Some practice questions for MDE; 14 April 2017;

Problem 1:

Write down expressions for boundary and initial conditions for the following: A musical string of length 400mm fixed at each end that will be plucked and released at x=200mm. Assume that it is held back 5mm before release.

Answer:

Boundary conditions: $u(0,t) = 0; u(400mm,t) = 0$

Initial conditions: $u(x,0) = \frac{x}{40}$ for x=0 to 200mm

$u(x,0) = 5mm - \frac{x-200}{40}$ for x= 200mm to 400mm

Problem 2:

We can use the method of Separation of Variables to find a solution to the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

In our lecture we made the following substitution for *u(x,t)*:

$u(x,t) = X(x)T(t)$

and we got the following two differential equations:

$$\frac{d^2X}{dx^2} + p^2X = 0 \; ; \; \frac{d^2T}{dt^2} + p^2c^2T = 0$$

Show how we got these equations.
Label the in-between steps.

Answer:

See notes

Problem 3

Derive an expression for the steady-state temperature distribution $u_s(x)$ across a wall of thickness L for the following set of boundary conditions:

$$u(0,t) = 100^0 \qquad u(L,t) = 50^0$$

Assume that the wall in question has an initial temperature distribution of *u(x,0)*=150ºC. Calculate an expression for the initial conditions affecting the transient solution *U(x,0)*.

Answer:

Find steady state solution-

$$u_s(x) = Fx + G$$
$$u_s(0) = G = 100^0$$
$$u_s(L) = 50^0 = FL + 100^0$$
$$F = \frac{-50}{L}$$
$$therefore$$
$$u_s(x) = 100^0 - 50^0 \frac{x}{L}$$

The solution will be the sum of the steady state part and the transient part:

$$u(x,t) = u_s(x) + U(x,t)$$

Write the equation for t=0:

$$u(x,0) = u_s(x) + U(x,0)\_$$
$$\text{Re}\,arrange:$$
$$U(x,0) = u(x,0) - u_s(x)$$
$$U(x,0) = 150^0 - \left(100^0 - 50^0 \frac{x}{L}\right) = 50^0\left(1 + \frac{x}{L}\right)$$

You now use these initial conditions with homogeneous boundary conditions (*U(0,t)=0 and U(L,t)=0*) to solve for *U(x,t)* following the method in Model 5 of your notes.

Problem 4:

Another wall with non-zero boundary conditions: Derive an expression for the steady-state temperature distribution $u_s(x)$ across a wall of thickness *1 m* for the following set of boundary conditions:

$$u(0,t) = 120^0 \quad u(1,t) = 70^0$$

Answer:

We can fit this to the boundary data:

$u_s(x) = Fx + G;\ u_s(0) = \ 120^0 = G;\ u_s(1) = \ 70^0 = F + 120\ so\ F = -50;$

$$u_s(x) = 120 - 50x$$

Check:

$\boldsymbol{u(0,t) = 120^0 + U(0,t) = 120^0 \quad Therefore\ U(0,t) = 0}$
$\boldsymbol{u(1,t) = 120^0 - 50^0 + U(1,t) = 70^0 \quad Therefore\ U(1,t) = 0}$

Note that the boundary conditions for the transient part must be zero!

ii. Assume that the wall in question has an initial temperature distribution of *u(x,0)*=120^0 C. Calculate an expression for the initial conditions for the transient part of the solution, *U(x,0).*

Answer:

$$u(x,0) = u_s(x) + U(x,0)$$

$$U(x,0) = u(x,0) - u_s(x) \; so \;\; U(x,0) = 120^0 - (120^0 - 50x) = 50x$$

$$U(x,0) = 50x$$

<u>Problem 5)</u> In Model 6 how did we get Laplace's Equation in 2D from Poisson's Equation in 3D? What assumptions/ conditions apply?

<u>Problem 6</u> Why would you expect the central difference formula for $\partial u/\partial x$ to be more accurate than a forward difference or reverse difference formula?

<u>Problem 7)</u> Referring to the recurrence relation in the final section of our notes:
a) If diffusivity can be increased (by using a different material) what influence will this have on the permissible range of time-step Δt?

Problem 8 **Extra for experts: Write Laplace's 2D equation using finite differences. Find a finite difference form for this assuming that *Δx=Δy*.**

Answer:

Remember that:

$$\frac{\partial^2 u(x_i)}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

1) The equation:

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

2) Find a solution: discretise in x and y:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2},$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2},$$

$$So \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad gives$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = 0$$

If we make $\Delta x = \Delta y = h$,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = 0$$

So

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

Note that:

$$u_{i,j} = [u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}]/4$$