An alternative proof of the Bessy-Thomassé Theorem

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Abstract

In this note we give an alternative proof of the Bessy and Thomassé Theorem which states that the vertices of any graph G can be covered by a cycle C_1 in G and disjoint cycle C_2 in the complement of G.

1 Introduction

Lehel made the following conjecture.

Conjecture 1.1 (Lehel). *The vertices of every graph G can be covered by a cycle C_1 in G and a vertex-disjoint cycle C_2 in the complement of G.*

In this conjecture, a single edge, a single vertex, and the empty set are all considered to be cycles. This is to avoid some trivial counterexamples. This convention will be used for the rest of this paper.

This conjecture attracted a lot of attention in the '90s and early '00s. The conjecture first appeared in Ayel's PhD thesis [2] where it was proved for some special types of colourings of K_n. Gerencsér and Gyárfás [4] showed that the conjecture is true if C_1 and C_2 are required to be paths rather than cycles. Gyárfás [5] showed that the conjecture is true if C_1 and C_2 are allowed to intersect in one vertex. Łuczak, Rödl, and Szemerédi [6] showed that the conjecture holds for sufficiently large graphs. Later, Allen [1] gave an alternative proof that works for smaller (but still large) graphs. Lehel's Conjecture was finally shown to be true for all graphs by Bessy and Thomassé [3].

Theorem 1.2 (Bessy and Thomassé, [3]). *The vertices of every graph G can be covered by a cycle C_1 in G and a vertex-disjoint cycle C_2 in the complement of G.*

In this note we give an alternative proof of this theorem. The following theorem from [8] is the main tool which we will need.

Theorem 1.3 ([8]). *Every graph G has a cycle C with wither $|G \setminus C| = 0$ or*

$$\Delta(G \setminus C) \leq \frac{1}{2}(|G \setminus C| - 1). \tag{1}$$

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2 Proof

In this section we use Theorem 1.3 to find an alternative proof of the Bessy-Thomassé Theorem. Recall that Theorem 1.3 produces a partition of a 2-edge-coloured K_n into a red cycle and a blue graph H satisfying.

$$\delta(H) \geq \frac{1}{2}(|H| - 1) \tag{2}$$

Notice that if H satisfied $\delta(H) \geq \frac{1}{2}|H|$ then the Bessy-Thomassé Theorem would follow since, by Dirac's Theorem, any graph satisfying $\delta(H) \geq \frac{1}{2}|H|$ is Hamiltonian. There are non-Hamiltonian graphs which satisfy (2), but the structure such graphs can have is quite limited. The following theorem allows us to classify non-Hamiltonian graphs satisfying (2).

Theorem 2.1 (Nash-Williams, [7])**.** *Let G be a 2-connected graph satisfying*

$$\delta(G) \geq \max\left(\frac{1}{3}(|G| + 2), \alpha(G)\right).$$

Then G is Hamiltonian.

Theorem 2.1 has the following corollary which classifies non-Hamiltonian graphs satisfying (2).

Corollary 2.2. *If G is a graph satisfying $\delta(G) \geq \frac{1}{2}(|G| - 1)$ then one of the following holds.*

(i) *$V(G)$ can be partitioned into two sets A and B such that we have $|A| = |B| + 1$, all the edges between A and B are present, and there are no edges within A.*

(ii) *There is a vertex $v \in G$ such that $V(G) - v$ can be partitioned into two sets A and B such that $|A| = |B|$, there are no edges between A and B, and the subgraphs $G[A+v]$ and $G[B+v]$ are complete.*

(iii) *G is Hamiltonian.*

Proof. For $|G| \leq 6$, the corollary is easy to check by hand. Therefore, we may suppose that we have $|G| \geq 7$.

Suppose that G has a cut-vertex v. We have that $\delta(G - v) \geq \frac{1}{2}(|G| - 3)$. This, combined with the fact that $G - v$ is disconnected implies that $G - v$ consists of two complete graphs A and B with no edges between them and satisfying $|A| = |B|$. Also $\delta(G) \geq \frac{1}{2}(|G| - 1)$ implies that every vertex in $A \cup B$ is connected to v. Therefore, we are in case (ii).

Suppose that G is 2-connected. Since $|G| \geq 7$, the identity $\delta(G) \geq \frac{1}{2}(|G| - 1)$ implies that we have $\delta(G) \geq \frac{1}{3}(|G| + 2)$. Therefore, by Theorem 2.1, if G is not Hamiltonian, then we have $\alpha(G) > \delta(G) \geq \frac{1}{2}(|G| - 1)$. This implies that there is an independent set A of order $\geq \frac{1}{2}|G|$ in G. Let $B = V(G) \setminus A$. Since $\delta(a) \geq \frac{1}{2}(|G| - 1)$ for all $a \in A$, we have that all the edges between A and B are present. This combined with the fact that G is not Hamiltonian implies that $|A| = |B| + 1$, and so G has the structure of case (i). $\square$

Combining Theorem 1.3 and Corollary 2.2 implies that any 2-edge-coloured K_m has a partition into a red cycle and a blue graph G which is either a cycle or satisfies part (i) or (ii) of Corollary 2.2. Therefore, to prove the Bessy-Thomassé Theorem, we need to only consider the case when K_n has a partition into a red cycle and a blue graph satisfying (i) or (ii).

We will need another intermediate result, which allows us to forbid certain induced subgraphs from the red colour class of a graph.

Lemma 2.3. *Suppose that we have $n \in \mathbb{N}$ such that for $m < n$ every 2-edge-coloured K_m has a partition into two monochromatic cycles of different colours. Let K_n be a 2-edge-coloured complete graph satisfying one of the following.*

(a) *K_n contains an induced red cycle on 5 or more vertices.*

(b) K_n *contains two vertex-disjoint induced red cycles* C_1 *and* C_2*, such that* $|C_1|, |C_2| \geq 3$ *and all the edges between* C_1 *and* C_2 *are blue.*

Then K_n *has a partition into two monochromatic cycles of different colours.*

Proof. (a) Let C be a red induced cycle in K_n on 5 or more vertices. Let K_n/C be the graph K_n with the cycle C contracted into a single vertex v_c. We colour the edges of K_n/C as follows:

- For $x, y \neq v_c$, the edge xy in K_n/C receives the same colour it has in K_n.
- The edge xv_c is red if there are more red edges between x and C than blue edges.
- The edge xv_c is blue if there are more blue edges between x and C than red edges or if the numbers of red and blue edges between x and C is equal.

By the minimality of n, K_n/C has a partition into a red cycle R and a blue cycle B.

Suppose that we have $v_c \in R$. We claim that there is a red cycle in G with vertices $C \cup (R - v_c)$. Indeed, let x and y be the neighbours of v_c on R. By the Pigeonhole Principle, since x and y each have $> |C|/2$ red neighbours on C, x has a red neighbour $c_x \in C$ and y has a red neighbour $c_y \in C$, such that c_x and c_y are adjacent on C. The edges of C form a red path between c_x and c_y which covers C. This path can be joined to the red path $R - v_c$ using the edges xc_x and yc_y to produce the required cycle. Now the cycles $(R \cup V) - v_c$ and B partition K_n into two monochromatic cycles of different colours.

Suppose that we have $v_c \in B$. If there is a majority of blue edges from one of the neighbours of v_c to B, then we can procceed just as we did in the previous case (using the fact that the complement of a cycle on 5 or more vertices is Hamiltonian). Otherwise we have that k is even. It is easy to check that the complement of any even cycle on 6 or more vertices is Hamiltonian connected. By a similar argument to before, this allows us to find a blue cycle with vertices $C \cup (B - v_c)$. This gives us a partition of K_n into two monochromatic cycles of different colours $(C \cup B) - v_c$ and R.

(b) If we are not in part (a) of this lemma, we can suppose that C_1 and C_2 are cycles on either 3 or 4 vertices. Let $K_n/(C_1, C_2)$ be the graph K_n with the cycles C_1 and C_2 replaced by two vertices v_1 and v_2. We colour the edges of $K_n/(C_1, C_2)$ as follows:

- For $x, y \notin \{v_1, v_2\}$, the edge xy in $K_n/(C_1, C_2)$ receives the same colour it has in K_n.
- For $x \notin \{v_1, v_2\}$, the edge xv_i is red if x has two red neighbours which are adjacent on the cycle C_i.
- For $x \notin \{v_1, v_2\}$, the edge xv_i is blue if x does not have two red neighbours which are adjacent on the cycle C_i.
- The edge v_1v_2 is blue.

By the minimality of n, $K_n/(C_1, C_2)$ has a partition into a red cycle R and a blue cycle B.

We will deal only with the case $|C_1| = 3$, $|C_2| = 4$ here. The other cases are all very similar. Let x_1, x_2, x_3 be the vertices of C_1, and y_1, y_2, y_3, y_4 be the vertices of C_2 (in the order in which they occur along the cycles).

If both v_1 and v_2 are in R then we can proceed similarly to how we did in part (a) to cover $C_1 \cup C_2 \cup (R - v_1 - v_2)$ by a red cycle.

We consider three cases depending on which cycles v_1 and v_2 are in.

Case 1: Suppose that $v_1 \in R$ and $v_2 \in B$ hold. Let r_1 and r_2 be the vertices adjacent to v_1 in R and b_1 and b_2 the vertices adjacent to v_2 in B. Since r_1 and r_2 each have at least two red neighbours in C_1, they must have a common red neighbour in C_1. Without loss of generality we may suppose that x_1 is this common neighbour, and so the edges r_1x_1 and r_2x_1 are red. Therefore there is a red cycle in K_n with vertices $R - v_1 + x_1$. We will construct a blue cycle covering $V(K_n)\setminus(R-v_1+x_1)$, which together with the cycle on $R-v_1+x_1$ gives a partition of K_n into two monochromatic cycles.

Since b_1 and b_2 each have at least two blue neighbours in C_2, there must be blue edges b_1y_i and b_2y_j for two distinct vertices $y_i, y_j \in C_2$. There are two subcases depending on whether y_i and y_j are adjacent on C_2 or not.

Suppose that y_i and y_j are adjacent on C_2. Without loss of generality, we may suppose that $i = 1$ and $j = 2$. The blue path $b_1y_1y_3x_2y_4x_3y_2b_2$ can be joined to the blue path $B - v_c$ to obtain a blue cycle covering $B - v_2 + x_2 + x_3 + y_1 + y_2 + y_3 + y_4$.

Suppose that y_i and y_j are not adjacent on C_2. Without loss of generality, we may suppose that $i = 1$ and $j = 3$. The blue path $b_1y_1x_2y_2y_4x_3y_3b_2$ can be joined to the blue path $B - v_c$ to obtain a blue cycle covering $B - v_2 + x_2 + x_3 + y_1 + y_2 + y_3 + y_4$.

Case 2: Suppose that $v_1 \in B$ and $v_2 \in B$ hold. We will construct a blue cycle with vertices $C_1 \cup C_2 \cup (B - v_1 - v_2)$. Together with R, this cycle gives a partition of K_n into two cycles contradicting our initial assumption. There are two subcases depending on whether the edge v_1v_2 is in B or not.

Case 2.1: Suppose that B does not contain the edge v_1v_2. Let b_1 and b_2 be adjacent vertices of v_1 in B and b_1' and b_2' adjacent vertices of v_2 in B. Note that v_1 and v_2 have a common neighbour in C_1. This allows us to extend B to cover one vertex of C_1. The remainder of $C_1 \cup C_2$ can be covered by a blue path and joined to b_1' and b_2' by an identical argument to the one used in Case 1.

Case 2.2: Suppose that B contains the edge v_1v_2. It is easy to check that the complement of two disjoint induced cycles on 3 and 4 vertices contains a Hamiltonian path with one endpoint in each cycle. This allows us to extend B to find a blue cycle with vertices $C_1\cup C_2\cup(B-v_1-v_2)$.

Case 3: Suppose that $v_1 \in B$ and $v_2 \in R$ hold. Let r_1 and r_2 be adjacent vertices of v_2 in R and b_1 and b_2 adjacent vertices of v_1 in B. Without loss of generality the edges r_1y_1 and r_2y_2 are red and the edges b_1x_1 and b_2x_2 are blue. We have a red path $r_1y_1y_2r_2$ and a disjoint blue path $b_1x_1y_3x_3y_4x_2b_2$. These can be joined to the paths $R - v_2$ and $B - v_1$ to obtain a red cycle with vertices $R - v_2 + y_1 + y_2$ and a disjoint blue cycle with vertices $B-v_1+x_1+y_3+x_3+y_4+x_2$. This gives us a partition of K_n into two disjoint monochromatic cycles of different colours.

□

We now give an alternative proof of the Bessy-Thomassé Theorem.

Proof of Theorem 1.2. The proof is by induction on the order of the complete graph. Clearly every K_1 has a partition into two monochromatic cycles (one of which is empty). Suppose that for all $m < n$ every 2-edge-coloured K_2 has a partition into two monochromatic cycles of different colours. Let K_n be a 2-edge-coloured complete graph.

By Lemma 2.3 we are done if K_n has cycle(s) as in parts (a) or (b) of Lemma 2.3. Therefore, for the remainder of the proof, we can assume that neither of these occur.

Apply Theorem 1.3 to K_n in order to partition K_n into a red cycle C and a blue graph H with $\delta(H) \geq \frac{1}{2}(|H|-1)$. In addition, suppose that $|C|$ is as small as possible. If H is Hamiltonian then we have a partition of K_n into two monochromatic cycles C and H with different colours. If H is not Hamiltonian, then H must have one of the structures (i) or (ii) in Corollary 2.2. We split into two cases depending on which structure H has.

Suppose that H is partitioned into A and B as in part (i) of Corollary 2.2 i.e. $|A| = |B| + 1$, A is a red complete graph and all edges between A and B are blue.

We say that a vertex in C is *red* if it has a red neighbour in A, and *blue* otherwise. We consider a number of cases depending on what colours vertices along C have.

Case 1: Suppose that there is a sequence of vertices $c_1c_2\dots c_k$ along C such that $k \geq 3$, the edge c_1c_k is red, and the vertices $c_2, \dots c_{k-1}$ are all blue. We may suppose that $c_1, \dots, c_k$ is the shortest such sequence, and hence the vertices $c_1c_2\dots c_k$ lie on an induced red cycle in K_n.

By Lemma 2.3 we have that $k \leq 4$. This, combined with $c_2, \dots, c_{k-1}$ being blue implies that $\delta_B(K_n \setminus (C - c_2 - \cdots - c_{k-1})) \geq \frac{1}{2}(|K_n \setminus (C - c_2 - \cdots - c_{k-1})| - 1)$ contradicting the minimality of $|C|$.

Case 2: Suppose that there is at most one red vertex in C. Let c be this red vertex (if it exists) and any other vertex on C otherwise. If C has any chords then we are back to the first case. By Lemma 2.3, we know that $|C| \leq 4$. If $|C| = 4$, then for a neighbour c_1 of c on C, we set $C' = \{c, c_1\}$ in order to get a shorter cycle with $H' = K_n \setminus V(C')$ satisfying $\delta(H') \geq \frac{1}{2}(|H'| - 1)$. If $|C| = 2$ or 3, then letting $C' = \{c\}$ gives a shorter cycle with $H' = K_n \setminus V(C')$ satisfying $\delta(H') \geq \frac{1}{2}(|H'| - 1)$. If $|C| = 1$, and c is blue, then all the edges between $B \cup c$ and A are blue, so K_n has a spanning blue balanced complete bipartite graph. If $|C| = 1$, and c is red, then there is a partition of K_n into a red edge between c and A and a balanced complete bipartite graph.

Case 3: Suppose that there are two adjacent vertices c_1 and c_2 on C which have a common red neighbour $a \in A$. Notice that there is a red cycle with vertices $C + a$ (by replacing the edge c_1c_2 in C with the red path c_1ac_2). The colouring on $K_n \setminus (C + a)$ contains a blue balanced complete bipartite graph with classes $A - a$ and B and so contains a blue Hamitonian cycle.

Case 4: Suppose that there is a sequence of three vertices $c_1c_2c_3$ along C such that c_1 and c_3 are both red and have no common red neighbours in A. Let x and y be red neighbours of c_1 and c_3 respectively in A. Notice that we have $x \neq y$ and the edges c_1y and c_3x are both blue. If we are not in Case 3, we can assume that the edges c_2x and c_2y are both blue.

Therefore, since by Lemma 2.3 $c_1c_2c_3yx$ is not an induced 5-cycle, the edge c_1c_3 must be red. Notice that the blue graph on $A \cup B + c_2$ is Hamiltonian (the blue graph on $A \cup B$ is a complete bipartite graph and so has a Hamiltonian path between x and y. This path can be joined to the blue path xc_2y). This gives a partition of K_n into the red cycle on $C - c_2$ and the blue cycle on $A \cup B + c_2$.

Case 5: Suppose that all the vertices in C are red. We also suppose that no sequences of vertices as in cases 1 – 4 occur. Since we are not in Case 2, we have that $|C| \geq 2$.

First notice that since we are not in Case 3, then there is always a red cycle covering $C \cup A$ (constructed by joining two adjacent vertices on C to a path spanning A). This implies that we can assume that B is not Hamiltonian in blue, and hence we have $|B| \geq 2$ and $|A| \geq 3$.

Case 5.1: Suppose that there is a vertex $c \in C$ which has at least 2 blue neighbours in A and a blue neighbour $b \in B$. Let c_1 and c_2 be the neighbours of c on C. Notice that since we are not in Case 4, the vertices c_1 and c_2 have a common red neighbour $a_1 \in A$. The vertex c has a blue neighbour $a_2 \in A$ such that $a_2 \neq a_1$. We show that there is a partition of K_n into a red cycle on $C - c + a_1$ and a blue cycle on $A \cup B - a_1 + c$. The red graph on $C - c + a_1$ has a spanning cycle formed by replacing c by a_1. The blue graph on $A \cup B - a_1 + c$ is Hamiltonian since there is a blue path between b and a_2 spanning the balanced complete bipartite graph on $A \cup B - a_1$. The vertex c can be joined to this path with the edges cb and ca_2.

Case 5.2: Suppose that for every $c \in C$, either all the edges between c and B are red, or there is at most one blue edge between c and A.

First we deal with the case when C has at least 4 vertices. Let $c_1c_2c_3c_4$ be a sequence of four vertices along C. Since $|A| \geq 3$ and we are not in Case 3, one of the vertices c_1 or c_2 must have at least 2 blue neighbours in A. Without loss of generality suppose that this is c_2. In this case all the edges between c_2 and B are red.

Case 5.2.1: Suppose that c_3 has a red neighbour $b \in B$. Since we are not in Case 3, there are red edges c_1a_1 and c_2a_2 for distinct $a_1, a_2 \in A$. There is a red path $c_1a_1a_2c_2bc_3$ which can be joined to C in order to obtain a parition of K_n into a red cycle on $C + a_1 + a_2 + b$ and a blue balanced complete bipartite graph with classes $A - a_1 - a_2$ and $B - b$.

Case 5.2.2: Suppose that all the edges between c_3 and B are blue. By the assumption of Case 5.2, there is at most one blue edge between c_3 and A. Combining this with the fact that we are not in Case 3 and $|A| \geq 3$, we obtain that there are at least 2 blue edges between c_4 and A. Again using the assumption of Case 5.2, we obtain that all the edges between c_4 and B are red. Since $|A| \geq 3$ and there is at most one blue edge between c_3 and A, there are red edges c_1a_1 and c_3a_3 for distinct $a_1, a_3 \in A$. Let b be any vertex in B. There is a red path $c_1a_1a_3c_3c_2bc_4$ which can be joined to C in order to obtain a partition of K_n into a red cycle on $C + a_1 + a_2 + b$ and a blue balanced complete bipartite graph with classes $A - a_1 - a_2$ and $B - b$.

If $|C| \leq 4$, then it is easy to see that a similar argument can be used to partition the graph into a red cycle and a blue balanced complete bipartite graph (the only difference being that some of the vertices c_1, c_2, c_3, and c_4 in the above argument might be equal).

Case 6: Suppose that none of the above cases occur. Then, since Cases 2 and 5 don't occur, there is a sequence of vertices $c_1c_2 \dots c_k$ along C such that we have $k \geq 3$, c_1 and c_k are red and $c_2, \dots, c_{k-1}$ are blue. If there are any red edges between c_i and c_j for $|i - j| \geq 2$, then we can return to Case 1, so suppose there are no such edges. Notice that we have an induced red cycle passing through $c_1, \dots, c_k$ and neighbours of c_1 and c_k in A. By Lemma 2.3, this cycle can have at most 4 vertices, so we have that $k = 3$ and that c_1 and c_3 have a common red neighbour, $a \in A$. There are two subcases depending on how big A is.

Case 6.1: Suppose that $|A| \geq 3$. Note that since c_2 is blue, A and $B + c_2$ have only blue edges between them. Since A contains no blue edges and $|A| \geq 3$, A contains a spanning red cycle, and hence part (ii) of Lemma 2.3 implies that the red subgraph on $B + c_2$ is acyclic. Therefore $B + c_2$ contains a blue edge, and hence the blue graph with vertex set $A \cup B + c_2 - a$ is Hamiltonian (say, by Corollary 2.2). This blue cycle together with the red cycle on $C - c_2 + a$ give the required partition of K_n.

Case 6.2: Suppose that $|A| \leq 2$. If $|A| \leq 1$, then the theorem holds trivially since C and the one vertex in A form the required partition of K_n into two cycles. Therefore, we can assume that $|A| = 2$ and $|B| = 1$. Let b be the vertex in B. Let a' be the vertex in $A - a$.

If the edge bc_2 is blue then we obtain a partition of K_n into the blue cycle $\{a', b, c_2\}$ and the red cycle with vertices $C - c_2 + a$ formed by replacing c_2 by a in C. Therefore, assume that bc_2 is red.

Let $c_4, \dots, c_{|C|}$ be the sequence of vertices along C between c_3 and c_1.

Case 6.2.1: Suppose that there is a vertex $c_m \in \{c_4, \dots c_{|C|}\}$ which has a red edge to one of the vertices in the set $\{a, a', b, c_2, \dots c_{m-2}\}$. We can assume that m is the smallest index for which such a vertex exists and hence for $i < m$ all the edges between c_i and $\{a, a', b, c_2, \dots c_{i-2}\}$ are blue.

Suppose that there is a red edge between c_i and c_m for some $i = 3, \dots, m - 2$. We can assume in addition that i is the largest index for which such an edge exists. Since $c_i, \dots, c_m$ form an induced cycle, Lemma 2.3 implies that $i \geq m - 3$. Notice that the blue graph with vertices $a, a', b, c_{i+1}, \dots, c_{m-1}$ is Hamiltonian (since the only red edges in this graph are aa' and possibly $c_{m-2}c_{m-3}$). This gives us a partition into a blue cycle on $a, a', b, c_{i+1}, \dots, c_{m-1}$ and a red cycle on $C \setminus \{c_{i+1}, \dots, c_{m-1}\}$.

Suppose that all the edges between c_m and $\{c_3, \dots, c_{m-2}\}$ are blue. This implies that c_m is connected to one of the vertices a, a', b, or c_2 by a red edge. Part (a) of Lemma 2.3 implies that $m = 4$ or 5. It is easy to see that there is a red cycle C' in K_n passing through the vertices $a, c_1, c_2, c_3, c_m, c_{m+1} \dots c_{|C|}$ and possibly a' and b in some order. The edges outside $V(C')$ are all blue giving a partition into two cycles of different colours.

Case 6.2.2: Suppose that no vertex $c_m \in \{c_4, \dots c_{|C|}\}$ has a red edge to $\{a, a', b, c_2, \dots c_{m-2}\}$. Then there is a partition of K_n into a red cycle on $\{c_1, c_2, c_3, a\}$ and a blue cycle on $K_n \setminus \{c_1, c_2, c_3, a\}$. To see that there is a blue cycle on $K_n \setminus \{c_1, c_2, c_3, a\}$ notice that the red graph on these vertices has maximum degree 2, and so by Dirac's Theorem there is a blue cycle on $K_n \setminus \{c_1, c_2, c_3, a\}$ as long as $|K_n \setminus \{c_1, c_2, c_3, a\}| \geq 5$. When $|K_n \setminus \{c_1, c_2, c_3, a\}| \leq 4$, then $K_n \setminus \{c_1, c_2, c_3, a\}$ consists of a', b, and possibly c_3 and c_4. The only red edge on these vertices is c_3c_4, and so the blue graph is Hamiltonian.

Suppose that H is partitioned into A, B, and v as in part (ii) of Corollary 2.2 i.e. $|A| = |B|$, there are no blue edges between A and B, and all the edges in $A + v$ and $B + v$ are blue. Notice that when $|A|, |B| \leq 1$ then H also has the structure of part (i) of Theorem 2.2, so this case was dealt with in the previous part of this proof. Therefore, we can suppose that $|A|, |B| \geq 2$. By applying part (b) of Lemma 2.3 to the graph formed from K_n by exchanging the two colours, we obtain that $|A|, |B| = 2$.

We say that a vertex in C is *red* if it has a red neighbour in $A \cup B$, and *blue* otherwise. We consider three cases.

Case 1: Suppose that there is a sequence of vertices $c_1c_2 \dots c_k$ along C such that $k \geq 3$, the edge c_1c_k is red, and the vertices $c_2, \dots c_{k-1}$ are all blue.

We may suppose that k is as small as possible, and hence the vertices $c_1c_2 \dots c_k$ form an induced red cycle in K_n. Lemma 2.3 implies that we have $k \leq 4$. Notice that the blue subgraph of K_n with vertex set $V(H) \cup \{c_2 \dots c_{k-1}\}$ satisfies the condition of Dirac's Theorem. Therefore this blue cycle together with the red cycle with vertices $C - c_2 - \dots - c_{k-1}$ give the required partition of K_n.

Case 2: Suppose that there is a sequence of vertices $c_1c_2 \dots c_k$ along C such that we have $k \geq 3$, c_1 and c_k are red and $c_2, \dots, c_{k-1}$ are blue.

If there are any red edges between c_i and c_j for $|i - j| \geq 2$, then we can return to the previous case, so suppose there are no such edges. Since $c_2, \dots, c_{k-1}$ are blue, there is an induced red cycle D passing through $c_1, \dots, c_k$ and some vertices in $A \cup B$. Lemma 2.3 implies that such a cycle can have at most one vertex, x, in $A \cup B$ and also that $k = 3$. Notice that the blue graph on $A \cup B - x + v + c_2$ is Hamiltonian giving a partition into two cycles of different colours.

Case 3: Suppose that all the vertices in C are red.

Case 3.1: Suppose that there is a vertex $x \in A \cup B$ such that every vertex in C has only blue neighbours in $A \cup B - x$. We'll assume for now that $|C| \geq 3$. Let c_1, c_2, c_3 be three adjacent vertices on C. Then the blue graph on $A \cup B - x + v + c_2$ is Hamiltonian as is the red subgraph on $C - c_2 + x$, giving the required partition. If $|C| \leq 2$, then the same argument works, by letting $c_1 = c_3$.

Case 3.2: Suppose that there are two vertices c_1, c_2 on C such that there are distinct vertices a_1,$a_2 \in A \cup B$ such that c_1a_1 and c_2a_2 are both red. By choosing c_1 and c_2 to be the closest pair of such vertices, we may assume that c_1 and c_2 are adjacent on C. Since $A \cup B$ is a red complete bipartite graph, there is a red path between a_1 and a_2 covering all except possibly one vertex in $A \cup B$. Joining this path to C using the edges c_1a_1 and c_2a_2 gives a partition into a red cycle and blue cycle consisting of v and possibly one other vertex.

Case 4: Suppose that none of the above cases occurs. If Case 3 doesn't occur then there must a blue vertex on C. If there was more than one red vertex in C, then we could find a sequence of vertices as in Case 2. Therefore, we can suppose that there is at most one red vertex in C.

Let c_1 be the red vertex of C if it exists and an arbitary vertex of C otherwise. Let $c_1, \dots, c_{|C|}$ be the vertices of C in the order in which they occur along C. If $|C| \geq 3$, then this sequence satisfies the condition of Case 1. Therefore, the only remaining case is when $|C| \leq 2$ (and hence $n \leq 7$), which is easy to check by hand.If $|C| = 2$, then the partition into $C_1 = \{c_1\}$ and $C_2 = A \cup B + v + c_2$ gives a partition into two disjoint monochromatic cycles. If $|C| = 1$ and c_1 has two red neighbours in A, then there is a partition into a red C_4 and a blue C_3. If $|C| = 1$ and C_1 has a blue neighbour in each of A and B, then K_n is Hamiltonian in blue. □

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