Problem Set 9

MIT CW Linear Algebra (18.06)

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Section 6.5, question 25

We are told that $C = \sqrt{D}L^T$. Also $S = L\sqrt{D}\sqrt{D}L^T = (\sqrt{D}L^T)^T\sqrt{D}L^T = C^TC$
So

$$S = C^TC = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$$

if

$$S = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix}$$

to diagonalize it we subtract 2 times the first row from the second row so we get -

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

and we are left with this upper triangular matrix:

$$\begin{bmatrix} 4 & 8 \\ 0 & 9 \end{bmatrix}$$

We see 4 and 9 on the diagonal so we want to factor them out and we get

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

and of course

$$L^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

So

$$C = \sqrt{D}L^T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

And indeed

$$\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} * \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix}$$

And also of course if we run C= chol(S) we get the same result.

Section 6.5, question 26

1. We start with

$$S = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}$$

   We find the pivots by subtracting twice the second row from the first row so the L matrix in this case is"

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

   and we get the upper triangular:

$$\begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

   We can factor out the diagonals such as to be left with only 1's there using

$$D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

   and then

$$C = \sqrt{D}L^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

2. We start with

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}$$

   We find the pivots by

(a) subtracting the first row from the second row
(b) subtracting the first row from the third row
(c) subtracting the second row from the third row

so the L matrix in this case is"

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and we get the upper triangular:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

We can factor out the diagonals such as to be left with only 1's there using

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

and then

$$C = \sqrt{D}L^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$$

Section 6.5, question 27

$$\frac{1}{a^2}(ax+by)^2 + \frac{ac-b^2}{a}y^2$$

For a=2, b=4 and c=10 we are talking about

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}(4x+16y)^2 + 2y^2$$

Section 6.5, question 29

1. For the equation $F_1(x, y) = \frac{1}{4}x^4 + x^2y + y^2$ we shall replace $z = x^2$ and get $\frac{1}{4}z^4 + zy + y^2$ for which we can establish the following matrix:

$$S = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{bmatrix} = \frac{1}{2}(z+y)^2 + \frac{3}{2}y^2$$

The most possible minimal point is when both squares are empty and that happens when $z + y = 0$ as well as $y = 0$ which means both z and y are 0 and which also means $(x, y) = (0, 0)$.

2. For the equation $F_2(x, y) = x^3 + xy - x$ the second derivative matrix looks as follows:

$$\begin{bmatrix} 6x+1 & 1 \\ 1 & 0 \end{bmatrix}$$

The test for positive definite is that both the determinant of the upper left 1x1 matrix is positive and the 2x2 matrix determinant is positive which means

$$6x + 1 > 0 6x + 1 - 1 > 0$$

which is true only if $x > 0$. So at $x = 0$ the matrix is

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

which translates into $(x + \frac{1}{2}y)^2 - \frac{1}{4}y^2$ So we get a saddle at point $(x, y) = (0, 0)$. So only when $x > 0$ and $y > 0$ do we see this function going above 0.

Section 6.5, question 32

a AB can take two positive definite matrices and send them to a non positive definite matrix. Here's the example:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix} = -14$$

$$-410$$

b orthogonal matrices stay in the group. Suppose we have A and B and they are orthogonal matrices. That means that $AA^T = I$ and $BB^T = I$.
We want to see what happens when we multiply... - $(AB) * (AB)^T = ABB^TA^T = AIA^T = AA^T = I$. Also $(AB)^{-1}((AB)^{-1})^T = (AB)^{-1}((AB)^T)^{-1}$.
As for Q^{-1} We can check if it too is orthogonal and indeed it is because $Q^{-1} * (Q^{-1})^T = Q^{-1}(Q^T)^{-1} = (QQ^T)^{-1} = I^{-1} = I$

c exponentials stay within the group because $e^{tA}e^{tB} = e^{t(A+B)}$ which is inside the family. Same for the $e^{tA^{-1}} = e^{t(-A)}$.

d using the same example that I used for positive definite matrices, one can see that matrices P with positive eigenvalues can go out of the group when undergoing mutual multiplication.

e Matrices D with determinant 1 stay in the group because $\det(D_1 * D_2) = \det(D_1)\det(D_2)$. Similar argument for reciprocal.

f Orthonormal matrices, meaning all columns are orthogonal but not only that but also their size equals 1.

g subgroup of Matrices D: Matrices with determinant 1 and real eigenvalues.

Section 6.5, question 33

If we follow the instructions (which Ididn't...) then we have to start with

$$ST\boldsymbol{x} = \lambda\boldsymbol{x}$$

If we do a dot product with $T\boldsymbol{x}$ then we get -

$$(ST\boldsymbol{x})^T(T\boldsymbol{x}) = (\lambda\boldsymbol{x})^T(T\boldsymbol{x})$$
$$(T\boldsymbol{x})^T S(T\boldsymbol{x}) = \lambda\boldsymbol{x}^T T\boldsymbol{x}$$

Since both $(T\boldsymbol{x})^T S(T\boldsymbol{x})$ and $\boldsymbol{x}^T T\boldsymbol{x}$ are positive (because S and T are positive definite matrices) then λ must be positive too.

Section 6.5, question 34

$$Q = \begin{bmatrix} \sin\frac{1}{6}\pi & \sin\frac{2}{6}\pi & \sin\frac{3}{6}\pi & \sin\frac{4}{6}\pi & \sin\frac{5}{6}\pi \\ \sin\frac{2}{6}\pi & \sin\frac{4}{6}\pi & \sin\frac{6}{6}\pi & \sin\frac{8}{6}\pi & \sin\frac{10}{6}\pi \\ \sin\frac{3}{6}\pi & \sin\frac{6}{6}\pi & \sin\frac{9}{6}\pi & \sin\frac{12}{6}\pi & \sin\frac{15}{6}\pi \\ \sin\frac{4}{6}\pi & \sin\frac{8}{6}\pi & \sin\frac{12}{6}\pi & \sin\frac{16}{6}\pi & \sin\frac{20}{6}\pi \\ \sin\frac{5}{6}\pi & \sin\frac{10}{6}\pi & \sin\frac{15}{6}\pi & \sin\frac{20}{6}\pi & \sin\frac{25}{6}\pi \end{bmatrix}$$

It's too much time consuming for me to calculate this manually so that could be done with octave. As for the eigenvalues they are described in the worked example as $2 - 2\cos k\pi h)$ and in our case $2 - 2\cos\frac{k}{6}\pi$ for $k = 1\ldots 5$. And specifically $2-2\cos\frac{1}{6}\pi$, $2-2\cos\frac{2}{6}\pi$, $2-2\cos\frac{3}{6}\pi$, $2-2\cos\frac{4}{6}\pi$, $2-2\cos\frac{5}{6}\pi$ and of course the sum is 10 because there are 5 times 2 in these lambdas and the rest are summing up to $\cos\frac{1}{6}\pi + \cos\frac{2}{6}\pi + \cos\frac{3}{6}\pi + \cos\frac{4}{6} + \cos\frac{5}{6}\pi$. The first and the fifth sum is 0, the second and the fourth sum is 0 and the middle one is already zero.
The product is $(2-2\cos\frac{1}{6}\pi)(2-2\cos\frac{2}{6}\pi)(2-2\cos\frac{3}{6}\pi)(2-2\cos\frac{4}{6}\pi)(2-2\cos\frac{5}{6}\pi) = 32(\cos 0 - \cos\frac{1}{6}\pi)(\cos 0 - \cos\frac{2}{6}\pi)(\cos 0 - \cos\frac{3}{6}\pi)(\cos 0 - \cos\frac{4}{6}\pi)(\cos 0 - \cos\frac{5}{6}\pi) = 32(2\sin^2\frac{1}{12}\pi)(2\sin^2\frac{2}{12}\pi)(2\sin^2\frac{3}{12}\pi)(2\sin^2\frac{4}{12}\pi)(2\sin^2\frac{5}{12}\pi) = 1024 * (\sin\frac{1}{12}\pi\sin\frac{2}{12}\pi\sin\frac{3}{12}\pi\sin\frac{4}{12}\pi\sin\frac{5}{12}\pi)^2 = 6$

former Section 6.5, question 35

Well if C is positive definite then it can be expressed as $Q^T \Lambda Q$. Since C eigenvalues are positive then they are on Λ diagonal which Λ is too a positive definite matrix. If we take an arbitrary $\boldsymbol{x} =$ then

$$\boldsymbol{x}^T A^T C A \boldsymbol{x} = (A\boldsymbol{x})^T C (A\boldsymbol{x}) > 0$$

So $A^T C A$ is positive definite as well.

section 10.2 question 3(former Section 8.1, question 3)