

Learning Model Predictive Control for Iterative Tasks: A Computationally Efficient Approach for Linear System

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Abstract: A Learning Model Predictive Controller (LMPC) for linear system is presented. The proposed controller builds on previous work on nonlinear LMPC and decreases its computational burden for linear system. The control scheme is reference-free and is able to improve its performance by learning from previous iterations. A convex safe set and a terminal cost function are used in order to guarantee recursive feasibility and non-increasing performance at each iteration. The paper presents the control design approach, and shows how to recursively construct the convex terminal set and the terminal cost from state and input trajectories of previous iterations. Simulation results show the effectiveness of the proposed control logic.

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1. INTRODUCTION

Iterative Learning Control (ILC) studies control design for autonomous systems performing repetitive tasks Bristow et al. (2006); Lee and Lee (2007); Wang et al. (2009). One task execution is often referred to as “iteration” or “trial”. In ILC, at each iteration, the system starts from the same initial condition and the controller objective is to track a given reference, rejecting periodic disturbances Bristow et al. (2006); Lee and Lee (2007). The tracking error from the previous iterations is used to improve the tracking performance of the closed loop system. Different strategies have been proposed to guarantee zero tracking error of the closed loop system Bristow et al. (2006); Lee and Lee (2007); Wang et al. (2009).

Several control frameworks which combine ILC and MPC strategies have been proposed in literature, Subbaraman and Benosman (2016); Lee and Lee (2000); Lee et al. (2000). In the classical ILC approach the goal of the controller is to track a reference trajectory, however, in some application such as autonomous racing Sharp and Peng (2011); Rucco et al. (2015) or for some manipulation tasks Tamar et al. (2016), it may be challenging to generate a priori a reference trajectory that maximize the system performance. For this reason, a very recent work Tamar et al. (2016) proposed a reference-free ILC scheme. The authors used a MPC controller with a terminal cost that allows to consider the long term planning. This terminal cost is computed using a neural network trained on data generated by offline simulations. The authors were able to improve the system performance over iterations. However, no guaranties about stability, recursive feasibility and performance improvement are provided.

Our objective is to design a reference-free iterative control strategy for linear system able to learn from previous iterations. At each iteration, the initial condition, the

constraints and the objective function do not change. The j -th iteration cost is defined as the objective function evaluated for the realized closed loop system trajectory. The iteration cost shall not increase over the iterations and state and input constraints shall be satisfied. Model Predictive Control is an appealing technique to tackle this problem for its ability to handle state and inputs constraints while minimizing a finite-time predicted cost Garcia et al. (1989). However, the receding horizon nature can lead to infeasibility and it does not guaranty improved performance at each iteration Mayne et al. (2000).

The contribution of this paper is the following. We present an extension to the learning MPC for iterative control task in Rosolia and Borrelli (2016). In particular, we introduce a new formulation for linear system that drastically reduces the computation burden of the controller without compromising the guaranties of the learning MPC. We show how to design a convex safe set and a terminal cost function in order to guarantee: (i): [asymptotic stability], the closed loop system converges asymptotically to the equilibrium point, (ii): [persistent feasibility], state and input constraints are satisfied if they were satisfied at iterations $j - 1$ (iii): [performance improvement], the j -th iteration cost does not increase compared with the $j-1$ -th iteration cost, (iv): [global optimality], if the steady state system converges to a closed-loop trajectory as the number of iterations j goes to infinity, then that closed-loop trajectory is globally optimal. We emphasize that (i)-(ii) are standard MPC design requirement and (iii)-(iv) are the core contribution of this work.

This paper is organized as follows: in Section II we introduce the notation used throughout the paper. Then, we define the convex safe set and the terminal cost function used in the design of the learning MPC. Section III describes the control design. We show the recursive feasibility and stability of the control logic and, afterwards, we prove

the convergence properties. Finally, in Section IV we test the proposed control logic on an infinite horizon linear quadratic regulator and we compare the computational efficiency with the learning MPC from Rosolia and Borrelli (2016).

2. PROBLEM FORMULATION

Consider the discrete time system

$$x_{t+1} = Ax_t + Bu_t, \quad (1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the system state and input, respectively, subject to the constraints

$$x_t \in \mathcal{X}, \quad u_t \in \mathcal{U}, \quad \forall t \in \mathbb{Z}_{0+}. \quad (2)$$

where \mathcal{X} and \mathcal{U} are convex sets.

At the j -th iteration the vectors

$$\mathbf{u}^j = [u_0^j, u_1^j, \dots, u_t^j, \dots], \quad (3a)$$

$$\mathbf{x}^j = [x_0^j, x_1^j, \dots, x_t^j, \dots], \quad (3b)$$

collect the inputs applied to system (1) and the corresponding state evolution. In (3), x_t^j and u_t^j denote the system state and the control input at time t of the j -th iteration, respectively. We assume that at each j -th iteration the closed loop trajectories start from the same initial state,

$$x_0^j = x_S, \quad \forall j \geq 0. \quad (4)$$

The goal is to design a controller which solves the following infinite horizon optimal control problem at each iteration:

$$J_{0 \rightarrow \infty}^*(x_S) = \min_{u_0, u_1, \dots} \sum_{k=0}^{\infty} h(x_k, u_k) \quad (5a)$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k, \quad \forall k \geq 0 \quad (5b)$$

$$x_0 = x_S, \quad (5c)$$

$$x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad \forall k \geq 0 \quad (5d)$$

where equations (5b) and (5c) represent the system dynamics and the initial condition, and (5d) are the state and input constraints. The stage cost, $h(\cdot, \cdot)$, in equation (5a) is continuous, jointly convex and it satisfies

$$h(x_F, 0) = 0 \text{ and } h(x_t^j, u_t^j) \succ 0 \quad \forall x_t^j \in \mathbb{R}^n \setminus \{x_F\}, \\ u_t^j \in \mathbb{R}^m \setminus \{0\}, \quad (6)$$

where the final state x_F is assumed to be a feasible equilibrium for the unforced system (1)

$$x_F = Ax_F. \quad (7)$$

Next we introduce the definition of the convex safe set and of the terminal cost. Both will be used later to guarantee stability and feasibility of the learning MPC for linear system.

2.1 Convex Safe Set

In the following we recall the definition of the sampled Safe Set from Rosolia and Borrelli (2016) which is necessary to construct the convex Safe Set used in the learning MPC for linear system.

The definition of the *sampled Safe Set* exploits the iterative nature of the control task to define an invariant control set,

using the realized system trajectories. At the j -th iteration the sampled safe set, \mathcal{SS}^j , is defined as

$$\mathcal{SS}^j = \left\{ \bigcup_{i \in M^j} \bigcup_{t=0}^{\infty} x_t^i \right\}. \quad (8)$$

\mathcal{SS}^j is the collection of all state trajectories at iteration i for $i \in M^j$. M^j in equation (8) is the set of indexes k associated with successful iterations k for $k \leq j$, defined as:

$$M^j = \left\{ k \in [0, j] : \lim_{t \rightarrow \infty} x_t^k = x_F \right\}. \quad (9)$$

Moreover, as \mathcal{X} and \mathcal{U} are convex, for each convex combination of the elements in \mathcal{SS}^j we can find a control sequence that steers the system (1) to x_F . Therefore, the *convex Safe Set*, defined as

$$\mathcal{CS}^j = \text{Conv}(\mathcal{SS}^j) = \left\{ \sum_{i=1}^{|\mathcal{SS}^j|} \alpha_i z_i : \alpha_i \geq 0, \sum_{i=1}^{|\mathcal{SS}^j|} \alpha_i = 1, \right. \\ \left. z_i \in \mathcal{SS}^j \right\}, \quad (10)$$

is a control invariant set. Note that $|\mathcal{SS}^j|$ is the cardinality of \mathcal{SS}^j . For further details on control invariant set we refer to Borrelli (2003).

From (9) we have that $M^i \subseteq M^j, \forall i \leq j$, which implies that

$$\mathcal{CS}^i \subseteq \mathcal{CS}^j, \quad \forall i \leq j. \quad (11)$$

2.2 Terminal Cost

At time t of the j -th iteration the cost-to-go associated with the closed loop trajectory (3b) and input sequence (3a) is defined as

$$J_{t \rightarrow \infty}^j(x_t^j) = \sum_{k=0}^{\infty} h(x_{t+k}^j, u_{t+k}^j), \quad (12)$$

where $h(\cdot, \cdot)$ is the stage cost of problem (5). We define the j -th iteration cost as the cost (12) of the j -th trajectory at time $t = 0$,

$$J_{0 \rightarrow \infty}^j(x_0^j) = \sum_{k=0}^{\infty} h(x_k^j, u_k^j). \quad (13)$$

$J_{0 \rightarrow \infty}^j(x_0^j)$ quantifies the controller performance at each j -th iteration.

Remark 1. In equations (12)-(13), x_k^j and u_k^j are the realized state and input at the j -th iteration, as defined in (3).

Finally we define the, barycentric function (Jones and Morari (2010))

$$P^j(x) = \begin{cases} p^{j,*}(x) & \text{If } x \in \mathcal{CS}^j \\ +\infty & \text{If } x \notin \mathcal{CS}^j \end{cases} \quad (14)$$

where

$$p^{j,*}(x) = \min_{\lambda_t \geq 0, \forall t \in [0, \infty)} \sum_{k=0}^j \sum_{t=0}^{\infty} \lambda_t^k J_{t \rightarrow \infty}^k(x_t^k) \quad (15a)$$

s.t.

$$\sum_{k=0}^j \sum_{t=0}^{\infty} \lambda_t^k = 1 \quad (15b)$$

$$\sum_{k=0}^j \sum_{t=0}^{\infty} \lambda_t^k x_t^k = x, \quad (15c)$$

where x_t^k is the realized state at time t of the j -th iteration, as defined in (3b).

Remark 2. The function $P^j(x)$ assigns to every point in \mathcal{CS}^j the minimum cost-to-go along the trajectories in \mathcal{CS}^j , in particular we have that $\forall x \in \mathcal{CS}^j$,

$$\begin{aligned} P^j(x) &= \sum_{k=0}^j \sum_{t=0}^{\infty} \lambda_t^{*,k} J_{t \rightarrow \infty}^k(x_t^k) = \\ &= \sum_{k=0}^j \sum_{t=0}^{\infty} \lambda_t^{*,k} \sum_{l=0}^{\infty} h(x_{t+l}^k, u_{t+l}^k) \end{aligned} \quad (16)$$

where $\lambda_t^{*,k}$ is the minimizer in (15).

Remark 3. In practical applications each j -th iteration has a finite time duration t_j , and therefore $p^{j,*}(x)$ is reformulated as

$$p^{j,*}(x) = \min_{\lambda_t \geq 0, \forall t \in [0, \infty)} \sum_{k=0}^j \sum_{t=0}^{t_j} \lambda_t^k J_{t \rightarrow \infty}^k(x_t^k) \quad (17a)$$

s.t.

$$\sum_{k=0}^j \sum_{t=0}^{t_j} \lambda_t^k = 1 \quad (17b)$$

$$\sum_{k=0}^j \sum_{t=0}^{t_j} \lambda_t^k x_t^k = x. \quad (17c)$$

3. LMPC FOR LINEAR SYSTEM

In this section we present the design of the proposed Learning Model Predictive Control (LMPC). We first assume that there exists a feasible input sequence that steers the system from the initial point x_S to terminal point x_F at the 0-th iteration. Then we prove that the proposed LMPC is guaranteed to be recursively feasible, i.e. feasible at all time instants of every successive iteration. Moreover, we show that the LMPC guaranties a non-increasing iterations cost between two successive executions of the task.

3.1 LMPC Control Design

The LMPC tries to compute a solution to the infinite time optimal control problem (5) by solving at time t of iteration j the finite time constrained optimal control problem

$$J_{t \rightarrow t+N}^{\text{LMPC},j}(x_t^j) = \min_{u_{t|t}, \dots, u_{t+N-1|t}} \left[\sum_{k=t}^{t+N-1} h(x_{k|t}, u_{k|t}) + P^{j-1}(x_{t+N|t}) \right] \quad (18a)$$

s.t.

$$x_{k+1|t} = Ax_{k|t} + Bu_{k|t}, \quad \forall k \in [t, \dots, t+N-1] \quad (18b)$$

$$x_{t|t} = x_t^j, \quad (18c)$$

$$x_{k|t} \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad \forall k \in [t, \dots, t+N-1] \quad (18d)$$

$$x_{t+N|t} \in \mathcal{CS}^{j-1}, \quad (18e)$$

where (18b) and (18c) represent the system dynamics and initial condition, respectively. The state and input constraints are given by (18d). Finally (18e) forces the terminal state into the set \mathcal{CS}^{j-1} defined in equation (10). Let

$$\begin{aligned} \mathbf{u}_{t:t+N|t}^{*,j} &= [u_{t|t}^{*,j}, \dots, u_{t+N-1|t}^{*,j}] \\ \mathbf{x}_{t:t+N|t}^{*,j} &= [x_{t|t}^{*,j}, \dots, x_{t+N|t}^{*,j}] \end{aligned} \quad (19)$$

be the optimal solution of (18) at time t of the j -th iteration and $J_{t \rightarrow t+N}^{\text{LMPC},j}(x_t^j)$ the corresponding optimal cost. Then, at time t of the iteration j , the first element of $\mathbf{u}_{t:t+N|t}^{*,j}$ is applied to the system (1)

$$u_t^j = u_{t|t}^{*,j}. \quad (20)$$

The finite time optimal control problem (18) is repeated at time $t+1$, based on the new state $x_{t+1|t+1} = x_{t+1}^j$ (18c), yielding a *moving or receding horizon* control strategy.

Remark 4. Problem (18) is a convex optimization problem as the terminal constraint (18a) enforces the terminal state in the convex set \mathcal{CS}^{j-1} and the terminal cost $P^{j-1}(\cdot)$ in (18e) is a convex function. This new formulation of the LMPC (18), (20) as a convex problem is the main contribution of this work compared to Rosolia and Borrelli (2016).

Assumption 1. At iteration $j = 1$ we assume that $\mathcal{CS}^{j-1} = \mathcal{CS}^0$ is a non-empty set and that the trajectory $\mathbf{x}^0 \in \mathcal{CS}^0$ is feasible and convergent to x_F .

In the next section we prove that, under Assumption 1, the LMPC (18) and (20) in closed loop with system (1) guarantees recursively feasibility and stability, and non-increase of the iteration cost at each iteration.

3.2 Recursive feasibility and stability

In this Section, the properties of \mathcal{CS}^j and $P^j(\cdot)$ are used to show recursive feasibility and asymptotic stability of the equilibrium point x_F .

Theorem 1. Consider system (1) controlled by the LMPC controller (18) and (20). Let \mathcal{CS}^j be the convex safe set at iteration j as defined in (10). Let assumption 1 hold, then the LMPC (18) and (20) is feasible $\forall t \in \mathbb{Z}_{0+}$ and iteration $j \geq 1$. Moreover, the equilibrium point x_F is asymptotically stable for the closed loop system (1) and (20) at every iteration $j \geq 1$.

Proof: The proof follows from standard MPC arguments. By assumption \mathcal{CS}^0 is non empty. From (11) we have that

$\mathcal{CS}^0 \subseteq \mathcal{CS}^{j-1} \forall j \geq 1$, and consequently \mathcal{CS}^{j-1} is a non empty set. In particular, there exists a trajectory $\mathbf{x}^0 \in \mathcal{CS}^0 \subseteq \mathcal{CS}^{j-1}$. From (4) we know that $x_0^j = x_s \forall j \geq 0$. At time $t = 0$ of the j -th iteration the N steps trajectory

$$[x_0^0, x_1^0, \dots, x_N^0] \in \mathcal{CS}^{j-1}, \quad (21)$$

and the related input sequence,

$$[u_0^0, u_1^0, \dots, u_{N-1}^0], \quad (22)$$

satisfy input and state constraints (18b)-(18c)-(18d). Therefore (21)-(22) is a feasible solution to the LMPC (18) and (20) at $t = 0$ of the j -th iteration.

Assume that at time t of the j -th iteration the LMPC (18) and (20) is feasible and let $\mathbf{x}_{t:t+N|t}^{*,j}$ and $\mathbf{u}_{t:t+N|t}^{*,j}$ be the optimal trajectory and input sequence, as defined in (19). From (18c) and (20) the realized state and input at time t of the j -th iteration are given by

$$\begin{aligned} x_t^j &= x_{t|t}^{*,j}, \\ u_t^j &= u_{t|t}^{*,j}. \end{aligned} \quad (23)$$

Moreover, the terminal constraint (18e) enforces $x_{t+N|t}^{*,j} \in \mathcal{CS}^{j-1}$ and, from (15) and (18a),

$$x_{t+N|t}^{*,j} = \sum_{k=0}^{j-1} \sum_{t=0}^{\infty} \lambda_t^{*,k} x_t^k. \quad (24)$$

We define

$$\bar{u} = \sum_{k=0}^{j-1} \sum_{t=0}^{\infty} \lambda_t^{*,k} u_t^k, \in \mathcal{U}, \quad (25)$$

and

$$\begin{aligned} \bar{x} &= Ax_{t+N|t}^{*,j} + B\bar{u} = \sum_{k=0}^{j-1} \sum_{t=0}^{\infty} \lambda_t^{*,k} (Ax_t^k + Bu_t^k) = \\ &= \sum_{k=0}^{j-1} \sum_{t=0}^{\infty} \lambda_t^{*,k} x_{t+1}^k \in \mathcal{CS}^{j-1}. \end{aligned} \quad (26)$$

Since the state update in (1) and (18b) are assumed identical we have that

$$x_{t+1}^j = x_{t+1|t}^{*,j}. \quad (27)$$

At time $t + 1$ of the j -th iteration the input sequence and the related feasible state trajectory

$$[u_{t+1|t}^{*,j}, u_{t+2|t}^{*,j}, \dots, u_{t+N-1|t}^{*,j}, \bar{u}], \quad (28a)$$

$$[x_{t+1|t}^{*,j}, x_{t+2|t}^{*,j}, \dots, x_{t+N-1|t}^{*,j}, x_{t+N|t}^{*,j}, \bar{x}] \quad (28b)$$

satisfy input and state constraints (18b)-(18c)-(18d). Therefore, (28) is a feasible solution for the LMPC (18) and (20) at time $t + 1$.

We showed that at the j -th iteration, $\forall j \geq 1$, (i): the LMPC is feasible at time $t = 0$ and (ii): if the LMPC is feasible at time t , then the LMPC is feasible at time $t + 1$. Thus, we conclude by induction that the LMPC in (18) and (20) is feasible $\forall j \geq 1$ and $t \in \mathbb{Z}_{0+}$.

Next we use the fact the Problem (18) is time-invariant at each iteration j and we replace $J_{t \rightarrow t+N}^{\text{LMPC},j}(\cdot)$ with $J_{0 \rightarrow N}^{\text{LMPC},j}(\cdot)$. In order to show the asymptotic stability of x_F we have to show that the optimal cost, $J_{0 \rightarrow N}^{\text{LMPC},j}(\cdot)$, is a Lyapunov function for the equilibrium point x_F (7) of the closed loop system (1) and (20) Borrelli (2003). Continuity of $J_{0 \rightarrow N}^{\text{LMPC},j}(\cdot)$ can be shown as in Mayne et al. (2000). Moreover from (5a), $J_{0 \rightarrow N}^{\text{LMPC},j}(x) \succ 0 \forall x \in \mathbb{R}^n \setminus \{x_F\}$ and $J_{0 \rightarrow N}^{\text{LMPC},j}(x_F) = 0$.

Thus, we need to show that $J_{0 \rightarrow N}^{\text{LMPC},j}(\cdot)$ is decreasing along the closed loop trajectory.

From (27) we have $x_{t+1|t}^{*,j} = x_{t+1}^j$, which implies that

$$J_{0 \rightarrow N}^{\text{LMPC},j}(x_{t+1|t}^{*,j}) = J_{0 \rightarrow N}^{\text{LMPC},j}(x_{t+1}^j). \quad (29)$$

Given the optimal input sequence and the related optimal trajectory in (19) and the definition of the $P^{j-1}(\cdot)$ (16), the optimal cost is given by

$$\begin{aligned} J_{0 \rightarrow N}^{\text{LMPC},j}(x_t^j) &= \min_{u_{t|t}, \dots, u_{t+N-1|t}} \left[\sum_{k=0}^{N-1} h(x_{k|t}, u_{k|t}) + \right. \\ &\quad \left. + P^{j-1}(x_{N|t}) \right] = \\ &= h(x_{t|t}^{*,j}, u_{t|t}^{*,j}) + \sum_{k=1}^{N-1} h(x_{t+k|t}^{*,j}, u_{t+k|t}^{*,j}) + P^{j-1}(x_{t+N|t}^{*,j}) = \\ &= h(x_{t|t}^{*,j}, u_{t|t}^{*,j}) + \sum_{k=1}^{N-1} h(x_{t+k|t}^{*,j}, u_{t+k|t}^{*,j}) + \\ &\quad + \sum_{k=0}^{j-1} \sum_{t=0}^{\infty} \lambda_t^{*,k} \sum_{l=0}^{\infty} h(x_{t+l}^k, u_{t+l}^k). \end{aligned} \quad (30)$$

We can further simplify the above expression using (15c), (24)-(26) and the fact that $h(\cdot, \cdot)$ is jointly convex in the arguments,

$$\begin{aligned} J_{0 \rightarrow N}^{\text{LMPC},j}(x_t^j) &= h(x_{t|t}^{*,j}, u_{t|t}^{*,j}) + \sum_{k=1}^{N-1} h(x_{t+k|t}^{*,j}, u_{t+k|t}^{*,j}) + \\ &+ \sum_{k=0}^{j-1} \sum_{t=0}^{\infty} \lambda_t^{*,k} h(x_t^k, u_t^k) + \sum_{k=0}^{j-1} \sum_{t=0}^{\infty} \lambda_t^{*,k} \sum_{l=1}^{\infty} h(x_{t+l}^k, u_{t+l}^k) \\ &\geq h(x_{t|t}^{*,j}, u_{t|t}^{*,j}) + \sum_{k=1}^{N-1} h(x_{t+k|t}^{*,j}, u_{t+k|t}^{*,j}) + \\ &+ h\left(\sum_{k=0}^{j-1} \sum_{t=0}^{\infty} \lambda_t^{*,k} x_t^k, \sum_{k=0}^{j-1} \sum_{t=0}^{\infty} \lambda_t^{*,k} u_t^k\right) + \\ &\quad + \sum_{k=0}^{j-1} \sum_{t=0}^{\infty} \lambda_t^{*,k} J_{t \rightarrow \infty}^k(x_{t+1}^k) \geq \\ &\geq h(x_{t|t}^{*,j}, u_{t|t}^{*,j}) + \sum_{k=1}^{N-1} h(x_{t+k|t}^{*,j}, u_{t+k|t}^{*,j}) + h(x_{t+N|t}^{*,j}, \bar{u}) + \\ &\quad + P^{j-1}(\bar{x}) \geq \\ &\geq h(x_{t|t}^{*,j}, u_{t|t}^{*,j}) + J_{0 \rightarrow N}^{\text{LMPC},j}(x_{t+1|t}^{*,j}). \end{aligned} \quad (31)$$

Note that, in the above derivation, we used the fact that $\lambda_t^{*,k}, \forall k \in \{0, j-1\}, t \in \mathbb{Z}_{0+}$ is a feasible solution to problem (15) and therefore $\sum_{t=0}^{\infty} \lambda_t^{*,k} J_{t \rightarrow \infty}^k(x_{t+1}^k)$ is a lower bound for $P^{j-1}(\bar{x})$. Finally, from equations (20), (23) and (29)-(31) we conclude that the optimal cost is a decreasing Lyapunov function along the closed loop trajectory,

$$\begin{aligned} J_{0 \rightarrow N}^{\text{LMPC},j}(x_{t+1}^j) - J_{0 \rightarrow N}^{\text{LMPC},j}(x_t^j) &\leq -h(x_t^j, u_t^j) < 0, \\ &\quad \forall x_t^j \in \mathbb{R}^n \setminus \{x_F\} \end{aligned} \quad (32)$$

Equation (32), the positive definiteness of $h(\cdot, \cdot)$ and the continuity of $J_{0 \rightarrow N}^{\text{LMPC},j}(\cdot)$ imply that x_F is asymptotically stable. ■

3.3 Convergence properties

In this Section we assume that the LMPC (18) and (20) converges to a steady state trajectory. We show two results. First, the j -th iteration cost $J_{0 \rightarrow \infty}^j(\cdot)$ does not worsen as j increases. Second, the steady state trajectory is a local optimal solution of the infinite horizon control problem (5).

Theorem 2. Consider system (1) in closed loop with the LMPC controller (18) and (20). Let \mathcal{CS}^j be the convex safe set at the j -th iteration as defined in (10). Let assumption 1 hold, then the iteration cost $J_{0 \rightarrow \infty}^j(\cdot)$ does not increase with the iteration index j .

Proof: Follows from *Theorem 2* in Rosolia and Borrelli (2016) ■

Theorem 3. Consider system (1) in closed loop with the LMPC controller (18) and (20) with $N > 1$. Let \mathcal{CS}^j be the convex safe set at the j -th iteration as defined in (10). Let assumption 1 hold and assume that the closed loop system (1) and (20) converges to a steady state trajectory \mathbf{x}^∞ , for iteration $j \rightarrow \infty$. Then, the steady state input $\mathbf{u}^\infty = \lim_{j \rightarrow \infty} \mathbf{u}^j$ and the related steady state trajectory $\mathbf{x}^\infty = \lim_{j \rightarrow \infty} \mathbf{x}^j$ is a global optimal solution for the infinite horizon optimal control problem (5), i.e., $\mathbf{x}^\infty = \mathbf{x}^*$ and $\mathbf{u}^\infty = \mathbf{u}^*$.

Proof: Follows from the convexity of (18), (20) and of Problem (5) and *Theorem 3* in Rosolia and Borrelli (2016). ■

4. EXAMPLE: CONSTRAINED LQR CONTROLLER

In this section, we test the proposed LMPC for linear system on the following infinite horizon linear quadratic regulator with constraints (CLQR)

$$J_{0 \rightarrow \infty}^*(x_S) = \min_{u_0, u_1, \dots} \sum_{k=0}^{\infty} \left[\|x_k\|_2^2 + \|u_k\|_2^2 \right] \quad (33a)$$

$$\text{s.t. } x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k, \quad \forall k \geq 0 \quad (33b)$$

$$x_0 = x_S, \quad (33c)$$

$$\begin{bmatrix} -4 \\ -4 \end{bmatrix} \leq x_k \leq \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad \forall k \geq 0 \quad (33d)$$

$$-1 \leq u_k \leq 1 \quad \forall k \geq 0. \quad (33e)$$

In Rosolia and Borrelli (2016) we showed that the LMPC converges to the solution of the infinite horizon control problem (33), whenever we have can compute a feasible trajectory \mathbf{x}^0 . However, the LMPC in Rosolia and Borrelli (2016) is implemented using the sampled Safe Set (8) as a terminal constraint, instead of the proposed convex Safe Set (10). Therefore, also for linear systems, the LMPC presented in Rosolia and Borrelli (2016) involves the solution of a Mixed Integer Programming (MIP) program which is computationally expensive. In the following, we show that the proposed convex formulation of the LMPC for linear systems reduces the computational burden by several order of magnitude and it converges to the solution of the infinite horizon control problem (33).

The LMPC (18), (20) is implemented with the quadratic running cost $h(x_k, u_k) = \|x_k\|_2^2 + \|u_k\|_2^2$, an horizon of \bar{N} steps, and the states and input constraints (33d)-(33e). The LMPC (18) and (20) is reformulated as a Quadratic Programming and it is implemented in YALMIP (Lofberg (2004)) using the solver quadprog. In order to implement the terminal cost (17) we defined the time t_j at which the iterations is completed,

$$t_j = \min \left\{ t \in \mathbb{Z}_{0+} : J_{0 \rightarrow \infty}^{\text{LMPC}, j}(x_t^j) \leq \epsilon \right\}. \quad (34)$$

with $\epsilon = 10^{-8}$.

For $x_S = [-3.95, -0.05]^T$ and $\bar{N} = 4$, the LMPC converges a to steady state solution $\mathbf{x}^\infty = \mathbf{x}^8$, after 8 iterations, with a error of $\gamma = 10^{-10}$:

$$\max_{t \in [0, t_9]} \|\mathbf{x}_t^9 - \mathbf{x}_t^8\|_2 < \gamma. \quad (35)$$

Table 1 shows the evolution of the iterations cost. We notice that accordingly with *Theorem 2* the cost is non-increasing over the iterations.

Table 1. Optimal cost of the LMPC at each j -th iteration

Iteration	Iteration Cost
$j = 0$	57.1959612235
$j = 1$	49.9313760793
$j = 2$	49.9166091658
$j = 3$	49.9163668042
$j = 4$	49.9163602472
$j = 5$	49.9163600537
$j = 6$	49.9163600469
$j = 7$	49.9163600468
$j = 8$	49.9163600464

Furthermore, the solution of the LMPC for linear system is compared with the exact solution of the CLQR (33), which is computed using the algorithm in Borrelli (2003). Given the optimal solution to the infinite horizon optimal control problem (33),

$$\begin{aligned} \mathbf{x}^* &= [x_0^*, x_1^*, \dots, x_t^*, \dots], \\ \mathbf{u}^* &= [u_0^*, u_1^*, \dots, u_t^*, \dots], \end{aligned} \quad (36)$$

we define the approximation error as

$$\sigma_t = \|x_t^\infty - x_t^*\|_2. \quad (37)$$

σ_t quantifies, at each time step t , the distance between the optimal trajectory of the CLQR (33) and steady state trajectory at of the LMPC (18) and (20). The maximum approximation error is

$$\bar{\sigma} = \max[\sigma_0, \dots, \sigma_{t_\infty}] = 8.6 \times 10^{-6}. \quad (38)$$

Moreover, the 2-norm of the normalized difference between the exact optimal cost and the cost associated with the steady state trajectory is

$$\Delta J = \frac{\|J_{0 \rightarrow \infty}^*(x_S) - J_{0 \rightarrow \infty}^*(x_0^\infty)\|_2}{J_{0 \rightarrow \infty}^*(x_S)} \times 100 = 1.8 \times 10^{-20}. \quad (39)$$

The LMPC for linear system (18) and (20) has converged to global optimal solution.

We tested the LMPC (18), (20) with different initial conditions x_S and horizon length $\bar{N} > 1$ to experimentally validate *Theorems 1-3*. Table 2 shows the maximum approximation error, $\bar{\sigma}$, and ΔJ . We underline that for all

the tested scenarios, regardless of the horizon length, the proposed LMPC converged to the global optimal solution of the infinite horizon control problem. It is interesting to notice that the LMPC (18), (20) with a longer horizon \bar{N} has more freedom to explore the state space and therefore it converges faster to the steady state trajectory.

Table 2. Convergence of the LMPC for different initial conditions

x_S	\bar{N}	$\bar{\sigma}$	ΔJ	Iterations
$[-3.95, -0.05]^T$	2	2.6×10^{-1}	1.3×10^{-1}	44
$[-3.95, -0.05]^T$	3	1.9×10^{-5}	1.7×10^{-17}	26
$[-3.95, -0.05]^T$	4	8.6×10^{-6}	1.8×10^{-20}	8
$[-4, 0]^T$	2	3.6×10^{-1}	4.2×10^{-1}	74
$[-4, 0]^T$	3	1.6×10^{-5}	5.9×10^{-18}	26
$[-4, 0]^T$	4	5.2×10^{-6}	1.2×10^{-20}	8
$[-2, 2]^T$	3	7.8×10^{-2}	5.1×10^{-3}	80
$[-2, 2]^T$	3	1.7×10^{-5}	3.4×10^{-17}	22
$[-2, 2]^T$	4	7.3×10^{-6}	5.7×10^{-20}	8
$[0, 1.5]^T$	2	1.0×10^{-1}	1.5×10^{-2}	45
$[0, 1.5]^T$	3	1.8×10^{-5}	3.4×10^{-17}	27
$[0, 1.5]^T$	4	6.6×10^{-6}	2.0×10^{-19}	8

Finally, we compare the computational burden associated with the LMPC (18), (20) and with the LMPC in Rosolia and Borrelli (2016). The proposed LMPC (18), (20) applied to Problem (33) converged in 40s to a steady state trajectory. On the other hand, the LMPC in Rosolia and Borrelli (2016) applied to Problem (33) took 2hr to reach convergence. Therefore, we conclude that the proposed approach significantly reduces the computational burden of the control logic preserving the properties of the LMPC.

5. CONCLUSIONS

In this paper, an extension to the learning Model Predictive Control (LMPC) is presented. The controller is designed for linear system and it significantly reduces the computational burden associated with the LMPC. A convex safe set and a terminal cost, learnt from previous iterations, allow to guarantee the recursive feasibility and stability of the closed loop system. Furthermore, the LMPC is guaranteed to improve the performance of the close-loop system over the iterations. We tested the proposed control logic on an infinite horizon linear quadratic regulator with constraints (CLQR) to show that the proposed control logic converges to the optimal solution of the infinite optimal control problem. Finally, we compared the computation time of the proposed strategy with the computational time of the LMPC for nonlinear system, and we showed that the proposed control logic reduces the computational burden by several order of magnitudes.

REFERENCES

- Borrelli, F. (2003). *Constrained optimal control of linear and hybrid systems*, volume 290. Springer.
- Bristow, D.A., Tharayil, M., and Alleyne, A.G. (2006). A survey of iterative learning control. *IEEE Control Systems*, 26(3), 96–114.
- Garcia, C.E., Prett, D.M., and Morari, M. (1989). Model predictive control: theory and practice—a survey. *Automatica*, 25(3), 335–348.
- Jones, C.N. and Morari, M. (2010). Polytopic approximation of explicit model predictive controllers. *IEEE Transactions on Automatic Control*, 55(11), 2542–2553.
- Lee, J.H. and Lee, K.S. (2007). Iterative learning control applied to batch processes: An overview. *Control Engineering Practice*, 15(10), 1306–1318.
- Lee, J.H., Lee, K.S., and Kim, W.C. (2000). Model-based iterative learning control with a quadratic criterion for time-varying linear systems. *Automatica*, 36(5), 641–657.
- Lee, K.S. and Lee, J.H. (2000). Convergence of constrained model-based predictive control for batch processes. *IEEE Transactions on Automatic Control*, 45(10), 1928–1932.
- Lofberg, J. (2004). Yalmip: A toolbox for modeling and optimization in matlab. In *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, 284–289. IEEE.
- Mayne, D.Q., Rawlings, J.B., Rao, C.V., and Sokaert, P.O. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, 36(6), 789–814.
- Rosolia, U. and Borrelli, F. (2016). Learning model predictive control for iterative tasks. *Submitted, available at <https://arxiv.org/abs/1609.01387>*.
- Rucco, A., Notarstefano, G., and Hauser, J. (2015). An efficient minimum-time trajectory generation strategy for two-track car vehicles. *IEEE Transactions on Control Systems Technology*, 23(4), 1505–1519.
- Sharp, R. and Peng, H. (2011). Vehicle dynamics applications of optimal control theory. *Vehicle System Dynamics*, 49(7), 1073–1111.
- Subbaraman, A. and Benosman, M. (2016). Extremum seeking-based iterative learning model predictive control (esilc-mpc). *IFAC-PapersOnLine*, 49(13), 193–198.
- Tamar, A., Thomas, G., Zhang, T., Levine, S., and Abbeel, P. (2016). Learning from the hindsight plan-episodic mpc improvement. *arXiv preprint arXiv:1609.09001*.
- Wang, Y., Gao, F., and Doyle, F.J. (2009). Survey on iterative learning control, repetitive control, and run-to-run control. *Journal of Process Control*, 19(10), 1589–1600.