Week 10_Course online

Elements of Numerical Integration

1-Trapezoidal and Simpson’s rule (4.3)

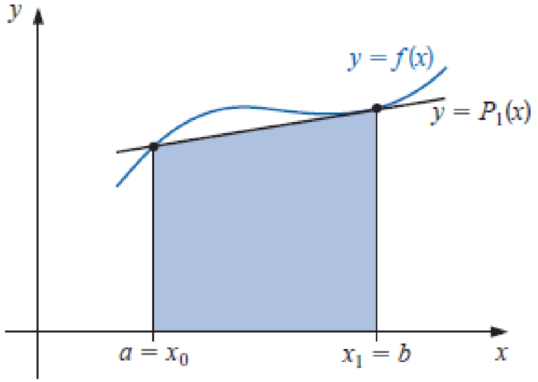
2- Closed and open Newton-cotes formula

3-Composite Numerical Integration (4.4)

The Trapezoidal Rule

To derive the Trapezoidal rule for approximating $\int_a^b f(x)\,dx$, let $x_0 = a, x_1 = b, h = b - a$ and use the linear Lagrange polynomial:

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1).$$



Then

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} \left[\frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1) \right] dx$$

$$\int_a^b f(x)\,dx = \left[\frac{(x - x_1)^2}{2(x_0 - x_1)} f(x_0) + \frac{(x - x_0)^2}{2(x_1 - x_0)} f(x_1) \right]_{x_0}^{x_1} = \frac{(x_1 - x_0)}{2}[f(x_0) + f(x_1)] = \frac{h}{2}[f(x_0) + f(x_1)]$$

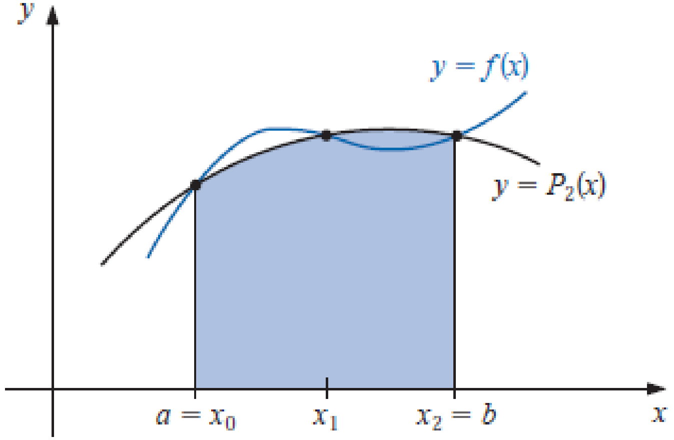
Similarly $\int_{x1}^{x2} f(x)dx = \frac{h}{2}[f(x1) + f(x2)]$, *and* $\int_{x(n-1)}^{xn} f(x)dx = \frac{h}{2}[f(x_{n-1}) + f(xn)]$,

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) + E_n$$

Is called trapezoidal rule

Simpson's Rule

Simpson's rule results from integrating over $[a, b]$ the second Lagrange polynomial with equally-spaced nodes $x_0 = a$, $x_2 = b$, and $x_1 = a + h$, where $h = (b - a)/2$. (See Figure 4.4.)



Therefore

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2}\left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)\right]dx$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

similarly $\int_{x2}^{x4} f(x)dx = \frac{h}{3}[f(x2) + 4f(x3) + f(x4)]$ and

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots + y_{2N-1}) + 2(y_2 + y_4 + \cdots + y_{2N-2}) + y_{2N}] + \text{Error term}$$

Is called Simpson's 1/3 rule

Quadrature formulas:

TRAPEZOIDAL RULE

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) + E_n$$

SIMPSON'S 1/3 RULE

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots + y_{2N-1}) + 2(y_2 + y_4 + \cdots + y_{2N-2}) + y_{2N}] + \text{Error term}$$

Simpson's 3/8 rule is

$$\int_a^b f(x)dx = \frac{3}{8}h[y(a) + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y(b)]$$

Closed-Newton-Cotes (Quadrature formulas)

Theorem 4.2 Suppose that $\sum_{i=0}^{n} a_i f(x_i)$ denotes the $(n+1)$-point closed Newton-Cotes formula with $x_0 = a$, $x_n = b$, and $h = (b-a)/n$. There exists $\xi \in (a, b)$ for which

Some of the common **closed Newton-Cotes formulas** with their error terms are listed. Note that in each case the unknown value ξ lies in (a, b).

$n = 1$: Trapezoidal rule

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi), \quad \text{where} \quad x_0 < \xi < x_1. \tag{4.25}$$

$n = 2$: Simpson's rule

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi), \quad \text{where} \quad x_0 < \xi < x_2. \tag{4.26}$$

$n = 3$: Simpson's Three-Eighths rule

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3h^5}{80} f^{(4)}(\xi), \qquad (4.27)$$

where $x_0 < \xi < x_3$.

$n = 4$:

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] - \frac{8h^7}{945} f^{(6)}(\xi),$$

where $x_0 < \xi < x_4$. $\qquad (4.28)$

Open-Newton-Cotes (Quadrature formulas)

Theorem 4.3 Suppose that $\sum_{i=0}^{n} a_i f(x_i)$ denotes the $(n+1)$-point open Newton-Cotes formula with $x_{-1} = a$, $x_{n+1} = b$, and $h = (b-a)/(n+2)$. There exists $\xi \in (a, b)$ for which

$n = 0$: Midpoint rule

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2hf(x_0).$$

$n = 1$:

$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3h}{2}[f(x_0) + f(x_1)]$$

$n = 2$:

$$\int_{x_{-1}}^{x_3} f(x)\,dx = \frac{4h}{3}[2f(x_0) - f(x_1) + 2f(x_2)].$$

$n = 3$:

$$\int_{x_{-1}}^{x_4} f(x)\,dx = \frac{5h}{24}[11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)].$$

Example 2 Compare the results of the closed and open Newton-Cotes formulas listed as (4.25)–(4.28) and (4.29)–(4.32) when approximating

$$\int_0^{\pi/4} \sin x \, dx = 1 - \sqrt{2}/2 \approx 0.29289322.$$

Solution For the closed formulas we have

$$n = 1: \quad \frac{(\pi/4)}{2}\left[\sin 0 + \sin \frac{\pi}{4}\right] \approx 0.27768018$$

$$n = 2: \quad \frac{(\pi/8)}{3}\left[\sin 0 + 4\sin \frac{\pi}{8} + \sin \frac{\pi}{4}\right] \approx 0.29293264$$

$$n = 3: \quad \frac{3(\pi/12)}{8}\left[\sin 0 + 3\sin \frac{\pi}{12} + 3\sin \frac{\pi}{6} + \sin \frac{\pi}{4}\right] \approx 0.29291070$$

$$n = 4: \quad \frac{2(\pi/16)}{45}\left[7\sin 0 + 32\sin \frac{\pi}{16} + 12\sin \frac{\pi}{8} + 32\sin \frac{3\pi}{16} + 7\sin \frac{\pi}{4}\right] \approx 0.29289318$$

and for the open formulas we have

$$n=0: \quad 2(\pi/8)\left[\sin\frac{\pi}{8}\right] \approx 0.30055887$$

$$n=1: \quad \frac{3(\pi/12)}{2}\left[\sin\frac{\pi}{12}+\sin\frac{\pi}{6}\right] \approx 0.29798754$$

$$n=2: \quad \frac{4(\pi/16)}{3}\left[2\sin\frac{\pi}{16}-\sin\frac{\pi}{8}+2\sin\frac{3\pi}{16}\right] \approx 0.29285866$$

$$n=3: \quad \frac{5(\pi/20)}{24}\left[11\sin\frac{\pi}{20}+\sin\frac{\pi}{10}+\sin\frac{3\pi}{20}+11\sin\frac{\pi}{5}\right] \approx 0.29286923$$

Example: Compute the integral $I=\sqrt{\frac{2}{\pi}}\int_0^1 e^{-x^2/2}dx$ using Simpson's 1/3 rule,

Taking h = 0.125.

EXERCISE SET 4.3

1. Approximate the following integrals using the Trapezoidal rule.

a. $\int_{0.5}^{1} x^4\,dx$

b. $\int_{0}^{0.5} \frac{2}{x-4}\,dx$

c. $\int_{1}^{1.5} x^2 \ln x\,dx$

d. $\int_{0}^{1} x^2 e^{-x}\,dx$

e. $\int_{1}^{1.6} \frac{2x}{x^2-4}\,dx$

f. $\int_{0}^{0.35} \frac{2}{x^2-4}\,dx$

g. $\int_{0}^{\pi/4} x \sin x\,dx$

h. $\int_{0}^{\pi/4} e^{3x} \sin 2x\,dx$

2. Approximate the following integrals using the Trapezoidal rule.

a. $\int_{-0.25}^{0.25} (\cos x)^2\,dx$

b. $\int_{-0.5}^{0} x \ln(x+1)\,dx$

c. $\int_{0.75}^{1.3} \left((\sin x)^2 - 2x \sin x + 1\right)\,dx$

d. $\int_{e}^{e+1} \frac{1}{x \ln x}\,dx$

EXERCISE SET 4.3

5. Repeat Exercise 1 using Simpson's rule.
6. Repeat Exercise 2 using Simpson's rule.
7. Repeat Exercise 3 using Simpson's rule and the results of Exercise 5.
8. Repeat Exercise 4 using Simpson's rule and the results of Exercise 6.
9. Repeat Exercise 1 using the Midpoint rule.
10. Repeat Exercise 2 using the Midpoint rule.

22. Given the function f at the following values,

x	1.8	2.0	2.2	2.4	2.6
$f(x)$	3.12014	4.42569	6.04241	8.03014	10.46675

approximate $\int_{1.8}^{2.6} f(x)\,dx$ using all the appropriate quadrature formulas of this section.

4.4 Composite Numerical Integration

Theorem 4.4 Let $f \in C^4[a,b]$, n be even, $h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$. There exists a $\mu \in (a,b)$ for which the **Composite Simpson's rule** for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\mu).$$

■

Theorem 4.5 Let $f \in C^2[a,b]$, $h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$. There exists a $\mu \in (a,b)$ for which the **Composite Trapezoidal rule** for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu).$$ ■

The Trapezoidal Rule (Composite Form)

The Newton-Cotes formula is based on approximating y = f (x) between (x_0, y_0) and $(x_1, y_1)$by a straight line, thus forming a trapezium, is called trapezoidal rule. In order to evaluate the definite integral

$$I = \int_a^b f(x)dx$$

we divide the interval [a, b] into n sub-intervals, each of size $h = (b - a)/n$ and denote the sub-intervals by $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, x_n]$, such that $x_0 = a$ and $x_n = b$ and $x_k = x_0 + k_h$, $k = 1, 2, \ldots, n - 1$.

Thus, we can write the above definite integral as a sum. Therefore,

$$I = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) + E_n$$

Simpson's Rules (Composite Forms)

the definite integral I can be written as

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{2N-2}}^{x_{2N}} f(x)dx$$

$$I = \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{2N-2} + 4y_{2N-1} + y_{2N})]$$

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots + y_{2N-1}) + 2(y_2 + y_4 + \cdots + y_{2N-2}) + y_{2N}] + \text{Error term}$$

This formula is called composite Simpson's 1/3 rule.

Similarly in deriving composite Simpson's 3/8 rule, we divide the interval of integration into n sub-intervals, where n is divisible by 3, and applying the integration formula

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \cdots + \int_{x_{n-3}}^{x_n} f(x)dx$$

$$\int_{x_0}^{x_3} f(x)dx = \frac{3}{8}h(y_0 + 3y_1 + 3y_2 + y_3)$$

We obtain the composite form of Simpson's 3/8 rule as

$$\int_a^b f(x)dx = \frac{3}{8}h[y(a) + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots$$

$$+ 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y(b)]$$

Is called Simpson's 3/8 rule

Example 1 Use Simpson's rule to approximate $\int_0^4 e^x\,dx$ and compare this to the results obtained by adding the Simpson's rule approximations for $\int_0^2 e^x\,dx$ and $\int_2^4 e^x\,dx$. Compare these approximations to the sum of Simpson's rule for $\int_0^1 e^x\,dx$, $\int_1^2 e^x\,dx$, $\int_2^3 e^x\,dx$, and $\int_3^4 e^x\,dx$.

$$\boxed{\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}$$

Solution Simpson's rule on $[0, 4]$ uses $h = 2$ and gives

$$\int_0^4 e^x\,dx \approx \frac{2}{3}(e^0 + 4e^2 + e^4) = 56.76958.$$

$$\boxed{\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}$$

Applying Simpson's rule on each of the intervals $[0, 2]$ and $[2, 4]$ uses $h = 1$ and gives

$$\begin{aligned}
\int_0^4 e^x\,dx &= \int_0^2 e^x\,dx + \int_2^4 e^x\,dx \\
&\approx \frac{1}{3}\left(e^0 + 4e + e^2\right) + \frac{1}{3}\left(e^2 + 4e^3 + e^4\right) \\
&= \frac{1}{3}\left(e^0 + 4e + 2e^2 + 4e^3 + e^4\right) \\
&= 53.86385.
\end{aligned}$$

$$\boxed{\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}$$

For the integrals on $[0,1]$,$[1,2]$,$[3,4]$, and $[3,4]$ we use Simpson's rule four times with $h = \frac{1}{2}$ giving

$$\begin{aligned}
\int_0^4 e^x\,dx &= \int_0^1 e^x\,dx + \int_1^2 e^x\,dx + \int_2^3 e^x\,dx + \int_3^4 e^x\,dx \\
&\approx \frac{1}{6}\left(e_0 + 4e^{1/2} + e\right) + \frac{1}{6}\left(e + 4e^{3/2} + e^2\right) \\
&\quad + \frac{1}{6}\left(e^2 + 4e^{5/2} + e^3\right) + \frac{1}{6}\left(e^3 + 4e^{7/2} + e^4\right) \\
&= \frac{1}{6}\left(e^0 + 4e^{1/2} + 2e + 4e^{3/2} + 2e^2 + 4e^{5/2} + 2e^3 + 4e^{7/2} + e^4\right) \\
&= 53.61622.
\end{aligned}$$

EXERCISE SET 4.4

1. Use the Composite Trapezoidal rule with the indicated values of n to approximate the following integrals.

   a. $\int_1^2 x \ln x \, dx, \quad n = 4$

   b. $\int_{-2}^2 x^3 e^x \, dx, \quad n = 4$

   c. $\int_0^2 \frac{2}{x^2+4} \, dx, \quad n = 6$

   d. $\int_0^{\pi} x^2 \cos x \, dx, \quad n = 6$

   e. $\int_0^2 e^{2x} \sin 3x \, dx, \quad n = 8$

   f. $\int_1^3 \frac{x}{x^2+4} \, dx, \quad n = 8$

   g. $\int_3^5 \frac{1}{\sqrt{x^2-4}} \, dx, \quad n = 8$

   h. $\int_0^{3\pi/8} \tan x \, dx, \quad n = 8$

2. Use the Composite Trapezoidal rule with the indicated values of n to approximate the following integrals.

   a. $\int_{-0.5}^{0.5} \cos^2 x \, dx, \quad n = 4$

   b. $\int_{-0.5}^{0.5} x \ln(x+1) \, dx, \quad n = 6$

   c. $\int_{.75}^{1.75} (\sin^2 x - 2x \sin x + 1) \, dx, \quad n = 8$

   d. $\int_e^{e+2} \frac{1}{x \ln x} \, dx, \quad n = 8$

3. Use the Composite Simpson's rule to approximate the integrals in Exercise 1.

4. Use the Composite Simpson's rule to approximate the integrals in Exercise 2.