Graphs with $2n-2$ edges

Lothar Narins, Alexey Pokrovskiy, Tibor Szabó

Methods for Discrete Structures,
Berlin.
alja123@gmail.com

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Why study graphs with $2n - 2$ edges?

Theorem (Nash-Williams)

For a graph G with $2|G| - 2$ edges, the following are equivalent

- *G is the union of two spanning trees.*
- *Every subgraph $H \leq G$ has at most $2|H| - 2$ edges.*

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For a graph G with $2|G| - 3$ edges, the following are equivalent

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Examples

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The conjecture

Conjecture (Erdős, Faudree, Gyárfás, and Schelp)

There is an increasing function $C(n)$ such that the following holds. Every graph G with n vertices, $2n-2$ edges, and no proper induced subgraphs of minimal degree 3 contains cycles of lengths $3, 4, 5, 6, \ldots, C(n)$.

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Every graph G with n vertices, $2n-2$ edges, and no proper induced subgraphs of minimal degree 3 contains cycles of lengths $3, 4, 5$.

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Theorem (Erdős, Faudree, Gyárfás, and Schelp)

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Theorem (Bollobás and Brightwell)

Every graph G with n vertices, $2n-2$ edges, and no proper induced subgraphs of minimal degree 3 contains a cycle of length at least $4\log_2 n - o(\log n)$.

There are graphs with n vertices, $2n-2$ edges, and no proper induced subgraphs of minimal degree 3 and no cycles of length at least $4\log_2 n + O(1)$.

Known results

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Proof uses the following characterization.

Lemma

Let G be a graph on n vertices with $2n-2$ edges and no proper, induced subgraphs with minimum degree 3. Then there is an ordering of the vertices of G $x_1, \ldots, x_n$ such that the following hold.

(i) *The edge x_1x_2 is present.*

(ii) *For $i = 2, 3, \ldots, n-1$, the vertex x_i has exactly 2 neighbours in $\{x_1, \ldots, x_{i-1}\}$.*

(iii) $d(x_n) = 3$.

Counterexamples

Theorem (Narins, P., and Szabó)

There is an infinite sequence of graphs $(G_n)_{n=1}^{\infty}$ with the following properties.

(i) $e(G_n) = 2|G_n| - 2.$

(ii) *G_n contains no proper, induced subgraphs with minimum degree 3.*

(iii) *G_n contains no cycle on 23 vertices.*

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(iii) G_n *contains no cycle on* 23 *vertices.*

The number 23 can be replaced by any *odd* number larger than 23.

Binary trees

Given a tree T, define $G(T)$ to be the graph formed from T by adding two new vertices x and y, the edge xy as well as every edge between $\{x, y\}$ and the leaves of T.

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Lemma

Let T be a binary tree such that all the leaves of T are in the same bipartition of T.

(i) *The graph $G(T)$ contains an odd cycle on $2k + 1$ vertices $\iff$ T contains a leaf-leaf path of length $2k - 2$.*

(ii) *The graph $G(T)$ contains an even cycle on $2k$ vertices $\iff$ T contains two vertex-disjoint leaf-leaf path P_1 and P_2 such that $|P_1| + |P_2| = 2k - 2$ or T contains a leaf-leaf path of length $2k - 2$.*

Binary trees

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Theorem

There are arbitrarily large binary trees T such that all the leaves of T are in the same bipartition of T and T contains no leaf-leaf paths of length 20.

Constructing the trees from sequences

We construct our binary trees from certain sequences of numbers.

Sequence

We construct a sequence of numbers a_n such that $a_i + a_j + |i - j| \neq 20$ for all i and j.

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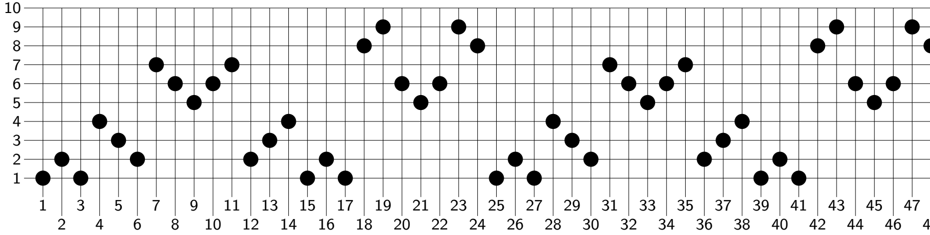

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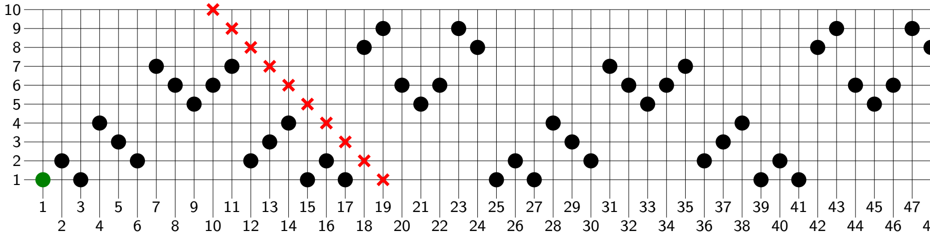

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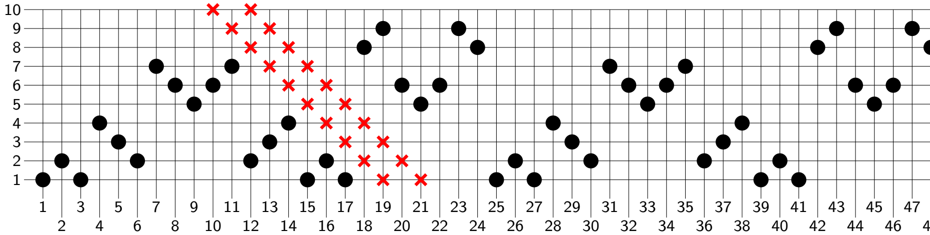

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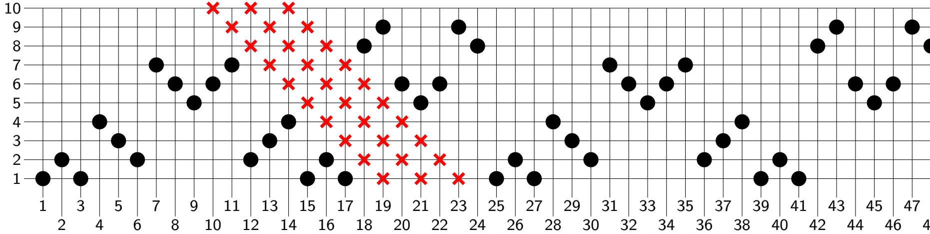

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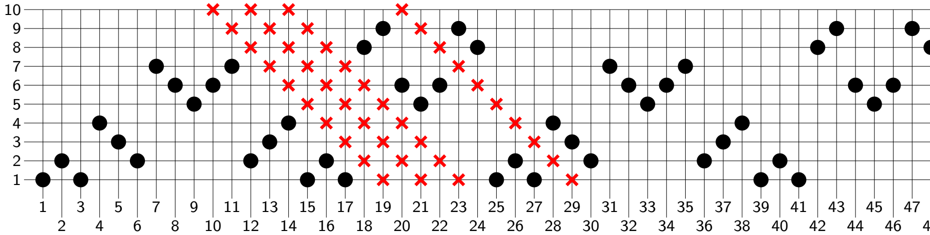

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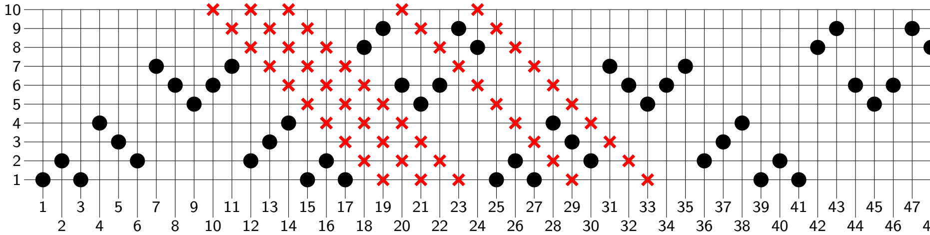

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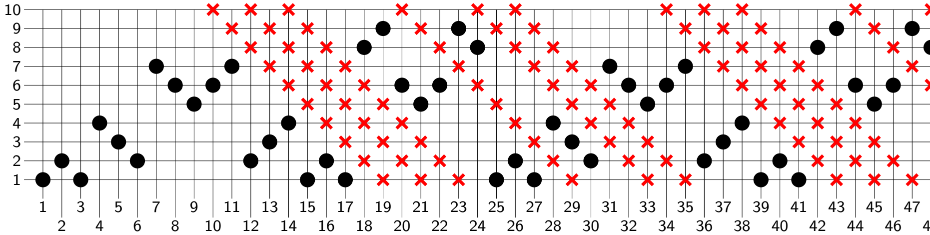

Positive results

Theorem (Narins, P., and Szabó)

There is a number N_0 such that the following holds. Let T be a binary tree, such that $|T| \geq N_0$ and all the leaves of T are in the same class of the bipartition of T. Then T contains leaf-leaf paths of lengths $0, 2, 4, \ldots, 18$.

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Theorem (Narins, P., and Szabó)

Let G be a graph with n vertices, $2n-2$ edges and no (not necessarily induced) subgraphs with minimum degree 3. Then G contains cycles of lengths $3, 4, 5, \ldots, n$.

Open problems

Problem

Is there an increasing function $C(n)$ such that the following holds. Every graph G with n vertices, $2n-2$ edges, and no proper induced subgraphs of minimal degree 3 contains cycles of lengths $4, 6, 8, \ldots, 2C(n)$.

Conjecture

There is a constant $\alpha > 0$, and an increasing function $C(n)$ such that the following holds. Every binary tree T of order n contains at least $\alpha C(n)$ of distinct leaf-leaf path lengths between 0 and $C(n)$.

Open problems

Conjecture

Let G be a graph with n vertices, $2n - 2$ edges, and having no proper induced subgraphs of minimal degree 3. Then there are cycles in G of at least $\log_2 n$ distinct lengths.

Conjecture

Let T be a binary tree. Then there are are leaf-leaf paths in T of at least $\log_2 n$ distinct lengths.