1. $\lim\limits_{x \to +\infty} \arctan x = \dfrac{\pi}{2}$, $\lim\limits_{x \to -\infty} \arctan x = -\dfrac{\pi}{2}$ $\Rightarrow$ $\lim\limits_{x \to \infty} \arctan x$ 不存在

$\lim\limits_{x \to 0^+} \arctan \dfrac{1}{x} = \dfrac{\pi}{2}$, $\lim\limits_{x \to 0^-} \arctan \dfrac{1}{x} = -\dfrac{\pi}{2}$ $\Rightarrow$ $\lim\limits_{x \to 0} \arctan \dfrac{1}{x}$ 不存在

$\lim\limits_{x \to 1^+} \arctan \dfrac{1}{x-1} = \dfrac{\pi}{2}$, $\lim\limits_{x \to 1^-} \arctan \dfrac{1}{x-1} = -\dfrac{\pi}{2}$ $\Rightarrow$ $\lim\limits_{x \to 1} \arctan \dfrac{1}{x-1}$ 不存在

2. 已知 $x_1 = 1$，$x_{n+1} = 1 + \dfrac{x_n}{1+x_n}$，$n = 1, 2, \cdots$，证明：$\lim\limits_{n \to \infty} x_n$ 存在

证明： 因为

$$x_{n+1} - x_n = \left(1 + \frac{x_n}{1+x_n}\right) - \left(1 + \frac{x_{n-1}}{1+x_{n-1}}\right) = \frac{x_n - x_{n-1}}{(1+x_n)(1+x_{n-1})}$$

所以 $(x_{n+1} - x_n)$ 与 $(x_n - x_{n-1})$ 同号，又因为 $x_2 > x_1$，故得 $\{x_n\}$ 单调增加

因为 $x_{n+1} = 1 + \dfrac{x_n}{1+x_n} = 2 - \dfrac{1}{1+x_n} < 2$，所以 $\{x_n\}$ 有上界

根据单调有界收敛定理，$\lim\limits_{n \to \infty} x_n$ 存在. 令 $\lim\limits_{n \to \infty} x_n = A$，

对 $x_{n+1} = 1 + \dfrac{x_n}{1+x_n}$ 两边取极限得： $A = 1 + \dfrac{A}{1+A}$ ，即 $A^2 - A - 1 = 0$

$A = \dfrac{1 \pm \sqrt{5}}{2}$，由保号性 $\Rightarrow A \geq 0$， 所以 $\lim\limits_{n \to \infty} x_n = \dfrac{1+\sqrt{5}}{2}$

3. $\lim\limits_{x \to +\infty} (\sin\sqrt{x+1} - \sin\sqrt{x}) = 0$

证明：$\forall \varepsilon > 0$，由于

$$\left| (\sin\sqrt{x+1} - \sin\sqrt{x}) - 0 \right| = \left| 2\cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right) \sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right) \right| \leq \left| \sqrt{x+1} - \sqrt{x} \right|$$

$$= \left| \frac{1}{\sqrt{x+1}+\sqrt{x}} \right| \leq \frac{1}{2\sqrt{x}} < \varepsilon \Rightarrow \sqrt{x} > \frac{1}{2\varepsilon} \Rightarrow x > \frac{1}{4\varepsilon^2}$$

取 $M = \dfrac{1}{4\varepsilon^2}$， 当 $x > M$ 时，有 $\left| \sin\sqrt{x+1} - \sin\sqrt{x} \right| < \varepsilon$，由极限定义知

$$\lim\limits_{x \to +\infty} (\sin\sqrt{x+1} - \sin\sqrt{x}) = 0$$

4. $\lim\limits_{x\to 0^+} e^{-\frac{1}{x}} = 0$

证明：$\forall \varepsilon > 0$，不妨设 $\varepsilon < 1$，由于

$$\left| e^{-\frac{1}{x}} - 0 \right| = \left| \frac{1}{e^{\frac{1}{x}}} \right| < \varepsilon \Rightarrow e^{\frac{1}{x}} > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x} > \ln\left(\frac{1}{\varepsilon}\right) = -\ln\varepsilon \Rightarrow x < -\frac{1}{\ln\varepsilon}$$

取 $\delta = -\frac{1}{\ln\varepsilon}$ 当 $0 < x - 0 < \delta$，$\left| e^{-\frac{1}{x}} - 0 \right| < \varepsilon$

由极限定义知，$\lim\limits_{x\to 0^+} e^{-\frac{1}{x}} = 0$

5. 确定 a，b 的值，使下列各式成立

(1) $\lim\limits_{x\to -\infty}\left(\sqrt{x^2 - x + 1} - ax - b \right) = 0$

解：原式 $= \lim\limits_{x\to -\infty}(-x)\left(\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + a + \frac{b}{x} \right) = 0 \Rightarrow a = -1$

$$\lim\limits_{x\to -\infty}\left(\sqrt{x^2 - x + 1} + x - b \right) = 0$$

$$b = \lim\limits_{x\to -\infty}\left(\sqrt{x^2 - x + 1} + x \right) = \lim\limits_{x\to -\infty}\frac{-x + 1}{\sqrt{x^2 - x + 1} - x} = \lim\limits_{x\to -\infty}\frac{1 - \frac{1}{x}}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + 1} = \frac{1}{2}$$

(2) $\lim\limits_{x\to +\infty}\left(\frac{x^2 + 1}{x + 1} - ax - b \right) = 0$

解：原式 $= \lim\limits_{x\to +\infty}\left(\frac{x^2 + 1 - ax^2 - ax - bx - b}{x + 1} \right)$

$$= \lim\limits_{x\to +\infty}\left(\frac{(1-a)x^2 - (a+b)x + 1 - b}{x + 1} \right) = 0 \Rightarrow a = 1,\ b = -1$$

6. 求 $\lim\limits_{x \to 0} x\left[\dfrac{2}{x}\right]$

解：$\quad\dfrac{2}{x} - 1 < \left[\dfrac{2}{x}\right] \leq \dfrac{2}{x}$，

当 $x > 0$ 时，$\quad x\left(\dfrac{2}{x} - 1\right) < x\left[\dfrac{2}{x}\right] \leq x \cdot \dfrac{2}{x} = 2$

$\lim\limits_{x \to 0^+} x\left(\dfrac{2}{x} - 1\right) = 2$，$\quad \lim\limits_{x \to 0^+} 2 = 2$，故 $\lim\limits_{x \to 0^+} x\left[\dfrac{2}{x}\right] = 2$

当 $x < 0$ 时，$\quad x\left(\dfrac{2}{x} - 1\right) > x\left[\dfrac{2}{x}\right] \geq x \cdot \dfrac{2}{x} = 2$

$\lim\limits_{x \to 0^-} x\left(\dfrac{2}{x} - 1\right) = 2$，$\quad \lim\limits_{x \to 0^-} 2 = 2$，故 $\lim\limits_{x \to 0^-} x\left[\dfrac{2}{x}\right] = 2$　从而　$\lim\limits_{x \to 0} x\left[\dfrac{2}{x}\right] = 2$

7. $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$

(1) $\lim\limits_{x \to 1} \dfrac{x^m - 1}{x - 1}$　$(m \in \mathbf{N}_+)$

解：$\quad \lim\limits_{x \to 1} \dfrac{x^m - 1}{x - 1} = \lim\limits_{x \to 1} \dfrac{(x-1)(x^{m-1} + x^{m-2} + \cdots + x + 1)}{x - 1} = m$