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Math 426.2SY
Calculus II

University of New Hampshire

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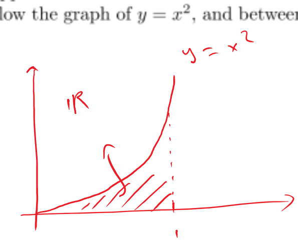
Outline

5.1-Area and Estimating with Finite Sums

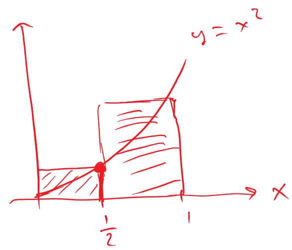
- Suppose we want to find the area of the region R above the $x-$ axis, below the graph of $y = x^2$, and between the vertical lines $x = 0$ and $x = 1$.



- We don't yet have a method to compute the exact area of R (which $=$ $1/3$), but we can estimate.

Area and Estimating with Finite Sums

- Idea: estimate the area of R by breaking the region into rectangles.
- Example: upper sum with 2 rectangles of equal width. $y = x^2 = f(x)$



$$U_2 = \frac{1}{2}\left(f(1/2) + f(1)\right)$$
$$= \frac{1}{2}\left(\frac{1}{4} + 1\right)$$
$$= \frac{1}{2}\left(\frac{5}{4}\right)$$
$$= \frac{5}{8}$$

Area and Estimating with Finite Sums

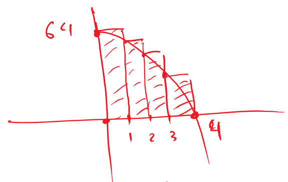
- Example: upper sum with 4 for $y = 64 - x^3$ on the interval $[0, 4]$

lenght of the base of each rectangle $= \frac{4-0}{4} = 1$

$U_4 = 1\,(f(0)+f(1)+f(2)+f(3))$

$= 64 + 63 + 56 + 37$

$= \sim$



$64 - x^3 = 0 \Rightarrow x^3 = 64 \Rightarrow x = \sqrt[3]{64} = 4$

Area and Estimating with Finite Sums

- Example: Estimate the area under $y = sin(\pi x) + 1$ on the interval $[0, 1]$ using 4 rectangle and the right end points.

Lenght of the base $= \frac{1-0}{4} = \frac{1}{4}$

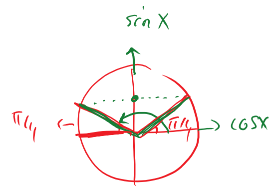
$0 \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \quad 1$

$$S_4 = \frac{1}{4}\left(f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4}) + f(1)\right) =$$

$$\frac{1}{4}\left(\sin(\pi\cdot\tfrac{1}{4})+1 + \sin(\tfrac{\pi}{2})+1 + \sin(\tfrac{3\pi}{4})+1 + \sin(\pi)+1\right) =$$

$$\frac{1}{4}\left(\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0 + 4\right) =$$

$$\frac{1}{4}\left(\sqrt{2} + 5\right)$$



Area and Estimating with Finite Sums

- Some things to notice about these estimates with finite sums:
  - Lower sums are always underestimates to the actual area.
  - Upper sums are always overestimates to the actual area.
  - The more rectangles we use, the better our estimate becomes

approximate area under $y = x^2$

n	L_n	U_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328355	0.3338335

5.2-Sigma Notation and Limits of Finite Sum

- Sigma notation: convenient notation for writing sums with lots of terms.

$$\sum_{k=n}^{m} a_k = a_n + a_{n+1} + a_{n+2} + \cdots + a_m$$

Ex

$$\sum_{k=-1}^{2} a_k = a_{-1} + a_0 + a_1 + a_2$$

Sigma Notation

- Examples: evaluate the following sums:

a_k

- $\displaystyle\sum_{k=1}^{5} k = 1+2+3+4+5 = 15$

a_k

- $\displaystyle\sum_{k=0}^{3} (-1)^k k = (-1)^0(0) + (-1)^1(1) + (-1)^2(2) + (-1)^3(3)$

$= 0 - 1 + 2 - 3 = -2$

- $\displaystyle\sum_{k=4}^{6} \frac{k^2}{k-1} = \frac{4^2}{4-1} + \frac{5^2}{5-1} + \frac{6^2}{6-1} = \frac{16}{3} + \frac{25}{4} + \frac{36}{5}$

$= \ldots$

k even

for($i=4$; $\cancel{i=k+1/2}$; $i<6$)

Sigma Notation

Example

Express the sum $1 + 3 + 5 + 7 + 9$ in sigma notation.

X $\sum_{k=1}^{9} (k+2) = 3 + 4$

✓ $\sum_{k=0}^{4} 2k+1 = 1 + 3 + 5 + 7 + 9$

Sigma Notation

Algebra rules for finite sums:

1. Sum Rule:
$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$
2. Difference Rule:
$$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$
3. Constant Multiple Rule:
$$\sum_{k=1}^{n} c a_k = c\sum_{k=1}^{n} a_k \quad (c \text{ is a constant})$$
4. Constant Value Rule:
$$\sum_{k=1}^{n} c = nc$$

proof of 1

$$\sum_{k=1}^{n}(a_k+b_k) = (a_1+b_1)+(a_2+b_2)+\cdots+(a_n+b_n) =$$
$$= (a_1+a_2+\cdots a_n)+(b_1+b_2+\cdots+b_n)$$
$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

Sigma Notation

Example

Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of:

- $\sum_{k=1}^{n} 8a_k = 8 \sum_{k=1}^{n} a_k = 8 \cdot 0 = 0$
- $\sum_{k=1}^{n} a_k + 1 = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} 1 = 0 + n(1) = n$
- $\sum_{k=1}^{n} b_k - \frac{1}{n} = \sum_{k=1}^{n} b_k - \sum_{k=1}^{n} \frac{1}{n} = 1 - \left(\frac{1}{n}\right) n = 1 - 1 = 0$

Sigma Notation

Helpful sum formulas:

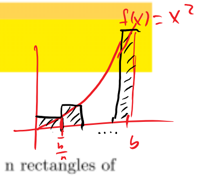
- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Ex let $n=3$

$\sum_{k=1}^{3} k^2 = 1^2+2^2+3^2 = 14$ LHS

$\frac{3(4)(7)}{6} = 14$ RHS

Riemann Sums



- Let's return to our estimates for the area of the region R
- How do we find the exact area?
  - We compute a lower (or upper) sum approximation using n rectangles of equal width
  - Then we take the limit as $n \to \infty$

use the right end points
using n rectangles of equal base.

$$0,\ \frac{b}{n},\ \frac{2b}{n},\ \frac{3b}{n},\ \ldots,\ b=\frac{nb}{n}$$

lenght of each segment = ?

$$\frac{b-0}{n} = \frac{b}{n}$$

$$S_n = \frac{b}{n}\left(f\left(\frac{b}{n}\right)+f\left(\frac{2b}{n}\right)+\cdots+f\left(\frac{nb}{n}\right)\right)$$

$$= \frac{b}{n}\sum_{k=1}^{n} f\left(\frac{kb}{n}\right)$$

$$= \frac{b}{n}\sum_{k=1}^{n} \frac{k^2b^2}{n^2}$$

$$= \frac{b}{n}\cdot\frac{b^2}{n^2}\sum_{k=1}^{n} k^2$$

$$= \frac{b^3}{n^3}\sum_{k=1}^{n} k^2 .$$

$$= \frac{b^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

next we let $n \to \infty$

$$\lim_{n\to\infty} \frac{b^3\, n(n+1)(2n+1)}{6n^3} =$$

$$\lim_{n\to\infty} \frac{b^3\,(n^2+n)(2n+1)}{6n^3} =$$

$$\lim_{n\to\infty} \frac{b^3\,(2n^3+n^2+2n^2+n)}{6n^3}$$

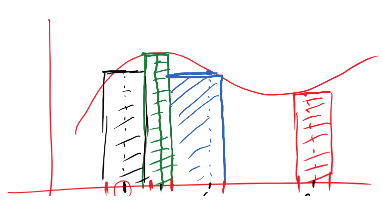
$$= \frac{b^3}{6}\lim_{n\to\infty}\frac{\cancel{n^3}\left(2+\frac{1}{n}+\frac{1}{n^2}\right)}{\cancel{n^3}}$$

$$= \frac{b^3}{6}\lim_{n\to\infty} 2+\underbrace{\frac{1}{n}}_{0}+\underbrace{\frac{1}{n^2}}_{0}$$

$$= \frac{b^3}{6}\cdot 2 = \boxed{\frac{b^3}{3}}$$

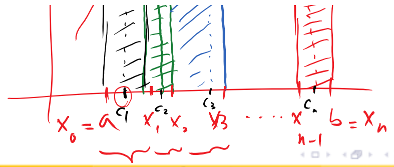
Riemann Sums

- The finite sum approximations we've computed are examples of a more general notion called a Riemann sum
- The Riemann sum for a function f on the interval $[a,b]$ is computed as follows:



RS for f

$$RS(f) = \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k$$



$$\boxed{\sum_{k=1}^{n} f(c_k)\cdot \Delta x_k}$$

$\Delta x_1 = x_1 - x_0$

$\Delta x_2 = x_2 - x_1$

$\vdots$

$\Delta x_n = x_n - x_{n-1}$