1．设 $f(x)$ 为连续函数，求证：

（1）$\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x$

（2）$\int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x = \pi\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x$，

并由此计算 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\mathrm{d}x$

证明：（1）设 $x=\frac{\pi}{2}-t$，则 $\mathrm{d}x=-\mathrm{d}t$，当 $x=0$ 时，$t=\frac{\pi}{2}$；当 $x=\frac{\pi}{2}$ 时，$t=0$

于是

$$\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = -\int_{\frac{\pi}{2}}^{0}\left(f(\sin(\frac{\pi}{2}-t)\right)\mathrm{d}t = \int_0^{\frac{\pi}{2}} f(\cos t)\mathrm{d}t = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x$$

（2）设 $x=\pi-t$，则 $\mathrm{d}x=-\mathrm{d}t$，当 $x=0$ 时，$t=\pi$；当 $x=\pi$ 时，$t=0$

于是 $\int_0^{\pi} xf(\sin x)\mathrm{d}x = -\int_{\pi}^{0}(\pi-t)f(\sin(\pi-t))\mathrm{d}t = \int_0^{\pi}(\pi-t)f(\sin t)\mathrm{d}t$

$$= \pi\int_0^{\pi} f(\sin t)\mathrm{d}t - \int_0^{\pi} tf(\sin t)\mathrm{d}t$$

$$= \pi\int_0^{\pi} f(\sin x)\mathrm{d}x - \int_0^{\pi} xf(\sin x)\mathrm{d}x$$

$$\int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x$$

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\mathrm{d}x = -\frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos^2 x}\mathrm{d}\cos x$$

$$= -\frac{\pi}{2}[\arctan(\cos x)]_0^{\pi} = \frac{\pi^2}{4}$$

现在来证：$\int_0^{\pi} f(\sin x)\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x$

$$\int_0^{\pi} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\mathrm{d}x$$

令 $x = \frac{\pi}{2} + t$　　则 $\int_{\frac{\pi}{2}}^{\pi} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\sin(\frac{\pi}{2}+t))\mathrm{d}t = \int_0^{\frac{\pi}{2}} f(\cos t)\mathrm{d}t$

$$= \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x$$

所以　　$\int_0^{\pi} f(\sin x)\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x$

2．　计算 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x}{1+e^x}\sin^4 x\mathrm{d}x$

解：　　$\int_{-a}^{a} f(x)\mathrm{d}x = \int_0^{a}\left(f(x)+f(-x)\right)\mathrm{d}x$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x}{1+e^x}\sin^4 x\mathrm{d}x = \int_0^{\frac{\pi}{2}}\left(\frac{e^x}{1+e^x}\sin^4 x + \frac{e^{-x}}{1+e^{-x}}\sin^4(-x)\right)\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}}\left(\frac{e^x}{1+e^x} + \frac{e^{-x}}{1+e^{-x}}\right)\sin^4 x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^4 x\mathrm{d}x = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{16}$$

3．设 $f(x)$ 在 $(-\infty,+\infty)$ 内连续，以 T 为周期，证明：

$\int_a^{a+T} f(x)\mathrm{d}x=\int_0^T f(x)\mathrm{d}x$ ，$\int_a^{a+nT} f(x)\mathrm{d}x=n\int_0^T f(x)\mathrm{d}x$ （a 为任意实数）

证明：

$$\int_a^{a+T} f(x)\mathrm{d}x=\int_a^0 f(x)\mathrm{d}x+\int_0^T f(x)\mathrm{d}x+\int_T^{a+T} f(x)\mathrm{d}x$$

令 $x=T+t$

$$\int_T^{a+T} f(x)\mathrm{d}x=\int_0^a f(T+t)\mathrm{d}t=\int_0^a f(t)\mathrm{d}t=\int_0^a f(x)\mathrm{d}x=-\int_a^0 f(x)\mathrm{d}x$$

$$\int_a^{a+T} f(x)\mathrm{d}x=\int_0^T f(x)\mathrm{d}x$$

$$\int_a^{a+nT} f(x)\mathrm{d}x=\int_a^0 f(x)\mathrm{d}x+\int_0^T f(x)\mathrm{d}x+\int_T^{2T} f(x)\mathrm{d}x+\cdots$$

$$+\int_{(n-1)T}^{nT} f(x)\mathrm{d}x+\int_{nT}^{a+nT} f(x)\mathrm{d}x=n\int_0^T f(x)\mathrm{d}x$$

令 $x=nT+t$

4. $\int_{\frac{\pi}{2}}^{\frac{21}{2}\pi}\sin^6 x\mathrm{d}x$

解：

$$\int_{\frac{\pi}{2}}^{\frac{21}{2}\pi}\sin^6 x\mathrm{d}x=\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+10\pi}\sin^6 x\mathrm{d}x=10\int_0^{\pi}\sin^6 x\mathrm{d}x=20\int_0^{\frac{\pi}{2}}\sin^6 x\mathrm{d}x=20\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

5. 设 $f(x)$ 在 $[a,b]$ 上连续，证明 $\int_a^b f(x)\mathrm{d}x=\int_a^b f(a+b-x)\mathrm{d}x$，并由此计算 $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^2 x}{x(\pi-2x)}\mathrm{d}x$

解：

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin^2 x}{x(\pi-2x)}\mathrm{d}x=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos^2 x}{x(\pi-2x)}\mathrm{d}x=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{x(\pi-2x)}\mathrm{d}x=\frac{1}{\pi}\ln 2$$

6．设 $f(x)$ 在 $[-\pi,\pi]$ 上连续，当

$$f(x)=\frac{x}{1+\cos^2 x}+\int_{-\pi}^{\pi} f(x)\sin x\mathrm{d}x，求 f(x)$$

解：令 $A=\int_{-\pi}^{\pi} f(x)\sin x\mathrm{d}x$

$$f(x)\sin x=\frac{x\sin x}{1+\cos^2 x}+A\sin x$$

$$A=\int_{-\pi}^{\pi} f(x)\sin x\mathrm{d}x=\int_{-\pi}^{\pi}\frac{x\sin x}{1+\cos^2 x}\mathrm{d}x+A\int_{-\pi}^{\pi}\sin x\mathrm{d}x$$

$$=2\int_{0}^{\pi}\frac{x\sin x}{1+\cos^2 x}\mathrm{d}x=\frac{\pi^2}{2}$$ （由上面第 1 题）

$$f(x)=\frac{x}{1+\cos^2 x}+\frac{\pi^2}{2}$$

7. 求 $\int_0^{\pi}\sqrt{1-\sin x}\mathrm{d}x$

解：

$$\int_0^{\pi}\sqrt{1-\sin x}\mathrm{d}x=\int_0^{\pi}\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{2}}(\cos\frac{x}{2}-\sin\frac{x}{2})\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}(\sin\frac{x}{2}-\cos\frac{x}{2})\mathrm{d}x=4(\sqrt{2}-1)$$

8．$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\cos(x-t)^2\mathrm{d}t$　　　$\left(\int_a^x f(t)\mathrm{d}t\right)'=f(x)$

解：令 $x-t=u$

$$\int_0^x\cos(x-t)^2\mathrm{d}t=\int_x^0\cos u^2(-\mathrm{d}u)=\int_0^x\cos u^2\mathrm{d}u$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\cos(x-t)^2\mathrm{d}t=\cos x^2$$

9．设 $f(x)$ 连续，则 $\dfrac{\mathrm{d}}{\mathrm{d}x}\int_1^2 f(x+t)\mathrm{d}t$

解：令 $x+t=u$

$$\int_1^2 f(x+t)\mathrm{d}t=\int_{x+1}^{x+2} f(u)\mathrm{d}u=\int_{x+1}^{x+2} f(t)\mathrm{d}t$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^2 f(x+t)\mathrm{d}t=f(x+2)-f(x+1)$$

$F(x)=\int_{u(x)}^{v(x)} f(t)\mathrm{d}t$　　$F'(x)=f\left(v(x)\right)v'(x)-f\left(u(x)\right)u'(x)$

10．设 $f(x)$ 连续，且 $\int_0^x tf(x-t)\mathrm{d}t=1-\cos x$，求 $\int_0^{\frac{\pi}{2}} f(x)\mathrm{d}x$

解：令 $x-t=u$

$$\int_0^x tf(x-t)\mathrm{d}t=-\int_x^0 (x-u)f(u)\mathrm{d}u=\int_0^x (x-u)f(u)\mathrm{d}u$$

$$=x\int_0^x f(u)\mathrm{d}u-\int_0^x uf(u)\mathrm{d}u$$

求导　$\int_0^x f(u)\mathrm{d}u+xf(x)-xf(x)=\sin x$

$$\int_0^x f(u)\mathrm{d}u=\sin x\Rightarrow\int_0^{\frac{\pi}{2}} f(u)\mathrm{d}u=\sin\frac{\pi}{2}=1，\ \int_0^{\frac{\pi}{2}} f(x)\mathrm{d}x=1$$