1. $\int \sin^n x\mathrm{d}x = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\mathrm{d}x$

证明：$\int \sin^n x\mathrm{d}x = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\mathrm{d}x$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\mathrm{d}x$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\mathrm{d}x - (n-1)\int \sin^n x\mathrm{d}x$$

$$\int \sin^n x\mathrm{d}x = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\mathrm{d}x$$

2. $\int \frac{1}{\sin^n x}\mathrm{d}x = -\frac{1}{n-1}\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{1}{\sin^{n-2} x}\mathrm{d}x$ （$n \geq 2$）

证明：

$$\int \frac{1}{\sin^{n-2} x}\mathrm{d}x = \int \frac{\sin x}{\sin^{n-1} x}\mathrm{d}x = -\frac{\cos x}{\sin^{n-1} x} - (n-1)\int \frac{\cos^2 x}{\sin^n x}\mathrm{d}x$$

$$= -\frac{\cos x}{\sin^{n-1} x} - (n-1)\int \frac{1-\sin^2 x}{\sin^n x}\mathrm{d}x$$

$$= -\frac{\cos x}{\sin^{n-1} x} - (n-1)\int \frac{1}{\sin^n x}\mathrm{d}x + (n-1)\int \frac{1}{\sin^{n-2} x}\mathrm{d}x$$

$$\int \frac{1}{\sin^n x}\mathrm{d}x = -\frac{1}{n-1}\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{1}{\sin^{n-2} x}\mathrm{d}x$$

3. 求 $\int \frac{1}{1+e^x} \mathrm{d}x$.

解： $\int \frac{1}{1+e^x} \mathrm{d}x = \int \frac{e^{-x}}{1+e^{-x}} \mathrm{d}x = -\int \frac{1}{1+e^{-x}} \mathrm{d}(e^{-x}+1) = -\ln(1+e^{-x}) + C$

也可以如下求解

$$\int \frac{e^x+1-e^x}{1+e^x} \mathrm{d}x = \int \left(1 - \frac{e^x}{1+e^x}\right) \mathrm{d}x = x - \int \frac{1}{1+e^x} \mathrm{d}(e^x+1) = x - \ln(e^x+1) + C.$$

4. $\int \frac{x^3}{(1+x^8)^2} \mathrm{d}x = \frac{1}{4} \int \frac{1}{(1+x^8)^2} \mathrm{d}x^4 = \frac{1}{4} \int \frac{1}{(1+u^2)^2} \mathrm{d}u$ （$u = x^4$）

解：$\int \frac{1}{(1+u^2)^2} \mathrm{d}u = \int \frac{1+u^2-u^2}{(1+u^2)^2} \mathrm{d}u = \int \left(\frac{1}{1+u^2} - \frac{u^2}{(1+u^2)^2}\right) \mathrm{d}u$

$$\int \frac{1}{1+u^2} \mathrm{d}u = \arctan u$$

$$\int \frac{u^2}{(1+u^2)^2} \mathrm{d}u = -\frac{1}{2} \int u \mathrm{d} \frac{1}{(1+u^2)} = -\frac{1}{2}\left(\frac{u}{1+u^2} - \int \frac{1}{1+u^2} \mathrm{d}u\right) \text{（分部积分）}$$

$$= -\frac{1}{2}\left(\frac{u}{1+u^2} - \arctan u\right) = \frac{1}{2} \arctan u - \frac{u}{2(1+u^2)}$$

$$\int \frac{1}{(1+u^2)^2} \mathrm{d}u = \frac{1}{2} \arctan u + \frac{u}{2(1+u^2)} + c$$

$$\int \frac{x^3}{(1+x^8)^2} \mathrm{d}x = \frac{1}{4} \int \frac{1}{(1+x^8)^2} \mathrm{d}x^4 = \frac{1}{4} \int \frac{1}{(1+u^2)^2} \mathrm{d}u = \frac{1}{8} \arctan x^4 + \frac{x^4}{8(1+x^8)} + c$$