Outperforming Library Solvers on Nonlinear & Stiff IVPs with a Wavelet Collocation Method

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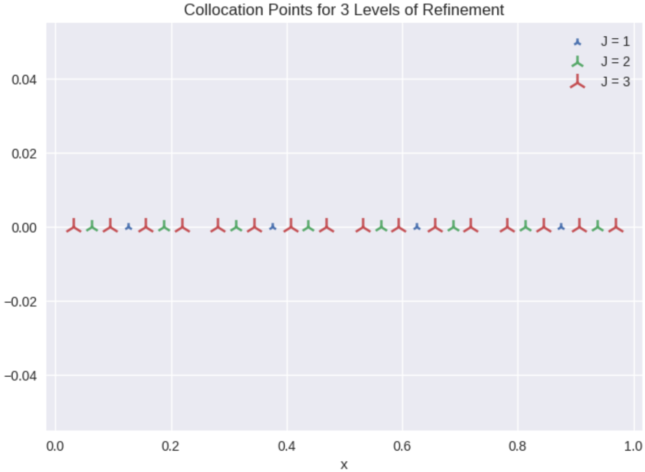
MRA of $L_2(\mathbb{R})$ with Haar Wavelets

- Any function $f(x) \in L_2(\mathbb{R})$ can be expressed as an infinite sum of Haar wavelets
- The MRA produces a sequence of subspaces $V_j, V_{j+1}, \ldots$ such that the projections of f onto these spaces give finer and finer approximations of f as $j \to \infty$
- Since the union of all V_j is dense in $L_2(\mathbb{R})$, the approximation from projections can be arbitrarily close

Collocation Points

On $(0,1)$, $x_j = \frac{j-0.5}{2J}$, $\quad j = 1, 2, \ldots, 2J$ where J is the highest level of the wavelet



Collocation with Wavelets

Let

- $m = 2^j, j \in \mathbb{Z}_{J+1}$ indicates the wavelet level
- $k \in \mathbb{Z}_m$ is the translation parameter
- $i := m + k + 1 \in [1, 2, \ldots, 2M]$ is the index of the ith scaled & translated Haar wavelet where $M = 2^{J+1}$

For each m, k, define

$$\alpha = \frac{k}{m}, \quad \beta = \frac{k+0.5}{m}, \quad \gamma = \frac{k+1}{m}. \tag{1}$$

and the wavelet

$$h_i(x_j) = \begin{cases} 1 & \text{for } x_j \in [\alpha, \beta), \\ -1 & \text{for } x_j \in [\beta, \gamma), \\ 0 & \text{elsewhere.} \end{cases} \tag{2}$$

Explicit Integration

For $\nu = 1, 2, \ldots,$ let

$$p_{i,1}(x) = \int_0^x h_i(s)ds, \quad p_{i,\nu+1}(x) = \int_0^x p_{i,\nu}(s)ds \tag{3}$$

$$p_{i,1}(x) = \begin{cases} x - \alpha & \text{for } x \in [\alpha, \beta) \\ \gamma - x & \text{for } x \in [\beta, \gamma) \\ 0 & \text{else} \end{cases} \tag{4}$$

$$p_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2 & \text{for } x \in [\alpha, \beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2 & \text{for } x \in [\beta, \gamma) \\ \frac{1}{4m^2} & \text{for } x \in [\gamma, 1) \\ 0 & \text{else.} \end{cases} \tag{5}$$

Incorporating Initial Value Conditions (IVPs)

Given $y''(x) = \phi(x, y, y'), y(0) = \alpha, y'(0) = \beta$,

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{6}$$

$$\implies y'(x) = \beta + \sum_{i=1}^{2M} a_i p_{i,1}(x) \tag{7}$$

$$\implies y(x) = \alpha + \beta x + \sum_{i=1}^{2M} a_i p_{i,2}(x) \tag{8}$$

by integrating from 0 to x twice.
Substituting these values into the given differential equation yields a system of equations for a_i.

Overall Algorithm

Algorithm 2.1 Wavelet Approximation to ODE IVP

Input Boundary conditions, ODE $y''(x) = \phi(x, y, y')$, level of resolution M

Output Approximations $y(x_j)$ on collocation points

For $j = 1, 2, \ldots, 2M$, set $x_j = \frac{j-0.5}{2M}$

Let $\mathbf{a} = \mathbf{0}$ as initial guess for Newton's method

for $j = 1, 2, \ldots, 2M$ **do**

Apply Newton's method to the system

$$\sum_{i=1}^{2M} a_i h_i(x_j) = \phi\left(x_j, \alpha_1 + \beta_1 x + \sum_{i=1}^{2M} a_i p_{i,2}(x), \beta_1 + \sum_{i=1}^{2M} a_i p_{i,1}(x)\right)$$

with unknowns $a_1, \ldots, a_{2M}$

end for

for $j = 1, 2, \ldots, 2M$ **do**

Set

$$y(x_j) = \alpha_1 + \beta_1 x_j + \sum_{i=1}^{2M} a_i p_{i,2}(x_j).$$

end for

return $(y(x_j))$

Error Estimate

Theorem (Siraj '10)

Let $y(x) \in L_2(\mathbb{R})$ have bounded first derivative on $(0,1)$, then for $C \in \mathbb{R}^+$, the norm of the error at the J-th level satisfies

$$||e_J(x)|| := \left\|y(x) - \sum_{i=1}^{2^J+1} a_i h_i(x)\right\| \leq C2^{-3J}. \tag{9}$$

This involves determining a bound on $|a_i|$ since

$$||e_J(x)||^2 = \sum_{i=2^{J+1}+1}^{\infty} \sum_{\ell=2^{J+1}+1}^{\infty} a_i a_\ell \int_{-\infty}^{\infty} h_i(x)h_\ell(x)dx \tag{10}$$

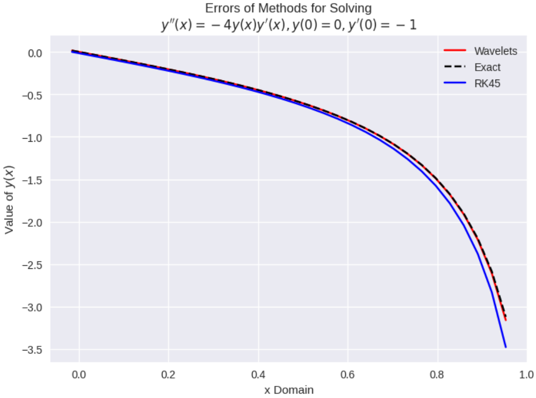
$$\leq \sum_{i=2^{J+1}+1}^{\infty} |a_i|^2. \tag{11}$$

A Simple 2nd Order Non-linear IVP

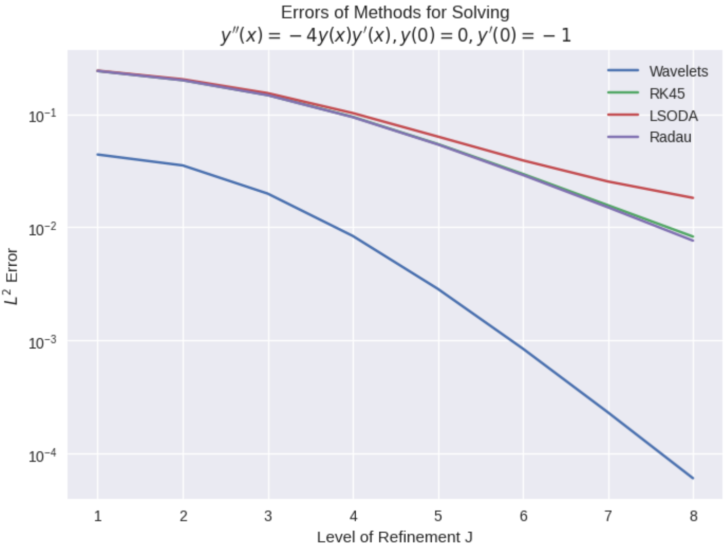
Consider

$$\begin{cases} y''(x) = -4y(x)y'(x) & \text{on } (0,1) \\ y(0) = 0, y'(0) = -1 \end{cases}$$

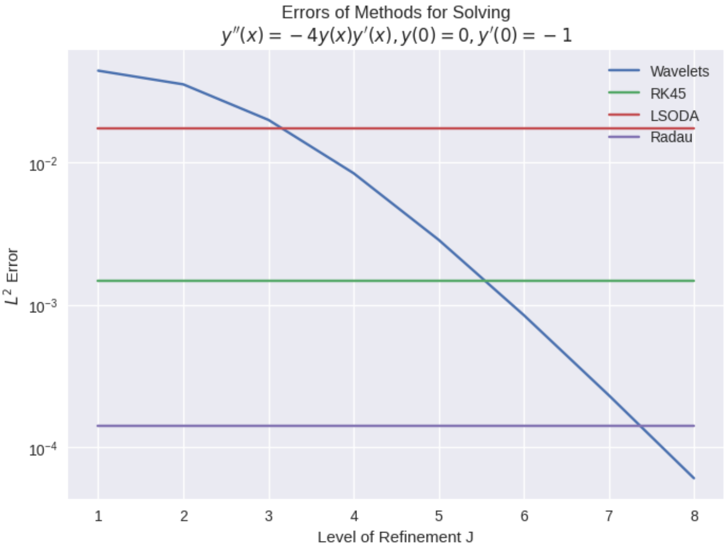
Exact solution: $y(x) = -\frac{\tan \sqrt{2}x}{\sqrt{2}}$



Error Comparison on Collocation Points



Error Comparison on Integrator's Choice

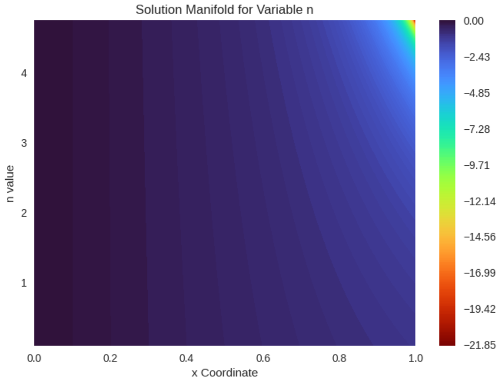


Parametric Version

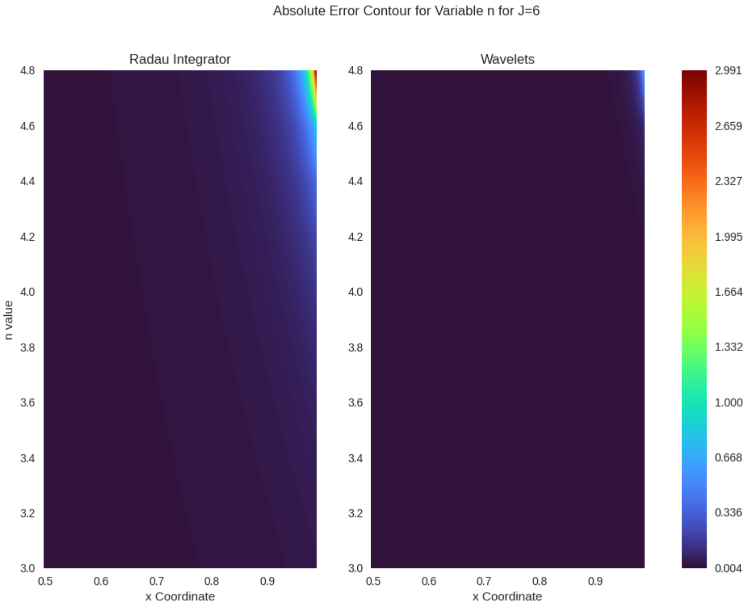
For $n \in \mathbb{R}^+$, consider

$$\begin{cases} y_n''(x) = -n y_n(x) y_n'(x) & \text{on } (0,1) \\ y_n(0) = 0, y_n'(0) = -1 \end{cases}$$

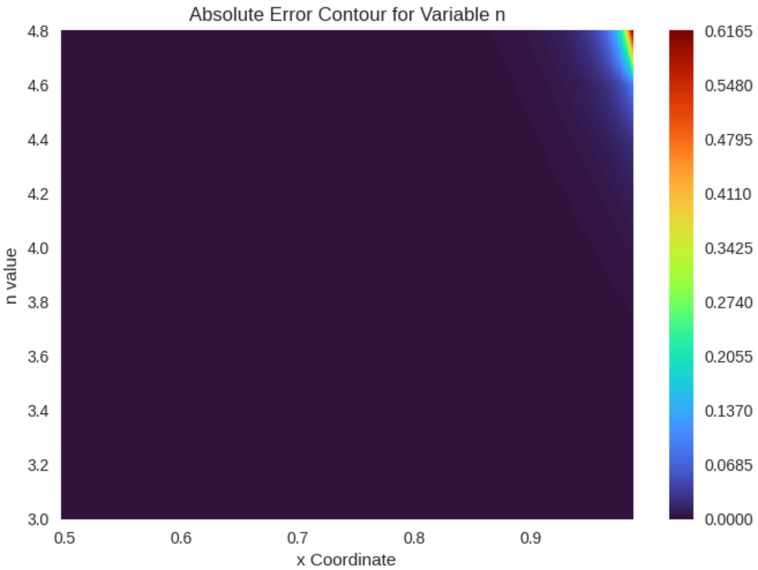
Exact solution: $y_n(x) = -\sqrt{\frac{2}{n}} \tan\left(\sqrt{\frac{n}{2}} x\right).$



Associated Error Contours



Wavelets Error...?



A Classic Stiff System

A mass-spring system can be formulated as the IVP

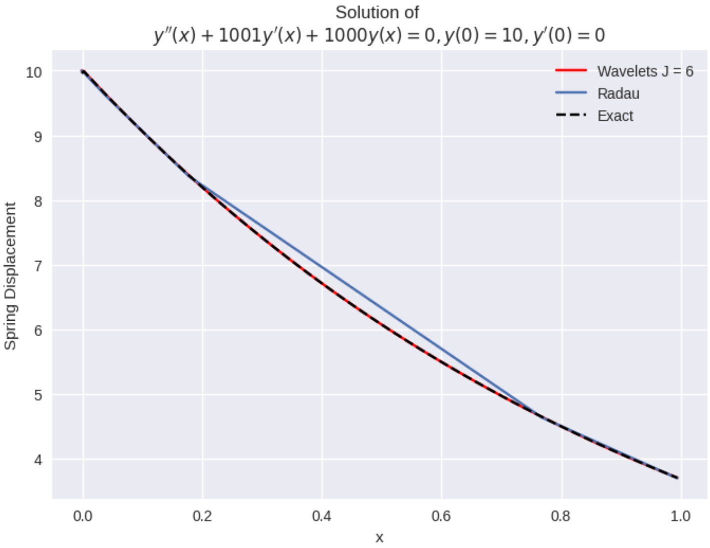
$$y''(x) + 1001y'(x) + 1000y(x) = 0, y(0) = 10, y'(0) = 0 \quad (12)$$

has exact solution

$$y(x) = 10\left(\frac{-1}{999}e^{-1000x} + \frac{1000}{999}e^{-x}\right). \quad (13)$$

The e^{-1000x} term makes numerical computation very sensitive to step size, meaning integrators must choose a small integration step size.

Tricked Integrator



Different Boundary Conditions

This works with any of the following:

1. $y(0) = \alpha_1, \quad y'(0) = \beta_1,$
2. $y'(0) = \alpha_2, \quad y'(1) = \beta_2,$
3. $y(0) = \alpha_3, \quad y(1) = \beta_3,$
4. $y'(0) = \alpha_4, \quad y(1) = \beta_4,$
5. $y(0) = \alpha_5, \quad y'(1) = \beta_5,$
6. $y(0) = y(1), \quad y'(0) = y'(1)$ (periodic boundary conditions),
7. $y(0) = \alpha_6, \quad y(c) = y(1)$ for $c \in (0, 1)$

The derivation is the same and BVPs do not require a transformation to an IVP, unlike for an RK solver.

References

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