Problem Set 7

MIT CW Linear Algebra (18.06)

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Section 5.2, question 16

$Fn = Fn_{1,1} * Fn - 1 - Fn_{1,2} * Fn - 2 = 1 * Fn - 1 - (-1) * Fn - 2 = Fn - 1 + Fn - 2$

Section 5.2, question 21

$Fn = 3 * Fn - 1 - Fn_{1,2} * Fn - 2 = 1 * Fn - 1 - (-1) * Fn - 2 = Fn - 1 + Fn - 2$

Section 5.2, question 32

$$S_n = \begin{bmatrix} 3 & 1 & 0 & 0\dots \\ 1 & 3 & 1 & 0\dots \\ 0 & 1 & 3 & 1\dots \\ 0 & 0 & 1 & 3\dots \\ \vdots & & & \end{bmatrix} = 3 * S_{n-1} - 1 * T_{n-1} = 3 * S_{n-1} - S_{n-2}$$

$$T_n = \begin{bmatrix} 1 & 1 & 0 & 0\dots \\ 0 & 3 & 1 & 0\dots \\ 0 & 1 & 3 & 1\dots \\ 0 & 0 & 1 & 3\dots \\ \vdots & & & \end{bmatrix} = 1 * S_{n-1} - 1 * T_{n-2} + 1 * T_{n-2} = S_{n-1}$$

Section 5.2, question 33

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 20\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} + 6\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 10 \end{bmatrix} + 3\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 6 & 10 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{bmatrix} \tag{1}$$

$$\det \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\\1 & 3 & 6 & 10\\1 & 4 & 10 & 20\end{bmatrix} = 19\det \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 3 & 6\end{bmatrix} + 6\det \begin{bmatrix}1 & 1 & 1\\1 & 2 & 4\\1 & 3 & 10\end{bmatrix} + 3\det \begin{bmatrix}1 & 1 & 1\\1 & 3 & 4\\1 & 6 & 10\end{bmatrix} + \det \begin{bmatrix}1 & 1 & 1\\2 & 3 & 4\\3 & 6 & 10\end{bmatrix} \tag{2}$$

Subtracting (2) from (1) we get

$$\det \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\\1 & 3 & 6 & 10\\1 & 4 & 10 & 20\end{bmatrix} - \det \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\\1 & 3 & 6 & 10\\1 & 4 & 10 & 19\end{bmatrix} = 20\det \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 3 & 6\end{bmatrix} - 19\det \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 3 & 6\end{bmatrix} \tag{3}$$

$$1 - \det \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\\1 & 3 & 6 & 10\\1 & 4 & 10 & 19\end{bmatrix} = 20 * 1 - 19 * 1 = 1 \tag{4}$$

$$\det \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\\1 & 3 & 6 & 10\\1 & 4 & 10 & 19\end{bmatrix} = 0 \tag{5}$$

Section 5.3, question 8

$$C = \begin{bmatrix}6 & -3 & 0\\3 & 1 & -1\\-6 & 2 & 1\end{bmatrix}$$

$$AC^T = \begin{bmatrix}1 & 1 & 4\\1 & 2 & 2\\1 & 2 & 5\end{bmatrix} \begin{bmatrix}6 & 3 & -6\\-3 & 1 & 2\\0 & -1 & 1\end{bmatrix} = \begin{bmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 3\end{bmatrix}$$

So $\det A = 3$ As can be seen above $detA = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + a_{1,3}C_{1,3}$. Since $C_{1,3} = 0$ it doesn't matter what is the value of $a_{1,3}$ is because after multiplication by 0 its contribution to $detA$ is anyway zero.

Section 5.3, question 28

$$\det \begin{bmatrix}\sin\phi * cos\theta & \sin\phi * sin\theta & \cos\phi\\p\cos\phi * cos\theta & p\cos\phi * sin\theta & -p * \sin\phi\\-p\sin\phi * sin\theta & p\sin\phi * cos\theta & 0\end{bmatrix}$$

$$=$$
$$\frac{1}{2} * p * sin(2 * \phi) - p^2 \sin\phi^2(\cos\phi^2 + \sin\phi^2) =$$
$$\frac{1}{2} * p * sin(2 * \phi) - p^2 \sin\phi^2$$

Section 5.3, question 40

$$\det \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} * \det \begin{bmatrix} A_{3,3} & A_{3,4} & A_{3,5} \\ A_{4,3} & A_{4,4} & A_{4,5} \\ A_{5,3} & A_{5,4} & A_{5,5} \end{bmatrix} + \tag{6}$$
$$\det \begin{bmatrix} A_{1,1} & A_{1,3} \\ A_{3,1} & A_{3,3} \end{bmatrix} * \det \begin{bmatrix} A_{1,1} & A_{1,4} & A_{1,5} \\ A_{4,1} & A_{4,4} & A_{4,5} \\ A_{5,1} & A_{5,4} & A_{5,5} \end{bmatrix} + \tag{7}$$
$$\cdots + \begin{bmatrix} A_{4,4} & A_{4,5} \\ A_{5,4} & A_{5,5} \end{bmatrix} * \det \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} \tag{8}$$
$$\tag{9}$$

Section 5.3, question 41

$$\det AB = \det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} * \det \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} +$$
$$\det \begin{bmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{bmatrix} * \det \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{3,1} & b_{3,2} \end{bmatrix} +$$
$$\det \begin{bmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{bmatrix} * \det \begin{bmatrix} b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix}$$

I checked with octave: $2^2 + 4^2 + 2^2 = 24$

Section 6.1, question 19

a we can say that the rank is smaller than n because with $\lambda = 0$ we can say $A\boldsymbol{x} = 0 * \boldsymbol{x} = 0$. Which means there exists a vector in the null space of A. As for whether the base of the null space can be greater than 1. That - I don't know! (answer not full)

b We know that $\det B - \lambda * I = 0$ Since B has $\lambda = 0$ as one of its eigenvalues then we can thus say that $\det B = 0$. Also $\det B^T = \det B = 0$ So $textdetB^T * B = 0$.

c Since the eigenvalues of a Matrix and it's transpose are the same then we can say $\det B^T * \det B = \det B^2$ so the eigenvalues must be 0, 1 & 4. (Wrong answer) d The eigenvalues of A, A^n, $m{*}A$ and A^{-1} are λ, λ^n, $m{*}\lambda$ and $1/lambda$. So any polynomials on A works the same on its eigenvalues. So in our case we get $\frac{1}{0^2+1} = 1$, $\frac{1}{1^2+1} = \frac{1}{2}$ and $\frac{1}{2^2+1} = \frac{1}{5}$

Section 6.1, question 29

To find the eigenvalues of a matrix A we go back to the definition and say that we want to find λ and $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$. Or in other words to provide the non trivial solution to this: $(A - \lambda * I = \mathbf{0}$. $A - \lambda * I$ is singular if and only if its determinant is 0.

1. to calculate the determinant for the first matrix minus $\lambda * I$ is easy - just multiply all the elements on the diagonal and check when they are equal 0. So $(1 - \lambda_1) * (4 - \lambda_2) * (6 - \lambda_3) = 0$ So the eigenvalues are $\boxed{1}$, $\boxed{4}$ and $\boxed{6}$.

2. The determinant in the case of matrix $B - \lambda * I$ is

$$\det \begin{bmatrix} -\lambda_1 & 0 & 1 \\ 0 & 2 - \lambda_2 & 0 \\ 3 & 0 & -\lambda_3 \end{bmatrix} =$$
$$\lambda_1 * (2 - \lambda_2) * \lambda_3 + 3 * (2 - \lambda_2) =$$
$$(\lambda_1 * \lambda_3 + 3) * (2 - \lambda_2) =$$

   which translates into stating that $lambda_2 = 2$ and $\lambda_1 * \lambda_3 = -3$. We also know that the sum of traces is the sum of eigenvalues so $\lambda_1 + 2 + \lambda_3 = 2$. And it means that λ_1 and λ_3 equal $\sqrt{3}$ and $-\sqrt{3}$. We don't know who is who but finally the eigenvalues are $\boxed{2}$, $\boxed{\sqrt{3}}$ and $\boxed{-\sqrt{3}}$.

3. As for matrix C, equating the determinant to 0 we get:

$$\det \begin{bmatrix} 2 - \lambda & 2 & 2 \\ 2 & 2 - \lambda & 2 \\ 2 & 2 & 2 - \lambda \end{bmatrix} = 0$$

$$(2 - \lambda)[(2 - \lambda)(2 - \lambda) - 4] - 2 * [2 * (2 - \lambda) - 4] + 2 * [4 - 2 * (2 - \lambda)] = 0$$
$$(2 - \lambda)(\lambda^2 - 4\lambda) + 4 * \lambda + 4 * \lambda = 0$$
$$2 * \lambda^2 - 8\lambda - \lambda^3 + 4\lambda^2 + 8\lambda = 0 - \lambda^3 + 6\lambda^2 = 0$$
$$\lambda^2(\lambda - 6) = 0$$

   So the eigenvalues are $\boxed{0}$, $\boxed{0}$ and $\boxed{6}$.

Section 6.2, question 6

$$(4-\lambda)(2-\lambda)=0$$

So the eigenvalues are 2 and 4 To find the eigenvectors... - we are looking for an $\boldsymbol{x}$ that's in the null space of

$$\begin{bmatrix}4 & 0\\1 & 2\end{bmatrix}$$

that is

$$\begin{bmatrix}4 & 0\\1 & 2\end{bmatrix} * \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2x_1\\2x_2\end{bmatrix}$$

So $x_1 = 0$ and $x_2 = 1$ so the first set is $\lambda_1 = 2$ and $\boldsymbol{x_1} = (0, 1)$. As for the second set - we want

$$\begin{bmatrix}4 & 0\\1 & 2\end{bmatrix} * \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}4x_1\\4x_2\end{bmatrix}$$

So $x_1 = 1$ and $x_2 = 0.5$ so the second set is $\lambda_1 = 4$ and $\boldsymbol{x_2} = (1, 0.5)$. So we want to find X, X^{-1} and Λ such as to say $A = X\Lambda X^{-1}$. So first of all X is the matrix that's composed of the eigenvectors:

$$X = \begin{bmatrix}0 & 1\\1 & 0.5\end{bmatrix}$$

and

$$X^{-1} = \begin{bmatrix}-0.5 & 1\\1 & 0\end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}$$

So

$$\begin{bmatrix}0 & 1\\1 & 0.5\end{bmatrix} * \begin{bmatrix}2 & 0\\0 & 4\end{bmatrix} * \begin{bmatrix}-0.5 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}4 & 0\\1 & 2\end{bmatrix}$$

To diagonalize A^-1 we use the same X and X^{-1} matrices together with

$$\Lambda = \begin{bmatrix}\frac{1}{4} & 0\\0 & \frac{1}{2}\end{bmatrix}$$

Section 6.2, question 16

First let's find the eigenvalues of

$$A1 = \begin{bmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{bmatrix}$$

$$\begin{aligned} (0.06 - \lambda)(0.1 - \lambda) - 0.36 &= 0 \\ 0.06 - 0.7\lambda + \lambda^2 - 0.36 &= 0 \\ \lambda^2 - 0.7\lambda - 0.3 &= 0 \end{aligned}$$

So $\lambda = \frac{0.7 \pm \sqrt{0.49 + 1.2}}{2} = \frac{0.7 \pm 1.3}{2}$ and $\lambda_1 = 1$ while $\lambda_2 = -0.3$ To find the eigenvector of $\lambda_1 = 1$, we have to solve:

$$\begin{bmatrix} -0.4 & 0.9 \\ 0.4 & -0.9 \end{bmatrix} * \boldsymbol{x_1} = 0$$

And we get $\boldsymbol{x_1} = 9, 4$ To find the eigenvector of $\lambda_1 = 1$, we have to solve:

$$\begin{bmatrix} 0.9 & 0.9 \\ 0.4 & 4 \end{bmatrix} * \boldsymbol{x_2} = 0$$

And we get $\boldsymbol{x_2} = 1, -1$ So

$$X = \begin{bmatrix} 9 & 1 \\ 4 & -1 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -0.3 \end{bmatrix}$$

and

$$X^{-1} = \frac{1}{13} * \begin{bmatrix} 1 & 1 \\ 4 & -9 \end{bmatrix}$$

$$\lim_{k \to +\infty} \Lambda^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\lim_{k \to +\infty} X \Lambda^k X^{-1} = \frac{1}{13} \begin{bmatrix} 9 & 9 \\ 4 & 4 \end{bmatrix}$$

The columns contain the eigenvector corresponding to the surviving eigenvalue: 1: $(9, 4)$.