Problem Set 8

MIT CW Linear Algebra (18.06)

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Section 6.3, question 14

1. Using octave I managed to get a hold of the eigenvalues and eigenvectors here. $\lambda_1 = \frac{1}{2} + c * i$ and

$$\boldsymbol{x_1} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} * i \\ 0 \end{bmatrix}$$

$\lambda_2 = \frac{1}{2} - c * i$ and

$$\boldsymbol{x_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} * i \\ 0 \end{bmatrix}$$

$\lambda_3 = -1$ and

$$\boldsymbol{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and so

$$\boldsymbol{u(t)} = c_1 * \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} * i \\ 0 \end{bmatrix} * e^{(\frac{1}{2}+c*i)t} + c_2 * \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} * i \\ 0 \end{bmatrix} * e^{(\frac{1}{2}-c*i)t} + c_3 * \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * e^{-t}$$

to make $mybfu'(t) = 0$ we shall have to have $c_1 * \frac{1}{2} + c * i + c_2 * \frac{1}{2} - c * i - c_3 = 0$ which is possible if $c_1 = c_2$ or if $c_3 = 0$.

2.

$$\begin{aligned} Q = e^{At} = I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots = \\ I + (-A^T t) + \frac{1}{2}(-A^T t)^2 + \frac{1}{6}(-A^T t)^3 + \cdots = \\ e^{-A^T t} Q^T Q = \end{aligned}$$

Section 6.3, question 22

$$\begin{aligned} e^{At} = I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots = \\ I + (At) + \frac{1}{2}At^2 + \frac{1}{6}At^3 + \cdots = \\ I + A(t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots = \\ I + A(-1 + 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots = \\ I + A(-1 + e^t) = \end{aligned}$$

Section 6.3, question 24

To find the lambdas we want:

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(3-\lambda) = 0\lambda_1 = 1\lambda_2 = 3$$

and then

$$\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \boldsymbol{x_1} = 0$$

So for $\lambda_1 = 1$ we have $\boldsymbol{x_1} = (1, 0)$. And then

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x_2} = 0$$

So for $\lambda_2 = 3$ we have $\boldsymbol{x_2} = (1, 2)$. So

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

and

$$X^{-1} = \begin{bmatrix} 1 & -0.5 \\ 0 & 0.5 \end{bmatrix}$$

and of course

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

And let's calculate

$$e^{At} = Xe^{\Lambda * t}X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} * \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} * \begin{bmatrix} 1 & -0.5 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} e^t & \frac{e^{3t}-e^t}{2} \\ 0 & e^{3t} \end{bmatrix}$$

at $t = 0$ we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Section 6.3, question 28

$$U = \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & -(\Delta t)^2 + 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 1 \\ -\Delta t & -(\Delta t)^2 + 1 \end{bmatrix} = -(\Delta t)^2 + 1 + (\Delta t)^2 = 1$$

for $\Delta t = 1$ we get

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = -(\Delta t)^2 + 1 + (\Delta t)^2 = 1$$

Sum of λ's is 1 and multiplication of λ's is 1. $\lambda_1 = \frac{1+\sqrt{3}i}{2}$ and $\lambda_2 = \frac{1-\sqrt{3}i}{2}$

Section 6.3, question 29

For $\Delta t = \sqrt{2}$ we get

$$A = \begin{bmatrix} 1 & 1 \\ -\Delta t & -(\Delta t)^2 + 1 \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}$$

Sum of λ's is 0 and multiplication of them is 1, so $\lambda_1 = i$ and $\lambda_2 = -i$. $\boldsymbol{x_1} = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $\boldsymbol{x_2} = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ For $\Delta t = 2$ we get

$$A = \begin{bmatrix} 1 & 1 \\ -\Delta t & -(\Delta t)^2 + 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$$

Sum of λ's is -2 and multiplication of them is 1, so $\lambda_1 = -1$ and $\lambda_2 = -1$. $\boldsymbol{x_1} = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $\boldsymbol{x_2} = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

$$\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}^4 = \begin{bmatrix} -7 & -8 \\ 8 & 9 \end{bmatrix}$$

Section 6.4, question 9 (former Section 6.4, question 7)

To find the eigenvalues of the matrix

$$\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$$

we have to solve

$$\det \begin{bmatrix} 1-\lambda & b \\ b & 1-\lambda \end{bmatrix} = 0$$

$$\begin{aligned} &(1-\lambda)^2 = b^2 \\ &\lambda_1 = 1+b \\ &\lambda_2 = 1-b \end{aligned}$$

For λ_2 to be negative... b has to meet $b > 1$. So for example we have the matrix where $b = 2 > 1$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

And when we want to do some row action in order to find the pivots we get -

$$\begin{bmatrix} 1 & b \\ 0 & 1-b^2 \end{bmatrix}$$

Since $b > 1$ then we know the second pivot is negative. if $b > 1$ then for sure $\lambda_1 = 1 + b > 2$ and cannot be negative.

Section 6.4, question 12 (former Section 6.4, question 10)

The problem with this proof is that it assumes that both the nominator and the denominator are real. However if $\boldsymbol{x}^T\boldsymbol{x}$ is real at the denominator then $\boldsymbol{x}^T A\boldsymbol{x} = \boldsymbol{x}^T\lambda\boldsymbol{x} = \lambda\boldsymbol{x}^T\boldsymbol{x}$ and since λ can be complex then even if $\boldsymbol{x}^T\boldsymbol{x}$ is real, still the nominator is going to be complex.

Section 6.4, question 25 (former Section 6.4, question 23)

A is invertible because all of its columns are linearly independent.
A is orthogonal because its columns are orthogonal
A is a projection matrix because two of its eigenvectors are 1 which means there exist x such that $Ax = x$. Which means x is in the column space and thus applying the projection matrix on this vector doesn't change it and it stays where it is.
A is of course also a permutation matrix.
A is diagonalizable because it's symmetric.

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & -1 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}^{-1} * \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & -1 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

A is Markov because the sum of all the entries in each of the columns is 1
A cannot undergo LU decomposition because the inversion process involves rows swapping.
A is already Q so QR decomposition should actually be Q=A and R=I.
A is invertible so it can go $X\Lambda X^{-1}$ decomposition.
A is a real symmetric matrix so it can go through $X\Lambda X^{-1}$ B is not invertible since the lines are identical B is not orthogonal B is a projection matrix because one of its eigenvalues is 1. B is not a permutation matrix B is symmetrical but it can't be diagonalized because its singular. B has LU decomposition - 2 row operations to create upper triangular.
B has QR decomposition as follows:

$$\frac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & 0 & 0\\1 & 0 & 0\\1 & 0 & 0\end{bmatrix} * \begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

B is not invertible so it can not have $X\Lambda X^{-1}$ decomposition.
and as result cannot have this composition $Q\Lambda Q^{-1}$

Section 6.4, question 25 (former Section 6.4, question 23)

eigenvalues are 1 and 3.

Section 6.4, question 32 (former Section 6.4, question 30)

$$\begin{aligned}(Q\Lambda Q^T)_{1,1} = \lambda_1 * q_1[0]^2 + \lambda_2 * q_2[0] + \cdots + \lambda_n * q_n[0] \leq \\ \lambda_{max} q_1[0]^2 + \lambda_{max} q_2[0] + \cdots + \lambda_{max} q_n[0] = \\ \lambda_{max}(q_1[0]^2 + q_2[0]^2 + \cdots + q_3[0]^2 = \\ \lambda_{max} q_1^T q_1 = \\ \lambda_{max}\end{aligned}$$

Section 10.3, question 9 (former Section 8.3, question 9)

To multiply a Markov matrix with a Markov matrix, each column of the result is a multiplication of a Matrix by a column whose sum of entries is 1. So if that is the case result column by definition has a sum of 1 for all the entries. If we keep doing this, we get a matrix where the sum of entries in each of the columns is 1. So this is true for every A and B Markov matrices... AB is also Markov. Specifically this is true for A^2.

Section 10.3, question 12 (former Section 8.3, question 12)

The sum of each of the columns of $B = A - I$ is 0. That means that there's a vector $\boldsymbol{x}$ such that $BI = 0$ which means that B is singular and therefore it must have at least one $\lambda = 0$. I understand that $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are eigenvectors. So

$$e^{\lambda_1 t} * \boldsymbol{x_1} + e^{\lambda_2 t}\boldsymbol{x_2} = \boldsymbol{x_1} + e^{-0.5}\boldsymbol{x_2}$$

At the limit this goes to $\boldsymbol{x_1}$.

Section 10.3, question 16 (former Section 8.3, question 13)

If $\lambda_3 = 0$ as given. and this is a Markov matrix which means there's $\lambda_1 = 1$ and trace is 1.2 then the last $\lambda_2 = 0.2$

$$\boldsymbol{u(k)} = c_1 * 1^k \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} + c_2 * 0.2^k \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 * 0^k \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

For the case of $\boldsymbol{u_0} = (1, 0, 0)$ we get

$$\boldsymbol{u(k)} = 0.1 \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} - 0.5 * 0.2^k \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - 0.2 * 0^k \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

For the case of $\boldsymbol{u_0} = (100, 0, 0)$ we get

$$\boldsymbol{u(k)} = 10 \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} - 50 * 0.2^k \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - 20 * 0^k \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$