HW1 - EL2700

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Part I

Deriving the simplified system

Initially, with the state variables being $x_1 = x, x_2 = \dot{x}, x_3 = \theta, x_4 = \dot{\theta}$, the system is described by the following state-space equations:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1(x_2, x_3, x_4, F, w) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2(x_2, x_4, x_4, F, w)
\end{aligned}$$

Now, we ignore the dynamics of the cart and focus only on the dynamic of the inverted pendulum. To do that we set $x_1 = x_2 = w = 0$ in the equation above. Furthermore we redefine the notation of the states to be $x_1^* = \theta, x_2^* = \dot{\theta}$ and $u = F/M$. With the assumption that $M \gg m$ we get that every term containing $M + m$ becomes only M, ie. $M + m \approx M$, $\frac{m}{M+m} \approx \frac{1}{M}$ and $\frac{m^2}{M+m} \approx 0$.

The new systems becomes

$$\begin{aligned}
\dot{x}_1^* &= x_2^* \\
\dot{x}_2^* &= f_2^*(x_1^*, x_2^*, F)
\end{aligned}$$

Where f_2^* is a simplified version of the original non-linear function f_2 with the simplification done according to the aforementioned assumptions. Now we show how to get from f_2 to f_2^*.

$$\begin{aligned}
f_2(x_2, x_3, x_4, F, w) &= \frac{1}{I + ml^2 - \frac{m^2l^2cos^2(x_3)}{M+m}}\Bigg(\frac{mlcos(x_3)}{M+m}F + \frac{Mlcos(x_3)}{M+m}w - b_p x_4 + \\
&+ \quad mglsin(x_3) - \frac{m^2l^2x_4^2sin(x_3)cos(x_3)}{M+m} - \frac{mlb_c x_2 cos(x_3)}{M+m}\Bigg)
\end{aligned}$$

With $x_1 = x_2 = w = 0$ and M this becomes:

$$f_2(x_2, x_3, x_4, F, w) = \frac{1}{I + ml^2}\left(mlcos(x_3)\frac{F}{M} - b_p x_4 + mglsin(x_3)\right)$$

After changing variables the equation becomes

$$\dot{x}_2^* = f_2^*(x_1^*, x_2^*, F) = \frac{1}{I + ml^2}(mlcos(x_1^*)u - b_p x_2^* + mglsin(x_1^*))$$

Now we ommit the asterix (*) in the notation. This finally leads to

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -a_0 sin(x_1) - a_1 x_2\end{bmatrix} + \begin{bmatrix} 0 \\ b_0 u cos(x_1)\end{bmatrix}$$

$$y = x_1$$

where

$$a_0 = -\frac{mgl}{I+ml^2}, a_1 = \frac{b}{I+ml^2}, b_0 = \frac{ml}{I+ml^2}$$

Linearized Model

To check for equilibrium we set $\dot{x}_1 = \dot{x}_2 = 0$ and $u = 0$. This gives us

$$\begin{aligned} x_2 &= 0 \\ a_0 sin(x_1) &= 0 \implies x_1 = i\pi \quad \forall i \in \mathbf{Z} \end{aligned}$$

So, the upright pendulum position at rest, $x_{ref} = [0, 0]^T$ is indeed an equilibrium point.

We can linearize the system using Taylor expansion around the points $x_{ref} = u_{ref} = 0$. We have:

$$\begin{aligned} \Delta\dot{x}_1 &= \frac{\partial \dot{x}_1}{\partial x_1}\bigg|_{x_{ref}=0,u=0} \Delta x_1 + \frac{\partial \dot{x}_1}{\partial x_2}\bigg|_{x_{ref}=0,u=0} \Delta x_2 + \frac{\partial \dot{x}_1}{\partial u}\bigg|_{x_{ref}=0,u=0} \Delta u \\ &= 0 + \Delta x_2 + 0 \\ \Delta\dot{x}_2 &= \frac{\partial \dot{x}_2}{\partial x_1}\bigg|_{x_{ref}=0,u=0} \Delta x_1 + \frac{\partial \dot{x}_2}{\partial x_2}\bigg|_{x_{ref}=0,u=0} \Delta x_2 + \frac{\partial \dot{x}_2}{\partial u}\bigg|_{x_{ref}=0,u=0} \Delta u \\ &= -a_0 cos(0)\Delta x_1 - a_1 \Delta x_2 + b_0 cos(0) \Delta u \\ &= -a_0 \Delta x_1 - a_1 \Delta x_2 + b_0 \Delta u \end{aligned}$$

Thus

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}}_{A_c} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ b_0 \end{bmatrix}}_{B_c} \Delta u$$

$$\Delta y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C_c} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

with $\Delta\mathbf{x(t)} = \mathbf{x(t)} - \mathbf{x}_{ref}$, $\Delta u(t) = u(t) - u_{ref}$ and $\Delta y(t) = y(t) - C_c \mathbf{x}_{ref}$.

Discretization

We know that the continuous-time linear system can be converted to a discrete-time system using zero-order-hold, where $u(t) = u_k$ for $t \in [kh, kh + h)$. We get

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned}$$

with $A = e^{A_c h}$ and $B = \int_{s=0}^{h} e^{A_c s} B_c ds$, where A_c and B_c are the continuous-time matrices and h the sampling time. We calculate the matrix exponential using the Laplace-transform.

$$\begin{aligned} A &= e^{A_c h} \\ &= \mathcal{L}^{-1}((sI - Ac)^{-1}) \\ (sI - Ac)^{-1} &= \begin{bmatrix} \frac{s+a_1}{s^2+a_1 s+a_0} & \frac{1}{s^2+a_1 s+a_0} \\ \frac{-a_0}{s^2+a_1 s+a_0} & \frac{s}{s^2+a_1 s+a_0} \end{bmatrix} \end{aligned}$$

first element

$$Q1 = \mathcal{L}^{-1}(\frac{s+a_1}{s^2+a_1 s+a_0}) = \mathcal{L}^{-1}(\frac{s - \frac{a_1}{2} + 1.5a_1}{(s - \frac{a_1}{2})^2 + a_0 - \frac{a_1^2}{4}})$$

From Laplace table

$$e^{at}sin(bt) = \frac{b}{(s-a)^2+b^2}$$

and

$$e^{at}cos(bt) = \frac{s-a}{(s-a)^2+b^2}$$

Using these

$$Q1(h) = e^{\frac{a_1}{2}h}cos(Kh) + \frac{1.5a_1}{K}e^{\frac{a_1}{2}h}sin(Kh)$$

where

$$K = \sqrt{a_0 - \frac{a_1^2}{4}}$$

Similarly

$$A = \begin{bmatrix} e^{\frac{a_1}{2}h}cos(Kh) + \frac{1.5a_1}{K}e^{\frac{a_1}{2}h}sin(Kh) & \frac{1}{K}e^{\frac{a_1}{2}h}sin(Kh) \\ \frac{-a_0}{K}e^{\frac{a_1}{2}h}sin(Kh) & e^{\frac{a_1}{2}h}cos(Kh) + \frac{a_1}{2K}e^{\frac{a_1}{2}h}sin(Kh) \end{bmatrix}$$

The B matrix can be found using

$$\begin{aligned} B &= \int_0^h e^{A_c s}Bds \\ &= \int_0^h \begin{bmatrix} Q1(s) & Q2(s) \\ Q3(s) & Q4(s) \end{bmatrix} \begin{bmatrix} 0 \\ b_0 \end{bmatrix} ds \\ &= \int_0^h \begin{bmatrix} Q2(s) * b_0 \\ Q4(s) * b_0 \end{bmatrix} \end{aligned}$$

We get

$$\begin{bmatrix} Q2(s) * b_0 \\ Q4(s) * b_0 \end{bmatrix} = \begin{bmatrix} \frac{b_0}{K}e^{0.5a_1s}sin(Ks) \\ b_0e^{0.5a_1s}cos(Ks) + \frac{a_1b_0}{2K}e^{0.5a_1s}sin(Ks) \end{bmatrix}$$

We start by performing the integral using integration by parts

$$\int_0^h \frac{b_0}{K}e^{0.5a_1s}sin(Ks)ds \quad = \quad -e^{0.5a_1s}\frac{cos(Ks)}{K}\Big|_0^h + \int_0^h \frac{a_1}{2}e^{0.5a_1s}\frac{cos(Ks)}{K}ds$$

This leads to

$$\int_0^h e^{0.5a_1s}sin(Ks) = \frac{1}{1+\frac{a_1^2K^4}{4}}[-e^{0.5a_1s}\frac{cos(Ks)}{K} + \frac{a_1}{2K}e^{0.5a_1s}sin(Ks)]\Big|_0^h$$

Finally we get

$$B = \begin{bmatrix} \frac{b_0}{K^2(1+\frac{a_1^2K^2}{4})}\left[-e^{0.5a_1s}cos(Kh) + \frac{a_1}{2K}e^{0.5a_1s}sin(Kh) + 1\right] \\ \frac{b_0}{K(1+\frac{a^2K^2}{4})}\left[e^{0.5a_1s}sin(Kh) + \frac{a_1}{2K}e^{0.5a_1s}cos(Kh) - \frac{a_1}{2K}\right] + ... \\ ...\frac{a_1b_0}{2K^2(1+\frac{a_1^2K^2}{4})}\left[-e^{0.5a_1s}cos(Kh) + \frac{a_1}{2K}e^{0.5a_1s}sin(Kh) + 1\right] \end{bmatrix}$$

$$C = C_c = [1 \quad 0]$$

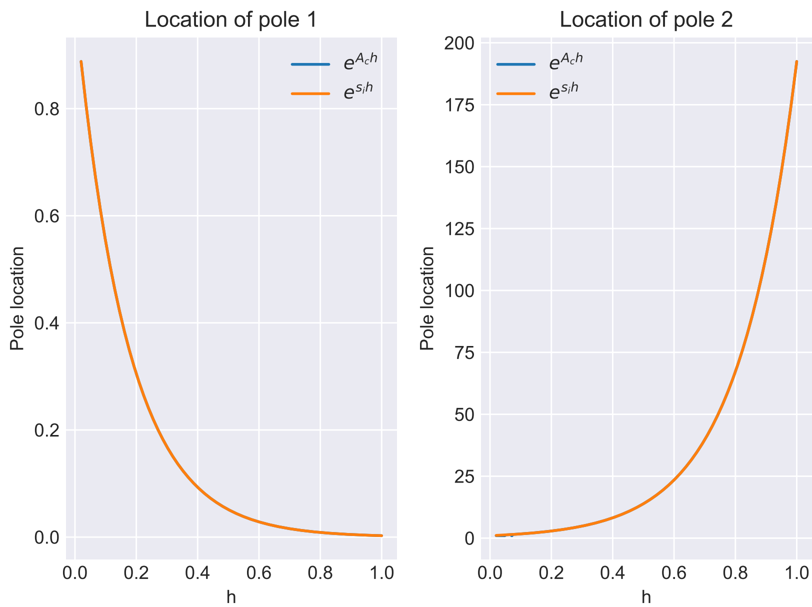

Figure 1: Pole location as a function of the sampling time for two of the system poles.

Sampling effect on pole location

In figure 1 we see a comparison between poles calculated using the eigenvalues of the discrete-time system matrix and the ones calculated using the relation $z_i = e^{s_i h}$, where z_i are the discrete-time poles and s_i the poles of the continuous-time poles. As we can see both methods yield matching pole locations, which verifies the given formula for calculating the poles of the discrete-time system by using the continuous-time system. On the left subplot we see the effect of a negative pole in the continuous time system. It is a stable pole in both representations of the system, whereas the right subplot shows an unstable pole in the continuous-time that also unstable in the discrete-time representation.

Part II

In this part we simulated the state feedback response of the system.
We used the feedback of the form

$$u_k = -Lx_k + l_r r_k$$

Where

$$l_r = \frac{1}{C(I - A + BL)^{-1}B}$$

Practically we should choose the sampling time such that the rise time has 4-10 samples. That corresponds to sampling time of between $1 - 2.5s$, if we consider the desired rise time to be of 10s. To be on the safe side we selected a faster sampling time of $0.5s$.

We then tuned the poles of the closed loop system so that the cart moves 90% of the distance between the initial position and the reference within 10 seconds, keeping the pendulum within $\pm 10^\circ$ around the upright position.

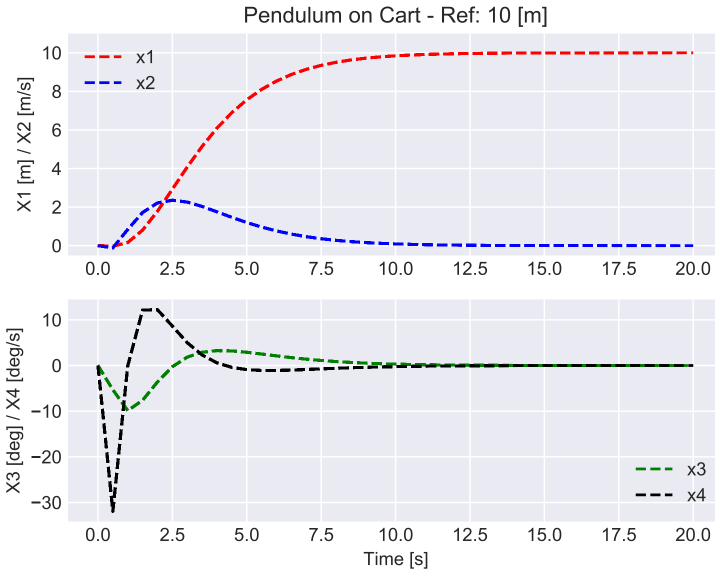

Figure 2: Linearized system response

The poles where set to

$$\begin{aligned} p_1 &= 0.3835 \\ p_2 &= 0.1718 \\ p_3 &= 0.7107 + 0.0103i \\ p_4 &= 0.7107 - 0.0103i \end{aligned}$$

The simulation of the linearized system response is shown in figure 2. We can see that the requirements of the rise time and percentage overshoot are met.

Next, we added a disturbance of $w = 0.01$. The simulation of the non-linear system response with disturbance is shown in figure 3.

We can observe that, the disturbance causes a steady state error at the output. But the systems remains stable and achieves a similar transient response to the undisturbed system.

At steady-state the error can be derived from

$$\begin{aligned} x_{ref} &= Ax_{ref} + Bu_{ref} + B_w w \\ &= Ax_{ref} + B(-Lx_{ref} + l_r r_{ref}) + B_w w \end{aligned}$$

where the *ref* subscript refers to the corresponding values at steady-state. This becomes

$$y_{ref} = Cx_{ref} = C(-A + BL)^{-1}(Bl_r r_{ref} + B_{ref} w) \iff$$
$$C(I - A + BL)^{-1}Bl_r r_{ref} - y_{ref} = -C(I - A + BL)^{-1}B_w w$$

since, $l_r = [C(I - A + BL)^{-1}B]^{-1}$. Hence we get that

$$\tilde{y}_{ss} = lim_{k\to\infty} r_k - y_k = r_{ref} - y_{ref} = -C(I - A + BL)^{-1}B_w w$$

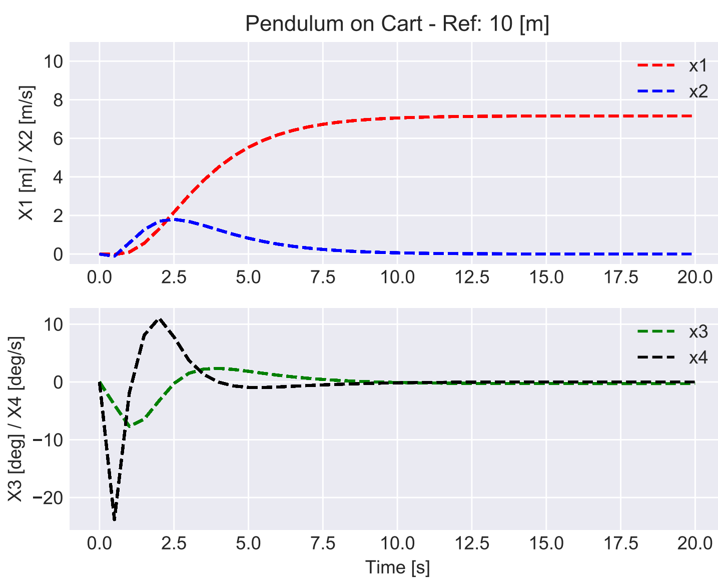

Figure 3: Performance in the presence of disturbance input