Problem Set 2

MIT CW Linear Algebra (18.06)

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February 13, 2021

Section 2.5

Problem 24

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \tag{1}$$

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & b & 1 & -a & ac \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \tag{3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \tag{4}$$

Problem 39

$$\left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{5}$$

$$\left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{6}$$

$$\left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{7}$$

$$\left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{8}$$

Section 2.6

Problem 13

$$\left[\begin{array}{cccc|cccc} a & a & a & a & 1 & 0 & 0 & 0 \\ a & b & b & b & 0 & 1 & 0 & 0 \\ a & b & c & c & 0 & 0 & 1 & 0 \\ a & b & c & d & 0 & 0 & 0 & 1 \end{array}\right] \tag{9}$$

$$\left[\begin{array}{cccc|cccc} a & a & a & a & 1 & 0 & 0 & 0 \\ 0 & b-a & b-a & b-a & -1 & 1 & 0 & 0 \\ 0 & b-a & c-a & c-a & -1 & 0 & 1 & 0 \\ 0 & b-a & c-a & d-a & -1 & 0 & 0 & 1 \end{array}\right] \tag{10}$$

$$\left[\begin{array}{cccc|cccc} a & a & a & a & 1 & 0 & 0 & 0 \\ 0 & b-a & b-a & b-a & -1 & 1 & 0 & 0 \\ 0 & 0 & c-b & c-b & 0 & -1 & 1 & 0 \\ 0 & 0 & c-b & d-b & 0 & -1 & 0 & 1 \end{array}\right] \tag{11}$$

$$\left[\begin{array}{cccc|cccc} a & a & a & a & 1 & 0 & 0 & 0 \\ 0 & b-a & b-a & b-a & -1 & 1 & 0 & 0 \\ 0 & 0 & c-b & c-b & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & d-c & 0 & 0 & -1 & 1 \end{array}\right] \tag{12}$$

so

$$U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} \tag{13}$$

And L is calculated as the inverse of the operations

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{14}$$

Problem 18

$$LDU = L_1 D_1 U_1 \tag{15a}$$
$$L_1^{-1} LDU = L_1^{-1} L_1 D_1 U_1 \tag{15b}$$
$$L_1^{-1} LDU = D_1 U_1 \tag{15c}$$
$$L_1^{-1} LDUU^{-1} = D_1 U_1 U^{-1} \tag{15d}$$
$$L_1^{-1} LD = D_1 U_1 U^{-1} \tag{15e}$$

First let's prove that L_1^{-1} is lower triangular. if we write

$$L^{-1} = \begin{bmatrix} \boldsymbol{y^1} & \boldsymbol{y^2} & \dots & \boldsymbol{y^n} \end{bmatrix} \tag{16}$$

where each $\boldsymbol{y^k}$ is an $nx1$ matrix. Then by definition,

$$LL^{-1} = I \begin{bmatrix} \boldsymbol{e^1} & \boldsymbol{e^2} & \dots & \boldsymbol{e^n} \end{bmatrix} \tag{17}$$

where $\boldsymbol{e^k}$ is the $nx1$ matrix with a 1 in the kth row and 0s everywhere else. So

$$L\boldsymbol{y^k} = \boldsymbol{e^k}(1 <= k <= n) \tag{18}$$

Let's look at the rows that are above k, for example the first row

$$e_1^k = \sum_{i=1}^{n} L_{1,i} * y_i^k = L_{1,1} * y_1^k \tag{19}$$

$e_1^k = 0$ and also $L_{1,1} \neq 0$ implies that $y_1^k = 0$ One more look at the second row... -

$$e_2^k = \sum_{i=1}^{n} L_{2,i} * y_i^k = L_{2,1} * y_1^k + L_{2,2} * y_1^k + 0 + \dots + 0 \tag{20}$$
$$= L_{2,1} * 0 + L_{2,2} * y_1^k + 0 + \dots + 0 \tag{21}$$
$$\tag{22}$$

So again $e_2^k = 0$ and also $L_{2,2} \neq 0$ implies that $y_2^k = 0$ We can continue this way till the k'th row; all $y_i^k = 0$ for $i < k$. Which means in other words that L^{-1} is a lower triangular matrix.

Now let's move on with the proof. $L_1^{-1}L$ is lower triangular because L is triangular and by multiplying it with a left side matrix that is lower triangular and thus doesn't mess with the upper part of L and leaves it zero. Multiplying it with D on the right side - we still maintain this property. So yes $L_1^{-1}LD$ is lower triangular. U^{-1} is upper triangular because of similar considerations. It's easy to show that multiplying two upper triangular matrices gives an upper triangular.

So the equation $L1^{-1}LD = D1U1U^{-1}$ is stating that a lower triangular matrix equals an upper triangular matrix. This is possible only if both sides are actually a diagonal matrix. Both side multiply with a diagonal matrix from which you can deduce that $L1^{-1}L$ is diagonal. Since both $L1^{-1}$ and L have 1's on the diagonal by definition then we have a diagonal matrix which has 1's on the diagonal - which means $L1^{-1}L = I$ which implies $L = L1$. In much the same way it can be deduced that also $U = U1$. And if that is the case then it's easy to replace $L1^{-1}LD = D1U1U^{-1}$ by $ID = D1I$ and deduce that $D = D1$.

Problem 25

I used the following code:

Listing 1: Insert code directly in your document

```
import numpy as np
from scipy.linalg import toeplitz

K = toeplitz([2, -1, 0, 0, 0], [2,-1,0,0,0]).astype('float64')
print(K)

U=K
L=np.zeros((5,5)).astype('float64')
np.fill_diagonal(L,1)

for i in range(1,5):
    L[i,i-1] = U[i,i-1]/U[i-1,i-1];
    U[i, i] = U[i,i] + U[i,i-1]/U[i-1,i-1]
    U[i, i-1] = 0

print("U = {}".format(U))
print("L = {}".format(L))

LU = np.matmul(L,U)
print("LU = {}".format(LU))


LM1 =np.linalg.inv(L)

print("LM1 = {}".format(LM1))

LM1F = np.zeros((5,5))
```

```
for i in range(0,5):
    for j in range (0,5):
        if i==j:
            LM1F[i,j]=1
        elif i<j:
            LM1F[i,j] = 0
        else:
            LM1F[i, j] = (j+1)/(i+1)

print("L1MF = {}".format(LM1F))
```

And the formula is (adjusting to 1..n format instead of 0..(n-1):

$$f(x) = \begin{cases} 1, & \text{if } i = j \\ \frac{j}{i}, & \text{if } i \geq j \\ 0, & \text{otherwise} \end{cases}$$

Problem 26

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 10 & 8 & 6 & 4 & 2 \\ 4 & 8 & 12 & 9 & 6 & 3 \\ 3 & 6 & 9 & 12 & 8 & 4 \\ 2 & 4 & 6 & 8 & 10 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix} \tag{23}$$

Section 2.7

Problem 13

a

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{24}$$

b

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

Problem 35 (former 36)

1. lower triangular matrix with 1's on the diagonal.
   I shall provide here than a lower triangular matrix stays in the group under multiplication and inverse operations. We shall divide the proof into two parts:

   (a) Proof that multiplying lower triangular matrix with 1 on the diagonal are still lower triangular with 1's on the diagonal.

      i. above the diagonal
         Let $L1$ and $L2$ be such lower triangular matrices. If we take a look at $L1 \cdot L2_{i,j}$ where i¡j then we see:

$$\begin{aligned}
(L1L2)_{i,j} &= sum_{k=1}^{k=n} L1_{i,k}L2_{k,j} && (26)\\
&= sum_{k=1}^{k=i-1} L1_{i,k}L2_{k,j} + sum_{k=i}^{k=j-1} L1_{i,k}L2_{k,j} + sum_{k=j}^{k=n} L1_{i,k}L2_{k,j} && (27)\\
&= sum_{k=1}^{k=i-1} L1_{i,k} * 0 + sum_{k=i}^{k=j-1} 0 * 0 + sum_{k=j}^{k=n} 0 * L2_{k,j} && (28)\\
= 0 & && (29)
\end{aligned}$$

         Which means that indeed that above the diagonal we have 0's.

      ii. on the diagonal
         If we take a look at $L1 \cdot L2_{i,i}$ then we see:

$$\begin{aligned}
(L1L2)_{i,i} &= sum_{k=1}^{k=n} L1_{i,k}L2_{k,j} && (30)\\
&= sum_{k=1}^{k=i-1} L1_{i,k}L2_{k,j} + L1_{i,i}L2_{i,i} + sum_{k=i+1}^{k=n} L1_{i,k}L2_{k,i} && (31)\\
&= sum_{k=1}^{k=i-1} L1_{i,k} * 0 + 1 * 1 + sum_{k=i+1}^{k=n} 0 * L2_{k,i} && (32)\\
&= 1 && (33)
\end{aligned}$$

         which means that each of the diagonal elements equals 1

   (b) Proof that the inverse of a low triangular matrix with 1 on the diagonal is still a lower triangualr with 1's on the diagonal.
      Proof by induction

      i. it's true for every 2x2 lower diagonal with 1's on the diagonal.
         let's see

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ a & 1 & 0 & 1 \end{array}\right] \quad (34)$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -a & 1 \end{array}\right] \tag{35}$$

So the inverse exists and it's a lower triangular with 1's on the diagonal

ii. suppose it's true for a all matrices of size nxn. Let's see the induction step for matrices of size (n+1)x(n+1).
So suppose that L and L^{-1} are such lower triangular matrices with 1's on the diagonal. Let's build the the following $(n+1)x(n+1)$ dimension matrix as a lower triangular with 1's on the diagonal.

$$\begin{bmatrix} 1 & \boldsymbol{0^T} \\ \boldsymbol{b} & L \end{bmatrix} \cdot \begin{bmatrix} x & \boldsymbol{c^T} \\ \boldsymbol{d} & M \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{0^T} \\ \boldsymbol{0} & I_{nxn} \end{bmatrix} \tag{36}$$

where both $\boldsymbol{b}$ and $\boldsymbol{0}$ are vectors of dimensions $nx1$ When calculating the top left element of the $I_{n+1,n+1}$ matrix on the right, we noticed that x must equal 1. When calculating the top right vector of the $I_{n+1,n+1}$ matrix on the right, we noticed that $/mybfc^T$ must equal 0. This means that the inverse is lower triangular with 1's on the diagonal as well.

2. symmetric matrices
multiplying symmetric matrices not necessarily yield a symmetric matrix. For example

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 20 & 11 \end{bmatrix} \tag{37}$$

which is not a symmetric matrix

3. positive matrices
Are not a closed group. For example inverse of such matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{38}$$

is a positive matrix. However the inverse matrix here is

$$\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} \tag{39}$$

Which has negative values

4. diagonal matrices
When multiplying two diagonal matrices $D1$ and $D2$ we get D which is diagonal because $D_{i,j}$ is a dot product between row i of matrix $D1$ and column j of matrix $D2$ and when $i/neqj$ the result is 0. Also if the inverse of a diagonal matrix D is D^{-1} then trivially $D^{-1}_{i,i} = \frac{1}{D_{i,i}}$

5. permutation matrices
   Stay inside the group because by definition a permutation matrix, when multiplied by another permutation matrix - gives a different permutation matrix.

6. matrices where $Q^{-1} = Q^T$
   The first question is whether Q^{-1} stays inside the group. That is if we take the inverse and take the transpose of Q^{-1} then we get the same result.

$$Q^{-1^{-1}} = Q^{T^{-1}} = Q^{-1^T} \tag{40}$$

   The second question is if Q and R are such matrices. Does QR meets the same?

$$QR^{-1} = R^{-1} \cdot Q^{-1} = R^T \cdot Q^T = (QR)^T \tag{41}$$

7. two more matrix groups:
   (a) upper triangular with 1's on the diagonal
   (b) I matrices

Problem 39 (former 40)

a $(QQ^T)_{i,i}$ are by definition $||\boldsymbol{q_i}||^2$. However in our case $QQ^T = QQ^{-1} = I$ where all the elements on the diagonal equal 1.

b $(QQ^T)_{i,j}$ are the dot product between row i and row j of the Q matrix. In much the same way - $QQ^T = QQ^{-1} = I$ which indicates that for every $i \neq j$ the dot product is 0, that is $\boldsymbol{q_i}^T \boldsymbol{q_j} = 0$.

c

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -cos\theta \end{bmatrix} \tag{42}$$

Section 3.1

Problem 18

a True. If M is a symmetric matrix and $A = c \cdot M$ then $A_{i,j} = c \cdot M_{i,j} = c * M_{j,i} = A_{j,i}$.
Also if $M1$ and $M2$ are symmetric matrices and $A = M1 + M2$ then $A_{i,j} = M1_{i,j} + M2_{i,j} = M1_{j,i} + M2_{j,i} = A_{j,i}$.

b True. If M is a skew symmetric matrix and $A = c \cdot M$ then for $i \neq j$, $A_{i,j} = c \cdot M_{i,j} = c * -M_{j,i} = -A_{j,i}$.
Also if $M1$ and $M2$ are both skew symmetric matrices and $A = M1 + M2$ then $A_{i,j} = M1_{i,j} + M2_{i,j} = -M1_{j,i} - M2_{j,i} = -A_{j,i}$.

c False. For example, we add two such matrices and get a matrix that's outside of the sub space:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix} \tag{43}$$

Problem 18

If we add an extra column $\boldsymbol{b}$ to a matrix A, then the column space gets larger unless the additional column is not a linear combination of the existing columns of the matrix.
An example where the column space grows as result of the addition of an extra column

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \tag{44}$$

An example where the column space doesn't as result of the addition of an extra column

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{45}$$

When the additional column is a linear combination of the other columns of A the additional column doesn't add to the space covered by the existing columns. At that very moment, the $\boldsymbol{x}$ is the linear combination that leads to that $\boldsymbol{b}$. This is the case for A and this is also the case for $[\,A\ \boldsymbol{b}\,]$ where $\boldsymbol{b}$ can be either added to the linear combination multiplied by 0 or all the X's are 0 except for 1 that multiplies the $\boldsymbol{b}$.

Problem 30

a Suppose $\boldsymbol{u}$ is a vector in the $\boldsymbol{sum\, S + T}$ space. It means that there is is a pair of vectors $\boldsymbol{s}$ in subspace S and $\boldsymbol{t}$ in subspace T such that $\boldsymbol{s}+\boldsymbol{t}$ is in $\boldsymbol{sum\, S + T}$ space. Now $c \cdot \boldsymbol{u} = c \cdot \boldsymbol{s} + c \cdot \boldsymbol{t}$ where both $c \cdot \boldsymbol{s}$ and $c \cdot \boldsymbol{t}$ are by definition in subspace S and subspace T respectively. Also suppose that there are two vectors $\boldsymbol{u1}$ and $\boldsymbol{u2}$ in the $\boldsymbol{sum\, S + T}$ space. So there exist $\boldsymbol{s1}$ and $\boldsymbol{s2}$ in subspace S and $\boldsymbol{t1}$ and $\boldsymbol{t2}$ in subspace T such that $\boldsymbol{s1} + \boldsymbol{t1} = \boldsymbol{u1}$ and $\boldsymbol{s2} + \boldsymbol{t2} = \boldsymbol{u2}$. Adding $\boldsymbol{u1} + \boldsymbol{u2} = \boldsymbol{s1} + \boldsymbol{t1} + \boldsymbol{s2} + \boldsymbol{t2} = \boldsymbol{s1} + \boldsymbol{s2} + \boldsymbol{t1} + \boldsymbol{t2}$ which is an addition of two vectors: $\boldsymbol{s1} + \boldsymbol{s2}$ which belongs by definition to S space and $\boldsymbol{t1} + \boldsymbol{t2}$ which belongs by definition to T space. Which proves that $\boldsymbol{u1} + \boldsymbol{u2}$ belongs to $\boldsymbol{sum\, S + T}$

b $\boldsymbol{s}$ is a vector of size m (that crosses $\mathbf{0}$). There are many such vectors on the same line and this group of vectors is called S. Same goes for $\boldsymbol{t}$ belonging to a line T that could be a totally different line. There's no meaning to 'add lines' so all I can think of is the addition of a vector in S with a vector in T creating what could belong to a totally different line $\boldsymbol{sum\,S+T}$. As for $S \cup T$, it contains all the lines S and all the lines T. So the span should be the set of all vectors that could be created by adding vectors from $S \cup T$.

Problem 32

$AB = A \cdot [\boldsymbol{b^{(1)}}, \boldsymbol{b^{(2)}}, \ldots \boldsymbol{b^{(b)}}] = [A \cdot \boldsymbol{b^{(1)}}, A \cdot \boldsymbol{b^{(2)}}, \ldots A \cdot \boldsymbol{b^{(n)}}]$

and each $A \cdot \boldsymbol{b^{(k)}}$ is a linear combination of the columns of A so it doesn't add any new information and thus doesn't extend the space covered by A.

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{46}$$

So originally the column space in the matrix A was 2 and it was reduced to 1 when squaring up.
An n by n matrix has $\boldsymbol{C}(A) = R^n$ exactly when A is an invertible matrix.