1 A)

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 3 & 6 & 3 & 9 \end{pmatrix}$$

$$\xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 6 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix} = L$$

1 B)

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

TWO INDEPENDENT COLUMNS

1 C)

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(1 d)

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + D \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(2a)

$$C+D=6$$
$$C+2D=4$$
$$C+4D=0$$

(2b)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} x = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$C=8 \quad D=-2$$

$$y = 8 - 2\cdot 2^{t} = 8 - 2^{(t+1)}$$

(2c) WE ACTUALLY WANT TO SOLVE

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

SINCE THERE'S ONLY ONE SOLUTION

AND $C=D=0$ SOLVES

THEN $y = 0 + 0\cdot 2^{t} = 0$

(3a) MATRIX FORM:

$AV = B$

TO FIND A WE NEED TO MULTIPLY BOTH SIDES BY V^{-1} SO V SHOULD BE INVERTIBLE. WHICH MEANS v_i SHOULD BE INDEPENDENT

SO $A = BV^{-1}$

(3b) ~~THE COLUMN SPACE OF A IS SPAN~~

~~BY THE COLUMNS VECTORS~~

~~WHICH THAT ARE IN~~

~~SO THE VECTORS ARE~~

THE COLUMN SPACE OF A IS SPAN BY ALL $b_i \neq 0$

(3c) IF ALL $b_i \neq 0$ THEN THE COLUMN SPACE DIMENSION OF $A = n$ WHICH MEANS IT'S INVERTIBLE

(4a) $$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{9}{10} \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}$$

$$\lambda_1 \cdot \lambda_2 = \frac{7}{10}$$

$$\lambda_1 + \lambda_2 = \frac{17}{10}$$

$\lambda_1 = 1$ BECAUSE MARKOV

$$\lambda_2 = \frac{7}{10}$$

$$x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} x_k \\ y_k \end{bmatrix} = c_1 \lambda_1^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \lambda_2^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}} \quad \cancel{\lim_{k \to \infty} A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} =}$$

$$\cancel{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = c_1 \lambda_1^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \lambda_2^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lim_{k \to \infty} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

WHICH MEANS THAT AT INFINITY $\frac{1}{3}$ PREFER LA AND $\frac{2}{3}$ PREFER CA

(4b)

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda_1 \cdot \lambda_2 = -1$$

$$\lambda_1 + \lambda_2 = 0$$

$$\lambda_1 = 1 \quad x_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \lambda_2 = -1 \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 =$$

$$= c_1 \cdot e^{t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v(0) = c_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{matrix} 1 \\ 0 \end{matrix}$$

$$c_1 = 1 \qquad c_2 = -1$$

$$\begin{matrix} x(t) \\ y(t) \end{matrix} = v(t) = e^{t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(4c) $y(t) = c_1 e^{t} + c_2 e^{-t}$

$x(t) = 2c_1 e^{t} + c_2 e^{-t}$

WE DONT WANT ANY e^{t} OR e^{-t}:

$$\begin{bmatrix} c_1 & c_2 \\ 2c_1 & c_2 \end{bmatrix} \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

IF $c_1 = 0$ THEN $y = x$

IF $c_2 = 0$ THEN $y = 2x$

(5a) $j - i$ IS PERPENDICULAR TO 111

$-(1,0,0)+(0,1,0)=(-1,1,0)$

$(-1,1,0)\cdot(1,1,1)=0$

(5b) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$A^2 = A \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ROTATING $3 \times 120^0 = 360^0$ = BACK TO INITIAL

~~THIS MATRIX~~
ANY VECTOR PARALLEL TO $(1,1,1)$
IS GOING TO END UP

$AX = X$

WHICH MEANS $\lambda = 1$ $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(5b) cont

$\lambda_1 = 1$

$\lambda_2, \lambda_3 = 0$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 1 = 0$$

$$(\lambda - 1)(\lambda^2 + \lambda + 1) = 0$$

$$\lambda_{2,3} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

(5c)

$$A = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

P MUST HAVE $\lambda = 0$ AND $\lambda = 1$

BECAUSE $P^2 = P$

$$P(P-1) = 0$$

FOR $\lambda = 0$ WE HAVE TWO EIGEN VECTORS THAT ARE IN THE NULL SPACE OF A

FOR EXAMPLE $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

FOR $\lambda = 1$ IF WE TAKE $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ THEN ITS IN THE ROW SPACE SO INDEPEN

(6b ~~CONT) THE SECOND COLUMN IS PARALLEL TO THE~~

~~THE SECOND VECTOR THAT IS BASED ON~~
~~THE SECOND COLUMN HAS 0 ERROR~~

(6b CONT) GRAM SHMIDT SUBTRACTS THE PROJECTION OF THE SECOND VECTOR $\begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$ ON THE FIRST VECTOR $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ AND LEAVES OUT THE PERPENDICULAR PART: WHICH IS ~~0~~

SO $B = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$

(6c) THE RANK OF A IS $r <= n$
SO THE RANK OF ~~A~~ AA^T CANT BE LARGER THAN n BUT IT HAS m ROWS
SO IT'S SINGULAR SO ~~IT~~ IT HAS $\lambda = 0$
CANT BE POSITIVE DEFINITE
COULD BE SEMI DEFINITE.

6) $A^T A = \begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 4\\ 3 & 6\end{bmatrix} = \begin{bmatrix}14 & 28\\ 28 & 56\end{bmatrix}$

$\lambda_1 = 0$ $\qquad\lambda_2 = 5$

$X_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\ -1\end{bmatrix}$ $\qquad X_2 = \begin{bmatrix}1\\ 2\end{bmatrix}$

$AA^T = \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \begin{bmatrix}5 & 10 & 15\\ 10 & 20 & 30\\ 15 & 30 & 45\end{bmatrix}$

SINGULAR MATRIX WE CAN FIND TWO ~~EIGENVALUES~~ VECTORS IN THE NULL SPACE, INDEPENDENT FOR THE $\lambda = 0$

$X_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}$ $\qquad X_2 = \frac{1}{\sqrt{10}}\begin{bmatrix}-3\\ 0\\ 1\end{bmatrix}$

THIRD EIGEN VECTOR IS IN THE ROW SPACE

$X_3 = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$

(6b) GRAM SCHMIDT STARTS WITH THE FIRST COL AND ONLY ORTHONORMALIZE IT

SO $A = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$

(6c CONT) A^TA SHALL HAVE THE SAME RANK AS A ~~S~~ FOR A^TA TO BE POSITIVE DEFINITE IT HAS TO HAVE $R = n$

A^TA IS NOT ALWAYS POSITIVE DEFINITE

SEE $A^TA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 28 \\ 28 & 56 \end{bmatrix}$ ABOVE

WITH ~~NEGATIVE~~ LAMBDA $= 0$
WHICH MEANS SEMI DEFINITE

~~IF A IS SINGULAR THEN ITS~~

THE TEST ON A IS WEATHER IT'S ~~NON SINGULAR~~ INVERTIBLE. BECAUSE THEN A^TA SHALL BE INVERTIBLE $\lambda \neq 0$, ~~AND~~ WITHOUT THIS CONDITION MET - THE MOST WE CAN GET FROM A^TA IS SEMI DEFINITE

(7)(a) det A = 0 BECAUSE ~~THE~~ SINGULAR BECAUSE TWO IDENTICAL ROWS.

(7 b) ~~S.H = det~~

COFACTOR $C11 = 0 \cdot \det \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} =$

$0 \cdot (0 - 0 - 1) = 0 \cdot -1 = 0$

$\det B = 0$

$-1 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

$+1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$

$-1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= 1$

THIS IS THE VOLUME OF A BOX IN R^4

$$\det C = x \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$- 1 \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$+ 1 \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$- 1 \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} =$$

$$-x + 1 = 1 - x$$

(8a) $B^k = (M^{-1} A M)^k = (M^{-1} X \Lambda X^{-1} M)^k =$

$= M^{-1} X \Lambda^k X^{-1} M$

since $\Lambda^k \to 0$ then $B^k \to 0$

(8b) $S^{-1}(A+B)S =$

$S^{-1}AS + S^{-1}BS =$

$\Lambda_1 + \Lambda_2$

$= \Lambda$

Λ is diagonal since it is addition of two diagonal

$S^{-1}(cA)S^{-1} = c.S^{-1}AS = \underbrace{c.\Lambda}_{\text{DIAGONAL}}$

c) BASE FOR 3x3 DIAGONALS:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

~~THE MATRICES~~ (8c - CONT)

IF A IS DIAGONALIZED BY S THEN IT MEANS

$$S^{-1}AS = \Lambda$$

SO

$$A = S\Lambda S^{-1}$$

~~IF S IS COMPOSED OF 3 COLUMN~~

LET'S DENOTE THE 3 COLUMNS OF S AS S_1, S_2, S_3.

So

$$A = S\Lambda S^{-1} = \lambda_1 S_1 S^{-1} + \lambda_2 S_2 S^{-1} + \lambda_3 S_3 S^{-1}$$

WHICH MEANS THAT

$$\{S_1 S^{-1}, S_2 S^{-1}, S_3 S^{-1}\}$$

IS THE BASE.

(2a) $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

$R(A) = 3$

$N(A^T) = 3$

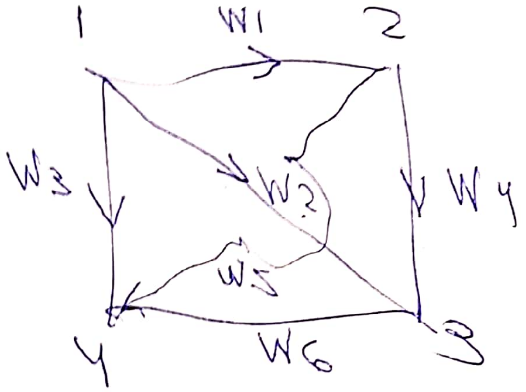
(2b) $A^T A = \begin{matrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{matrix}$

$R(A^T A) = 3$

$N(A^T A) = 1$

3c)

$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ V_2 \\ V_3 \\ 0 \end{bmatrix} = 0$$



$$\begin{bmatrix} -1 & -1 \\ 3 & -1 \\ -1 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$V_2 = V_3 = 1 \text{ V}$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ -1 \end{bmatrix} = W$$

NODE 4 GETS $(W_3, W_5, W_6) = (-1, -1, -1)$

3 AMPERS IN TOTAL