extra condtions

$$\begin{bmatrix} (2h_0+2h_1) & h_1 & & & \\ h_1 & (2h_1+2h_2) & h_2 & & \\ \vdots & \vdots & \vdots & & \\ & & & & h_{N-2} \\ & & & h_{N-2} & (2h_{N-2}+2h_{N-1}) \end{bmatrix} \begin{bmatrix} s_1'' \\ s_2'' \\ \vdots \\ \vdots \\ s_{N-1}'' \end{bmatrix} = 6 \begin{bmatrix} \frac{y_2-y_1}{h_1} - \frac{y_1-y_0}{h_0} \\ \frac{y_3-y_2}{h_2} - \frac{y_2-y_1}{h_1} \\ \vdots \\ \vdots \\ \frac{y_N-y_{N-1}}{h_{N-1}} - \frac{y_{N-1}-y_{N-2}}{h_{N-2}} \end{bmatrix}$$

(2) **Clamped Spline:** Specified first derivatives

Given $f'(x_0) = A$ and $f'(x_n) = B$.

$$\text{A} = Y_0'(x_0) = c_0 = \frac{y_1-y_0}{h_0} - \frac{2h_0 s_0'' + h_0 s_1''}{6} = \text{A} \Rightarrow \boxed{2h_0 s_0'' + h_0 s_1'' = 6\left(\frac{y_1-y_0}{h_0} - \text{A}\right)}$$

$$\begin{aligned} \text{B} = Y_{N-1}'(x_n) &= 3a_{n-1}h_{n-1}^2 + 2b_{n-1}h_{n-1} + c_{n-1} \\ &= 3\left(\frac{s_n - s_{n-1}}{6h_{n-1}}\right)h_{n-1}^2 + s_{n-1}h_{n-1} + \frac{y_n - y_{n-1}}{h_{n-1}} - \left(\frac{2h_{n-1}s_{n-1}'' + h_{n-1}s_n''}{6}\right) \end{aligned}$$

$$6\left(B - \frac{y_n - y_{n-1}}{h_{n-1}}\right) = 3h_{n-1}s_n'' - 3h_{n-1}s_{n-1}'' + 6h_{n-1}s_{n-1}'' - 2h_{n-1}s_{n-1}'' - h_{n-1}s_n''$$

$$\boxed{6\left(B - \frac{y_n - y_{n-1}}{h_{n-1}}\right) = 2h_{n-1}s_{n-1}'' + 2h_{n-1}s_n''}$$

But $h_0 s_0'' + (2h_0 + 2h_1)s_1'' + h_1 s_2'' = 6\{[f[x_1, x_2] - f[x_0, x_1]\}$

$$\begin{bmatrix} 2h_0 & h_0 & & \\ h_0 & (2h_0+2h_1) & h_1 & \\ & & & \\ & h_{n-2} & (2h_{n-2}+2h_{n-1}) & h_{n-1} \\ & & h_{n-1} & 2h_{n-1} \end{bmatrix} \begin{bmatrix} s_0'' \\ s_1'' \\ \\ s_{N-1}'' \\ s_N'' \end{bmatrix} = 6 \begin{bmatrix} (y_1-y_0)/h_0 - \text{A} \\ (y_2-y_1)/h_1 - \frac{y_1-y_0}{h_0} \\ \vdots \\ \vdots \\ (y_n - y_{n-1})/h_{n-1} - (y_{n-1}-y_{n-2})/h_{n-2} \\ \text{B} - \frac{y_n - y_{n-1}}{y\,h_{n-1}} \end{bmatrix}$$

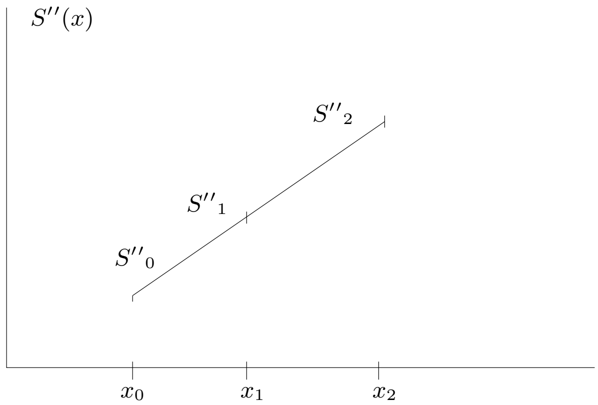
3. **Quadratic boundary interval representation**

$$s_0'' = s_1'' \qquad s_{N-1}'' = s_N''$$

$$\begin{array}{cc} a_0 = \left(\frac{s_1''-s_0''}{6h_0}\right) = 0 & a_{n-1} = \frac{s_N''-s_{N-1}''}{6h_{N-1}} \\ \therefore \text{ a quadratic} & \text{a quadratic} \end{array}$$

$$\begin{bmatrix} (3h_0+2h_1) & h_1 & 0 & & \\ h_1 & (2h_1+2h_2) & h_2 & 0 & 0 \\ & & & (2h_{N-3}+2h_{N-2})h_{N-2} & \\ & & & h_{N-2}(2h_{N-2}+3h_{N-1}) & \end{bmatrix} \begin{bmatrix} s_1'' \\ s_2'' \\ \vdots \\ s_{N-1}'' \end{bmatrix} = 6 \begin{bmatrix} f[x_1,x_2]-f[x_0,x_1] \\ \\ \\ f[x_{N-1},x_N]-f[x_{N-2},x_{N-1}] \end{bmatrix}$$

4. **Linear extrapolation**



$$\begin{aligned} &\frac{s_1''-s_0''}{h_0} = \frac{s_2''-s_1''}{h_1} \Rightarrow s_0'' = s_1'' - (s_2''-s_1'')\frac{h_0}{h_1} = s_1''\frac{(h_1+h_0)}{h_1} - s_2''\frac{h_0}{h_1} \\ &h_0 s_0'' + (2h_0+2h_1)s_1'' + h_1 s_1'' = 6(f[x_1,x_2]-f[x_0,x_1]) \\ &h_0\frac{(h_0+h_1)}{h_1}s_1'' - s_2''\frac{h_0^2}{h_1} + \frac{h_1}{h_1}2(h_0+h_1)s_1'' + h_1 s_2'' = \text{ RHS} \\ \therefore\ &\frac{(h_0+h_1)}{h_1}(h_0+2h_1)s_1'' + \left(\frac{h_1^2-h_0^2}{h_1}\right)s_2'' = \text{ RHS} \end{aligned}$$

$$\begin{bmatrix} \frac{(h_0+h_1)}{h_1}(h_0+2h_1) & \frac{(h_0^2-h_0^2)}{h_1} & 0 & \dots \\ h_1 & (2h_1+2h_2) & h_2 & 0 \\ 0 & & & \end{bmatrix} \begin{bmatrix} s_1'' \\ s_2'' \\ \vdots \end{bmatrix}$$

5. **Not-a-knot condition**

$Y_0(x) \equiv Y_1(x) \qquad Y_{N-2}(x) \equiv Y_{N-1}(x)$

Unknowns $= 4(N-2)$
Constraints $= 3(N-3)$

$4(N-2) - 3(N-3) = N+1 \qquad$ unknowns $\longleftrightarrow$ the values at $x_0, \ldots, x_N$.

$$s''_{i-1}h_{i-1} + 2(h_{i-1}+h_i)s''_i + h_i s''_{i+1} \quad = \quad 6(f[x_1,x_2] - f[x_0,x_1])$$

$$Y_1(x) = a_1(x-x_1)^3 + b_1(x-x_1)^2 + c_1(x-x_1) + d_1$$

$$Y'_1(x) = 3a_1(x-x_1)^2 + b_1(x-x_1) + c_1$$

$$Y''_1(x) = 6a_1(x-x_1) + 2b_1, \qquad x_1 - x_0 = h_0$$

$$s''_0 = Y''_1(x_0) = 6\left(\frac{s''_2 - s''_1}{6h_1}\right)(x_0 - x_1) + 2(s''_1/2)$$

$$= -(s''_2 - s''_1)\frac{h_0}{h_1} + s''_1$$

$$\therefore s''_0 = s''_1\frac{(h_1+h_0)}{h_1} - s''_2\frac{h_0}{h_1}$$

But $\; h_0 s''_0 + 2(h_0+h_1)s''_1 + h_1 s''_2 = 6(f[x_1,x_2] - f[x_0,x_1])$

$$\therefore \left(2 + \frac{h_0}{h_1}\right)(h_0+h_1)s''_1 + \left(1 - \frac{h_0}{h_1}\right)(h_0+h_1)s''_2 = 6(f[x_1,x_2] - f[x_0,x_1])$$

Also $\left(2 + \frac{h_{N-1}}{h_{N-2}}\right)(h_{N-2}+h_{N-1})s''_{N-1} + \left(1 - \frac{h_{N-1}}{h_{N-2}}\right)(h_{N-2}+h_{N-1})s''_{N-2} = 6(f[x_{N-1},x_N] - f[x_{N-2},x_{N-1}])$

$$\begin{bmatrix} (2+\frac{h_0}{h_1})(h_0+h_1) & (1-\frac{h_0}{h_1})(h_1+h_0) & 0 & \ldots & 0 & \\ h_1 & 2(h_1+h_2) & h_2 & 0 & \ldots & 0 \\ 0 & & & & & \\ & & & & & \\ & & & h_{N-2} & & \\ & & (1-\frac{h_{N-1}}{h_{N-2}})(h_{N-2}+h_{N-1}) & (2+\frac{h_{N-1}}{h_{N-2}})(h_{N-2}+h_{N-1}) & & \end{bmatrix} \begin{bmatrix} s''_1 \\ s''_2 \\ \vdots \\ s''_{N-2} \\ s''_{N-1} \end{bmatrix}$$

6. **A periodic spline**:

$$s''(x_{0^+}) = s''(x_{N^-})$$

$$s''(x_{0^+}) = s''(x_{N^-}).$$

(III) **Smoothness property of a natural spline** $\quad s''(x_0) = 0 = s''(x_N)$

In order to impose this condition it is convenient to consider the s''_k as unknowns in which case the equations become:

$$\Delta x_k s''_{k-1} + 2(\Delta x_k + \Delta x_{k+1}) s''_k + \Delta x_{k+1} s''_{k+1} = 6(f[x_k, x_{k+1}] - f[x_{k-1}, x_k]) \qquad k = 1, \ldots, N-1.$$

Important identity: $(y'' - s'')[(y'' - s'') + 2s''] = (y'' - s'')(y'' + s'') = (y'')^2 - (s'')^2$

NOTE:

(1) Let $y(x)$ be any other interpolant of $f(x)$ at $(x_0, \ldots, x_N)$ then

$$\int_a^b (y'')^2 dx - \int_a^b (s''^2) dx = \int_a^b (y'' - s'')^2 dx + 2 \int_a^b s''(y'' - s'') dx.$$

$$\text{Now} \int_a^b s''(y'' - s'') dx = s''(y' - s')\Big|_a^b - \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} s'''(y' - s') dx \qquad s''' = \text{ const on each subinterval}$$

$$= s''(y' - s')\Big|_a^b - \sum_{k=1}^{N} c_k (y - s)\Big|_{x_{k-1}}^{x_k} \quad \text{both interpolants}$$

$$= s''(y' - s')\Big|_a^b$$

If we choose $s : s''(a) = 0 = s''(b)$ for example or if y and s both interpolate f' at a and b then $\int_a^b s''(y'' - s'') dx = 0$.

$$\therefore \int_a^b y''^2 dx = \int_a^b (y'' - s'')^2 dx + \int_a^b (s'')^2 dx \geq \int_a^b s''^2 dx.$$

Thus s is the interpolant with the **MINIMUM CURVATURE** (i.e. smoothest)

(2) **Error involved in spline interpolation**:

$$|f(x) = s(x)| \leq ||f^{(4)}||_\infty 5 \frac{\Delta \bar{x}^4}{384} \qquad \Delta \bar{x} = \max \Delta x : \qquad \text{for nonuniform points and all } x \in [x_0, x_N]$$

$$|f'(x) - s'(x)| \leq ||f^{(4)}||_\infty \frac{\Delta \bar{x}^3}{24} \qquad \text{for nonuniform points and all } x \in [x_0, x_N]$$

$$|f'(x_i) - s'(x_i)| \leq ||f^{(5)}||_\infty \frac{\Delta x^4}{60} \qquad \text{for uniform sample points } x_i$$

$\therefore$ cubic spline provides an excellent technique for **NUMERICAL DIFFERENTIATION**.

How do we solve for the $\{s_k''\}$?

$$\begin{bmatrix} d_1 & c_1 & & & & \\ a_2 & d_2 & c_2 & & 0 & \\ 0 & a_3 & d_3 & & & \\ & & & \ddots & \ddots & \\ & & 0 & & d_{n-1} & c_{n-1} \\ & & & \ddots & & \\ & & & & a_n & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \\ \\ \\ \\ b_n \end{bmatrix}$$

$$d_1 \neq 0 \qquad d_1 x_1 + c_1 x_2 = b_1 \tag{4.1}$$

$$a_2 x_1 + d_2 x_2 + c_2 x_3 = b_2 \tag{4.2}$$

$$(2) - \frac{a_2}{d_1}(1) \Rightarrow \qquad \left(d_2 - \frac{a_2}{d_1} c_1\right) x_2 + c_2 x_3 = b_2 - \frac{a_2}{d_1} b_1$$

$$d_2' x_2 + c_2 x_3 = b_2' \qquad d_2' = d_2 - \frac{a_2}{d_1} c_1$$

$$b_2' = b_2 - \frac{a_2}{d_1} b_1$$

Similarly

$$d_{k+1}' x_{k+1} + c_{k+1} x_{k+2} = b_{k+1}' \tag{4.3}$$

$$\text{where } \ d_{k+1}' = d_{k+1} - \frac{a_{k+1}}{d_k'} c_k \qquad\qquad b_{k+1}' = b_{k+1} - \frac{a_{k+1}}{d_k'} b_k' \tag{4.4}$$

$$\begin{bmatrix} d_1' & c_1 & & & \\ & d_2' & c_2 & & \\ & & & d_{n-1}' & c_{n-1} \\ & & & & d_n' \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1' \\ b_2' \\ \\ b_n' \end{bmatrix}$$

Back Substitution

$$x_n = b_n'/d_n'$$

$$x_{n-1} = (b_{n-1}' - c_{n-1} x)/d_{n-1}'$$

$$x_n = b_n'/d_n' \tag{4.5}$$

$$x_k = (b_k' - c_k x_{k+1})/d_k' \qquad k = n-1, \ldots, 1 \tag{4.6}$$

Note, if we let $m_k = \frac{a_k}{d'_{k-1}}$ then

$$A = \begin{bmatrix} 1 & \ddots & 0 \\ m_2 & \ddots & \\ & & \\ 0 & & \\ & & m_n 1 \end{bmatrix} \begin{bmatrix} d'_1 & c_1 & 0 \\ & \ddots & \\ 0 & \ddots & c_{n-1} \\ & & d'_n \end{bmatrix} = LU \begin{bmatrix} & LU & \\ \text{Forward} & Ly = b & y = L^{-1}b \\ \text{Back} & Ux = y & \\ \text{Substitution} & & \end{bmatrix}$$