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# Secrecy, Computational Loads and Rates in Practical Quantum Cryptography\*

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## Abstract

A number of questions associated with practical implementations of quantum cryptography systems having to do with unconditional secrecy, computational loads and effective secrecy rates in the presence of perfect and imperfect sources are discussed. The different types of unconditional secrecy, and their relationship to general communications security, are discussed in the context of quantum cryptography. In order to actually carry out a quantum cryptography protocol it is necessary that sufficient computational resources be available to perform the various processing steps, such as sifting, error correction, privacy amplification and authentication. We display the full computer machine instruction requirements needed to support a practical quantum cryptography implementation. We carry out a numerical comparison of system performance characteristics for implementations that make use of either weak coherent sources of light or perfect single photon sources, for eavesdroppers making individual attacks on the quantum channel characterized by different levels of technological capability. We find that, while in some circumstances it is best to employ perfect single photon sources, in other situations it is preferable to utilize weak coherent sources. In either case the secrecy level of the final shared cipher is identical, with the relevant distinguishing figure-of-merit being the effective throughput rate.

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# 1 Introduction

Recent progress in telecommunications and optoelectronics is making it possible for serious consideration to be devoted to the prospect of practical quantum key distribution systems. One possible use to which quantum key distribution may be put is as a precursor to the real-time encryption of plaintext using the method of the Vernam cipher, or one-time pad. Such an end-to-end secret system, which would offer the promise of undecipherable communications, is attractive for a wide class of applications in spite of the fact that Vernam encryption requires a cipher that is bit-for-bit as long as the plaintext. However, in order to make progress in implementing real systems that will work in actual physical environments, it is necessary to determine to what extent predictions regarding secrecy in ideal situations can be realized with practical devices and components. To that end, in this paper we clarify the important but often overlooked distinction between secrecy and security in communications. This is a distinction *with* a difference, as it illuminates the proper constraints that must be satisfied in order to achieve privacy in realistic communications scenarios. We then focus on the specific types of secrecy that are relevant to quantum cryptography, distinguishing between what may actually be achieved in practical implementations in physical environments and what is restricted to formal discussions that are pertinent only to idealized models. We point out that there are actually three “levels” of unconditional secrecy that one may discuss. We provide a careful description of a particular implementation of sifting, error correction and privacy amplification for three reasons: it is of intrinsic interest, it identifies the points at which authentication is required to prevent man-in-the-middle attacks by an eavesdropper, and it provides the necessary input for the detailed result we subsequently present on computational loads. These are the requirements on classical computing machinery that must be satisfied in order to actually carry out the quantum cryptographic transmission. Finally, we consider practical implementations of quantum cryptography in which optical fiber is used as the quantum channel and compare the use of perfect single photon sources with the use of weak coherent sources, and we in particular consider this in distinct scenarios in which the eavesdropping attacks are constrained to lesser and greater degrees. Based on this we are able to identify situations in which it is preferable to use a perfect single photon source instead of a weak coherent source, and *vice versa*.

# 2 Quantum Key Distribution Using the BB84 Protocol

Here we provide a very brief description of the basic elements of quantum key distribution. We will illustrate this with the original four-state QKD protocol developed by Bennett and Brassard in 1984 known as the “BB84” protocol [1]. For definiteness in this illustration we will assume that individual photons serve as the quantum bits for the protocol, or more precisely, the polarization states of individual photons. To carry out the protocol one of the parties transmits a sequence of photons to the other party. The parties publicly agree to make use of two distinct polarization bases which are chosen to be maximally non-orthogonal. In a

completely random order, a sequence of photons are prepared in states of definite polarization in one or the other of the two chosen bases and transmitted by one of the parties to the other through a channel that preserves the polarization. The photons are measured by the receiver in one or the other of the agreed upon bases, again chosen in a completely random order. The choices of basis made by the transmitter and receiver thus comprise two independent random sequences. Since they are independent random sequences of binary numbers, about half of the basis choices will be the same and are called the “compatible” bases, and the other half will be different and are called the “incompatible” bases. The two parties compare publicly, making use for this purpose of a classical communications channel, the two independent random sets of polarization *bases* that were used, without revealing the polarization *states* that they observed. The bit values of those polarization states measured in the compatible bases furnish the “sifted key.” Note that, if the two parties used classical signals to send the key, an eavesdropper could simply measure the signals to obtain complete knowledge of the key. If, on the other hand, the two parties use single photons to transmit the key, the Heisenberg Indeterminacy Principle guarantees that an eavesdropper cannot measure the polarizations without being detected. The sifted keys possessed by each of the parties will in general be slightly different from each other due to errors caused by the use of imperfect equipment. A classical error correction procedure, carried out through the classical communication channel, is executed in order to produce identical, error-free keys at both ends. It is possible that an enemy may have obtained some information about the key during the publicly-discussed error correction phase of the protocol. In addition, it is also possible for the enemy to have obtained information due to the presence in the sequence of quantum bits of multiple photon states. The process of “privacy amplification” is therefore applied to the sifted, error-free key. This has the effect of reducing the information available to the enemy to an arbitrarily low value with extremely high probability.

### 3 Security and Secrecy

Much of the current research in quantum cryptography is devoted to the construction of proofs of the “unconditional security” of quantum cryptographic protocols. The first such proofs assumed idealized equipment [2, 3, 4, 5]. More recent investigations have explored the consequences of the noise, errors, and losses that inevitably occur in a practical implementation [6, 7, 8]. Our concern here is to understand what, precisely, is meant by “unconditional security”, to develop a nomenclature that is more consistent with established definitions, and to point out the fundamental way in which quantum cryptographic security differs from what is achievable with classical cryptographic techniques.

### 3.1 The Distinction between Communications Security and Cryptographic Secrecy

First, we note that the distinction between “security” and “secrecy” is frequently glossed over. Shannon’s original definition of the term “perfect secrecy” is found in his seminal work on the subject: *Communication Theory of Secrecy Systems* (*cf*[9]). The basic requirement for secrecy is that, in comparing the situation *before* the enemy has intercepted the transmission with the situation *after* any such interception (and analysis) has occurred, the *a posteriori* and *a priori* probabilities for the enemy to know the content of the transmission must be identical. Under this definition, the transmission of a key  $K$  from the transmitter (Alice) to the receiver (Bob) is perfectly secret when

$$I(K, X_1) = I(K, X_2) , \quad (1)$$

where  $I$  denotes the mutual information,  $X_1$  is the data the eavesdropper (Eve) has obtained prior to the execution of the key distribution protocol and  $X_2$  is the data she has after the protocol. We see that the term “secrecy” applies solely to the protection provided by the cryptographic protocol alone. The system taken as a whole may not be *secure* even though the protocol is perfectly *secret*. This may happen in one of two ways. Either Eve has prior knowledge of the key, so that  $I(K, X_1) > 0$ , or Eve manages to add to her information the key at some later time, thus obtaining a string  $X_3$  with  $I(K, X_3) > 0$ . In either case, Eve has obtained information without compromising the protocol itself.

The notion of “security” or, more precisely, communications security [10], is defined in a broader context to which Shannon’s “perfect secrecy” is a contributing factor. According to the standard scheme advocated by the U.S. National Security Agency in [11], communications security is split into four separate categories:

- (1) cryptosecurity - [The] component of communications security that results from the provision of *technically sound cryptosystems* (*emphasis added*) and their proper use.
- (2) emission security - Protection resulting from all measures taken to deny unauthorized persons information of value which might be derived from intercept and analysis of compromising emanations from crypto-equipment, computer and telecommunications systems.
- (3) physical security - [The] component of communications security that results from all physical measures necessary to safeguard classified equipment, material, and documents from access thereto or observation thereof by unauthorized persons.
- (4) transmission security - [The] component of communications security that results from the application of measures designed to protect transmissions from interception and exploitation by means other than cryptanalysis.

In this paper we will use the word *secrecy* as synonymous with cryptosecurity, in the sense of

definition (1) above, in recognition of the fact that the security of the entire cryptographic system depends on several factors in addition.

*Unconditional secrecy* refers to secrecy that remains intact when the cryptosystem is subjected to attacks by an enemy equipped with unlimited time and - within the constraints dictated by the laws of physics - unlimited computing machinery. We therefore assume that Eve has complete access to the specifications of the protocols used by Alice and Bob as well as physical access to both the quantum and classical channels. Secrecy is assured by fundamental restrictions of quantum mechanics. Note that unconditional secrecy is completely beyond the reach of classical cryptographic systems. Although Vernam encryption is of course unbreakable without explicit knowledge of the key, there is no classical, purely *cryptographic* method of distributing the key which provides perfect unconditional secrecy (the use of a trustworthy courier, for instance, does not correspond to cryptographic secrecy, but rather is an example of transmission security). With regard to classical cryptographic protection applied to key distribution, if Eve has free access to the classical communications channel and unlimited computing power, there is no way to guarantee an upper bound to the information she is able to obtain. Such bounds can be obtained for quantum cryptographic systems that rely on the distribution of Vernam keys over a quantum channel. Information bounds resulting from proofs of secrecy fall into three categories. We will call these perfect secrecy, asymptotic secrecy, and secrecy in the sense of privacy amplification.

### 3.2 Three Categories of Secrecy

In this section we describe the secrecy of quantum key distribution in terms of the mutual information  $I(K, X)$  of the key  $K$  and a string containing the information Eve has obtained  $X$ . Note that the uniformity of the key, that is, the uniformity of the probability distribution of the key in the space of all possible keys, is also a requirement for successfully using the key in Vernam encryption. In practice, both secrecy and uniformity can be established by bounding Eve's entropy:

$$H(K|X) \geq \mathcal{L}(K) - \epsilon \quad (2)$$

where  $\mathcal{L}(K)$  is the length of the key string in bits. Given this bound on the entropy, we have

$$I(K, X) \equiv H(K) - H(K|X) \leq \mathcal{L}(K) - H(K|X) \leq \epsilon \quad (3)$$

which establishes the secrecy bound and

$$H(K) = I(K, X) + H(K|X) \geq H(K|X) \geq \mathcal{L}(K) - \epsilon \quad (4)$$

which establishes uniformity of the key in the space of binary strings of length  $\mathcal{L}(K)$ . In this paper we will restrict our attention to the secrecy of the key, with the implicit understanding that the corresponding statements of uniformity also apply.

### 3.2.1 Perfect secrecy

Proofs of *perfect secrecy* often proceed by assuming that Eve has no initial knowledge of the key, so that  $I(K, X_1) = 0$ , and then attempting to show that  $I(K, X_2) = 0$ . It should be noted that, in view of the distinction between secrecy and security as clarified in Section 3.1 above, such an approach, although common, is not completely general. Instead, one should proceed by not assuming anything about the *specific* amount of information that Eve may have previously obtained. Thus, a proof of perfect secrecy should begin merely with the assumption that Eve has *some* amount of *a priori* knowledge of the key, say an amount  $\hat{I}$  which may be zero or greater than zero, so that  $I(K, X_1) = \hat{I}$ . (Of course, if  $\hat{I} > 0$  then there has been some non-cryptographic violation of the security of the communication system, arising from failure to observe proper “technically sound cryptosystem design and practice,” as discussed in [8]. The communications system in this case is already insecure, even though the cryptographic protocol that it employs may be provably secret.) A proof of perfect secrecy then consists in showing that  $I(K, X_2) = \hat{I}$  as well, so that

$$I(K, X_2) - I(K, X_1) \equiv \Delta I = 0 . \quad (5)$$

*Unconditional* perfect secrecy consists in further demonstrating that  $\Delta I = 0$  in the absence of any assumptions about the computational resources and capabilities of Eve. This more general approach includes as a special case those proofs which rely on the restrictive initial assumption that  $I(K, X_1) = 0$ . The point here is that one may indeed demonstrate that a *cryptographic protocol* (in this case, key distribution) is perfectly secret, *i.e.*  $\Delta I = 0$ , and even show that it is unconditionally perfectly secret, *i.e.*  $\Delta I = 0$  with no assumed conditions, and at the same time have the actual *communications* be entirely insecure. However, all this is moot since, in any event, perfect unconditional secrecy (*i.e.*, having  $\Delta I = 0$  hold *exactly*) is not achievable for *any* cryptographic protocol in which strings of finite length are transmitted, even for perfect quantum systems (*i.e.*, systems utilizing perfect singlet state quantum bits), since there is a finite non-vanishing probability that Eve can eavesdrop on the quantum channel without introducing errors, thus gaining information without being detected.

### 3.2.2 Asymptotic secrecy

In practice, the mutual information is not strictly zero, but is bounded by a quantity that is exponentially small. For the case of *asymptotic secrecy*, the bound is of the form (for brevity here and in Section 3.2.3 we now *do* assume that  $I(K, X_1) = 0$ )

$$I(K, X_2) \leq 2^{-\mathcal{O}(N_s)}, \quad (6)$$

where the exponent is of the order of the size in bits of the resulting key,  $N_s$ , and the inequality holds with probability close to 1.

For practical purposes there is little difference between asymptotic and perfect secrecy, since the exponential quantity that bounds  $I(K, X_2)$  is extremely small. Note that in the limit of an infinitely long cipher we have  $\lim_{N_s \rightarrow \infty} 2^{-\mathcal{O}(N_s)} = 0$  and we recover perfect secrecy as defined above (here in the special case  $I(K, X_1) = 0$ ).

### 3.2.3 Secrecy in the sense of privacy amplification

Discussions of the secrecy of practical implementations of quantum cryptography [6, 7, 8] typically establish *secrecy in the sense of privacy amplification*. This is because information leaked to Eve during error correction and other phases of the protocol is removed by a privacy amplification protocol [12]. This results in a bound of the form

$$\langle I(K, X_2) \rangle < 2^{-g_{pa}}, \quad (7)$$

where the angle brackets indicate an expectation value over the class of hash functions used to carry out privacy amplification, and where  $g_{pa}$  is a security parameter determined by the protocol, independent of the size of the string,  $N_s$ . The price we pay for this is a shortening of the string by a number of bits equal to the upper bound on Eve's mutual information prior to privacy amplification plus the security parameter  $g_{pa}$ .

Privacy amplification was first applied to the problem of quantum cryptography by Bennett *et al.* [1]. Closely allied techniques are used in security proofs by Mayers [2] and by Biham *et al.* [4] which treat quantum key distribution systems using idealized single photon sources, but which allow Eve to make any possible quantum mechanical attack. Gilbert and Hamrick [8] apply privacy amplification to practical systems using realistic photon sources subject to “individual” attacks, in which Eve attacks each photon independently of the others.

Secrecy in the sense of privacy amplification is different from asymptotic secrecy in three important respects. First, there is no necessary relationship between the security parameter  $g_{pa}$  and the size of the key,  $N_s$ . Second, and corollary to the first difference, secrecy in the sense of privacy amplification does not necessarily become perfect secrecy in the limit of large keys. (This would happen if  $g_{pa}$  were arbitrarily chosen to be some fraction of the key size. Since  $g_{pa}$  bits are removed from the string by privacy amplification, this approach is costly in terms of the rate of generation of key material. Note that bounds of this form are in fact obtained by Biham *et al.* [4].) Third, we improve the secrecy bound in privacy amplification by increasing  $g_{pa}$ , thus producing a *shorter* key. In contrast, we improve the secrecy bound of an asymptotically secret string simply by constructing a *longer* key.

Finally, note that the asymptotic secrecy bound is expressed as an absolute bound on the mutual information, while the secrecy bound for privacy amplification is given as an average over the set of hash functions. This is really an artificial distinction, since absolute bounds can be derived for the privacy amplification result as well [13, 14].

There is no qualitative distinction between the secrecy achieved by systems using pure single photon sources and that achieved by systems whose sources generate a mixture of single-photon and multi-photon pulses. In both cases information will be leaked to Eve at some point in the protocol, and any protocol that removes the leaked information by classical privacy amplification falls into the category of secrecy in the sense of privacy amplification. We shall return to this point later.

It must be emphasized that the distinction between security and secrecy is not merely a matter of semantics, but has a definite impact on what is required in order to obtain proofs of unconditional secrecy. In particular, unconditional proofs of secrecy need not be concerned with eavesdropping attacks that compromise the physical security of the cryptographic system. As an example, attacks in which Eve manipulates the efficiency of Bob’s detection device have been discussed in the literature [15]. This manipulation might be achieved by modifying the wavelength of the pulses received by Bob, in which case the attack can be countered by inserting a narrowband filter in the optical path. Otherwise the attack would require direct access to Bob’s detection devices, which is a breach not of cryptosecurity but of physical security. Note also that, given such access to Bob’s physical installation, Eve could compromise the security of the entire system much more easily by using other parts of Bob’s equipment, notably the mass storage where the secret key material is kept prior to use. Such attacks are clearly beyond the scope of what is required for proofs of unconditional secrecy.

## 4 Error Correction, Authentication, Privacy Amplification and Computational Loads

The BB84 protocol is really of family of protocols that can be implemented in many different ways. The details of the quantum transmissions depend on the way Alice and Bob choose to represent qubits, *e.g.* by choosing to use photons or electrons, and by the quantum basis states they pick to encode the information. Similarly the classical transmissions used to identify basis choices, correct errors, and authenticate the communications channels can be implemented in a variety of ways. This section describes details of a specific implementation to give an idea of the kind of classical communications and computational algorithms that are required. See reference [8] for a more detailed analysis of the computational and communications resources required to carry out this implementation of BB84.

The complete implementation of the protocol occurs in three phases. The first phase is the production of a sifted string of bits shared, except for some errors, by Alice and Bob. This is

achieved by applying the BB84 protocol previously described. If the equipment were perfect and there were no possibility of errors, the sifted strings would be identical and any errors would be due to an attempt by Eve to measure the polarizations of the photons. Since real equipment is never perfect, it is essential to include mechanisms to correct the errors due to the equipment and to eliminate information leaked to Eve in the process. The second phase is thus error correction. Alice and Bob agree on a systematic protocol to identify and correct the errors. Since Eve can eavesdrop on this discussion, some additional amount of key information is leaked to her at this time. The third phase is privacy amplification, during which Alice and Bob apply a hash transformation to the error-corrected string. This results in a shorter string about which Eve's expected information is vanishingly small. At various points during these three phases, Alice and Bob must authenticate their communications to ensure that Eve is not making a man-in-the-middle attack.

In addition, Alice and Bob check that the observed error rate is below a threshold value. The information that Eve can obtain by directly measuring single photon pulses is bounded by the error rate on the channel, and the protocol uses privacy amplification to protect against information losses up to that bound. By testing the error rate, Alice and Bob can detect any attempt by Eve to obtain additional information by a stronger direct-measurement attack on the quantum channel.

## 4.1 Sifting

The first phase of the key distribution protocol is the generation of an initial sifted string that is shared between Alice and Bob, but which may contain errors and about which Eve may have partial information. We describe here a specific implementation of the sifting protocol. Alice generates two blocks of  $m$  random bits. The first block is the raw key material, and the second block determines the choice of basis she uses to transmit the bits over the quantum channel. Bob generates a single block of  $m$  bits that reflect his choice of basis in measuring the incoming qubits. Bob must now identify to Alice those pulses for which he detected a qubit and inform her of his choice of basis for those pulses. Bob has several choices available in deciding how he wants to encode this information. For purposes of estimating the computational load it is necessary to choose a specific implementation. Accordingly, we choose an implementation in which Bob sends to Alice two pieces of information for each photon he detects. The first piece indicates which of the  $m$  bits sent by Alice resulted in the detected photon, and the second gives Bob's choice of basis for that photon. This requires that Bob send  $2n(1 + \log_2 m)$  bits for each block of key material. (The factor of 2 arises from the fact that Bob measures about one half of the qubits in the wrong basis. These qubits are discarded to produce the sifted string. Since the sifted string is of length  $n$ , the number of qubits detected by Bob is  $2n$ .)

Once Alice has received Bob's information, she compares Bob's basis choices with her own and informs Bob of the results. Alice can accomplish this by sending Bob a single bit corresponding to each of the photons Bob detected, resulting in a total of  $2n$  bits of information

sent to Bob.

## 4.2 The Need for Authentication during Sifting

We must now augment the protocol with provisions that will prevent Eve from making the so called man-in-the-middle attack. In this attack, Eve interposes herself between Alice and Bob, measuring Alice's pulses on the quantum channel as though she were Bob, and generates a distinct set of pulses to send to Bob as though she were Alice. In all her subsequent correspondence with Alice over the classical channel, she responds just as Bob would, and in all correspondence with Bob she plays the role of Alice. After the first phase of the protocol, Eve has two blocks of sifted keys, one of which she shares with Alice and the other with Bob. Assuming she can continue this attack through the error correction and privacy amplification phases, she will have completely compromised Alice and Bob's ability to use the keys to transmit secret information. At this point Eve is able to decipher any encrypted information sent between Alice and Bob, always passing the re-encrypted text to the intended recipient so that neither Alice nor Bob is any the wiser.

In order to prevent this state of affairs, it is necessary to provide an authentication mechanism to guarantee that the transmissions received by Bob were sent by Alice, and not by Eve, and to guarantee that the transmissions received by Alice were sent by Bob. Wegman and Carter [16] describe an authentication technique based on “*almost universal*<sub>2</sub>” sets of hash functions that are well suited to this problem. The authentication works as follows. Alice and Bob first agree upon a suitable space of hash functions to be used for authentication. All details of their agreement may be revealed to Eve without compromising the authentication. For each message that is to be authenticated, Alice picks a hash function from the space that is known to Bob, but not to Eve. She does this by using a string of secret bits that is known only to herself and Bob as an index to select the hash function. She uses some of the secret key generated by previous iterations of the protocol to provide this secret index, with the result that some of the key material is sacrificed in order to achieve authentication. She then applies the hash function to the block of raw data to produce an authentication key. This authentication key is transmitted to Bob along with the message. Bob uses the same string of secret bits to pick the same hash function, applies it to the message, and compares the result with the authentication key sent by Alice. If they match, Bob concludes that Alice, and not Eve was, the sender of the message. Wegman and Carter describe a class of hash functions such that the probability that Eve can generate the correct authentication key without knowing the index used is vanishingly small. Let  $\mathcal{M}_1$  denote the precondition that Eve has obtained a copy of the message to be authenticated and  $\mathcal{T}_1^{(E)}$  denote the outcomes in which Eve guesses the tag for the message. The probability of such an outcome is

$$\mathcal{P}(\mathcal{T}_1^{(E)} | \mathcal{M}_1) = 2^{-g_{auth}} , \quad (8)$$

where  $g_{auth}$  depends on the space of hash functions Alice and Bob have chosen to use for the

protocol. It can be made as large as desired by making the space sufficiently large. Alice and Bob do pay a price for increased confidence. A larger space of functions requires a larger set of indices, and thus a longer string of secret bits must be sacrificed to perform the authentication. The other restriction on the protocol is that a new hash function, and thus a new index, must be used for each message to be authenticated if we desire to maintain this upper bound on Eve’s ability to spoof the authentication process. If we allow Eve to obtain one prior message and tag, denoted as  $\mathcal{M}_1\mathcal{T}_1$ , and then allow her to obtain the next message, denoted as  $\mathcal{M}_2$ , as well as the information that Alice and Bob intend to use the same hash function for both, her chances of guessing the second tag improve only slightly to

$$\mathcal{P}(\mathcal{T}_2^{(E)}|\mathcal{M}_1\mathcal{T}_1\mathcal{M}_2) = 2^{1-g_{auth}}. \quad (9)$$

If we allow additional messages to be authenticated using the same hash function, Wegman and Carter’s analysis provides no upper bound on Eve’s ability to produce a correct authentication tag. Although it would be more efficient to allow the same hash function to be applied exactly twice, we will consider the simpler case in which a new hash function is picked for each message.

Before we consider which transmissions require authentication, it is important to realize that any man-in-the-middle attack that results in differences between Bob’s and Alice’s strings of key material can be detected by an equivalence check following error correction. Alice and Bob perform the equivalence check by applying the same authentication hash function to each of their strings and comparing the result. If the results match, they conclude that the strings are identical with a high probability. This check is discussed in more detail in the next section. For the purposes of discussing authentication and sifting, it is sufficient that we require Alice and Bob to perform the equivalence check as a part of the protocol.

The transmissions on the quantum channel do not require authentication, since a man-in-the-middle attack by Eve on the quantum channel will become evident when the error correction process reveals that there is no correlation between Alice’s and Bob’s sifted strings. (Note that the strings would also fail the equivalence check.)

It is not strictly necessary for Alice and Bob to authenticate the classical messages they exchange during sifting, since the equivalence check will eventually detect a man-in-the-middle attack on the sifting protocol. Nevertheless it *is* advantageous for Alice and Bob to authenticate these messages. Note that Eve can compromise the secrecy of the final key by a man-in-the-middle attack only if she makes the attack both on the quantum transmission and on the classical messages used for sifting. An attack on the classical messages alone results only in a “denial-of-service” attack, that is, an attack that makes it difficult or impossible for Alice and Bob to complete the protocol successfully, but that does not compromise the secrecy of the key material. Authentication of sifting renders these attacks ineffective early in the protocol and thus localizes the attack to the quantum transmission and sifting phases of the protocol. This is useful for Alice and Bob to know in formulating a response to the attack. Authentication guarantees Alice and Bob that they are working with the same subset

of the pulses sent by Alice and that any remaining errors are due to physical imperfections of the equipment or attempts by Eve to measure, and therefore disturb, the pulses sent by Alice. Any further interference by Eve is restricted to denial-of-service attacks during error correction.

The authentication of the classical discussion results in a cost to the overall rate of quantum key generation, since some of the secret bits produced by previous iterations of the protocol must be sacrificed to generate an authentication tag that Alice or Bob can validate but that Eve cannot forge. Wegman and Carter [16] show that the size of the secret index required to select a hashing function is

$$w(g, c) = 4(g + \log_2 \log_2 c) \log_2 c \quad (10)$$

where  $c$  is the length in bits of the message to be authenticated and  $g$  is the length in bits of the authentication tag. The full and complete expression for the quantity that we denote by  $w$  and refer to as the Wegman-Carter function, which is of crucial importance in practical quantum cryptography, does not appear to have been properly analyzed previously in the context of QC (nor apparently even *named* by any authors). Surprisingly, the closed-form function, as such, doesn't appear as a numbered equation in [16]. In fact, it must be obtained instead by combining quantities that appear in lines 3 and 17 in the first paragraph of section 3 in [16]. See [8] for a complete discussion of the cost of authentication and its effect on the secrecy capacity of the quantum key distribution system.

### 4.3 Error Correction Phase

At this point Bob and Alice move on to the error correction phase. We will estimate the authentication, communication, and computational costs for a modified version of the error correction protocol described by Bennett *et. al.*, [17]. More efficient techniques have been developed, for example the “Shell” and “Cascade” protocols described in [18], but the method described here is more suitable for our purposes since it is simpler to analyze.

At the beginning of the error correction phase, Alice and Bob each have a string of  $n$  bits. The strings are expected to be nearly identical, but they will also contain errors for which Alice and Bob disagree on the value of the bit. It is the goal of error correction to identify and remove all of these errors, so that Alice and Bob can proceed with a high degree of certainty that the strings are identical. Error correction consists of three steps. The first step is the error detection and correction step, which eliminates all or almost all of the errors. The validation step which follows eliminates any residual errors and iteratively tests randomly chosen subsets of the string to generate a high degree of confidence that the strings are identical. The final step is authentication, which protects against a man-in-the-middle attack by Eve during the error correction process.

At the beginning of the error detection and correction step, Alice and Bob each shuffle the

bits in their string using a random shuffle upon which they have previously agreed. The purpose of this shuffle is to separate bursts of errors so that the errors in the shuffled string are uniformly distributed. Alice and Bob may use the same shuffle each time they process a new string of sifted bits, and security is not compromised if Eve has complete prior knowledge of the shuffling algorithm, even including any random numbers used as parameters.

The error detection and correction step is an iterative process. Alice and Bob begin each iteration  $i$  by breaking their strings into shorter blocks. The block length is chosen so that the expected number of errors in each block is given by a parameter  $\varrho$ . This is achieved by breaking the string into  $J^{(i)}$  blocks for the  $i^{\text{th}}$  iteration, where

$$J^{(i)} = \left\lceil \frac{e_T^{(i-1)}}{\varrho} \right\rceil, \quad (11)$$

and  $e_T^{(i)}$  is the expected number of errors remaining after the  $i^{\text{th}}$  iteration or at the beginning of the  $i + 1^{\text{st}}$  iteration. In principle the parameter  $\varrho$  could change from iteration to iteration. We assume that it is a constant to simplify the analysis. Alice and Bob compute the parity of each of the blocks and exchange their results. Blocks for which the parities do not match necessarily contain at least one error. For each of the blocks in which Alice and Bob have detected an error, they isolate the erroneous bit by a bisective search, which proceeds as follows. Alice and Bob bisect one of the blocks containing an error, that is, they divide it as evenly as possible into 2 smaller blocks. Alice and Bob each pick one of the smaller blocks for the next parity check. For definiteness, assume they pick the block that lies closer to the beginning of the shuffled string, which we will call the “lower” block. The other block is then the “upper” block. Alice and Bob then compute the parity of the lower block and compare their results. If the parities do not match, the error is in the lower block. If they do match, the error is in the upper block. Alice and Bob then bisect the block that contains the error and proceed recursively until they find an erroneous bit. Bob then inverts that bit in his string, and thus the error is removed.

We have described the bisective search as though the search were completed for any block containing a detected error before beginning the bisection on the next block. In fact, it is more efficient from a communications standpoint to apply each bisection to all the blocks with detected errors at the same time, exchange parities for all of the sub-blocks, and then to proceed recursively to the next bisection. This results in fewer, but larger, packets of data for each exchange between Bob and Alice, thus reducing the overall frame overhead.

When the bisective search is completed for all blocks in which an error is detected, a new blocksize is computed based on the expected number of errors remaining, the string is broken up into a new set of larger blocks, parity checks are compared for the blocks, and bisective searches are made in those blocks containing detected errors. This process is repeated until there would be only one or two blocks in the string for the next iteration, that is, until

$$J^{(N_1+1)} \leq 2 , \quad (12)$$

where  $N_1$  is the number of iterations in the error correction and detection step.

In this version of the algorithm, the bits are not shuffled between successive iterations of the error detection and correction step. There is some value in performing the shuffle, since it separates pairs of errors that may survive previous iterations, thus making it more likely that they are found before the validation step. However, this step is not essential and has not been included in the estimate of computational loads to follow.

The second step in the error correction phase, validation, is also iterative. During each iteration, Alice and Bob select the same random subset of their blocks. They compute the parities and exchange them. If the parities do not match, Alice and Bob execute a bisective search to find and eliminate the error. Iterations continue until  $N_2$  consecutive matching parities are found. At this point, Alice and Bob conclude that their strings are error free.

The last step in the error correction phase is authentication. Up until now, Alice and Bob have made no attempt to authenticate their exchange of parity information on the classical channel. Eve could mount a man-in-the-middle attack during the error correction phase that would fool Alice and Bob into correcting the wrong set of bits. This would not give Eve any additional information about the secret string, but it could result in Alice and Bob believing that their strings are identical when in fact they are not. Even if one bit is different, the privacy amplification phase will produce strings that are completely uncorrelated, and Alice and Bob will still believe that their strings are identical. The solution to this problem is for Alice and Bob to verify that their strings are the same at the end of the error correction phase. This effectively authenticates their prior communications, since any successful attempt by Eve to steer the error correction process will be immediately apparent.

This approach presupposes that Alice and Bob can verify that their strings are the same without leaking too much additional information to Eve. This can be accomplished if Alice and Bob apply the same hash function to their strings and compare the resulting tag. This does not provide an absolute guarantee that the strings are the same, but if the hash function is chosen as described in [16], the probability that two different strings will yield the same tag is

$$\mathcal{P}(\text{same tag, two strings}) = 2^{-g_{EC}} , \quad (13)$$

where  $g_{EC}$  is the length of the tag. This gives a high degree of confidence that the strings are identical even for relatively short ( $g_{EC} \sim 30$ ) tags. The price Alice and Bob have to pay for this is that they must use a portion of the secret bits obtained from previous iterations of the protocol to select the hash function, indicate whether the keys match, and authenticate their transmissions.

## 4.4 Privacy Amplification Phase

The general scheme of privacy amplification is described in [12] and [19]. The hash functions map a sifted, error corrected string of length  $n$  to a string of length  $L_{pa}$ , where

$$L_{pa} \equiv n - e_T^{(0)} - q - t - \nu - g_{pa}, \quad (14)$$

where  $n$  is the length of the sifted string,  $e_T^{(0)}$  is the number of errors removed during error correction,  $q$  is the additional information leaked during error correction,  $t$  is the amount of information Eve obtains by attacks on single photon pulses, and  $\nu$  is the amount of information she obtains by attacks on multi-photon pulses. The resulting string is thus shorter than the sifted string by the number of bits that Eve may have obtained by listening to the classical discussion, plus an additional security parameter  $g_{pa}$ . As shown in [12], this parameter determines an upper bound on the expected amount of information,  $I$ , that Eve can retain following privacy amplification:

$$\langle I(\tilde{K}, X) \rangle \leq \frac{2^{-g_{pa}}}{\ln 2}, \quad (15)$$

where  $\tilde{K}$  is the key after privacy amplification and  $X$  is the information Eve has obtained from all phases of the protocol. The expectation value is over the set of functions from which Alice and Bob choose their hash function. See [13, 14] for a discussion of secrecy bounds that are not conditioned on an average over the hash functions.

Hash functions appropriate for privacy amplification are described by Carter and Wegman [19]. The class of hash functions used for authentication and equivalence checking is not practical for privacy amplification due to the much larger size of the output string. The authentication hash functions are designed to produce output strings that are no more than half as long as the input string. Since we wish to retain as much information as possible, it is clearly advantageous to use hash functions that can produce an output string that is nearly as long as the input string. Furthermore, recall that the length of the index for choosing an authentication hash function is [16, 8]

$$w(g, c) = 4(g + \log_2 \log_2 c) \log_2 c \quad (16)$$

where  $c$  and  $g$  are the lengths of the input and output strings, respectively. For purposes of authentication and error correction, an output string of length  $g \leq 50$  is adequate, and the length of the index is relatively short even for long input strings due to the logarithmic factors. In privacy amplification, where the output string is nearly as long as the input string, this index is roughly 4 times as long as the string to be hashed. In contrast, the hash functions suitable for privacy amplification are described by two parameters, each as long as the input string, so that the total size of the index is only twice as long as the input string. The Carter-Wegman functions described in [19] are therefore a much better choice for privacy

amplification since they are capable of producing keys nearly as long as the input and since they require shorter indices for their definition than do the Wegman-Carter functions given the large size of the output strings.

The error correction phase guarantees that the strings Alice and Bob have obtained are identical with high probability. Bob and Alice implement privacy amplification by agreeing on an index and applying the hash functions separately to their strings. The resulting strings are identical and secret in the sense of privacy amplification (*cfeq.(15)*). Note that the sifting protocol itself supplies random strings of sufficient length to define the required hash index. Bob's choice of basis for the  $2n$  pulses he receives is one such source. Another alternative is to compute the parities of the indices Bob sends to Alice by which he identifies which pulses were detected by his equipment.

## 4.5 Computational Load

The total computational load implied by the protocols described here is analyzed in detail in [8]. The result of the analysis is an approximate upper bound on the number of instructions per block of key material:

$$\begin{aligned}
\mathcal{L}_B \leq & \mathcal{L}_B^{(0)} \\
& + \left( 50 + \frac{220}{g_{auth}} \right) n (1 + \log_2 m) \\
& + \left[ 200 + 25N_1 + 12.5 \left( 1 - e^{-2\varrho} \right) N_1 + 25\varrho + 37.5 \left( N_2^{(n)} + N_2^{(f)} \right) \right. \\
& \quad \left. + \frac{43}{w} + \frac{220}{g_{auth}} + \frac{110}{g_{EC}} \right] n \\
& + \frac{46}{w^2} n^2 .
\end{aligned} \tag{17}$$

In this expression,  $\mathcal{L}_B^{(0)}$  is the “non-iterative” portion of the load, representing code that executes once for each block of data without iterating bit-by-bit through the string of key material.  $m$  is the block size in bits of the raw key material sent by Alice to Bob over the quantum channel.  $n$  is the block size of the sifted key material.  $g_{auth}$  and  $g_{EC}$  are the security parameters for authentication and error correction, respectively.  $\varrho$  is the parameter that determines the blocksizes used in successive iterations of the first step of error correction.  $N_1$  is the number of iterations required in the first step of error correction. The term  $N_2^{(n)} + N_2^{(f)}$  is the total number of iterations in the second step of error correction. Expressions for these iteration counts in terms of more fundamental parameters are found in [8]. Finally,  $w$  is the wordsize in bits of the processing element that performs elementary integer arithmetic.

It is instructive to evaluate this expression for a practical example. Assume a substantial non-iterative contribution:

$$\mathcal{L}_B^{(0)} = 10^6 \text{ operations per block ,} \tag{18}$$

and take the wordsize of the processor to be 64 bits. The other parameters are chosen to have reasonable operational values and to give a reasonable value for the computation rate as computed below ( $m = 2 \times 10^8$  bits,  $n = 2 \times 10^5$  bits,  $e_T^{(0)} = 2 \times 10^3$  bits,  $\varrho = 0.5$ ,  $g_{EC} = g_{auth} = N_2 = 30$ ). The resulting estimate of the load is 1.1 billion operations per block. The quadratic term contributes 450 million operations to the total. Of the other terms, the dominant contributions are the term in  $N_2^{(n)} + N_2^{(f)}$ , which is due to parity checks and random block extractions during the validation step of error correction, and the term in  $(1 + \log_2 m)$ , which is due to sifting. Note that the non-iterative overhead load is negligible in comparison with the other contributions. This indicates that a substantial amount of “bookkeeping” code can be included along with the core software that is essential to arriving at the final secret key without significantly affecting the processing requirements. One of the uses of eq.(17) is to establish a load budget for such code during software design and implementation to ensure that the bulk of the processing resources are available for the core software functions.

The computation rate  $\mathcal{R}_B^{comp}$  required to support key distribution is found by dividing the load per block by the time required to transmit one block over the quantum channel:

$$\mathcal{R}_B^{comp} = \frac{\mathcal{L}_B}{m\tau} , \quad (19)$$

where  $m$  is the raw block size, and  $\tau$  is the bit cell period for sending each bit. Note that the processing load is quadratic in the sifted block size, so that the computation rate increases roughly linearly with the block size. This means that the computations required for the full protocol imply upper bounds on the sifted block size  $n$  and, by extension, the raw block size  $m$  in order to keep the computational load within the capabilities of the equipment.

In order to obtain a quantitative result for the computational rate, we must choose a value for  $\tau$ , which is the inverse of the pulse repetition frequency of the photon source. Experimental demonstrations of quantum key distribution have used attenuated lasers with pulse rates up to 1 MHz [20], and photon transmission and detection has been demonstrated at 400 MHz [21]. As discussed in [8], the limiting physical factors include the switching speed of the optical equipment and the time required for the single photon detectors to reset for the next pulse. A rate of 10 GHz is feasible with high quality optoelectronics, but current photon detectors are much too slow to support this. It is possible that new technologies [22] will enable single photon detectors to support these rates. We will accordingly set  $\tau = 10^{-10}$  sec. The computation rate for our example is then

$$\mathcal{R}_B^{comp} = 56 \text{ billion operations/sec .} \quad (20)$$

This is rather high for a single general purpose processor, but should be achievable in a parallel architecture in which each block of the input data is allocated to a single processor as it becomes available. Recall also that general purpose computers are far from optimal for this type of operation. Most of the processing steps involving the packing and unpacking of the bits would not be necessary in a special purpose device, and many of the other processing steps, notably block parity computations and random selection of substrings, could be accomplished much more efficiently using special purpose hardware. In any case, it should be clear that computational power is not a limitation on the practicality of quantum cryptography even at very high rates of operation.

## 5 Effective Secrecy Capacity and Rate of Key Generation

In this section we apply the analysis key generation rate analysis of [8] to the consideration of system performance characteristics in representative practical implementations of quantum cryptography for which optical fiber is used as the quantum channel. In particular we wish

to compare the use of perfect single photon sources with the use of weak coherent sources, and we consider this in distinct scenarios in which the eavesdropping attacks are constrained to lesser and greater degrees.

The motivation for this discussion is the common misconception that pure single photon sources inherently provide a higher degree of secrecy than do sources with some admixture of multi-photon pulses. This misconception arises from the following argument. If Eve measures single photon states, she necessarily introduces a disturbance that can be detected by Alice and Bob. The secrecy of the key is thus protected by the laws of quantum mechanics. However, if some of the pulses encoding the key contain multiple photons, Eve can measure a fraction of the photons in any given multi-photon pulse without disturbing the rest, thus obtaining information without being detected. This appears to circumvent the guarantee of secrecy for single photon sources. The need to carry out privacy amplification, however, applies whether the protocol is implemented with a pure single photon source *or* with a source that produces both single and multiple photon pulses. Even if there are no multi-photon pulses *at all* amongst the signals sent from Alice to Bob, the fact that the physical hardware generates errors, combined with the fact that Eve may be present, results in the need for privacy amplification. *Precisely the same degree of secrecy* is realized whether the protocol is implemented with pulsed lasers generating weak coherent pulses or with single photon sources. The chief difference is the rate of generation of key material. In principle the rates achievable with single photon sources should be higher, all other things being equal, since Eve cannot gain information from attacks on multi-photon pulses in this case. In reality, none of the single photon sources now in existence can produce pulses at a rate comparable with currently available pulsed lasers. In this section we show that the key generation rates are much higher for weak coherent sources (*i.e.* for sources producing some fraction of multi-photon pulses) than for single photon sources for implementations that are likely to be feasible in the near future.

We first summarize the results of [8] for key generation rates achieved with weak coherent sources. We next present results suitably modified for ideal single photon sources. Following [8], we consider only individual attacks, that is, attacks in which Eve attempts to obtain information by making measurements of individual photons. The case of a more general quantum attack, in which Eve may entangle probes with arbitrarily chosen groups of photons is a subject for a subsequent analysis.

## 5.1 Definitions

We define the secrecy capacity  $\mathcal{S}$  as the ratio of the length of the final key to the length of the original string of pulses sent from Alice to Bob over the quantum channel:

$$\mathcal{S} = \frac{L}{m} . \quad (21)$$

This quantity is useful for two reasons. First, it can be used in proving the secrecy of specific practical quantum cryptographic protocols by establishing that the inequality

$$\mathcal{S} > 0 \quad (22)$$

holds for the protocol. Second, it can be used to establish the rate of generation of key material according to

$$\mathcal{R} = \frac{\mathcal{S}}{\tau}, \quad (23)$$

where  $\tau$  is the pulse period of the initial sequence of photon transmissions. We refer to  $\mathcal{R}$  as the effective secrecy rate.

The length of the final key is given by

$$L = n - (e_T + q + t + \nu) - (a + g_{pa}), \quad (24)$$

which is the same as the length of the key after privacy amplification (*cf* eq.(14)) except that an additional amount  $a$  has been subtracted to account for the secret bits required for authentication during the next iteration of the protocol. Recall that  $n$  is the length of the sifted string,  $e_T \equiv e_T^{(0)}$  is the number of errors removed during error correction,  $q$  is the additional information leaked during error correction,  $t$  is the amount of information Eve obtains by attacks on single photon pulses, and  $\nu$  is the amount of information she obtains by attacks on multi-photon pulses. See [8, 23] for a detailed quantitative treatment of the terms appearing in this result. We apply this analysis to a number of scenarios involving key distribution system operating over a fiber optic quantum channel in section 5.2.

## 5.2 Effective Secrecy Rates

There is extensive research activity [24, 25, 26, 27, 28] currently taking place devoted to the development of sources of single photons to serve as quantum bits in various applications, including in quantum cryptography. At the present time, though, there are no robust, *perfect* sources of single photons available that can be used for practical quantum cryptography purposes. At the same time a number of groups have carried out demonstrations of quantum key distribution in which filtered, pulsed lasers are used as the source of the quantum bits. The use of such weak coherent sources results in the production of both single-photon pulses and multiple-photon pulses in the transmission stream from Alice to Bob. The presence of multi-photon pulses allows Eve to execute a set of attacks that may (depending on precisely how Bob monitors his photon detector) require a significant amount of privacy amplification compression in order to assure secrecy, resulting in a considerably reduced throughput rate. However, under the conditions described in Section 3 above, privacy amplification is also required in any practical implementation even if we utilize a perfect single photon source. Thus with the use of either type of source we see that the final, shared key is characterized by unconditional secrecy in the sense of privacy amplification (*cf* eq.(7)). Then there is no

secrecy advantage that inures to the use of a single photon source, and the only figure-of-merit that distinguishes between the use of a single photon source (SPS) and a weak coherent source (WCS) is the effective secrecy *rate*.

To analyze and compare the effective secrecy rates due to single photon sources and weak coherent sources we must employ the appropriate expression for the number of sifted bits shared between Alice and Bob in each case. This is given for a WCS by [8]

$$\begin{aligned} n_{WCS} &= \frac{m}{2} \left[ (1 - r_d) \psi_{\geq 1}(\eta\mu\alpha) + r_d \right] \\ &\simeq \frac{m}{2} \left[ \psi_{\geq 1}(\eta\mu\alpha) + r_d \right], \end{aligned} \quad (25)$$

where  $\mu$  is the average number of photons per pulse,  $\eta$  is the efficiency of Bob's detector,  $\alpha$  is the transmission probability in the quantum channel, and  $r_d$  is the probability of obtaining a dark count in Bob's detector during a single pulse period.  $\psi_{\geq 1}(X)$  is the probability of encountering 1 or more photons in a pulse selected at random from a stream of Poisson pulses having a mean of  $X$  photons per pulse:

$$\psi_{\geq 1}(X) = \sum_{l=1}^{\infty} e^{-X} \frac{X^l}{l!}, \quad (26)$$

and we have assumed that  $r_d \ll 1$  in the second line in eq.(25). The corresponding expression that arises in the case of a *perfect* source of single photon quantum bits is [29]

$$\begin{aligned} n_{SPS} &= \frac{m}{2} \left[ (1 - r_d) \eta\alpha + r_d \right] \\ &\simeq \frac{m}{2} (\eta\alpha + r_d). \end{aligned} \quad (27)$$

As expected, we see that  $n_{SPS}$  is independent of the quantity  $\mu$ . The corresponding expressions for the numbers of errors in the sifted strings generated by the two types of sources are

$$e_{T,WCS} \simeq \frac{m}{2} \left[ \psi_{\geq 1}(\eta\mu\alpha) r_c + \frac{r_d}{2} \right], \quad (28)$$

( $r_c$  is the intrinsic channel error fraction) and

$$e_{T,SPS} \simeq \frac{m}{2} \left( \eta\alpha r_c + \frac{r_d}{2} \right). \quad (29)$$

We recall that  $q$  and  $t$  are, respectively, the information leaked during error correction and an upper bound for the amount of information Eve can obtain by direct measurement on single photon pulses. Upon introducing the quantities  $Q$  and  $T$  through

$$q \equiv Qe_T, \quad (30)$$

and

$$t \equiv T e_T , \quad (31)$$

the expressions for the effective secrecy capacities for the two types of sources can be shown to be given by [8, 23, 29]

$$\begin{aligned} \mathcal{S}_{WCS} &\equiv \frac{n - e_T - q - t - \nu - g_{pa} - a}{m} \Big|_{WCS} \\ &= \frac{n - f e_T - \nu - g_{pa} - a}{m} \Big|_{WCS} \\ &= \frac{1}{2} \left[ \psi_{\geq 1} \cdot (1 - f_{WCS} r_c) + \left(1 - \frac{f_{WCS}}{2}\right) r_d - \tilde{\nu} \right] - \frac{g_{pa} + a}{m} , \end{aligned} \quad (32)$$

(here  $\psi_{\geq 1} \equiv \psi_{\geq 1}(\eta\mu\alpha)$  and  $\tilde{\nu} \equiv 2\nu/m$  with the rescaled quantity  $\tilde{\nu}$  independent of  $m$ ) and

$$\begin{aligned} \mathcal{S}_{SPS} &\equiv \frac{n - e_T - q - t - g_{pa} - a}{m} \Big|_{SPS} \\ &= \frac{n - f e_T - g_{pa} - a}{m} \Big|_{SPS} \\ &= \frac{1}{2} \left[ \eta\alpha \cdot (1 - f_{SPS} r_c) + \left(1 - \frac{f_{SPS}}{2}\right) r_d \right] - \frac{g_{pa} + a}{m} , \end{aligned} \quad (33)$$

where we have introduced

$$f_{WCS} \equiv 1 + Q_{WCS} + T_{WCS} \quad (34)$$

and

$$f_{SPS} \equiv 1 + Q_{SPS} + T_{SPS} , \quad (35)$$

with  $Q_{SPS}$  calculated solely as a function of  $n_{SPS}$  and  $e_{T,SPS}$ , and with  $Q_{WCS}$  calculated solely as a function of  $n_{WCS}$  and  $e_{T,WCS}$ , respectively (and likewise for the calculation of  $T_{WCS}$  and  $T_{SPS}$ ). Detailed expressions for  $Q$  and  $T$  may be found in [8, 23, 29].

We now proceed to compare the use of a perfect single photon source with the use of a weak coherent source. We also wish to consider two levels of technology available to the enemy, one in which the enemy can surreptitiously effectively eliminate the intrinsic attenuation along the quantum channel, which is usually to her advantage, and another in which this is not possible for her to do. In the former case we may imagine two ways in which, at least in principle, it could be possible for this to occur. In one situation, we may imagine that the enemy somehow has the capability to remove the installed fiber optic cable and replace it with a new cable that is effectively lossless, all without being detected “in the act.” Of course, it is clear that this is an extremely unlikely situation, due both to the near-impossibility of undetectably removing and replacing a long length of cable, and, even moreso, the implausibility of being able to produce lossless cable at all. We will underscore

this implausibility by referring to such a hypothetical lossless fiber optic cable as “magic” cable. The other situation in which the enemy may effectively eliminate the attenuation along the quantum channel arises if she has both an accomplice near Bob and access to prior shared entanglement. In that case Eve and the accomplice located near Bob prepare pairs of entangled photons in advance. Eve then entangles one of these pairs with a photon emitted by Alice. Her accomplice can then make measurements on the entangled state, gaining information about the photons at Eve’s location without losing photons to the attenuation in the channel.

For the calculation of  $\mathcal{S}_{WCS}$  and  $\mathcal{R}_{WCS}$  we will need to use the correct expression for  $\nu$ , the function that specifies the amount of privacy amplification compression that is associated to the presence of multi-photon pulses in the transmission from Alice to Bob. The appropriate form for  $\nu$  is determined by whether or not we presume that it is possible for Eve to effectively eliminate the line attenuation along the quantum channel, as described above. As discussed in detail in [8, 23], the correct expressions for  $\nu$  that will ensure unconditional secrecy (in the sense of privacy amplification) are given follows: If we presume that Eve *can* somehow effectively eliminate the line attenuation along the quantum channel we have

$$\nu^{max} = \frac{m}{2} \left[ \left[ \psi_{\geq 2}(\mu) - (1-y)^{-1} \cdot \left\{ e^{-y\mu} - e^{-\mu} \left[ 1 + \mu(1-y) \right] \right\} \right] \right], \quad (36)$$

with the parameter  $y$  given by  $y = \eta$ , subject to the constraint  $y > 1 - \frac{1}{\sqrt{2}}$  (*i.e.*,  $y \gtrsim 0.293$ ), which is automatically satisfied in the examples we consider. If we presume that Eve *cannot* effectively eliminate the line attenuation along the quantum channel we have

$$\nu^{max} = \frac{m}{2} \left[ \psi_2(\mu)y + 1 - e^{-\mu} \left( \sqrt{2} \sinh \frac{\mu}{\sqrt{2}} + 2 \cosh \frac{\mu}{\sqrt{2}} - 1 \right) \right], \quad (37)$$

with the parameter  $y$  now given by  $y = \eta\alpha$ , subject to the constraint  $y < 1 - \frac{1}{\sqrt[3]{2}}$  (*i.e.*,  $y \lesssim 0.206$ ), which is also automatically satisfied in the examples we consider.

In the scenarios that follow we consider fiber-optic cable implementations of quantum key distribution making use of either good quality single-mode, polarization-preserving fiber characterized by an intrinsic attenuation characteristic of  $A_1 = 0.3$  dB per kilometer, or of high quality fiber characterized by  $A_2 = 0.2$  dB per kilometer. We take the photon detector device efficiency to be  $\eta = 50\%$ , and we assume that appropriate splicing and insertion of suitable dispersion-compensating fiber segments, as discussed in [8], has been carried out so as to mitigate the dispersion losses described and analyzed there. To account for the associated splicing loss and other insertion losses we assume that the quantum channel is characterized by a total bulk loss of  $\kappa = -5$  dB, in addition to the losses associated with the attenuation per unit length. Thus the transmission probability for the quantum channel is given by

$$\begin{aligned} \alpha &= \alpha(L_{fiber}, A, \kappa) \\ &= 10^{-\frac{AL_{fiber}+\kappa}{10}}, \end{aligned} \quad (38)$$

where  $L_{fiber}$  is the length of the fiber cable connecting Alice and Bob.

For all of our numerical examples we have taken a value for the Shannon deficit parameter  $x$  of  $x = 1.16$  (*cf* [8] for a discussion of  $x$ ), which means that we are assuming that an efficient method of error correction has been employed that approaches the Shannon limit to within 16%, and we use a raw bit processing block size of  $m = 200$  Megabits. In addition, we have also set all of the continuous authentication security parameter values,  $g_i$ , as well as the privacy amplification security parameter  $g_{pa}$ , equal to 30, and we have employed a value of  $\epsilon = 10^{-9}$  for the selectable infinitesimal quantity that determines the success likelihood for attacks on single-photon pulses, (this applies for both authentic single photon qubits generated by a single photon source, and the single-photon *part* of the transmission stream generated by a filtered, pulsed laser). We are also assuming the use of a photon detector device that has a dark count rating of  $10^{-6}$  counts per bit cell. In addition, we make the further assumption that the intrinsic channel error rate due to pulse dispersion in the optical fiber is no greater than 1%.

It is important to point out a distinguishing feature in the computations of the effective secrecy capacities for the WCS and SPS cases. In calculating  $\mathcal{S}_{WCS}$  we want to determine the value of  $\mu$  that produces the maximum throughput, whereas in the calculation of  $\mathcal{S}_{SPS}$  this issue doesn't arise since each pulse carries precisely one photon. When we plot  $\mathcal{R}_{WCS}$  as a function of position along the fiber-optic quantum channel we utilize this optimized value of the mean photon number per pulse,  $\mu_{opt}$ , and in particular we must do so for *each point along the path*. This is because  $\alpha(L_{fiber}, A, \kappa)$  is explicitly a function of  $L_{fiber}$ , the position of Bob on the quantum channel (*cf* eq.(38)). As the effective secrecy capacity  $\mathcal{S}_{WCS}$  itself depends on  $\alpha$ , we must determine new values of  $\mu_{opt}$  for each value of  $L_{fiber}$  by obtaining a new solution of the optimization equation  $0 = \partial_\mu \mathcal{S}_{WCS}|_{\mu=\mu_{opt}}$ . In contrast,  $\mathcal{R}_{SPS}$  and  $\mathcal{S}_{SPS}$  have no  $\mu$ -dependence at all.

### 5.2.1 Scenario One: Eavesdropper cannot eliminate line attenuation

In Figure 1 we plot three effective secrecy rate curves. In this example, we *presume* that the enemy can neither (a) surreptitiously replace the installed cable with a different cable (in particular this means that the enemy is presumed to be unable to surreptitiously replace the installed cable with an effectively lossless, or “magic” cable), nor (b) make use of prior shared entanglement as a resource for cryptanalytic attacks. This means that the enemy cannot alter the actual value of the line attenuation that exists along the cable. In turn, this means that [8] we should set the privacy amplification parameter  $y$  to  $y = \eta\alpha$  in the *multi-photon part* of the privacy amplification function to be used in the calculation of  $\mathcal{S}_{WCS}$  and  $\mathcal{R}_{WCS}$ . There is no analogous issue in the case of the calculation of  $\mathcal{S}_{SPS}$  and  $\mathcal{R}_{SPS}$ , since there is no multi-photon term in the privacy amplification function in the case of a single photon source. Thus, for the computation of  $\mathcal{S}_{WCS}$  and  $\mathcal{R}_{WCS}$  we make use of the form of  $\nu^{max}$  given in eq.(37) above.

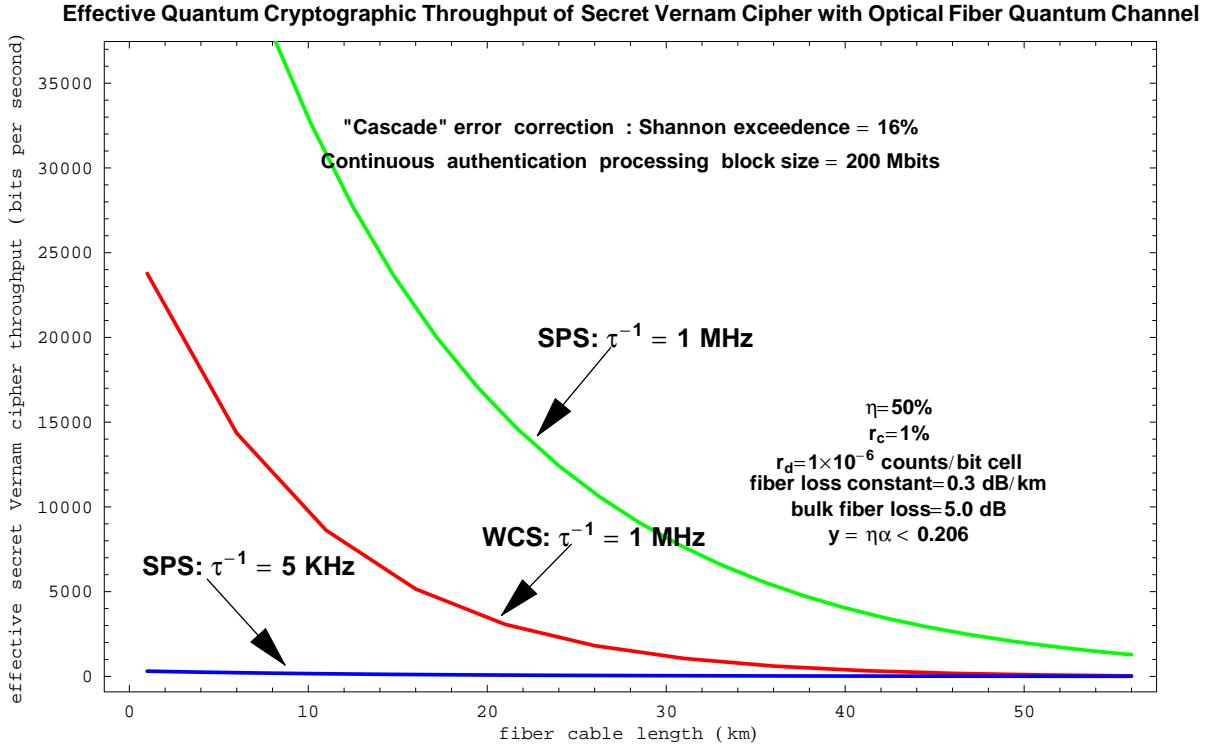


Figure 1: Effective Rate Curves for Fiber-Optic Link *without* Surreptitious Cable Replacement or use of Prior Shared Entanglement by Eve

The upper curve displays the effective secrecy rate that would arise with a perfect single photon source operating at a pulse repetition frequency (PRF) of 1 MHz. The middle curve gives the effective secrecy rate corresponding to the use of a weak coherent source, also operating at a pulse repetition frequency of 1 MHz. In order to represent a more realistic situation based on the current state of the art in perfect single photon generation, the lower curve shows the rate that is obtained with the use of a perfect single photon source operating at a pulse repetition frequency of 5 KHz. Assuming that we compare systems with equal pulse repetition frequencies, there is a substantial gain realized with the use of a perfect single photon source compared to the use of a weak coherent source. For instance, inspection of the graph reveals that at a separation distance of  $L_{fiber} = 10$  kilometers an effective secrecy rate of about 9840 bits per second can be realized with the use of a weak coherent source with a 1 MHz PRF. A perfect single photon source operating at a PRF of 1 MHz achieves this same secrecy rate at a distance of about 27.5 kilometers. We may also compare rates between the two systems at a fixed separation distance between Alice and Bob. For example, at a separation distance of 10 kilometers, in going from a WCS system to a SPS system the rate increases from 9840 bits per second to 32900 bits per second, a gain of about 5.2 dB.

Although on the scale of the graph in Figure 1 the curve for the SPS system operating at a

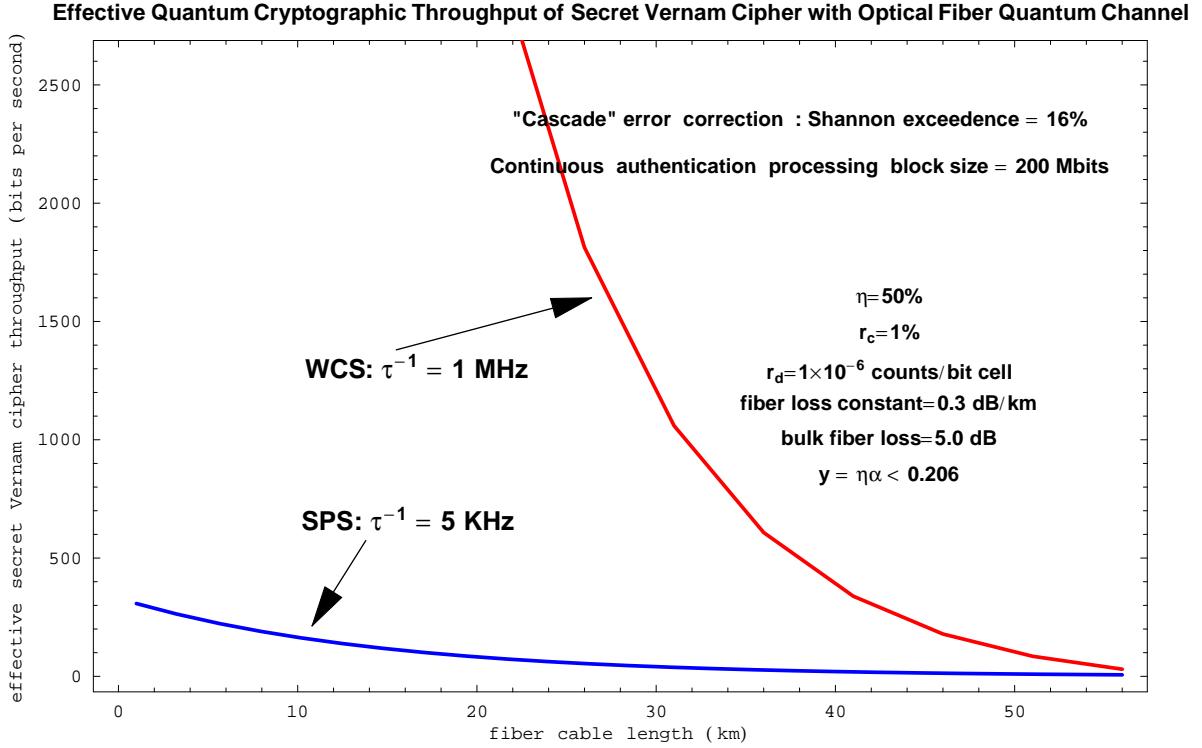


Figure 2: Effective Rate Curves for Fiber-Optic Link *without* Surreptitious Cable Replacement or use of Prior Shared Entanglement by Eve

PRF of 5 KHz appears to yield no throughput at all, inspection of Figure 2 reveals that this is not the case. We note that, in spite of the *substantial* reduction in the amount of privacy amplification compression that is realized upon going from a WCS system to a SPS system (*i.e.*, replacing  $\nu^{max}$  with 0), there is no location, up to a separation distance of 56 kilometers, at which the 1 MHz WCS curve and 5 KHz SPS curve cross each other. Therefore, if we take a putative 5 KHz SPS system as representative of what might be achieved in the near future, we see that it is nevertheless advantageous to employ a 1 MHz WCS system compared to the 5 KHz SPS system.

### 5.2.2 Scenario Two: Eavesdropper can eliminate line attenuation

In Figure 3 we consider a scenario in which the enemy *is* somehow able to effectively eliminate the attenuation along the quantum channel, either by surreptitiously replacing the installed cable with a “magic” cable that is lossless, or by employing prior shared entanglement distributed between two operating locations adjacent to the Alice and Bob locations (but somehow unobserved by Alice and Bob).

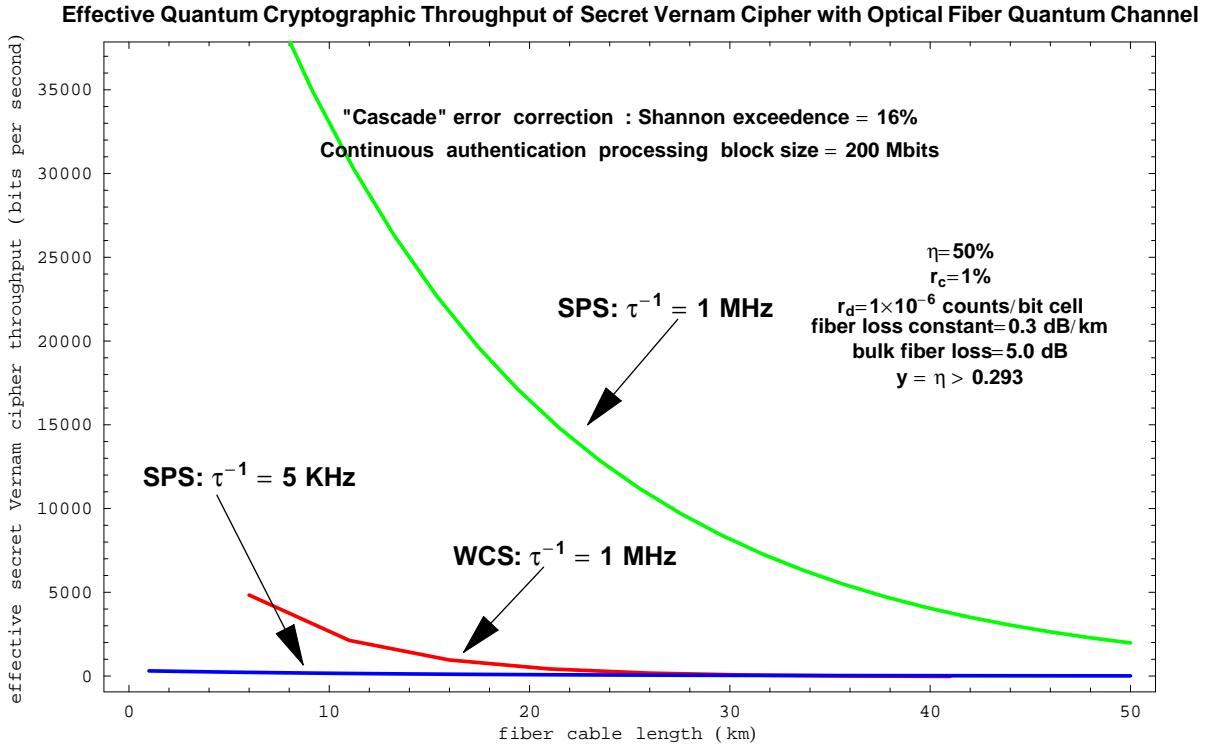


Figure 3: Effective Rate Curves for Fiber-Optic Link *with* Surreptitious Cable Replacement or use of Prior Shared Entanglement by Eve

As in Figure 1, the upper curve in Figure 3 displays the effective secrecy rate that would arise with a perfect single photon source operating at a pulse repetition frequency of 1 MHz. The middle curve gives the effective secrecy rate corresponding to the use of a weak coherent source, also operating at a pulse repetition frequency of 1 MHz. The lower curve shows the rate that would arise with the use of a perfect single photon source operating at a pulse repetition frequency of 5 KHz. Assuming as before that we compare systems with equal pulse repetition frequencies, we see that there is a substantial gain realized with the use of a perfect single photon source compared to the use of a weak coherent source. Inspection of the graph reveals that at a separation distance of  $L_{fiber} = 10$  kilometers one obtains an effective secrecy rate of about 2130 bits per second with the use of a weak coherent source operating at a PRF of 1 MHz. A perfect single photon source also operating at a PRF of 1 MHz achieves the same effective secrecy rate at a distance of about 50 kilometers, a substantial increase. We may also compare rates between the two systems at a fixed separation distance between Alice and Bob. For example, at a separation distance of 10 kilometers, upon going from a WCS system to a SPS system the rate increases from 2130 bits per second to 30740 bits per second, a gain of about 11.6 dB.

As with the example given in Scenario One above, the scale employed in Figure 3 makes

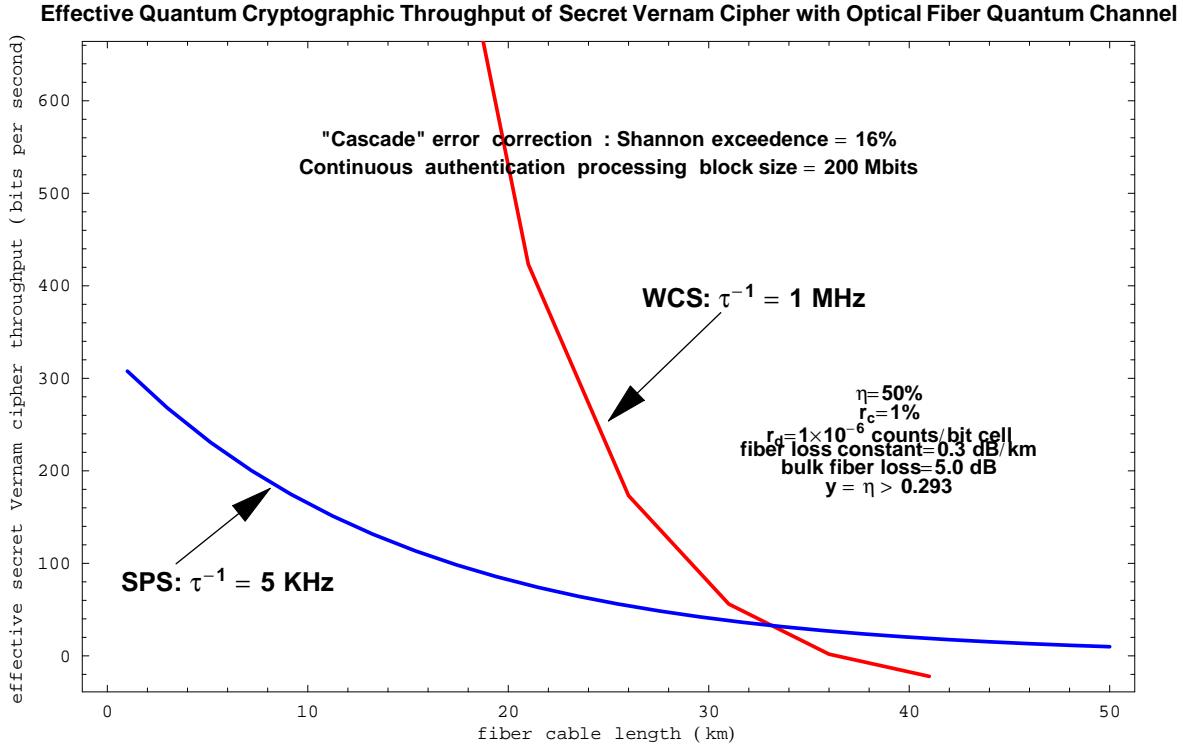


Figure 4: Effective Rate Curves for Fiber-Optic Link *with* Surreptitious Cable Replacement or use of Prior Shared Entanglement by Eve

it difficult to directly compare the cases of a WCS source operating at a PRF of 1 MHz and a SPS source operating at a PRF of 5 KHz. Inspection of Figure 4, however, reveals an interesting feature that distinguishes Scenario Two from Scenario One. For separation distances of less than 33 kilometers it is apparent that it is always preferable to employ a 1 MHz WCS source rather than a 5 KHz SPS source, as the throughput rate for the former is always greater than that for the latter. However, the two throughput curves cross each other at a separation distance of about 33 kilometers. Thus, for distances greater than about 33 kilometers we find that one obtains a better secrecy rate by using a SPS source instead of a WCS source, even though in this case the PRF of the WCS source is 200 times larger than the PRF of the SPS source.

### 5.2.3 Scenario Three: Comparison of Optical Fiber Quality

In Figure 5 we illustrate the effect of improving the intrinsic attenuation characteristic of the optical fiber, demonstrated here for the case of a SPS source operating at a PRF of 5 KHz. The two curves in the graph are plotted for two different fiber-optic cables, with intrinsic attenuation values of 0.2 dB per kilometer and 0.3 dB per kilometer, respectively,

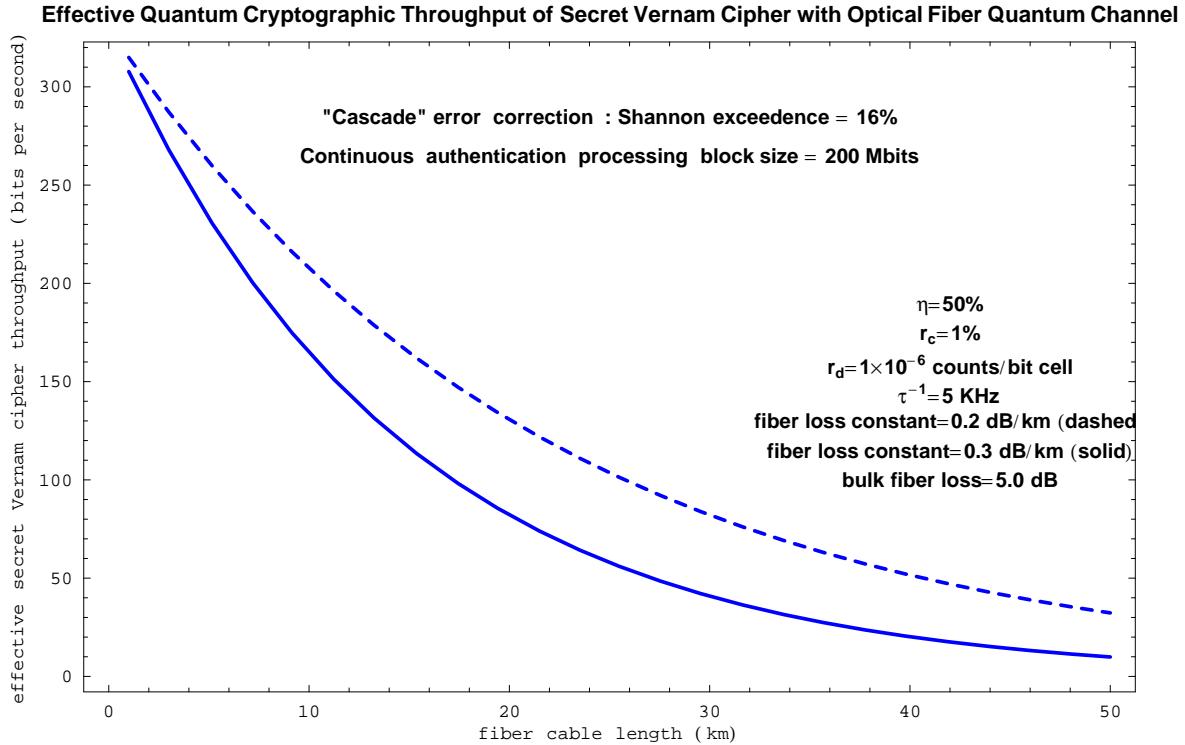


Figure 5: Effective throughput Rate Comparison for Single Photon Source with Fiber Cables of Two Different Qualities

in the upper and lower curves.

Inspection of the graph reveals that, at a separation distance between Alice and Bob of  $L_{fiber} = 10$  kilometers, the fiber with the attenuation characteristic of 0.3 dB per kilometer supports an effective secrecy rate of about 164 bits per second, while the fiber with the attenuation characteristic of 0.2 dB per kilometer supports this throughput rate out to a separation distance of 15 kilometers. We also note that at a separation distance of 25 kilometers, the lower quality fiber supports an effective secrecy rate of about 57.9 bits per second, while the higher quality fiber supports an effective secrecy rate of about 103.6 bits per second, corresponding to a gain of about 2.5 dB.

## 6 Conclusion

In this paper we have discussed several features of the practical implementation of quantum cryptography in real environments having to do with unconditional secrecy, computational loads and effective secrecy rates in the presence of perfect and imperfect sources. As progress in telecommunications and optoelectronics continues to make the insertion of quantum cryp-

tographic technology into real communications systems a more realistic prospect it will become increasingly important to uncover further details and subtleties that determine the optimum possible performance characteristics.

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