ENGSCI 313 - Mathematical Modelling
Fourier Assignment - 2017

Hand in your solutions to the questions that are starred (*). Staple your answers to your signed coversheet.

Due date: Wednesday May 24, 11:00am

Help on these specific questions will not be given in tutorials/problem clinics

Total mark: 100

4.1 Fourier series

Consider the following functions, each defined over the interval [$-\alpha$, α]:

$$f_a(x) = x^2 \quad -\alpha \le x < \alpha$$

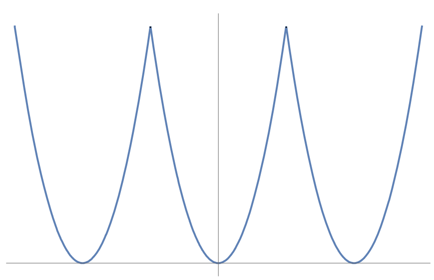
$$f_b(x) = \begin{cases} x^2 & -\alpha \le x < 0 \\ \dfrac{\alpha}{\alpha - x} & 0 \le x < \alpha \end{cases}$$

$$f_c(x) = \begin{cases} \alpha - \dfrac{x}{2\alpha} & -\alpha \le x < 0 \\ \alpha + \dfrac{x}{2\alpha} & 0 \le x < \alpha \end{cases}$$

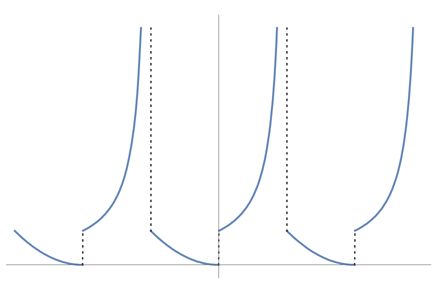
$$f_d(x) = \begin{cases} \dfrac{1}{x^2} & -\alpha \le x < 0 \\ 2x & 0 \le x < \alpha \end{cases}$$

a. Sketch the periodic extension of each of the above functions over the interval [-3α, 3α].

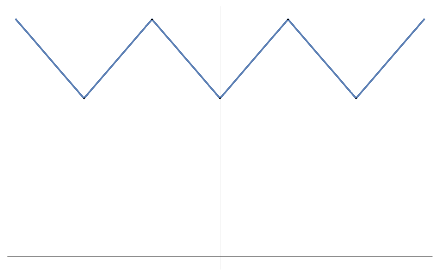
$f_a(x)$:



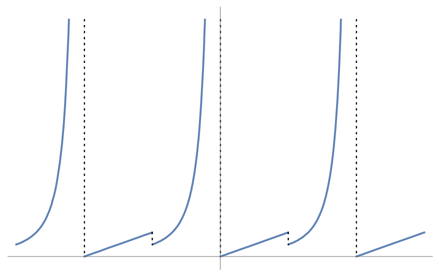
$f_b(x)$:



$f_c(x)$:



$f_d(x)$:



b. For each of the above functions, show whether the Fourier series representation of the function is convergent.

For $f_a(x)$:

$$\lim_{x\to-\alpha} f_a(x) = \alpha^2$$
$$\lim_{x\to\alpha} f_a(x) = \alpha^2$$

Since the function is finite at the limits, and there are no discontinuities, the Fourier series representation of the function is convergent.

For $f_b(x)$:

$$\lim_{x\to-\alpha} f_b(x) = \alpha^2$$
$$\lim_{x\to0-} f_b(x) = 0$$
$$\lim_{x\to0+} f_b(x) = 1$$
$$\lim_{x\to\alpha} f_b(x) = \infty$$

Since the function is not finite at the limit $x \longrightarrow \alpha$, the Fourier series representation of the function is not convergent.

For $f_c(x)$:

$$\lim_{x\to-\alpha} f_c(x) = \alpha + \frac{1}{2}$$
$$\lim_{x\to0-} f_c(x) = \alpha$$
$$\lim_{x\to0+} f_c(x) = \alpha$$
$$\lim_{x\to\alpha} f_c(x) = \alpha + \frac{1}{2}$$

Since the function is finite at the limits, and there are no discontinuities, the Fourier series representation of the function is convergent.

For $f_d(x)$:

$$\lim_{x\to-\alpha} f_d(x) = \frac{1}{\alpha^2}$$
$$\lim_{x\to0-} f_d(x) = \infty$$
$$\lim_{x\to0+} f_d(x) = 0$$

$$\lim_{x \to \alpha} f_d(x) = 2\alpha$$

Since the function is not finite at the limit $x \longrightarrow 0-$, the Fourier series representation of the function is not convergent.

c. For those functions where the Fourier series representation of the function is convergent:

i. give the fundamental frequency of the Fourier series approximation;

For all of the above functions, the period, T, is 2α.
Since $f = \frac{1}{T} = \frac{1}{2\alpha}$.

ii. give the integral expressions for a_0, a_n and b_n;

$$f_a(x) = x^2 \quad -\alpha \le x < \alpha$$

$f_a(x)$ is an even function, so bn = 0.

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f_a(x)dx$$
$$= \frac{1}{\alpha}\int_{-\alpha}^{\alpha} x^2 dx$$
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_a(x)\cos\left(\frac{2\pi n x}{T}\right) dx$$
$$= \frac{1}{\alpha}\int_{-\alpha}^{\alpha} x^2 \cos\left(\frac{\pi n x}{\alpha}\right) dx$$

For $f_c(x)$:

$$f_c(x) = \begin{cases} \alpha - \dfrac{x}{2\alpha} & -\alpha \le x < 0 \\ \alpha + \dfrac{x}{2\alpha} & 0 \le x < \alpha \end{cases}$$

$f_c(x)$ is an even function, so bn = 0.

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f_c(x)dx$$
$$= \frac{1}{\alpha}\left(\int_{-\alpha}^{0}\left(\alpha - \frac{x}{2\alpha}\right)dx + \int_{0}^{\alpha}\left(\alpha + \frac{x}{2\alpha}\right)dx\right)$$
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_c \cos\left(\frac{2\pi n x}{T}\right) dx$$
$$= \frac{1}{\alpha}\left(\int_{-\alpha}^{0}\left(\alpha - \frac{x}{2\alpha}\right)\cos\left(\frac{\pi n x}{\alpha}\right)dx + \int_{0}^{\alpha}\left(\alpha + \frac{x}{2\alpha}\right)\cos\left(\frac{\pi n x}{\alpha}\right)dx\right)$$

iii. calculate the values for a_0, a_n and b_n, in terms of α;

For $f_a(x)$:

$$
\begin{aligned}
a_0 &= \frac{1}{\alpha}\int_{-\alpha}^{\alpha} x^2 dx \\
&= \frac{1}{\alpha}\left[\frac{x^3}{3}\right]_{-\alpha}^{\alpha} \\
&= \frac{1}{3\alpha}(\alpha^3 - (-\alpha)^3) \\
&= \frac{2\alpha^2}{3} \\
a_n &= \frac{1}{\alpha}\int_{-\alpha}^{\alpha} x^2 \cos\left(\frac{\pi n x}{\alpha}\right) dx \\
&= \frac{1}{\alpha}\int_{-\alpha}^{\alpha} x^2 \cos\left(\frac{\pi n x}{\alpha}\right) dx
\end{aligned}
$$

We need to use integration by parts:

$$
\int_a^b u\, v' dx = [u\, v]_a^b - \int_a^b u' v\, dx
$$

Here, we set

$$
\begin{aligned}
u &= x^2 & u' &= 2x \\
v' &= \cos\left(\frac{\pi n x}{\alpha}\right) & v &= \left(\frac{\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right)
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\alpha}\left(\left[x^2\left(\frac{\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right)\right]_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha} 2x\left(\frac{\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right) dx\right) \\
&= \frac{1}{\pi n}\left(\left[x^2 \sin\left(\frac{\pi n x}{\alpha}\right)\right]_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha} 2x \sin\left(\frac{\pi n x}{\alpha}\right) dx\right) \\
&= \frac{1}{\pi n}\left(\left(\alpha^2 \sin(\pi n) - \alpha^2 \sin(-\pi n)\right) - \int_{-\alpha}^{\alpha} 2x \sin\left(\frac{\pi n x}{\alpha}\right) dx\right) \\
&= \frac{-2}{\pi n}\left(\int_{-\alpha}^{\alpha} x \sin\left(\frac{\pi n x}{\alpha}\right) dx\right) \quad \text{because } \sin(\pi n) = 0
\end{aligned}
$$

We need to use integration by parts again. Here, we set

$$
\begin{aligned}
u &= x & u' &= 1 \\
v' &= \sin\left(\frac{\pi n x}{\alpha}\right) & v &= \left(\frac{-\alpha}{\pi n}\right)\cos\left(\frac{\pi n x}{\alpha}\right)
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{-2}{\pi n}\left(\left[x\left(\frac{-\alpha}{\pi n}\right)\cos\left(\frac{\pi n x}{\alpha}\right)\right]_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha} 1\left(\frac{-\alpha}{\pi n}\right)\cos\left(\frac{\pi n x}{\alpha}\right) dx\right) \\
&= \frac{2\alpha}{\pi^2 n^2}\left(\left[x \cos\left(\frac{\pi n x}{\alpha}\right)\right]_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha} \cos\left(\frac{\pi n x}{\alpha}\right) dx\right) \\
&= \frac{2\alpha}{\pi^2 n^2}\left(\left(\alpha \cos(\pi n) - (-\alpha)\cos(-\pi n)\right) - \left[\left(\frac{-\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right)\right]_{-\alpha}^{\alpha}\right) \\
&= \frac{2\alpha}{\pi^2 n^2}\left(\left(2\alpha\,(-1)^n\right) - \left(\frac{-\alpha}{\pi n}\right)\left(\sin(\pi n) - \sin(-\pi n)\right)\right) \\
&= \frac{4\alpha^2(-1)^n}{\pi^2 n^2} \\
&= \left(\frac{2\alpha}{\pi n}\right)^2 (-1)^n
\end{aligned}
$$

For $f_c(x)$:

$$a_0 = \frac{1}{\alpha}\left(\int_{-\alpha}^{0}\left(\alpha - \frac{x}{2\alpha}\right)dx + \int_{0}^{\alpha}\left(\alpha + \frac{x}{2\alpha}\right)dx\right)$$
$$= \frac{1}{\alpha}\left(\left[\alpha x - \frac{x^2}{4\alpha}\right]_{-\alpha}^{0} + \left[\alpha x + \frac{x^2}{4\alpha}\right]_{0}^{\alpha}\right)$$
$$= \frac{1}{\alpha}\left(\left(\alpha\,0 - \frac{0^2}{4\alpha} - \alpha(-\alpha) + \frac{(-\alpha)^2}{4\alpha}\right) + \left(\alpha\,\alpha + \frac{\alpha^2}{4\alpha} - \alpha\,0 - \frac{0^2}{4\alpha}\right)\right)$$
$$= \frac{1}{\alpha}\left(\left(\alpha^2 + \frac{\alpha^2}{4\alpha}\right) + \left(\alpha^2 + \frac{\alpha^2}{4\alpha}\right)\right)$$
$$= \frac{4\alpha + 1}{2}$$
$$a_n = \frac{1}{\alpha}\left(\int_{-\alpha}^{0}\left(\alpha - \frac{x}{2\alpha}\right)\cos\left(\frac{\pi n x}{\alpha}\right)dx + \int_{0}^{\alpha}\left(\alpha + \frac{x}{2\alpha}\right)\cos\left(\frac{\pi n x}{\alpha}\right)dx\right)$$

We need to use integration by parts:

$$\int_a^b u\,v'dx = [u\,v]_a^b - \int_a^b u'v\,dx$$

Here, we set

$$u = \left(\alpha - \frac{x}{2\alpha}\right) \qquad u' = -\frac{1}{2\alpha}$$
$$v' = \cos\left(\frac{\pi n x}{\alpha}\right) \qquad v = \left(\frac{\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right)$$

For the first integral, and

$$u = \left(\alpha + \frac{x}{2\alpha}\right) \qquad u' = \frac{1}{2\alpha}$$
$$v' = \cos\left(\frac{\pi n x}{\alpha}\right) \qquad v = \left(\frac{\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right)$$

For the second integral, so

$$a_n = \frac{1}{\alpha}\left(\left[\left(\alpha - \frac{x}{2\alpha}\right)\left(\frac{\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right)\right]_{-\alpha}^{0} - \int_{-\alpha}^{0}\frac{-1}{2\alpha}\left(\frac{\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right)dx \right.$$
$$\left. + \left[\left(\alpha + \frac{x}{2\alpha}\right)\left(\frac{\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right)\right]_{0}^{\alpha} - \int_{0}^{\alpha}\frac{1}{2\alpha}\left(\frac{\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right)dx\right)$$
$$= \frac{1}{\alpha}\left(\int_{-\alpha}^{0}\frac{1}{2\alpha}\left(\frac{\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right)dx - \int_{0}^{\alpha}\frac{1}{2\alpha}\left(\frac{\alpha}{\pi n}\right)\sin\left(\frac{\pi n x}{\alpha}\right)dx\right) \quad \text{because } \sin(\pi n) = 0$$
$$= \frac{1}{2\alpha\pi n}\left(\int_{-\alpha}^{0}\sin\left(\frac{\pi n x}{\alpha}\right)dx - \int_{0}^{\alpha}\sin\left(\frac{\pi n x}{\alpha}\right)dx\right)$$
$$= \frac{1}{2\alpha\pi n}\left(\left[\left(\frac{-\alpha}{\pi n}\right)\cos\left(\frac{\pi n x}{\alpha}\right)\right]_{-\alpha}^{0} - \left[\left(\frac{-\alpha}{\pi n}\right)\cos\left(\frac{\pi n x}{\alpha}\right)\right]_{0}^{\alpha}\right)$$
$$= \frac{1}{2\alpha\pi n}\frac{-\alpha}{\pi n}\left(\left[\cos\left(\frac{\pi n x}{\alpha}\right)\right]_{-\alpha}^{0} - \left[\cos\left(\frac{\pi n x}{\alpha}\right)\right]_{0}^{\alpha}\right)$$
$$= \frac{-1}{2\pi^2 n^2}\left(\left(\cos(0) - \cos(-\pi n)\right) - \left(\cos(\pi n) - \cos(0)\right)\right)$$
$$= \frac{-1}{2\pi^2 n^2}\left((1 - (-1)^n) - ((-1)^n - 1)\right)$$
$$= \frac{(-1)^n - 1}{\pi^2 n^2}$$

iv. give the expression for the Fourier series approximation, $S_N(x)$;

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi n x}{T}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{2\pi n x}{T}\right)$$

For fa(x):

$$a_0 = \frac{2\alpha^2}{3}$$
$$a_n = \left(\frac{2\alpha}{\pi n}\right)^2 (-1)^n$$
$$b_n = 0$$

Thus

$$S_N(x) = \frac{\alpha^2}{3} + \sum_{n=1}^{N} \left(\frac{2\alpha}{\pi n}\right)^2 (-1)^n \cos\left(\frac{2\pi n x}{T}\right)$$

For fc(x):

$$a_0 = \frac{4\alpha + 1}{2}$$
$$a_n = \frac{(-1)^n - 1}{\pi^2 n^2}$$
$$b_n = 0$$

Thus

$$S_N(x) = \frac{4\alpha + 1}{4} + \sum_{n=1}^{N} \frac{(-1)^n - 1}{\pi^2 n^2} \cos\left(\frac{2\pi n x}{T}\right)$$

v. give the coefficients a_n and power of each of the spectral components that exist in $S_N(x)$ in terms of α for $n = 0, 1, 2, 3, 4, 5, 6$;

Note that the power in a_0 is $\frac{{a_0}^2}{2}$, while power in a_n is ${a_n}^2$.

For $f_a(x)$:

n	0	1	2	3	4	5	6
a_n	$\frac{2\alpha^2}{3}$	$-\left(\frac{2\alpha}{\pi}\right)^2$	$\left(\frac{\alpha}{\pi}\right)^2$	$-\left(\frac{2\alpha}{3\pi}\right)^2$	$\left(\frac{\alpha}{2\pi}\right)^2$	$-\left(\frac{2\alpha}{5\pi}\right)^2$	$\left(\frac{\alpha}{3\pi}\right)^2$
Power	$\frac{2\alpha^4}{9}$	$\left(\frac{2\alpha}{\pi}\right)^4$	$\left(\frac{\alpha}{\pi}\right)^4$	$\left(\frac{2\alpha}{3\pi}\right)^4$	$\left(\frac{\alpha}{2\pi}\right)^4$	$\left(\frac{2\alpha}{5\pi}\right)^4$	$\left(\frac{\alpha}{3\pi}\right)^4$

For $f_c(x)$:

n	0	1	2	3	4	5	6
a_n	$\frac{4\alpha + 1}{2}$	$\frac{-2}{\pi^2}$	0	$\frac{-2}{9\pi^2}$	0	$\frac{-2}{25\pi^2}$	0
Power	$\frac{(4\alpha + 1)^2}{8}$	$\frac{4}{\pi^4}$	0	$\frac{4}{81\pi^4}$	0	$\frac{4}{625\pi^4}$	0

vi. if $\alpha = 1$ s, give numerical values, 4 decimal places, of the magnitude and the power for each harmonic, $n = 0, 1, 2, 3, 4, 5, 6$;

For $f_a(x)$:

n	0	1	2	3	4	5	6
a_n	0.6667	-0.4053	0.1013	-0.0450	0.0253	-0.0162	0.0113
$\|a_n\|$	0.6667	0.4053	0.1013	0.0450	0.0253	0.0162	0.0113
Power	0.2222	0.1643	0.0103	0.0020	0.0006	0.0003	0.0001

For $f_c(x)$:

n	0	1	2	3	4	5	6
a_n	2.5000	-0.2026	0.0000	-0.0225	0.0000	-0.0081	0.0000
$\|a_n\|$	2.5000	0.2026	0.0000	0.0225	0.0000	0.0081	0.0000
Power	3.1250	0.0411	0.0000	0.0005	0.0000	0.0001	0.0000

vii. Why is the power in these signals concentrated in the DC and first few harmonics?

Both $f_a(x)$ and $f_c(x)$ are relatively smooth functions with no jumps in value, having only discontinuities in first and higher order derivatives.
Smooth functions, such as these, can be well approximated using relatively few terms in the Fourier series because the non-zero coefficients, and hence energies, are significantly larger at low frequencies.

4.2 For each of the following statements, choose the response that is most correct.

i. If $F(\omega)$ is the Fourier transform of the function $f(t)$, then the Fourier transform of $\frac{d^2 f(t)}{dt^2}$ is:

a $\omega\, F(\omega)$;

b $\omega^2\, F(\omega)$;

c $-\omega\, F(\omega)$;

d $-\omega^2\, F(\omega)$; ✓

e none of the above.

ii. A function $g(t)$ has a DC offset of 10, then in the Fourier series approximation and the Complex Fourier series approximation we have:

a $a_0 = 10\ and\ c_0 = 5$

b $a_0 = 2\ and\ c_0 = 10$

c $a_0 = 5\ and\ c_0 = 5$

d $a_0 = 5\ and\ c_0 = 10$ ✓

d none of the above.

iii If c_n are components of the complex Fourier series of a function $g(t)$, then:

a c_0 is equal to $g(0)$;

b c_0 is equal to $g(\infty)$;

c c_0 is equal to the mean of $g(t)$; ✓

d c_0 is equal to $g(-\infty)$;

e none of the above.

vi If $F(\omega)$ is the Fourier transform of the function $f(t)$, then the Fourier transform of $f(t - t_0)$ is:

a $e^{i\omega t_0}\, F(\omega)$;

b $e^{-i\omega t_0}\, F(\omega)$; ✓

c $F(\omega + t_0)$;

d $F(\omega - t_0)$;

e none of the above.

v If $0 < \omega_L < \omega_H$, a bandpass filter:

a attenuates frequencies $-\omega_L < \omega < \omega_L$;

b attenuates frequencies $-\omega_H < \omega < -\omega_L$ and $\omega_L < \omega < \omega_H$;

c attenuates frequencies $\omega < -\omega_H$ and $\omega_H < \omega$;

d attenuates frequencies $\omega < -\omega_H$, $-\omega_L < \omega < \omega_L$, and $\omega_H < \omega$; ✓

e none of the above.

iv Consider the following functions defined on the interval $-2 \le t \le 2$:

$$f(t) = \frac{1}{t}$$

$$g(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \end{cases}$$

$$h(t) = \frac{\sin(t)}{t}$$

a $f(t)$, $g(t)$, and $h(t)$ are piecewise continuous;

b $f(t)$ and $h(t)$ are piecewise continuous;

c $g(t)$ and $h(t)$ are piecewise continuous; ✓

d $g(t)$ is piecewise continuous;

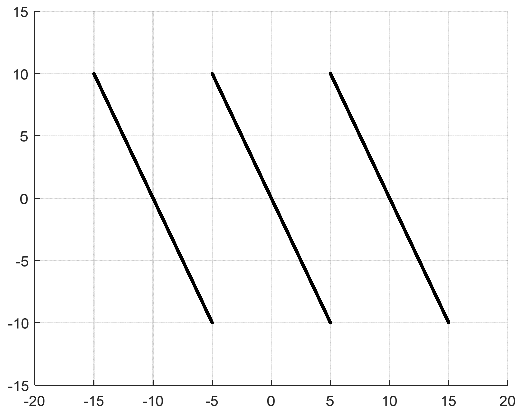
e none of the above.

4.3(*) Fourier series [34 marks]

Consider function $f(x)$ is defined as:

$$f(x) = -2x \quad 0 \le x \le 5$$

i. Sketch the odd periodic extension of the function over the range [–15, 15];



1 marks

ii. What is the period of the odd extension?

1 marks

$T = 10$

iii. What can we say about the convergence of the Fourier series? Justify your answer.

The Fourier series approximation will converge 1 marks

iv. Determine the Fourier series approximation of the odd periodic function and give the expression for the Fourier series approximation, SN(x);

$a_0 = 0$ no DC shift 1 marks

$a_n = 0$ No even terms. Cos terms' coefficients are all zero 2 marks

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{2\pi nx}{T}\right)dx = \frac{2}{10}\int_{-10/2}^{10/2} f(x)\sin\left(\frac{2\pi nx}{10}\right)dx$$

$** \; odd * odd = even \rightarrow$

$$= \frac{2\times 2}{10}\int_0^{10/2} f(x)\sin\left(\frac{\pi nx}{5}\right)dx$$

$$= \frac{2}{5}\int_0^5 f(x)\sin\left(\frac{\pi nx}{5}\right)dx$$

$$= \frac{2}{5}\int_0^5 (-2x)\sin\left(\frac{\pi nx}{5}\right)dx$$

$$= \frac{-4}{5}\int_0^5 (x)\sin\left(\frac{\pi nx}{5}\right)dx$$

3 marks

We need to use integration by parts:

$$\int_a^b u\,v'dx = [u\,v]_a^b - \int_a^b u'v\,dx$$

Here, we set

$$u = x \quad \rightarrow \quad u' = 1$$
$$v' = \sin\left(\frac{\pi n x}{5}\right) \quad \rightarrow \quad v = \left(\frac{-5}{\pi n}\right)\cos\left(\frac{\pi n x}{5}\right)$$

$$b_n = \frac{-4}{5}\left(\left[x\left(\frac{-5}{\pi n}\right)\cos\left(\frac{\pi n x}{5}\right)\right]_0^5 - \int_0^5 1\left(\frac{-5}{\pi n}\right)\cos\left(\frac{\pi n x}{5}\right)\,dx\right)$$
$$b_n = \frac{-4}{5}\left(\left[x\left(\frac{-5}{\pi n}\right)\cos\left(\frac{\pi n x}{5}\right)\right]_0^5 + \left(\frac{5}{\pi n}\right)\int_0^5 \frac{5}{\pi n}.\frac{\pi n}{5}\cos\left(\frac{\pi n x}{5}\right)\,dx\right)$$
$$= \frac{-4}{5}\left(\left[x\left(\frac{-5}{\pi n}\right)\cos\left(\frac{\pi n x}{5}\right)\right]_0^5 + \left(\frac{5}{\pi n}\right)^2\int_0^5 \frac{\pi n}{5}\cos\left(\frac{\pi n x}{5}\right)\,dx\right)$$
$$= \frac{-4}{5}\left(\left(\frac{-5}{\pi n}\right)(5\cos(\pi n) - (0)\cos(0)) + \left(\frac{5}{\pi n}\right)^2\left[\sin\left(\frac{\pi n x}{5}\right)\right]_0^5\right)$$
$$= \frac{-4}{5}\left(\left(\frac{-5}{\pi n}\right)(5(-1)^n) + \left(\frac{5}{\pi n}\right)^2(\sin(\pi n) - \sin(0))\right)$$
$$= \frac{-4}{5}\left(\left(\frac{-5}{\pi n}\right)(5(-1)^n) + 0\right)$$
$$= \frac{20}{\pi n}(-1)^n$$

3 marks

Now:

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n\cos\left(\frac{2\pi n x}{T}\right) + \sum_{n=1}^{N} b_n\sin\left(\frac{2\pi n x}{T}\right)$$

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{20}{\pi n}(-1)^n$$

Thus

$$S_N(x) = \frac{0}{2} + \sum_{n=1}^{N} 0\cos\left(\frac{\pi n x}{5}\right) + \sum_{n=1}^{N}\frac{20}{\pi n}(-1)^n\sin\left(\frac{\pi n x}{5}\right)$$
$$S_N(x) = \sum_{n=1}^{N}\frac{20}{\pi n}(-1)^n\sin\left(\frac{\pi n x}{5}\right)$$

2 marks

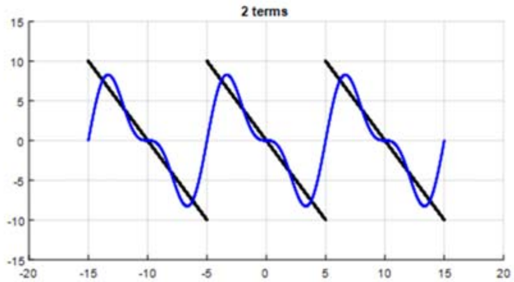
v. Write down the first five non-zero terms of your approximation.

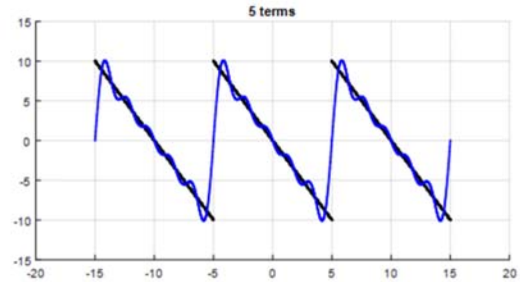
$$S_5(x) = \sum_{n=1}^{5}\frac{20}{\pi n}(-1)^n\sin\left(\frac{\pi n x}{5}\right)$$
$$= \frac{-20}{\pi}\sin\left(\frac{\pi x}{5}\right) + \frac{10}{\pi}\sin\left(\frac{2\pi x}{5}\right) - \frac{20}{\pi 3}\sin\left(\frac{3\pi x}{5}\right) + \frac{5}{\pi}\sin\left(\frac{4\pi x}{5}\right) - \frac{4}{\pi}\sin(\pi x)$$

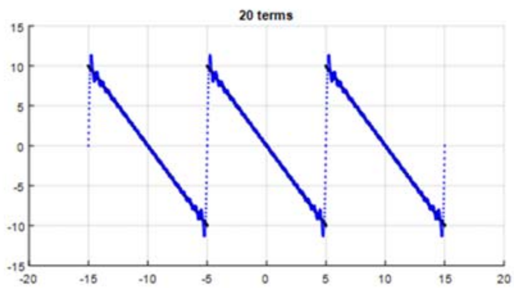
2.5 marks: 0.5 mark for each correct term

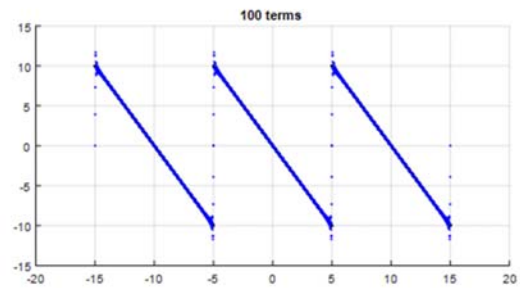
vi. Plot your Fourier series approximation on the top of the original odd function over the range [-15, 15] using 2 harmonic terms, 5 harmonic terms, 20 harmonic terms and 100 harmonic terms (4 plots) and justify which one is a better approximation of the function $f(x)$? Note: use $\Delta x = 0.01$ for your plots.

4 marks – 1 mark for each correct plot









The approximation using 100 terms is the best among these 4. 1 marks

vii. What is the value of the Fourier series approximation at the discontinuities?

Zero 1 marks

viii. Calculate the total power in the Fourier series approximation and plot the one-sided Power spectrum for both power (W) vs Freq (Hz) and power (W) vs n, for n=1 to 5.

$$S_5(x) = \sum_{n=1}^{5} \frac{20}{\pi n}(-1)^n \sin\left(\frac{\pi n x}{5}\right)$$

$$= \frac{20}{\pi(1)}(-1)^1 \sin\left(\frac{\pi x}{5}\right) + \frac{20}{\pi(2)}(-1)^2 \sin\left(\frac{\pi 2x}{5}\right) + \frac{20}{\pi 3}(-1)^3 \sin\left(\frac{\pi 3x}{5}\right) + \frac{20}{\pi 4}(-1)^4 \sin\left(\frac{\pi 4x}{5}\right) + \frac{20}{\pi 5}(-1)^5 \sin\left(\frac{\pi 5x}{5}\right)$$

$$= \underset{\substack{\uparrow \\ b_1}}{\frac{-20}{\pi}} \sin\left(\frac{\pi x}{5}\right) + \underset{\substack{\uparrow \\ b_2}}{\frac{10}{\pi}} \sin\left(\frac{\pi 2x}{5}\right) + \underset{\substack{\uparrow \\ b_3}}{\frac{-20}{\pi 3}} \sin\left(\frac{\pi 3x}{5}\right) + \underset{\substack{\uparrow \\ b_4}}{\frac{5}{\pi}} \sin\left(\frac{\pi 4x}{5}\right) + \underset{\substack{\uparrow \\ b_5}}{\frac{-4}{\pi}} \sin\left(\frac{\pi 5x}{5}\right)$$

$$\sin\left(\frac{\pi}{5}x\right) = \sin(\omega_1 x) \rightarrow \quad \omega_1 = \frac{\pi}{5} = 2\pi f_1 \rightarrow f_1 = \frac{1}{10} = 0.1\ Hz$$

$$\sin\left(\frac{2\pi}{5}x\right) = \sin(\omega_2 x) \rightarrow \quad \omega_2 = \frac{2\pi}{5} = 2\pi f_2 \rightarrow f_2 = \frac{2}{10} = 0.2Hz$$

$$\sin\left(\frac{3\pi}{5}x\right) = \sin(\omega_3 x) \rightarrow \quad \omega_3 = \frac{3\pi}{5} = 2\pi f_3 \rightarrow f_3 = \frac{3}{10} = 0.3Hz$$

2.5 marks -
0.5 mark for each
correct frequency

$$\sin\left(\frac{4\pi}{5}x\right) = \sin(\omega_4 x) \rightarrow \quad \omega_4 = \frac{4\pi}{5} = 2\pi f_4 \rightarrow f_4 = \frac{4}{10} = 0.4Hz$$

$$\sin\left(\frac{5\pi}{5}x\right) = \sin(\omega_5 x) \rightarrow \quad \omega_5 = \frac{5\pi}{5} = 2\pi f_5 \rightarrow f_5 = \frac{5}{10} = 0.5Hz$$

➔ Or simply $f_n = \frac{n}{T}, T = 10$ therefore in this case: $f_n = \frac{n}{10} Hz$

Now:

$$Power\ in\ S_N(x) = \frac{{a_0}^2}{2} + \sum_{n=1}^{N} {a_n}^2 + {b_n}^2$$

$$Power\ in\ S_5(x) = \frac{{a_0}^2}{2} + \sum_{n=1}^{5} {a_n}^2 + {b_n}^2$$

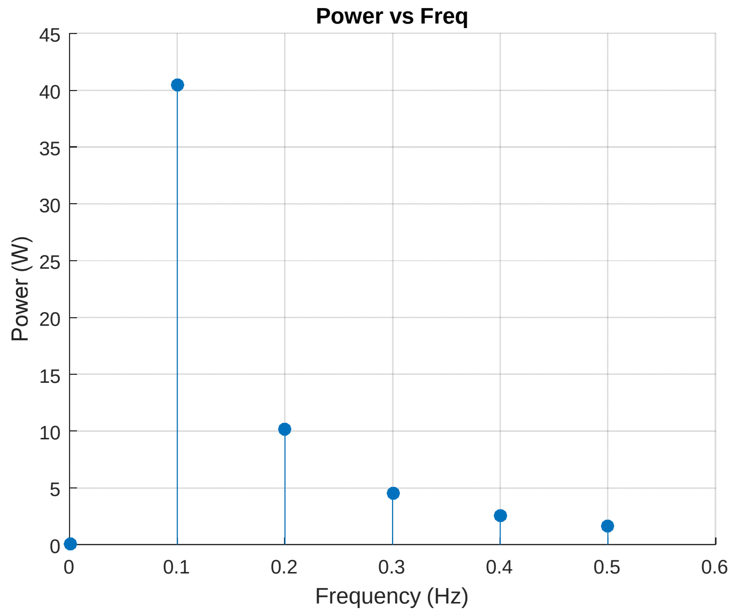
$$S_5(x) = \frac{0^2}{2} + \sum_{n=1}^{5} 0^2 + {b_n}^2 = \sum_{n=1}^{5} {b_n}^2 = \sum_{n=1}^{5} {b_1}^2 + {b_2}^2 + {b_3}^2 + {b_4}^2 + {b_5}^2$$

$$= \underbrace{\left(\frac{-20}{\pi}\right)^2}_{{b_1}^2} + \underbrace{\left(\frac{10}{\pi}\right)^2}_{{b_2}^2} + \underbrace{\left(\frac{-20}{3\pi}\right)^2}_{{b_3}^2} + \underbrace{\left(\frac{5}{\pi}\right)^2}_{{b_4}^2} + \underbrace{\left(\frac{-4}{\pi}\right)^2}_{{b_5}^2}$$

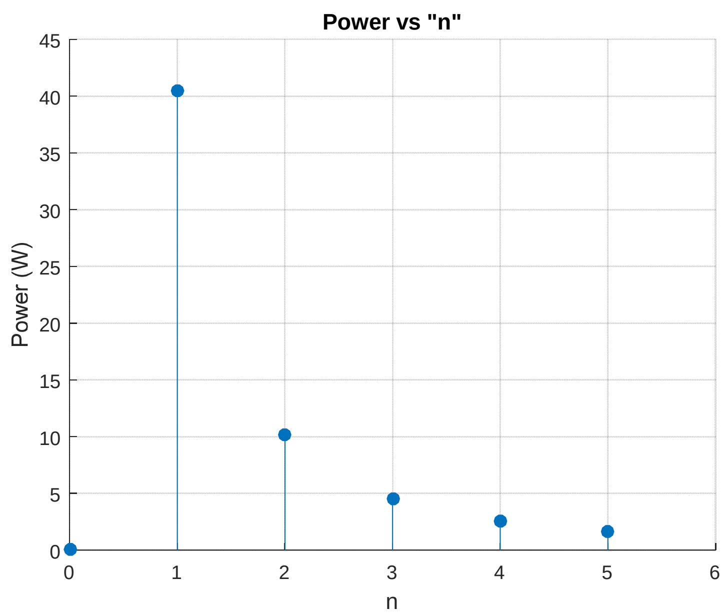
2.5 marks :
0.5 mark for each
correct power

1.5 mark

$$Power\ in\ S_N(x) = 59.32$$



2 marks



2 marks

ix. What is the power of at 0 frequency?

$Power\ at\ 0Hz\ is\ \ 0$. There's no power at 0Hz. 1 marks
In other words, the DC offset is 0, therefore no power at 0Hz

4.4 Fourier series – important practice question

Try the same steps in question 4.2 for when the function $f(x)$ is defined as:

$$f(x) = 2x \quad 0 \leq x \leq 5$$

and you are asked to extend $f(x)$ to be an even periodic function.

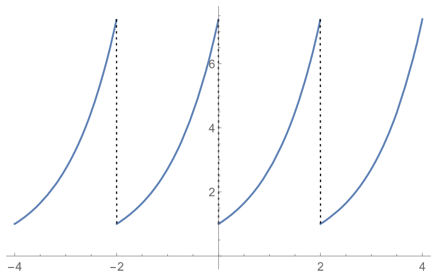
According to the solution provided for Q4.3, work this out yourself.

4.5(*) Fourier series [40 marks]

The function $f(x)$ is periodic and is defined such that:

$$f(x) = e^{x} \quad 0 \leq x < 2$$

x. sketch the periodic function over the range [–4, 4];



4 marks

xi. calculate the fundamental frequency, F, and fundamental angular frequency, ω0, of this signal;

Since $T = 2$

$$F = \frac{1}{T} = \frac{1}{2\text{ s}}$$

$F = 0.5\ \text{s}^{-1}$. 2 mark

$\omega_0 = 2\pi F = \frac{2\pi}{T} = \pi$ 2 mark

$\omega_0 = 3.141592653589793\ \text{s}^{-1}$!

:D

xii. calculate the coefficients, c_0 and c_n, of the complex Fourier series of this function;

Note that

$$S_N(x) = c_0 + \sum_{\substack{n=-N \\ n\neq 0}}^{N} c_n e^{in\omega_0 x}$$

where

$$c_0 = \frac{1}{T}\int_a^b f(x)dx$$

$$c_n = \frac{1}{T}\int_a^b f(x)\, e^{-in\omega_0 x}\, dx$$
$$T = b - a$$
$$\omega_0 = 2\pi f = \frac{2\pi}{T}$$

For this function
$a = 0$ and $b = 2$, so $T = 2$

$$c_0 = \frac{1}{T}\int_a^b f(x) dx$$
$$= \frac{1}{2}\int_0^2 e^x dx$$
$$= \frac{1}{2}[e^x]_0^2$$
$$= \frac{1}{2}(e^2 - e^0)$$
$$= \frac{1}{2}(e^2 - 1)$$

4 marks

$$c_n = \frac{1}{T}\int_a^b f(x)\, e^{-in\omega_0 x}\, dx$$
$$= \frac{1}{2}\int_0^2 e^x\, e^{-in\pi x}\, dx$$
$$= \frac{1}{2}\int_0^2 e^{x(1-in\pi)}\, dx$$
$$= \frac{1}{2}\left[\frac{e^{x(1-in\pi)}}{1-in\pi}\right]_0^2$$
$$= \frac{1}{2(1-in\pi)}\left(e^{2(1-in\pi)} - e^{0(1-in\pi)}\right)$$
$$= \frac{1}{2(1-in\pi)}\left(e^2 e^{-in2\pi} - 1\right)$$
$$= \frac{1}{2(1-in\pi)}(e^2 - 1)$$
$$= \frac{(e^2-1)(1+in\pi)}{2(1+n^2\pi^2)}$$

6 marks

xiii. express the complex Fourier series, $S_N(x)$, of this function in terms of the coefficients, c_0 and c_n;

$$S_N(x) = c_0 + \sum_{\substack{n=-N \\ n\neq 0}}^{N} c_n e^{in\omega_0 x}$$
$$= \frac{1}{2}(e^2 - 1) + \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{(e^2-1)(1+in\pi)}{2(1+n^2\pi^2)} e^{in\pi x}$$

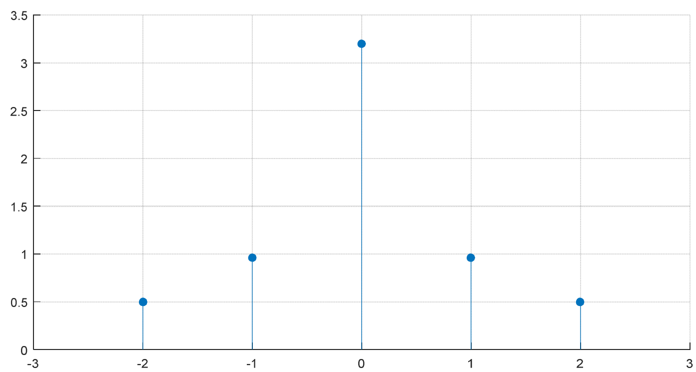
4 marks

xiv. tabulate, to 4 decimal places, the magnitude and phase (in radians) of the coefficients for $-2 \le n \le 2$;

n	c_n	$\lvert c_n \rvert$	$\angle c_n$
-2	$\dfrac{(e^2-1)(1-i2\pi)}{2(1+4\pi^2)}$	0.5021	-1.4130
-1	$\dfrac{(e^2-1)(1-i\pi)}{2(1+\pi^2)}$	0.9689	-1.2626
0	$\dfrac{(e^2-1)}{2}$	3.195	0.0000
1	$\dfrac{(e^2-1)(1+i\pi)}{2(1+\pi^2)}$	0.9689	1.2626
2	$\dfrac{(e^2-1)(1-i2\pi)}{2(1+4\pi^2)}$	0.5021	1.4130

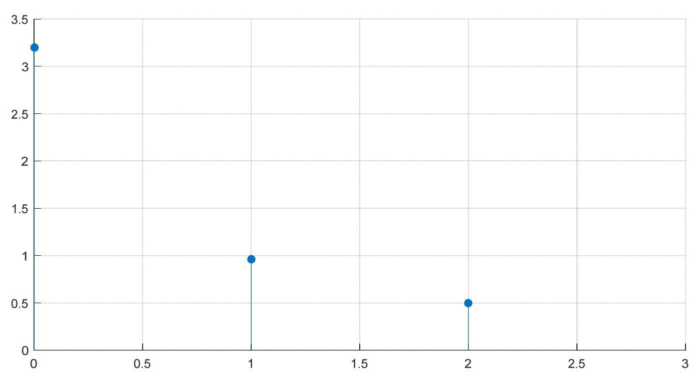
10 marks:
1 mark for each correct value of |Cn| and $\angle c_n$

xv. plot the two-sided magnitude spectrum (mag vs "n") for $-2 \le n \le 2$;



4 marks

xvi. plot the one-sided magnitude spectrum for $0 \le n \le 2$.



4 marks

4.6 Fourier transform

The functions $f_a(x)$ to $f_c(x)$ are defined such that:

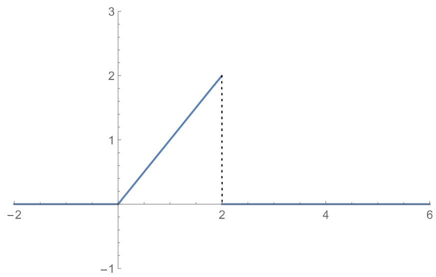
$$f_a(x) = \begin{cases} x & 0 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_b(x) = \begin{cases} 2 & 2 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$$

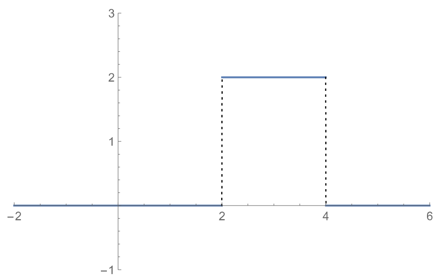
$$f_c(x) = \begin{cases} 5 - x & 4 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$$

i. sketch the above functions, $f_a(x)$, $f_b(x)$, and $f_c(x)$, on **separate** graphs over the range [–2, 6];

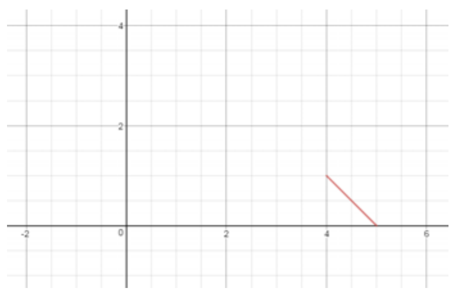
$f_a(x)$:



$f_b(x)$:



$f_c(x)$:



ii. calculate the Fourier transform of $f_a(x)$,;

$$F_a(\omega) = \int_{-\infty}^{\infty} f_a(x)\, e^{-i\omega x}\, dx$$

$$= \int_{0}^{2} x\, e^{-i\omega x}\, dx$$

We need to use integration by parts to evaluate the first integral:

$$\int_a^b u\, v' dx = [u\, v]_a^b - \int_a^b u' v\, dx$$

Here, we set

$$u = x \quad \rightarrow \quad u' = 1$$
$$v' = e^{-i\omega t} \quad \rightarrow \quad v = \frac{i}{\omega} e^{-i\omega t}$$

$$F_a(\omega) = \int_0^2 x\, e^{-i\omega x}\, dx$$
$$= \left[x \frac{i}{\omega} e^{-i\omega x}\right]_0^2 - \int_0^2 1\, \frac{i}{\omega} e^{-i\omega x}\, dx$$
$$= \frac{i}{\omega}\left(2e^{-i\omega 2} - 0\, e^{-i\omega 0}\right) - \left[\frac{-1}{\omega^2} e^{-i\omega x}\right]_0^2$$
$$= \frac{i2}{\omega} e^{-i2\omega} + \frac{1}{\omega^2}\left(e^{-i2\omega} - 1\right)$$
$$= \frac{i2}{\omega} e^{-i2\omega} - \frac{e^{-i\omega}}{\omega^2}\left(e^{i\omega} - e^{-i\omega}\right)$$
$$= \frac{i2}{\omega} e^{-i2\omega} - \frac{2ie^{-i\omega}}{\omega^2} \sin(\omega)$$

since $\sin(\omega) = \frac{1}{2i}\left(e^{i\omega} - e^{-i\omega}\right)$

iii. calculate the Fourier transform of $f_b(x)$,;

$$F_b(\omega) = \int_{-\infty}^{\infty} f_b(x)\, e^{-i\omega x}\, dx$$
$$= \int_2^4 2\, e^{-i\omega x}\, dx$$
$$= 2\left[\frac{i}{\omega} e^{-i\omega t}\right]_2^4$$
$$= \frac{2i}{\omega}\left(e^{-i\omega 4} - e^{-i\omega 2}\right)$$
$$= \frac{2i}{\omega}\left(e^{-i\omega 4} - e^{-i\omega 2}\right)$$
$$= \frac{2ie^{-i\omega 3}}{\omega}\left(e^{-i\omega} - e^{i\omega}\right)$$
$$= \frac{4e^{-i\omega 3}}{\omega} \sin(\omega)$$

Since $\sin(\omega) = \frac{1}{2i}\left(e^{i\omega} - e^{-i\omega}\right)$

iv. calculate the Fourier transform of $f_c(x)$,;

$$F_c(\omega) = \int_{-\infty}^{\infty} f_c(x)\, e^{-i\omega x}\, dx$$
$$= \int_4^5 (5 - x)\, e^{-i\omega x}\, dx$$

We need to use integration by parts to evaluate the first integral:

$$\int_a^b u\, v' dx = [u\, v]_a^b - \int_a^b u' v\, dx$$

Here, we set

$$u = (5 - x) \quad \to \quad u' = -1$$

$$v' = e^{-i\omega t} \quad \to \quad v = \frac{i}{\omega} e^{-i\omega t}$$

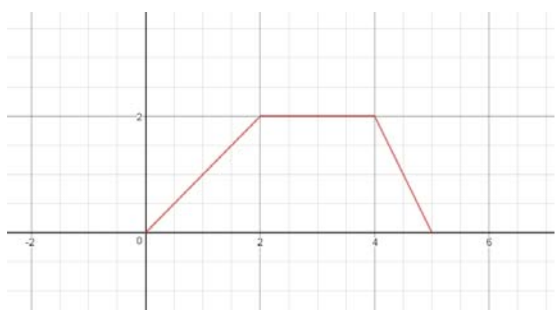
$$F_c(\omega) = \int_4^5 (5 - x)\, e^{-i\omega x}\, dx$$

$$= \left[(5 - x)\frac{i}{\omega} e^{-i\omega x}\right]_4^5 - \int_4^5 (-1) \times \frac{i}{\omega} e^{-i\omega x}\, dx$$

$$= \left((5 - 5)\frac{i}{\omega} e^{-i\omega 5} - (5 - 4)\frac{i}{\omega} e^{-i\omega 4}\right) - \left[\frac{1}{\omega^2} e^{-i\omega x}\right]_4^5$$

$$= \left(\frac{-i}{\omega} e^{-i\omega 4}\right) - \frac{1}{\omega^2}\left(e^{-i5\omega} - e^{-i4\omega}\right) = \left(\frac{-i}{\omega} + \frac{1}{\omega^2}\right) e^{-i\omega 4} - \frac{1}{\omega^2} e^{-i5\omega}$$

v. sketch the function $f_d(x) = f_a(x) + f_b(x) + 2\, f_c(x)$ over the range [–2, 6];



$f_d(x)$

vi. Hence, determine the Fourier transform of $f_d(x)$.

Since $f_d(x)$ is a linear combination of $f_a(x)$, $f_b(x)$, and $f_c(x)$, the Fourier transform of $f_d(x)$ is the same linear combination of the Fourier transforms of $f_a(x)$, $f_b(x)$, and $f_c(x)$:

$$F_d(\omega) = F_a(\omega) + F_b(\omega) + 2F_c(\omega)$$

$$= \frac{i2}{\omega} e^{-i2\omega} - \frac{2ie^{-i\omega}}{\omega^2}\sin(\omega) + \frac{4e^{-i\omega 3}}{\omega}\sin(\omega) + 2\left(\frac{-i}{\omega} + \frac{1}{\omega^2}\right)e^{-i\omega 4} - \frac{2}{\omega^2} e^{-i5\omega}$$

Now you may use $\sin(\omega) = \frac{1}{2i}\left(e^{i\omega} - e^{-i\omega}\right)$ to simplify the expression above.

4.7(*) Convolution **[26 marks]**

Consider these two discrete signals:

$$f[n] = \{0.5, 3, 1, 2, 4, 1.5, 6, 0, 5\}$$

$$\mathrm{g}[n] = \{2, 2.5, 5, 1\}$$

i. Find the convolution of these two discrete signals

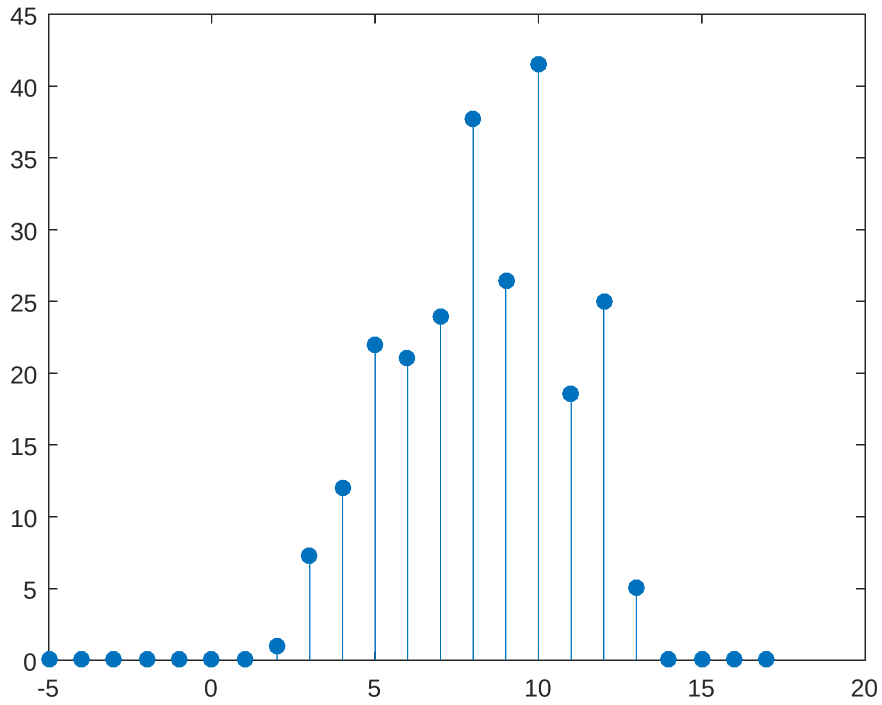
$$y[k] = \sum_{k=-\infty}^{\infty} f[n]g[k-n]$$

ii. Plot $y[k]$

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	F[n].g[k-n]
f[n]						0.5	3	1	2	4	1.5	6	0	5					
g[n]						2	2.5	5	1										
g[-n]	1	5	2.5	2															0
g[-1-n]		1	5	2.5	2														0
g[-2-n]			1	5	2.5	2													2*.5=1
g[-3-n]				1	5	2.5	2												2.5*.5+2*3=7.25
g[-4-n]					1	5	2.5	2											5*.5+2.5*3+2*1=12
g[-5-n]						1	5	2.5	2										1*.5+5*3+2.5*1+2*2=22
g[-6-n]							1	5	2.5	2									1*3+5*1+2.5*2+2*4=21
g[-7-n]								1	5	2.5	2								1*1+5*2+2.5*4+2*1.5=24
g[-8-n]									1	5	2.5	2							1*2+5*4+2.5*1.5+2*6=37.75
g[-9-n]										1	5	2.5	2						1*4+5*1.5+2.5*6+2*0=26.5
g[-10-n]											1	5	2.5	2					1*1.5+5*6+2.5*0+2*5=41.5
g[-11-n]												1	5	2.5	2				1*6+5*0+2.5*5+2*0=18.5
g[-12-n]													1	5	2.5	2			1*0+5*5+2.5*0+2*0=25
g[-13-n]														1	5	2.5	2		1*5+5*0+2.5*0+2*0=5
g[-14-n]															1	5	2.5	2	0

y[k]={1, 7.25, 12, 22, 21, 24, 37.75, 26.5, 41.5, 18.5, 25, 5}

20 marks



6 marks