Problem Set 3

MIT CW Linear Algebra (18.06)

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Worked example - 3.4A

a The vectors $v_1 = (1, 2, 0)$ and $v_2 = (2, 3, 0)$ are independent because the only combination $c_1 \cdot v_1 + c_2 \cdot v_2 = 0$ is $c_1 = c_2 = 0$

b they are a basis

c they span a plane $Z = 0$ in $\boldsymbol{R^3}$.

d they dimension of $\boldsymbol{V}$ is 2 because two independent vectors span it.

e the following is a matrix that has $\boldsymbol{V}$ as its column space:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 0 \end{bmatrix}$$

Also the following is a matrix that has $\boldsymbol{V}$ as its column space:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

But actually it could be any 3 by n matrix A of rank 2 where each column is a linear combination of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

f we are looking for a matrix that when multiplied with each of the vectors - gives $\mathbf{0}$. So the rows of this matrix should have a 0 dot product with both $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.It's actually perpendicular to the Z plane and it's:

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

But actually it could be any matrix of m by 3 with a rank of 1.

g An example for $\boldsymbol{v_3}$ that complete the basis $\boldsymbol{v_1}$, $\boldsymbol{v_1}$, $\boldsymbol{v_1}$ in $\boldsymbol{R^3}$ is $(0,0,1)$. Which is orthogonal to the Z space. But actually it can be any vector (a,b,c) as long as $c \neq 0$.

Worked example - 3.4B

If B is invertible then we know that the only $\boldsymbol{x}$ that gives $B\boldsymbol{x} = 0$ is $\boldsymbol{x} = \mathbf{0}$. Calculating $W * B\boldsymbol{x}$ for that $\boldsymbol{x}$ will give $\mathbf{0}$ which means that $(WB)\boldsymbol{x} = \mathbf{0}$ for non zero $\boldsymbol{x}$.
If B is not invertible then there exist a $\boldsymbol{x} \neq \mathbf{0}$ such that $B\boldsymbol{x} = \mathbf{0}$. Multiplying both side by W gives $WB\boldsymbol{x} = W\mathbf{0}$ or in other words $(WB)\boldsymbol{x} = \mathbf{0}$ or in other words $V\boldsymbol{x} = \mathbf{0}$ for a non zero $\boldsymbol{x}$ which means that the columns $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ are linearly independent.
So the test should be whether the B matrix is invertible or not.
if $c \neq 1$ then B is invertible. Which according to the above means that $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ are linearly independent.
If how ever c=1 then B is invertible. The second column of B is the sum of the first and the third column, and indeed the second equation for $mybfv_2$ is the sum of the equations for $\boldsymbol{v_1}$ and $\boldsymbol{v_3}$.

Worked example - 3.4B

If we say that a set of vectors is a basis then every vector in the space can be expressed by a linear combination of these vectors and also they are linearly independent. S we know that $\boldsymbol{v_1}, \ldots \boldsymbol{v_n}$ are a basis. So

1. for every vector $\boldsymbol{b}$ in $\boldsymbol{R^N}$ there's $\boldsymbol{x}$ such that $V\boldsymbol{x} = \boldsymbol{b}$
2. The only $\boldsymbol{x}$ that results in $V\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$

As for AV

1. for every vector $\boldsymbol{b}$ in $\boldsymbol{R^N}$ is there $\boldsymbol{x}$ such that $AV\boldsymbol{x} = \boldsymbol{b}$? There is because if we multiply both sides by A^{-1} then we get $V\boldsymbol{x} = A^{-1}mybfb$. $A^{-1}mybfb$ is a vector in $\boldsymbol{R^N}$ so by definition there's a vector $\boldsymbol{x}$ such that $V\boldsymbol{x}$ equals this vector.
2. Suppose there was a $\boldsymbol{x} \neq \mathbf{0}$ such that $AV\boldsymbol{x} = \mathbf{0}$. Multiplying both sides again we get that it means that there exists such a $\boldsymbol{x}$ such that $V\boldsymbol{x} = \mathbf{0}$, however by definition - there isn't.

Section 3.5, Problem 2 (Former section 3.5, Problem 2)

To find the largest possible number of independent vectors, we shall do elimination.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 3 pivots so we expect to find at most 3 independent vectors. Since we didn't have to replace the first 3 lines, then the 3 independent vectors are the first 3:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Let's see if we can express the rest of the vectors using these 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right] \tag{1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right] \tag{2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \tag{3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{5}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{6}$$

And indeed $\boldsymbol{v4} = -\boldsymbol{v_1} + \boldsymbol{v_2}$

Section 3.4, Problem 20 (Former section 3.5, Problem 20)

The plane is $x - 2y + 3z = 0$. A plane in $\boldsymbol{R^3}$ can be expressed by two vectors which are actually the base.

$$\begin{bmatrix} 1 \\ 0 \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{2}{3} \end{bmatrix} \tag{7}$$

$$\tag{8}$$

The intersection of $x - 2y + 3z = 0$ with $Z = 0$ (The XY plane) needs to meet $x - 2y = 0$ and since this is a line then there's only one vector in the base:

$$\begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} \tag{9}$$

The vectors that are perpendicular to the plane are by definition $(1, -2, 3)$

Section 3.4, Problem 37 (Former section 3.5, Problem 37)

Multiplying by S on the right side shifts A to the right. Multiplying by S on the left side shifts A upward. It means that each $(AS)_{i,j}$ (also $(SA)_{i,j}$) gets its value from below $(i+1, j)$ and from the left $(i, j-1)$. It means that $A_{i+1,j} = A_{i,j+1}$ or in other words - the values along the diagonals are the same. Moreover for the case of $i = n$ the shift up brings 0 because there's no line $i+1$. Same argument for $j = 1$. So we end up with a matrix on which all the diagonals have the same value and all diagonals below the primary diagonal have a value of 0. The subspace of matrices that commute with the shift S has thus the dimension n. In our case we can see a 3x3 case with free a, b and c variables.

Section 3.4, Problem 41 (Former section 3.5, Problem 41)

So we are saying that the I Matrix can be expressed as a linear combination of the other 5 permutations matrices. So to find the linear combination - let's put all of these matrices into one big matrix and solve. I put the matrices into columns: P_{32}, P_{31}, P_{21}, $P_{21}P_{32}$, $P_{32}P_{21}$, I.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \tag{10}$$

Looking at the table we can see that $P_{32} + P_{31} + P_{21} = P_{21}P_{32} + P_{32}P_{21} + I$
We can continue to get the R form of this matrix -

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \tag{11}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \tag{12}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right] \tag{13}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right] \tag{14}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right] \tag{15}$$

$$\left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{16}$$

The five matrices are linearly independent because there are 5 pivots.

Section 3.4, Problem 44 (Former section 3.5, Problem 44)

In an m by n matrix - we get a vector of size m but with a dimension of r. It's because it is a linear combination of the columns who span an r dimensional space. The dimension of the null space is $n-r$ and the dimension of the inputs... that is the $\boldsymbol{x}$ vector that we multiply with is n.

Section 3.5, Problem 11 (Former section 3.6, Problem 11)

a if $r < m$ then it means that the column space cannot reach all possible $\boldsymbol{b}$.

b if A is invertible

Section 3.5, Problem 24 (Former section 3.6, Problem 24)

$A^T\boldsymbol{y} = \boldsymbol{d}$ is solvable if $\boldsymbol{d}$ is in the column space of A^T or if $\boldsymbol{d}^T$ is in the row space of A. The solution is unique when the null space of A^T contains only the zero vector

Section 3.5, Problem 27 (Former section 3.6, Problem 28)

1. As for the check board and the chess board - the rank is 2 because by elimination all rows can become 0 except for the first 2 who are independent.

2. The basis for the row space should be the first two rows.

3. The basis for the left null space of checkers is the base for the null space of the transposed matrix which is the same: After some elimination

we have:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So the basis is:

$$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

4. The basis for the left null space of checkers is the base for the null space of the transposed matrix which after few steps of elimination gets to : After some elimination we have:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So the basis

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

5. The base of the null space of C:

$$\begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{n}{r} & \frac{b}{r} & \frac{q}{r} & \frac{k}{r} & \frac{b}{r} & \frac{n}{r} & 1 \\ 0 & 1 & \frac{r-b}{r-n} & \frac{r-q}{r-n} & \frac{r-k}{r-n} & \frac{r-b}{r-n} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} 1 & 0 & \frac{b-n}{r} & \frac{q-n}{r} & \frac{k-n}{r} & \frac{b-n}{r} & 0 & 1 \\ 0 & 1 & \frac{r-b}{r-n} & \frac{r-q}{r-n} & \frac{r-k}{r-n} & \frac{r-b}{r-n} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so the null space is:

$$\begin{bmatrix} \frac{n-b}{r} \\ \frac{b-r}{r-n} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{n-q}{r} \\ \frac{q-r}{r-n} \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{n-k}{r} \\ \frac{k-r}{r-n} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{n-b}{r} \\ \frac{b-r}{r-n} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

Section 3.5, Problem 29 (Former section 3.6, Problem 30)

if we take $\boldsymbol{u} = (1, b)$ and $\boldsymbol{v} = (1, a)$ then we get the rank 1 matrix:

$$\begin{bmatrix} 1 & a \\ b & ab \end{bmatrix}$$

top left: row space is $(1, a)$.
bottom left: null space is $(-a, 1)$
top right: column space is $(1, b)$.
bottom right: left null space is $(b, -1)$.
If B produces the same spaces then $B = C * A$ where C is a constant in R^1.

Section 3.5, Problem 30 (Former section 3.6, Problem 31)

a We notice that $(1, 1, 1)$ is in the null space of A so if we build a matrix on which all columns are in the column space then we shall have 0 on all columns of the multiplication AX. Here's an example:

$$\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$$

b the column space are the columns with pivots after elimination, having the specific vector: $(1, -1, 0)$ and $(0, 1, -1)$. So as an example:

$$\begin{bmatrix} d & f & h \\ -d+e & -f+g & -h+i \\ e & g & i \end{bmatrix}$$

The dimensions are 3 for the null space matrix and 6 for the column space. The dimension of the null space of the columns of A is 3. The dimension of each such column null space is 1. The dimension of each such column column space is 2 (=rank) and indeed 1+2=3. When dealing with 2 more such columns we are multiplying this by 3 and get 3+6=9.