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A Stochastic MPC Approach with Application to Iterative Learning

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***Abstract*— We present a Stochastic Model Predictive Controller for constrained uncertain systems. The proposed framework guarantees recursive feasibility of the controller while ensuring probabilistic constraint satisfaction. Furthermore, we show that the closed loop system converges to a neighborhood of the origin regardless of the disturbance realization. The main contribution of this work is to propose a deterministic reformulation of the chance constraints, where the constraint tightening is constant over the prediction horizon. Therefore, the proposed strategy can be integrated with the recently proposed Leaning Model Predictive Control (LMPC) scheme. The properties of the controller and its integration with the LMPC are discussed in the result section.**

I. Introduction

In the past three decades Model Predictive Control (MPC) has gained popularity in the control community [1]–[4]. MPC provides a systematic strategy to deal with state and input constraint while guarantee safety and closed-loop stability. The controller employs a model of the plant to forecast the system's evolution over a moving time horizon. This forecast is used to select the best control action with respect to a performance index. However, when the system model is inaccurate guarantees about safety and stability are in general lost [3]–[5].

One way to overcome these limitations is Robust Model Predictive Control [5]. In Robust MPC the controller accounts for all possible realizations of the model uncertainty. This strategy allows to deal with uncertain system, guaranteeing convergence and recursive feasibility. However, in applications which are not safety critical, this strategy may result in a conservative behavior. For this reason, Stochastic Model Predictive Control strategies have been recently proposed [6]–[13]. In Stochastic MPC the stochastic nature of the uncertainty is exploited and the controller is allowed to violate the constraint with some probability. Designing Stochastic MPC algorithm is challenging and is still an active field of research. There are several challenges in Stochastic MPC design, in the past decade researchers have mainly focused on *i)* chance constraint reformulation, *ii)* closed-loop stability and recursive feasibility (i.e. feasibility at all time instants).

Significant research in the past has focused on solving the associated chance constrained optimization problem, which is challenging due to the fact that computing a probability integral for general distributions requires multi-dimensional integration and does not admit a closed form solution. Furthermore, it turns out that even for the simple problems (e.g., linear systems with polyhedral constraints), the chance constraint renders the feasible set non-convex, which makes optimization under chance constraints difficult. Indeed, it is believed that problems that satisfy both property (i.e., efficient evaluation of probability integral and convex feasible set) are a "rare commodity" [14]. Therefore, chance constrained optimization problems must be approximated in all but the simplest cases. Since chance constraints are not the main focus of this paper, we refer the interested reader to the recent survey papers [2], [10] for in-depth discussion on approximating chance constraints in Stochastic MPC.

Approaches for ensuring recursive feasibility and stability of Stochastic MPC have been proposed in [6], [7], [12], [13], [15]. In [6] the authors have proposed a Stochastic MPC scheme based on a fixed shape cross section tube. The scaling of the cross section is computed offline and results in a *time-varying* constraint tightening along the prediction horizon. This strategy allowed the authors to guarantee recursive feasibility and convergence. In [15] the authors build on [6], they made explicit use of the disturbance distribution to compute offline the constraint tightening which is used in the MPC scheme. As a result, the authors in [15] improved the closed loop performance while guaranteeing recursive feasibility and convergence. In order to guarantee recursive feasibility, the authors in [12] used additional constrains on the *first predicted* step. The author demonstrated on a numerical example that their Stochastic MPC scheme satisfies the chance constraint while guaranteeing recursive feasibility and convergence. A similar idea has been used also in [13], where the authors computed the constraint tightening based on an offline sampling strategy suggested in [2]. In [7] the authors proposed a two modes controller which is able to guarantee recursive feasibility and convergence of the closed loop trajectory. In all of the aforementioned papers the constraint tightening along the horizon is not constant.

In this paper, we propose a Stochastic Model Predictive Control strategy with constant constraint tightening along the prediction horizon. The proposed strategy uses an error model approach, also referred to as tube approach [4], and it allows to decouple the planning of the nominal trajectory and the handling of the disturbance. As a result, the constraint tightening is *time-invariant* over the prediction horizon and no additional constraints on the *first predicted* step are required. The advantage of this formulation is that it can be integrated with nominal MPC strategies which cannot handle time-varying constraint tightening along the prediction horizon. In particular, we show that the proposed strategy can be integrated with the Learning Model Predictive Control (LMPC) framework from [16], [17]. Finally, we

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show that the proposed Stochastic MPC guarantees: *i)* recursive feasibility, *ii)* chance constraint satisfaction at all time instants and *iii)* convergence of the closed loop trajectory to a neighborhood of the origin regardless of the disturbance realization.

Notation: Throughout the paper we use the following notation. The positive (semi)definite matrix A is denoted as $(A \succeq 0) A \succ 0$ and we defined $||x||_Q = x^T Q x$. The Minkowski sum is denoted as $\mathcal{X} \oplus \mathcal{Y} = \{x + y \in \mathbb{R}^n : x \in \mathcal{X}, y \in \mathcal{Y}\}$, similarly the the Pontryagin difference is defined as $\mathcal{X} \ominus \mathcal{Y} = \{x \in \mathcal{X} : x + y \in \mathcal{X}, \forall y \in \mathcal{Y}\}$.

II. Technical Background

In this section we recall some definitions from set theory [3, Chapter 10], which will be used later on.

Definition 1 (Robust Positive Invariant Set): A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a robust positive invariant set for the autonomous system $x_{k+1} = Ax_k + w_k$, where $w_k \in \mathcal{W}$, if

$$x_0 \in \mathcal{O} \rightarrow x_k \in \mathcal{O},\ \forall w_k \in \mathcal{W}, k \geq 0.$$

The invariance properties of robust positive invariant sets will be used in the controller design to guarantee recursive feasibility.

Definition 2 (Robust Successor Set): For the autonomous system $x_{k+1} = Ax_k + w_k$ we will denote the robust successor set from the set $\mathcal{S}$ as

$$\mathrm{Suc}(\mathcal{S}, \mathcal{W}) = \{x \in \mathbb{R}^n : \exists \bar{x} \in \mathcal{S}, \exists w \in \mathcal{W}. \text{ such that } x = A\bar{x} + w_k\}.$$

Given the initial state $\bar{x}$, the robust successor set $\mathrm{Suc}(\bar{x}, \mathcal{W})$ collects the states that the uncertain autonomous system may reach in one time step. The robust successor set will be used to enforce the probability of constraint violation.

Definition 3 (Predecessor Set): For the system $x_{k+1} = Ax_k + Bu_k$ and the input constraint $u \in \mathcal{U}$, we denote the predecessor set from the set $\mathcal{S}$ as

$$\mathrm{Pre}(\mathcal{S}, \mathcal{U}) = \{x \in \mathbb{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu \in \mathcal{S}\}.$$

Definition 4 (N-Step Controllable Set $\mathcal{K}_N(\mathcal{S})$): For a given target set $\mathcal{S} \subseteq \mathcal{X}$, the N-step controllable set $\mathcal{K}_N(\mathcal{S})$ of the system $x_{k+1} = Ax_k + Bu_k$ and constraints set $\mathcal{X}$ is defined recursively as:

$$\mathcal{K}_j(\mathcal{S}) = \mathrm{Pre}\left(\mathcal{K}_{j-1}(\mathcal{S}), \mathcal{U}\right) \cap \mathcal{X}, \mathcal{K}_0(\mathcal{S}) = \mathcal{S},$$

$j \in \{1, \dots, N\}$.

Given a time invariant system, the N-Step controllable set $\mathcal{K}_N(\mathcal{S})$ collects the state which can be driven to the set $\mathcal{S}$ in N-steps, satisfying input and state constraint. $\mathcal{K}_N(\mathcal{S})$ will be used to compute the region of attraction of the proposed controller.

III. Problem Formulation

A. The system

Consider the uncertain linear time invariant system,

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^d$ are the state and input at time k, respectively. The disturbances w_k are independent and identically distributed (*i.i.d.*) with bounded support $\mathcal{W}$. Moreover, system (1) is subjected to the following constraints on states and inputs

$$x_k \in \mathcal{X} \text{ and } u_k \in \mathcal{U},\ \forall k \geq 0. \tag{2}$$

B. Control Objective

Our objective is to design a feedback policy

$$\pi(x_k) : \mathbb{R}^n \rightarrow \mathbb{R}^d,$$

such that for the closed-loop system

$$x_{k+1} = Ax_k + B\pi(x_k) + w_k$$

we have that, if x_0 belongs to the region of attraction $\mathcal{C}_\infty$, then:

1) the chance constraint

$$\Pr(x_{k+1} \in \mathcal{X}) \geq 1 - \epsilon, \tag{3}$$

is satisfied $\forall k \geq 0$.

2) the input constraint is robustly satisfied, namely

$$\pi(x_k) \in \mathcal{U},\ \forall k \geq 0, \forall w_k \in \mathcal{W}. \tag{4}$$

.

3) there exists $\mathcal{O}_e \subset \mathcal{C}_\infty \subseteq \mathbb{R}^n$ such that

$$\lim_{k \to \infty} x_k \in \mathcal{O}_e, \forall w_k \in \mathcal{W}. \tag{5}$$

IV. Proposed Approach

In this section we present the proposed control strategy which guarantees that objectives (3)-(5) are achieved.

A. Control Policy

Firstly, we introduce the nominal dynamics

$$s_{k+1} = As_k + Bv_k \tag{6}$$

where the nominal state $s_k \in \mathbb{R}^n$ and the nominal open loop input $v_k \in \mathbb{R}^d$. We will relate the input to the system u_k and the input to the nominal dynamics v_k through the feedback policy. Given the nominal state we defined the error

$$e_k = x_k - s_k, \tag{7}$$

which captures the uncertainty of the system. The above linear change of variables is standard in robust tube MPC [4, Chapter 3]. We consider the feedback policy

$$u_k = v_k - Ke_k \tag{8}$$

where the feedback gain K is chosen such that $A - BK$ is stable and $v_k \in \mathbb{R}^d$ is the open loop control input from (6). Finally, from (7) we have that $x_k = s_k + e_k$, and therefore from equations (1) and (8) follows that we can decouple the system dynamics as

$$s_{k+1} = As_k + Bv_k \tag{9a}$$
$$e_{k+1} = (A - BK)e_k + w_k. \tag{9b}$$

The above decoupling allows us to exploit the nominal dynamics for planing and the error dynamics for constraint tightening [4, Chapter 3].

B. Constraint Tightening

In the following, we present the constraint tightening strategy for input and state constraints.

Assumption 1: We assume that we are given a set $\mathcal{D} \subset \mathcal{W}$ such that $\Pr(w_k \in \mathcal{D}) \geq 1 - \epsilon$.

In Section VI, we discuss a sampling based method for computing such a set.

Let $\mathcal{O}_e \subseteq \mathbb{R}^n$ be a robust positive invariant set for the error dynamics (9b) with the disturbance $w_k \in \mathcal{W}$. We compute the one step robust successor set for the error dynamics (9b) $\mathcal{S}_e = \mathrm{Suc}(\mathcal{O}_e, \mathcal{D})$. Note that from definitions 1-2 and the fact that $\mathcal{D} \subset \mathcal{W}$, we have $\mathcal{S}_e \subset \mathcal{O}_e$.

Proposition 1: Let assumption 1 hold and let $\mathcal{O}_e$ be a positive robust invariant set for the error dynamics (9b) with the disturbance $w_k \in \mathcal{W}$, and let $\mathcal{S}_e = \mathrm{Suc}(\mathcal{O}_e, \mathcal{D})$, then we have that

$$\Pr\Big((A - BK)e_k + w_k \in \mathcal{S}_e | e_k \in \mathcal{O}_e\Big) \geq 1 - \epsilon.$$

Proof: By definition $\mathcal{S}_e = \mathrm{Suc}(\mathcal{O}_e, \mathcal{D})$ for the error dynamics (9b) and therefore if $e_k \in \mathcal{O}_e$ and $w_k \in \mathcal{D}$ then $(A - BK)e_k + w_k \in \mathcal{S}_e$. By assumption $\Pr(w_k \in \mathcal{D}) \geq 1 - \epsilon$, therefore $\Pr(w_k \in \mathcal{D}) = \Pr((A - BK)e_k + w_k \in \mathcal{S}_e | e_k \in \mathcal{O}_e) \geq 1 - \epsilon$. ■

In the controller design we will use the invariance property of $\mathcal{O}_e$ to constraint the evolution of the error dynamics. In particular, we will require $e_0 \in \mathcal{O}_e$ and this will ensure that $e_k \in \mathcal{O}_e$, $\forall k \geq 0$ (Def. 1). Moreover, as the feedback policy depends on the error at time k we define the tightened input constraint set

$$\mathcal{V} = \mathcal{U} \ominus \Big\{ - Ke : e \in \mathcal{O}_e \Big\}. \tag{10}$$

The above set guarantees that $u = v - Ke \in \mathcal{U}$, if $v \in \mathcal{V}$ and $e \in \mathcal{O}_e$. Therefore $\mathcal{V}$ will be used in the controller design to guarantee robust constraint satisfaction.

Finally, we use the one step robust reachable set $\mathcal{S}_e$ to define the tightened state constraint set

$$\mathcal{C} = \mathcal{X} \ominus \mathcal{S}_e, \tag{11}$$

where the constraint set $\mathcal{X}$ is defined as in (2). By construction of $\mathcal{C}$, we have that if $s \in \mathcal{C}$ and $e \in \mathcal{S}_e$, then $x = s + e \in \mathcal{X}$. The set $\mathcal{C}$ and Proposition 1 will guarantee that the chance constraint (3) is satisfied, as discussed in Section V.

C. MPC Reformulation

Now consider the following receding horizon strategy where, at time k, the controller solves the following finite time optimal control problem

$$\begin{aligned}
\min_{s_{0|k}, v_{0|k}, \dots, v_{N-1|k}} \quad & \sum_{t=0}^{N-1} \big(||s_{t|k}||_Q + ||v_{t|k}||_R\big) + ||s_{N|k}||_{Q_F} \\
\text{s.t.} \quad & s_{t+1|k} = As_{t|k} + Bv_{t|k},\ \forall t = 0, \dots, N-1 \\
& s_{t|k} \in \mathcal{C},\ \forall t = 1, \dots, N-1 \\
& v_{t|k} \in \mathcal{V},\ \forall t = 0, \dots, N-1 \\
& x_k - s_{0|k} \in \mathcal{O}_e \\
& s_{N|k} \in \mathcal{C}_F
\end{aligned} \tag{12}$$

where $Q \succeq 0$ and $R \succ 0$. The state and input constraint are defined as in (10) and (11). The matrix Q_F which defines the terminal cost is the solution to the Riccati equation for the nominal dynamics (9a) and the matrices Q and R. Finally, given the optimal LQR feedback gain K^* for the nominal dynamics (9a) and the matrices Q and R, we define the terminal set

$$\mathcal{C}_F \subseteq \big\{s \in \mathbb{R}^n : s \in \mathcal{C}, -K^*s \in \mathcal{V}\big\}, \tag{13}$$

as the positive invariant set for the autonomous system $s_{k+1} = (A - BK^*)s_k$.

Let

$$\begin{aligned} &[v^*_{0|k}, \cdots, v^*_{N-1|k}] \\ &[s^*_{0|k}, \cdots, s^*_{N|k}] \end{aligned} \tag{14}$$

be the optimal solution of (12) at time k. Then, at time k, the controller applies to system (1)

$$u_k = v^*_{k|k} - Ke_k = v^*_{k|k} - K(x_k - s^*_{0|k}). \tag{15}$$

The finite time optimal control problem (12) is repeated at time $k+1$, based on the new state x_{k+1}, yielding a *moving* or *receding horizon* control strategy.

V. Feasibility, Stability and Constraint Satisfaction

In this section we describe the properties of the proposed Stochastic MPC. In particular, we show that controllers satisfies the objectives (3)-(5).

First we introduce the set of states $\mathcal{C}_\infty$ for which problem (12) is feasible. Later we will prove that this is the region of attraction for the closed loop system (1) and (15).

Definition 5: Let $\mathcal{O}_e$ be a robust positive invariant set for the error dynamics (9b), $\mathcal{C}_F$ be defined as in (13) and $\mathcal{K}_N(\mathcal{C}_F)$ be the $(N$-1)-Step Controllable Set for system (9a), constraints (10)-(11) and target set $\mathcal{C}_F$. We define

$$\mathcal{C}_\infty = \mathrm{Pre}(\mathcal{K}_{N\text{-}1}(\mathcal{C}_F), \mathcal{V}) \oplus \mathcal{O}_e. \tag{16}$$

A. Recursive Feasibility

Theorem 1: Let Assumption 1 hold and $\mathcal{C}_\infty$ be defined as in Definition 5. If $x_0 \in \mathcal{C}_\infty$, then the Stochastic MPC (12) and (15) is feasible at all time $k \geq 0$.

Proof: Firstly, we show that (12) is feasible at $k = 0$. We have that $x_0 \in \mathcal{C}_\infty = \mathrm{Pre}(\mathcal{K}_{N\text{-}1}(\mathcal{C}_F), \mathcal{V}) \oplus \mathcal{O}_e$, therefore

$$\begin{aligned} &\exists s_{0|0} \in \mathrm{Pre}(\mathcal{K}_{N\text{-}1}(\mathcal{C}_F), \mathcal{V}),\ \exists e_{0|0} \in \mathcal{O}_e,\ \exists v_{0|0} \in \mathcal{V}: \\ &\qquad e_{0|0} + s_{0|0} = x_0, s_{1|0} = As_{0|0} + Bv_{0|0} \in \mathcal{K}_{N\text{-}1}(\mathcal{C}_F). \end{aligned}$$

From the above we have that $x_0 - s_{0|0} \in \mathcal{O}_e$. Moreover, as $s_{1|0} \in \mathcal{K}_{N\text{-}1}(\mathcal{C}_F)$, there exist a control sequence $v_{k|0} \in \mathcal{V}, k = [1, \dots, N-1]$ such that the $N-1$ steps trajectory $s_{k|0} \in \mathcal{C}, k = [1, \dots, N-1]$ and $s_{N|0} \in \mathcal{C}_F$. Therefore, $v_{k|0}, k = [0, \dots, N-1]$ and the related state sequence $s_{k|0}, k = [0, \dots, N]$ is a feasible solution to (12) at time $k = 0$. Now, we show that if (12) is feasible at time k then (12) is feasible at time $k+1$. Let (14) be the optimal solution of (12). Firstly, we notice that $x_k - s^*_{0|k} = e_k \in \mathcal{O}_e$, therefore $\forall w_k \in \mathcal{W}$ we have that $\hat{e} = (A - BK)e_k + w_k \in \mathcal{O}_e$ and

$\hat{e} = x_{k+1} - s^*_{1|k}$. Moreover, as $s^*_{N|k} \in \mathcal{C}_F$ we have that for $\hat{v} = -K^* s^*_{N|k} \in \mathcal{V}$ then $\hat{s} = As^*_{N|k} + B\hat{v} \in \mathcal{C}_F$. Therefore the shifted input sequence and the related trajectory

$$\begin{aligned} &[s^*_{1|k}, \ldots, s^*_{N|k}, \hat{s}] \in \mathcal{C} \\ &[v^*_{1|k}, \ldots, v^*_{N|k}, \hat{v}] \in \mathcal{V} \end{aligned} \tag{17}$$

is a feasible solution for (12) $\forall w_k \in \mathcal{W}$.
We have shown that *i)* the Stochastic MPC controller (12) and (15) is feasible at time $k = 0$, and *ii)* if the Stochastic MPC controller (12) and (15) is feasible at time k then the controller is feasible at time $k+1$. Therefore, we conclude by induction that the Stochastic MPC (12) and (15) is recursively feasible. ■

B. Constraint Satisfaction

Theorem 2: Let Assumption 1 hold and $\mathcal{C}_\infty$ be defined as in Definition 5. If $x_0 \in \mathcal{C}_\infty$, then the input sequence and the closed loop system (1) and (15) satisfy $u_k \in \mathcal{U}, \forall w_k \in \mathcal{W}, k \geq 0$ and $\Pr(x_k \in \mathcal{X}) \geq 1 - \epsilon, \forall k \geq 1$.

Proof: Firstly, we show that the hard constraint on the input is satisfied. By the recursive feasibility of (12) we have that $v_k \in \mathcal{V}\ \forall k \geq 0$ and $e_k = x_k - s^*_{0|k} \in \mathcal{O}_e,\ \forall k \geq 0$. Therefore, by the definition of $\mathcal{V}$ in (10) we have that $u_k \in \mathcal{U},\ \forall w_k \in \mathcal{W}, k \geq 0$. In the following we show that the chance constraint is satisfied. By the recursive feasibility of (12) we have that $e_k = x_k - s^*_{0|k} \in \mathcal{O}_e\ \forall k \geq 0$ and from Proposition 1

$$\Pr\big((A-BK)e_k + w_k \in \mathcal{S}_e | e_k \in \mathcal{O}_e\big) \geq 1-\epsilon,\ \forall k \geq 0 \tag{18}$$

From recursive feasibility of (12) we have that $s^*_{1|k-1} \in \mathcal{C},\ \forall k \geq 0$, therefore from (18) we have

$$\Pr\big(s^*_{1|k} + (A - BK)e_k + w_k \in \mathcal{S}_e \oplus \mathcal{C} | e_k \in \mathcal{O}_e\big) \geq 1 - \epsilon, \tag{19}$$

$\forall k \geq 0$. Finally we notice that by definition $x_{k+1} = s^*_{1|k} + (A - BK)e_k + w_k$ and from (11) $\mathcal{X} = \mathcal{C} \oplus \mathcal{S}_e$. Thus, as $e_k \in \mathcal{O}_e\ \forall k \geq 0$ from (19) we conclude that

$$\Pr\big(x_{k+1} \in \mathcal{X}\big) \geq 1 - \epsilon,\ \forall k \geq 0.$$

■

C. Convergence

Theorem 3: Let Assumption 1 hold and $\mathcal{C}_\infty$ be defined as in Definition 5. If $x_0 \in \mathcal{C}_\infty$, then the closed loop system (1) and (15) converges asymptotically to $\mathcal{O}_e$.

Proof: As the nominal and the error dynamics are decoupled and the running cost in (12) penalizes the nominal dynamics, the proof follows from standard MPC arguments. Firstly, we notice that the terminal cost $||\cdot||_{Q_F}$ is a control Lyapunov function in the terminal positive invariant set $\mathcal{S}_F$ and the shifted optimal solution (17) is feasible, therefore it follows from [3, Theorem 12.2] that $\lim_{k\to\infty} s_k = 0$. Moreover, by feasibility of (12) we have that $e_k \in \mathcal{O}_e,\ \forall k \geq 0$. Thus, we conclude

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} s_k + e_k \in \left\{ \lim_{k\to\infty} s_k \right\} \oplus \mathcal{O}_e = \mathcal{O}_e \tag{20}$$

■

The above theorems show that the proposed control design is able to achieve the objectives (3)-(5). In particular, we showed that if the starting position $x_0 \in \mathcal{C}_\infty$, then the proposed controller is able to guarantee robust contrasting satisfaction for the input and stochastic constraint satisfaction for the state. Finally, we showed that the closed loop trajectory converges to $\mathcal{O}_e$, regardless of the disturbance realization.

VI. Bounding the Uncertainty Set $\mathcal{D}$

Recall that problem (12) requires the availability of a set $\mathcal{D}$ that contains at least $1 - \epsilon$ probability mass, see also Assumption 1. Unfortunately, finding such a set is computationally hard for general distributions. Indeed, the authors of [18] point out that even computing the probability of a weighted sum of uniformly distributed variables being non-positive is already NP-hard. In the following, we describe a simple sampling-based method for finding $\mathcal{D}$ that has been proposed in [8], [19], [20], and refer to [21], [22] for other approaches.

Following [8], [19], [20], we restrict the uncertainty set $\mathcal{D} \subset \mathbb{R}^n$ to be an (axis-aligned) hyper rectangle[1] that we parametrize by $\underline{d}, \overline{d} \in \mathbb{R}^n$, i.e., $\mathcal{D} = \mathcal{D}(\underline{d}, \overline{d})$. Furthermore, let $w^{(1)}, w^{(2)}, \ldots, w^{(S)}$ be S i.i.d. samples, extracted according to $\mathbb{P}$, and consider the following *scenario program* [23], [24]

$$\begin{aligned} \min_{\underline{d},\overline{d}} \quad & \left\|\overline{d} - \underline{d}\right\|_1 \\ \text{s.t.} \quad & \underline{d} \leq w^{(i)} \leq \overline{d}, \quad \forall i = 1, \ldots, S. \end{aligned} \tag{21}$$

Intuitively speaking, (21) determines the "smallest" box that encapsulates the samples $w^{(1)}, w^{(2)}, \ldots, w^{(S)}$. Let $(\underline{d}^\star, \overline{d}^\star)$ be the optimal solution of (21), and consider $\mathcal{D}^\star(\underline{d}^\star, \overline{d}^\star) := \{w \in \mathcal{W} \colon \underline{d} \leq w \leq \overline{d}\} \subset \mathcal{W}$. Then, the following holds:

Proposition 2: If the sample size S in (21) is chosen such that $S \geq \frac{2}{\epsilon}(2n-1+\log\frac{1}{\beta})$, then $\mathbb{P}(w \in \mathcal{D}^\star(\underline{d}^\star, \overline{d}^\star)) \geq 1-\epsilon$ with confidence at least $1 - \beta$.

Proof: The number of decision variables in (21) is $2n$. The statement now immediately follows from [23, Theorem 1] and [24, Corollary 5.1]. ■

We conclude this section by pointing out that the confidence level β in Proposition 2 can be chosen very low in practice (e.g., $\beta = 10^{-7}$), without having a big effect on the sample size S. In other words, with "practical certainty", the set $\mathcal{D}^\star(\underline{d}^\star, \overline{d}^\star)$ will satisfy Assumption 1.

VII. Numerical Examples

A. Example I: Stochastic MPC

In this section we evaluate the proposed Stochastic MPC strategy on the example from [12, Section V]. The controller aims to regulate to the origin a linear time invariant system subject to bounded uncertainty, satisfying hard constraints on the input and chance constraints on the state. The system dynamics is

$$x_{k+1} = \begin{bmatrix} 1 & 0.0075 \\ -0.143 & 0.996 \end{bmatrix} x_k + \begin{bmatrix} 4.798 \\ 0.115 \end{bmatrix} u_k + w_k$$

[1]The choice of a hyper rectangle is not restrictive. In fact, other representations with a finite-dimensional (convex) parametrization such as ellipsoid chould be chosen instead.

where w_k is uniformly distributed on $\mathcal{W} = \{w \in \mathbb{R}^2 : ||w||_\infty \leq 0.02\}$. The state and input constraints are defined as follows,

$$\Pr\left(x_{k+1} \leq \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) \geq 0.8 \text{ and } ||u_k||_1 \leq 0.2,$$

$\forall k \geq 0$.

We implemented the Stochastic MPC (12) and (15) with $Q = \text{diag}(1, 10)$, $R = 1$ and we used the LQR feedback gain in the control policy (8). The set $\mathcal{D}$, which satisfies Assumption 1, is given as

$$\mathcal{D} = \left\{ w \in \mathbb{R}^2 : \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} \leq w \leq \begin{bmatrix} 0.2 \\ 0.12 \end{bmatrix} \right\}.$$

The robust positive invariant set $\mathcal{O}_e$ is computed using the strategy proposed in [4, Chapter 3]. This allows to compute an outer approximation of the minimal robust positive invariant set for autonomous system of the form (9b). Furthermore, we computed the robust one step reachable set $\mathcal{S}_e = \text{Suc}(\mathcal{O}_e, \mathcal{D})$ accordingly to definition 2. The Quadratic Program (QP) from the Stochastic MPC is solved using Gurobi through the Yalmip interface [25]. The set operation needed to compute $\mathcal{O}_e$ and $\mathcal{S}_e$ are performed using the MPT3 toolbox [26].

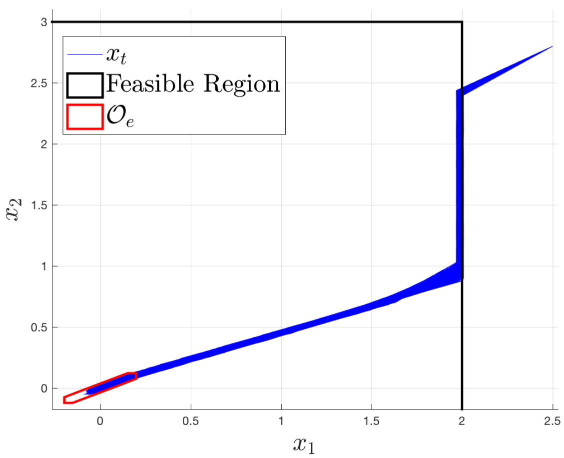

Fig. 1. Closed loop trajectories for 2000 random realizations.

Figure 1 reports the closed loop trajectories for 2000 randomly generated disturbance realizations. We underline that, accordingly with Theorem 1, the controller is feasible at all time instants. In the detail of Figure 2, we notice that the constraints were violated during the first 6 time steps. For these 2000 disturbance realizations, the empirical constraint violation averaged over the first 6 steps is 20.3%, which is very close to the desired one of 20%. Finally, we see from Figure 1 that the trajectories converge to the minimal robust positive invariant set $\mathcal{O}_e$, validating Theorem 3.

B. Example II: Integration with Learning MPC

In this section we show the application which motivated our work. We exploit the constant constraint tightening

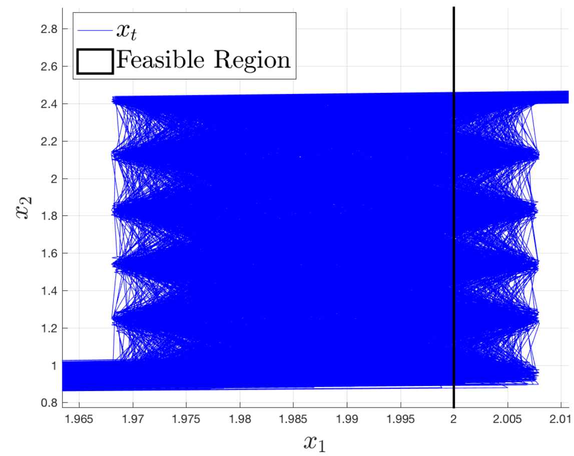

Fig. 2. Closed loop trajectories for 2000 random disturbance realizations. We notice that the closed loop trajectories violates the state constraints.

along the Stochastic MPC horizon to integrate the proposed strategy with the Learning Model Predictive Control (LMPC) [16]. The LMPC framework is designed for autonomous system performing the same task repeatedly. Each task execution is referred to as "iteration". The data from each iteration are stored and used to compute a terminal set and terminal cost used in the LMPC. We showed that when no model mismatch is present, the LMPC improves the overall closed loop performance at each iteration, while guaranteeing recursive feasibility and convergence [16].

In order to integrate the LMPC with the proposed Stochastic MPC, we use a strategy similar to the ones used in [17]. As the constraint tightening is constant for the nominal state, we used the LMPC for computing the open loop input v_k to the nominal dynamics in (9a). Then, at time k, the nominal open loop input v_k and the measured error e_k are combined to compute the input to the system u_k, through the feedback policy (8).

We implemented the Stochastic LMPC on the uncertain double integrator system from [17] where the support of the disturbance is $\mathcal{W} = \{w \in \mathbb{R}^2 : ||w||_\infty \leq 0.2\}$. Given the initial position $x_S = [-3.498, -0.4]^T$, the controller aims to solve the following infinite horizon optimal control problem

$$\begin{aligned}
\min_{\substack{e_0, s_0 \\ v_0, v_1, \dots}} \quad & \sum_{k=0}^{\infty} ||s_k||_1 + 10||v_k||_1 \\
\text{s.t} \quad & s_{k+1} = As_k + Bu_k, \ \forall k \geq 0 \\
& e_{k+1} = (A - KB)e_k + w_k, \ \forall k \geq 0 \\
& x_S - s_0 = e_0 \in \mathcal{O}_e \\
& \Pr(s_k + e_k \in \mathcal{X}) \geq 1 - \epsilon, \forall k \geq 1 \\
& v_k - Ke_k \in \mathcal{U}, \forall w_k \in \mathcal{W}, k \geq 0
\end{aligned} \tag{22}$$

where K is the optimal feedback gain from the LQR problem and $\mathcal{O}_e$ is the outer approximation of the minimal robust positive invariant for the error dynamic [15, Chapter 3]. The state and input constraints are

$$\mathcal{X} = \left\{ \begin{bmatrix} -4 \\ -4 \end{bmatrix} \leq x \leq \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right\} \text{ and } \mathcal{U} = \{-1 \leq u \leq 1\}.$$

The LMPC computes an approximate solution to the infinite time optimal control problem (22). The regulation task is performed repeatedly and, at each j-th iteration, the j-th nominal closed loop trajectory $[s_0^j, s_1^j, \ldots]$ and input sequence $[v_0^j, v_1^j, \ldots]$ are recorded. These data are used from the LMPC to improve the iteration cost, defined as $J^j(s_0^j) = \sum_{k=0}^{\infty} ||s_k^j||_1 + 10||v_k^j||_1$. We test the Stochastic LMPC controller on 50 simulations. At each LMPC simulation we run 10 iterations, meaning that the LMPC perform the regulation task 10 times. The data from each iteration are used from the LMPC to improve the the closed loop performance, until the controller converges to a steady state behavior. Figure 3 shows the 50 steady state trajectories for the 50 LMPC simulations. We notice that the closed-loop trajectory violated the constraint at the first time step, with an average violation of 20% as desired. Finally, in Table I, we report the iteration cost and we confirm that the controller is able to improve the closed-loop performance at each iteration.

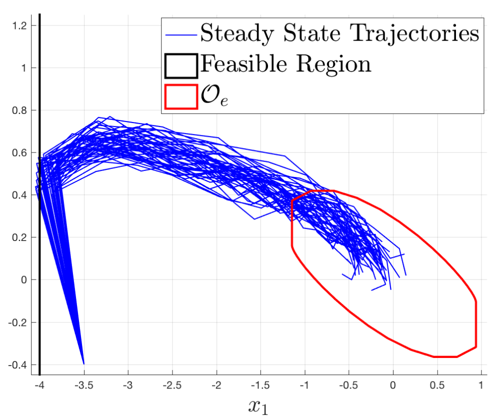

Fig. 3. Steady state closed loop trajectory for 50 random realizations.

TABLE I
OPTIMAL COST OF THE LMPC AT EACH j-TH ITERATION.

Iteration	Iteration Cost	Iteration	Iteration Cost
$j = 1$	208.6	$j = 6$	120.0
$j = 2$	163.1	$j = 7$	119.0
$j = 3$	142.2	$j = 8$	118.8
$j = 4$	130.9	$j = 9$	118.8
$j = 5$	122.1	$j = 10$	118.8

VIII. Conclusions

In this paper, we proposed a Stochastic Model Predictive Control scheme for linear system subject to additive disturbances. The proposed strategy results in a constant constraint tightening along the planning horizon. We showed that the controller guarantees recursive feasibility, chance constraint satisfaction and convergence, regardless of the disturbance realization. The effectiveness of the proposed strategies has been tested on an numerical example.

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