Problem Set 3

MIT CW Linear Algebra (18.06)

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Section 3.2

Problem 13 (former problem 18)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

Problem 18 (former problem 24)

I don't know.

Problem 30 (former problem 36)

This was my wrong solution: if A has a rank of r. That means that the null space of A can be represented as a linear combination of $n-r$ vectors. Any additional row that is contributed by B can be either dependent on the previous rows or not and thus either leaves the rank at r or increase it to $r+1$ which means that the sub space looses one degree of freedom. So $N(C) \subset N(A)$

The correct solution is that the null space of A should take

$$\begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix} = \mathbf{0} \tag{2}$$

so both $A\boldsymbol{x_1}$ AND $B\boldsymbol{x_2}$ should equal 0 and $N(C) = N(A) \cap N(B)$

Problem 32 (former problem 37)

$$\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

Here are the special vectors:

$$\begin{bmatrix} 1 \\ 1 \\ \mathbf{0} \\ -1 \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \mathbf{0} \\ -1 \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \mathbf{1} \\ 0 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{9}$$

Section 3.3

worked example 3.3 A

1.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right] \tag{10}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2 * b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3 * b_1 \end{array}\right] \tag{11}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2 * b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 5b_1 \end{array}\right] \tag{12}$$

2. $b_2 + b_3 - 5b_1 = 0$

3. if there was no $\boldsymbol{b}$ or in other words $\boldsymbol{b} = \mathbf{0}$ then we are talking about all linear combinations of $A \cdot \boldsymbol{x}$ without restrictions. However in this case there's a restriction $A \cdot \boldsymbol{x}$ must equal $\boldsymbol{b}$. Actually $\boldsymbol{b}$ is a linear combination and the only restriction that it imposes is $b_2+b_3-5b_1 = 0$. So the plane in $\boldsymbol{R}^3$ are all the $\boldsymbol{b}$ that meets $b_2 + b_3 - 5b_1 = 0$.

4. first let's continue from a u form to an R form of the matrix:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 1 & 1 & \frac{b_2 - 2 * b_1}{2} \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 5b_1 \end{array}\right] \tag{13}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & b_1 - \frac{3(b_2 - 2 * b_1)}{2} \\ 0 & 0 & 1 & 1 & \frac{b_2 - 2 * b_1}{2} \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 5b_1 \end{array}\right] \tag{14}$$

$$\tag{15}$$

first let's find the particular solution:

$$\boldsymbol{x_{particular}} = \begin{bmatrix} 4b_1 - \frac{3b_2}{2} \\ 0 \\ \frac{b_2 - 2 * b_1}{2} \\ 0 \end{bmatrix} \tag{16}$$

And now for the special solution:

$$s1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \tag{17}$$

And the null space

$$c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \tag{18}$$

5. I did before... and already got

$$R = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & b_1 - \frac{3(b_2 - 2*b_1)}{2} \\ 0 & 0 & 1 & 1 & \frac{b_2 - 2b_1}{2} \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 5b_1 \end{array}\right] \tag{19}$$

with the particular solution: first let's find the particular solution:

$$\boldsymbol{x_{particular}} = \begin{bmatrix} 4b_1 - \frac{3b_2}{2} \\ 0 \\ \frac{b_2 - 2*b_1}{2} \\ 0 \end{bmatrix} \tag{20}$$

6. if we assign $(b_1, b_2, b_3) = (0, 6, -6)$ to the $\boldsymbol{x_{particular}}$ then we get

$$\boldsymbol{x_{particular}} = \begin{bmatrix} 4b_1 - \frac{3b_2}{2} \\ 0 \\ \frac{b_2 - 2*b_1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} \tag{21}$$

The complete solution:

$$\boldsymbol{x} = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \tag{22}$$

worked example 3.3 B

1. $m >= n = r$.

2. m is arbitrary, $n = 2$, $r = 1$, and $\boldsymbol{b}$ is a vector of size m.
The particular solution requires that x_2 the free variable shall equal 0. So the whole solution can be expressed as:

$$\boldsymbol{x} = \boldsymbol{x_{particular}} + C\boldsymbol{X_{special}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{23}$$

We know that $R\boldsymbol{x_{particular}} = \boldsymbol{b}$. So we can deduce that R looks as follows:

$$R = \begin{bmatrix} \boldsymbol{b} & \boldsymbol{f} \end{bmatrix} \tag{24}$$

where we so far 'know' $\boldsymbol{b}$ which is the solution, but have no idea about f. We also know that the special solution has $x_2 = 1$:

$$\begin{bmatrix} b & f \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boldsymbol{0} \tag{25}$$

so from that we can learn that $\boldsymbol{f} = -\boldsymbol{b}$ so all in all we have here:

$$A = \begin{bmatrix} b & -b \end{bmatrix} \tag{26}$$

so basically the following tells the whole story:

$$\begin{bmatrix} \boldsymbol{b} & -\boldsymbol{b} \end{bmatrix} * [\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}] = \boldsymbol{b} \tag{27}$$

3. $m >= n$,$r < n$,Also $A\boldsymbol{x} != \boldsymbol{b}$

4. $n = 3$, m is arbitrary. if $C = 0$ then $\boldsymbol{b}$ is in the column space of A, otherwise $\boldsymbol{b}$ is not in the column space of A. $(1, 0, 1)$ should be in the null space of A - it means that column 1 of A is the negavie to column 3 of A. If the 2nd column is a multiple of columns 1 or 3 then $r = 1$, otherwise $r = 2$.

5. with infinitely many solutions, the null space must contain non zero solutions and $\boldsymbol{b}$ should be one of them. The rank r must be smaller than n.

worked example 3.3 C

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 2 & 4 & 4 & 8 & 2 \\ 4 & 8 & 6 & 8 & 10 \end{array}\right] \tag{28}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 4 & 2 & 8 & -6 \end{array}\right] \tag{29}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 1 & \frac{1}{2} & 2 & -1\frac{1}{2} \\ 0 & 0 & 1 & 4 & -3 \end{array}\right] \tag{30}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 7 \\ 0 & 1 & \frac{1}{2} & 2 & -1\frac{1}{2} \\ 0 & 0 & 1 & 4 & -3 \end{array}\right] \tag{31}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 7 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & -3 \end{array}\right] \tag{32}$$

$\boldsymbol{x_{particular}}$ is when the free variable $x_4 = 0$. So

$$\boldsymbol{x_{particular}} = \begin{bmatrix} 7 \\ 0 \\ -3 \end{bmatrix} \tag{33}$$

$\boldsymbol{x_{special}}$ is when the free variable $x_4 = 1$. So

$$\boldsymbol{x_{special}} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} \tag{34}$$

So,

$$\boldsymbol{x} = \boldsymbol{x_{particular}} + C \cdot \boldsymbol{X_{special}} = \begin{bmatrix} 7 \\ 0 \\ -3 \end{bmatrix} + C \cdot \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} \tag{35}$$

Let's denote $\boldsymbol{y} = (y_1, y_2, y_3)$. Instead of calculating $y_1(row1) + y_2(row2) + y_3(row3)$, we shall do $A^T\boldsymbol{y} = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 2 & 4 & 8 & b_2 \\ 1 & 4 & 6 & b_3 \\ 0 & 8 & 8 & b_4 \end{array}\right] \tag{36}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 2 & 2 & b_3 - b1 \\ 0 & 8 & 8 & b_4 \end{array}\right] \tag{37}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 2 & 2 & b_3 - b1 \\ 0 & 8 & 8 & b_4 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right] \tag{38}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & b_1 \\ 0 & 1 & 1 & \frac{b_3 - b1}{2} \\ 0 & 0 & 0 & b_4 - 4b_3 + 4b1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right] \tag{39}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2b_1 - b_3 \\ 0 & 1 & 1 & \frac{b_3 - b1}{2} \\ 0 & 0 & 0 & b_4 - 4b_3 + 4b1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{array}\right] \tag{40}$$

The solution for the transposed matrix can be expressed as

$$A^T = \boldsymbol{x_{particular}} + C \cdot \boldsymbol{x_{special}} \tag{41}$$
$$= \begin{bmatrix} 2b_1 - b_3 \\ \frac{b3-b1}{2} \\ 0 \end{bmatrix} + C \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \tag{42}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + C \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \tag{43}$$

Which means that for a zero solution $y_1 = -2$, $y_2 = -1$ and $y_3 = 1$. Let's check: $(4, 2, 10) \cdot (-2, -1, 1) = -8 - 2 + 10 = 0$ The $(4, 2, 10)$ is in the column space of A, so it's a linear combination of the other columns. So in much the same way it can be added as the 5th row to the A^T matrix and as such it should be a linear combination of the rows and thus adhere to $(r_1, r_2, r_3) \cdot (-2, -1, 1) = 0$ that every row (r_1, r_2, r_3) in the transposed matrix adheres to.

section 3.2 Problem 48 (former section 3.3 - problem 17)

1. The problem states that actually there is a vector $\boldsymbol{d}$ such that $B\boldsymbol{d} = B^{(j)}$ where $B^{(j)}$ is the j'th column of B. So

$$B\boldsymbol{d} = B^{(j)}$$
$$A(B\boldsymbol{d}) = AB^{(j)}$$
$$(AB)\boldsymbol{d} = AB^{(j)}$$
$$(AB)\boldsymbol{d} = (AB)^{(j)}$$

   or in other words the same linear combination $\boldsymbol{d}$ that was applied on the columns of $B$and created column j in B, is the same linear combination $\boldsymbol{d}$ that when applied on the columns of AB, creates column j in AB. So AB cannot indeed create a new pivot column out of thin air and indeed $rank(AB) <= rank(B)$.

2.

$$A1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{44}$$
$$A2 = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{45}$$

section 3.2 Problem 50 (former section 3.3 - problem 19)

Every column of AB is a linear combination of the columns of A. Thus, we can't find in AB a column which is not a linear combination of the columns

in A. So the column space of AB is a subset of the column space of A ($AB \subseteq A$) and the rank is thus lower equal.
$AB = I$ which means the rank of AB is n. Since the rank of A is greater equal then the rank of A is n as well which means A is invertible and thus B must be its two sided inverse, that is $AB = I$ as well as $BA = I$.

section 3.2 Problem 56 (former section 3.3 - problem 25)

A = (pivot columns of A) (first r rows of R) =

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix} \tag{46}$$

section 3.2 Problem 58 (former section 3.3 - problem 27)

a I_{rxr}, $F_{rx(n-r)}$, $0_{(m-r)xn}$

b

$$B = \begin{bmatrix}I\\0\end{bmatrix}$$

where I has dimensions r x r and 0 has dimensions n x r.

c

$$C = \begin{bmatrix}I & 0\end{bmatrix}$$

where I has dimensions r x r and 0 has dimensions $(m-r)$ x $(m-r)$.

section 3.2 Problem 60 (former section 3.3 - problem 28)

I don't understand the question

section 3.3 Problem 13 (former section 3.4 - problem 13)

a proof that it's not correct by example:

$$x_1 + 2x_2 = 3$$
$$\boldsymbol{x} = \boldsymbol{x_p} + C \cdot \boldsymbol{x_n} = \begin{bmatrix}3\\0\end{bmatrix} + C \cdot \begin{bmatrix}-2\\1\end{bmatrix}$$

however if we claim that

$$\boldsymbol{x} = C1 \cdot \boldsymbol{x_p} + C2 \cdot \boldsymbol{x_n} = C1 \cdot \begin{bmatrix}3\\0\end{bmatrix} + C2 \cdot \begin{bmatrix}-2\\1\end{bmatrix} \tag{47}$$

Then for $C1 = 2$ and $C2 = 1$ we shall get that the solution is:

$$\boldsymbol{x} = C1 \cdot \boldsymbol{x_p} + C2 \cdot \boldsymbol{x_n} = 2 \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{48}$$

or in other words: $x_1 = 4$ and $x_2 = 1$ and putting this back into the equation we get $x_1 + 2x_2 = 4 + 2 = 6 \neq 3$

b Again by using the same example:

$$x_1 + 2x_2 = 3 \tag{49}$$

One solution is $x_1 = 3$ and $x_2 = 0$ Another solution is $x_1 = 1$ and $x_2 = 1$. We demonstrated 2 solutions which is more than one. We can demonstrate endless more :)

c let's look at the following 2x2 set:

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{array}\right]$$

The solution for this is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

For the case of $C = 0$, that is with the free variable x_2 equals 0 - then $\boldsymbol{x_p} = (10)$ and the size is $\sqrt{1^2 + 0^2} = 1$ For the case of $C = \frac{1}{2}$ we have $x_1 = \frac{3}{4}$ and $x_2 = \frac{1}{2}$ with the size $= \sqrt{(\frac{3}{4})^2 + (\frac{1}{2})^2} = \sqrt{\frac{13}{16}} < 1$

d if A is invertible then there's on solution in the null space: $\boldsymbol{x_n} = \boldsymbol{0}$

section 3.3 Problem 25 (former section 3.4 - problem 25)

a no solutions can happen if $r < m$ (and the $\boldsymbol{b}$ elements are not zero for the rows without pivots.

b infinitely many solutions for every $\boldsymbol{b}$ is the case of $r = m < n$.

c exactly one solution for some $\boldsymbol{b}$ is the case of $r = n$ and $m > r$. If the $\boldsymbol{b}$ items beyond the first r are 0 then endless solutions. Otherwise no solution.

d exactly one solution for every b: $r = m = n$.

section 3.3 Problem 28 (former section 3.4 - problem 28)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \boldsymbol{x_n} = C \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

and then -

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -6 \\ 0 & 0 & 1 & 2 \end{array}\right] \boldsymbol{x_p} = \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix}$$

section 3.3 Problem 35 (former section 3.4 - problem 35)

Figure 1 charts the vector $\boldsymbol{x}$. I used the following python code:

Listing 1: Insert code directly in your document

```
import numpy as np
import matplotlib.pyplot as plt

K = np.zeros((9, 9))

K[0,0]=2
for i in range(0,8):
    K[i+1,i+1]=2
    K[i+1,i]=-1
    K[i, i+1] = -1
print(K)

b=10*np.ones((9,1))
print(b)

base =[i for i in range(1,10)]
x= np.matmul(np.linalg.inv(K),b)

plt.plot(base, x)
plt.show()
```

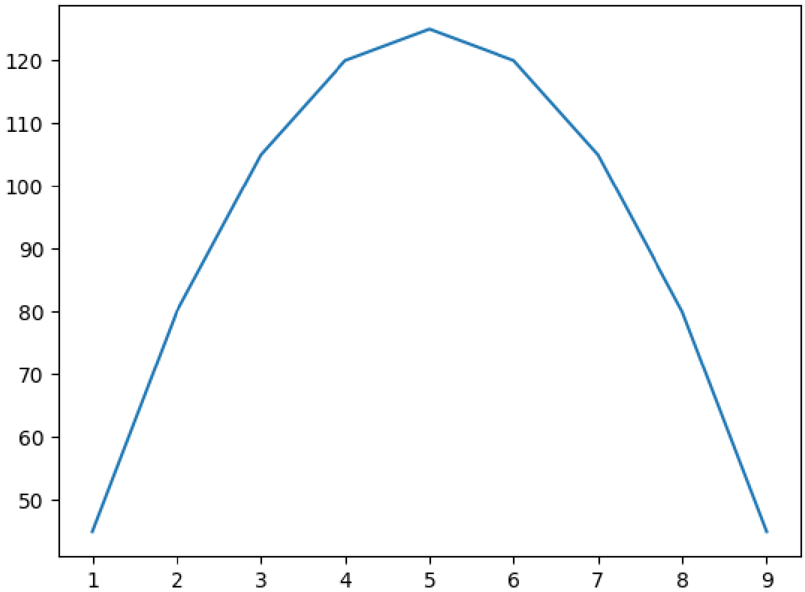

Figure 1: Drawing $\boldsymbol{x}$. Multiplying a 9x9 second difference matrix by $\boldsymbol{x}$ gives $\boldsymbol{b} = (10, 10, \ldots, 10)$

section 3.3 Problem 36 (former section 3.4 - problem 36)

My wrong answer: Not necessarily. Let's take for example

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \cdot \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{50}$$

vs.

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \cdot \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{51}$$

Same solutions however A doesn't equal C