ENGSCI 313 - Mathematical Modelling
Fourier Assignment - 2017

Hand in your solutions to the questions that are starred (*). Staple your answers to your signed coversheet.

Due date: Wednesday May 24, 11:00am

Help on these specific questions will not be given in tutorials/problem clinics

Total mark: 100

4.1 Fourier series

Consider the following functions, each defined over the interval [$-\alpha$, α]:

$$f_a(x) = x^2 \quad -\alpha \le x < \alpha$$

$$f_b(x) = \begin{cases} x^2 & -\alpha \le x < 0 \\ \dfrac{\alpha}{\alpha - x} & 0 \le x < \alpha \end{cases}$$

$$f_c(x) = \begin{cases} \alpha - \dfrac{x}{2\alpha} & -\alpha \le x < 0 \\ \alpha + \dfrac{x}{2\alpha} & 0 \le x < \alpha \end{cases}$$

$$f_d(x) = \begin{cases} \dfrac{1}{x^2} & -\alpha \le x < 0 \\ 2x & 0 \le x < \alpha \end{cases}$$

a. Sketch the periodic extension of each of the above functions over the interval [-3α, 3α].

b. For each of the above functions, show whether the Fourier series representation of the function is convergent.

c. For those functions where the Fourier series representation of the function is convergent:

i. give the fundamental frequency of the Fourier series approximation;

ii. give the integral expressions for a_0, a_n and b_n;

iii. calculate the values for a_0, a_n and b_n, in terms of α;

iv. give the expression for the Fourier series approximation, $S_N(x)$;

v. give the coefficients a_n and power of each of the spectral components that exist in $S_N(x)$ in terms of α for n = 0, 1, 2, 3, 4, 5, 6;

vi. if α = 1 s, give numerical values, 4 decimal places, of the magnitude and the power for each harmonic, n = 0, 1, 2, 3, 4, 5, 6;

vii. Why is the power in these signals concentrated in the DC and first few harmonics?

4.2(*) For each of the following statements, choose the response that is most correct.

[12 marks]

i. If $F(\omega)$ is the Fourier transform of the function $f(t)$, then the Fourier transform of $\frac{d^2f(t)}{dt^2}$ is:

a $\omega\ F(\omega)$;

b $\omega^2\ F(\omega)$;

c $-\omega\ F(\omega)$;

d $-\omega^2\ F(\omega)$;

e none of the above.

ii. A function $g(t)$ has a DC offset of 10, then in the Fourier series approximation and the Complex Fourier series approximation we have:

a $a_0 = 10\ and\ c_0 = 5$

b $a_0 = 2\ and\ c_0 = 10$

c $a_0 = 5\ and\ c_0 = 5$

d $a_0 = 5\ and\ c_0 = 10$

d none of the above.

iii If c_n are components of the complex Fourier series of a function $g(t)$, then:

a c_0 is equal to $g(0)$;

b c_0 is equal to $g(\infty)$;

c c_0 is equal to the mean of $g(t)$;

d c_0 is equal to $g(-\infty)$;

e none of the above.

vi If $F(\omega)$ is the Fourier transform of the function $f(t)$, then the Fourier transform of $f(t - t_0)$ is:

a $e^{i\omega t_0}\ F(\omega)$;

b $e^{-i\omega t_0}\ F(\omega)$;

c $F(\omega + t_0)$;

d $F(\omega - t_0)$;

e none of the above.

v If $0 < \omega_L < \omega_H$, a bandpass filter:

a attenuates frequencies $-\omega_L < \omega < \omega_L$;

b attenuates frequencies $-\omega_H < \omega < -\omega_L$ and $\omega_L < \omega < \omega_H$;

c attenuates frequencies $\omega < -\omega_H$ and $\omega_H < \omega$;

d attenuates frequencies $\omega < -\omega_H$, $-\omega_L < \omega < \omega_L$, and $\omega_H < \omega$;

e none of the above.

iv Consider the following functions defined on the interval $-2 \leq t \leq 2$:

$$f(t) = \frac{1}{t}$$

$$g(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \leq t \end{cases}$$

$$h(t) = \frac{\sin(t)}{t}$$

a $f(t)$, $g(t)$, and $h(t)$ are piecewise continuous;

b $f(t)$ and $h(t)$ are piecewise continuous;

c $g(t)$ and $h(t)$ are piecewise continuous;

d $g(t)$ is piecewise continuous;

e none of the above.

4.3(*) Fourier series **[25 marks]**

Consider function *f*(*x*) is defined as:

$$f(x) = -2x \quad 0 \leq x \leq 5$$

i. Sketch the odd periodic extension of the function over the range [–15, 15];

ii. What is the period of the odd extension?

iii. What can we say about the convergence of the Fourier series? Justify your answer.

iv. Determine the Fourier series approximation of the odd periodic function and give the expression for the Fourier series approximation, SN(x).

v. Write down the first five non-zero terms of your approximation.

vi. Plot your Fourier series approximation on the top of the original odd function over the range [-15, 15] using 2 harmonic terms, 5 harmonic terms, 20 harmonic terms and 100 harmonic terms (4 plots) and justify which one is a better approximation of the function $f(x)$? Note: use $\Delta x = 0.01$ for your plots.

vii. What is the value of the Fourier series approximation at the discontinuities?

viii. Calculate the total power in the Fourier series approximation and plot the one-sided Power spectrum for both power (W) vs Freq (Hz) and power (W) vs n, for n=1 to 5.

ix. What is the power of at 0 frequency?

4.4 Fourier series

Try the same steps in question 4.3 for when the function *f*(*x*) is defined as:

$$f(x) = 2x \quad 0 \leq x \leq 5$$

and you are asked to extend $f(x)$ to be an even periodic function.

4.5(*) Fourier series **[25 marks]**

The function $f(x)$ is periodic and is defined such that:

$$f(x) = e^{x} \quad 0 \leq x < 2$$

i. sketch the periodic function over the range [–4, 4];

ii. calculate the fundamental frequency, F, and fundamental angular frequency, ω0, of this signal;

iii. calculate the coefficients, c_0 and c_n, of the complex Fourier series of this function;

Note that

$$S_N(x) = c_0 + \sum_{\substack{n=-N \\ n\neq 0}}^{N} c_n e^{in\omega_0 x}$$

where

$$c_0 = \frac{1}{T}\int_a^b f(x)dx$$
$$c_n = \frac{1}{T}\int_a^b f(x)\, e^{-in\omega_0 x}\, dx$$
$$T = b - a$$
$$\omega_0 = 2\pi f = \frac{2\pi}{T}$$

iv. express the complex Fourier series, $S_N(x)$, of this function in terms of the coefficients, c_0 and c_n;

v. tabulate, to 4 decimal places, the magnitude and phase (in radians) of the coefficients for $-2 \leq n \leq 2$;

vi. plot the two-sided magnitude spectrum (mag vs "n") for $-2 \leq n \leq 2$;

vii. plot the one-sided magnitude spectrum for $0 \leq n \leq 2$.

4.6(*) Fourier transform **[25 marks]**

The functions $f_a(x)$ to $f_c(x)$ are defined such that:

$$f_a(x) = \begin{cases} x & 0 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_b(x) = \begin{cases} 2 & 2 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$f_c(x) = \begin{cases} 5 - x & 4 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$$

i. sketch the above functions, $f_a(x)$, $f_b(x)$, and $f_c(x)$, on **separate** graphs over the range [–2, 6];

ii. calculate the Fourier transform of $f_a(x)$,;

iii. calculate the Fourier transform of $f_b(x)$,;

iv. calculate the Fourier transform of $f_c(x)$,;

v. sketch the function $f_d(x) = f_a(x)+ f_b(x)+2\, f_c(x)$ over the range [–2, 6];

vi. Hence, determine the Fourier transform of $f_d(x)$.

Note: use: $\sin(\omega) = \frac{1}{2i}\left(e^{i\omega} - e^{-i\omega}\right)$ where needed.

4.7(*) Convolution **[13 marks]**

Consider these two discrete signals:

$$f[n] = \{0.5, 3, 1, 2, 4, 1.5, 6, 0, 5\}$$

$$\mathrm{g}[n] = \{2, 2.5, 5, 1\}$$

i. Find the convolution of these two discrete signals

$$y[k] = \sum_{k=-\infty}^{\infty} f[n]g[k-n]$$

ii. Plot $y[k]$