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Robust Learning Model Predictive Control for Iterative Tasks: Learning From Experience

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Abstract— **We present a Robust Learning Model Predictive Controller (RLMPC) for constrained uncertain systems performing iterative tasks. The proposed controller builds on earlier work of Learning Model Predictive Control (LMPC) for deterministic systems. The main idea behind RLMPC is to collect data from previous iterations and use it to estimate the current value function and build a robust safe set. We demonstrate that the proposed RLMPC algorithm is able to iteratively improve its control performance and robustly satisfy system constraints. The efficacy and limitations of the proposed RLMPC approach are illustrated on a numerical example.**

I. Introduction

Control design for systems performing repetitive tasks has been extensively studied in the literature since such problems often arise in practical applications [1]. Examples include autonomous cars racing around a given track [2]–[4], robotic system manipulators [5], and batch processes in general [6]. In the iterative learning control literature, one task execution is often referred to as "iteration". In general, the goal of ILC is to track a given reference while rejecting periodic disturbances [1] and improve performance at each iteration.

Recent studies have combined Model Predictive Control (MPC) with ILC. In [7] the authors proposed a Model-based Predictive Control for iterative task based on a time-varying MIMO system that has a dynamic memory of past batches tracking error. In [8] the authors used a linearized model based on the information of the previous iterations. The authors proved zero steady-state tracking error in presence of model mismatch. Also in [9] the authors successfully achieved zero tracking error using a MPC which uses measurements from previous iterations to modify the cost function. A nonlinear MPC based on iterative learning control is proposed in [10]. There, a MPC is designed for disturbance rejection and the ILC is designed to minimize errors occurring at each iteration. The authors proved that the steady state tracking error converges to zero as the iteration index goes to infinity.

Recently, a novel *reference-free* control strategy for iterative tasks known as *Learning Model Predictive Control (LMPC)* has been proposed for systems where a priori generation of reference trajectories can be difficult [4], [11]. The LMPC work in [11] improves the system performance using the data from the previous iterations; it assumes, however, perfect knowledge of the system dynamics. This work extends the LMPC of [11] to uncertain systems. In general, two main approaches exist for addressing uncertainty in constrained control design: robust control where the system constraints must be satisfied for all possible uncertainty realizations [12]–[14], and stochastic control where the constraints are allowed to be violated with a given probability [15]–[17]. A robust control approach is used in this paper. We build on the LMPC schema in [11] and we propose a *Robust LMPC (RLMPC)* algorithm that is able to improve the system performance for linear system subject to additive uncertainty.

The contribution of this paper is twofold. First, we present a reference-free Robust Learning MPC design for linear systems with additive uncertainty performing iterative tasks. It is assumed that the uncertainty set, the initial condition, the constraints and the control objective remain constant over the iterations, and that the uncertainty set is bounded. Second, we show how to design a terminal convex safe set and a terminal convex cost function in order to guarantee that: (i) the controller's performance is non-decreasing in the iterations; (ii) state and input constraints are robustly satisfied at iterations j if they were satisfied at iteration $j-1$; and (iii) the closed-loop system converges to a neighborhood of the origin whose size depends on the uncertainty set.

This paper is organized as follows: Section II introduces the problem setup and formulates the robust optimal control problem. Technical background is given in Section III to lay the foundation for this work. The Robust Learning MPC algorithm is presented in Section IV, where we also analyze its properties. An illustrative example is provided in Section V, and Section VI concludes the paper.

II. Problem Statement

Given an initial state x_S, we consider uncertain linear time-invariant systems of the form

$$x_{t+1} = Ax_t + Bu_t + Ew_t, \quad x_0 = x_S, \tag{1}$$

where $x_t \in \mathbb{R}^{n_x}$ is the state at time t, $u_t \in \mathbb{R}^{n_u}$ is the input, and A, B, and E are known system matrices. At each time step t, the system is affected by a random disturbance $w_t \in \mathbb{W} \subset \mathbb{R}^{n_w}$, where $\mathbb{W}$ is assumed to be a compact polytope that contains the origin. We consider joint state and input constraints of the form

$$Fx_t + Gu_t \leq f, \tag{2}$$

which must be satisfied for all uncertainty realizations $w_t \in \mathbb{W}$. The matrices $F \in \mathbb{R}^{n_f \times n_x}$, $G \in \mathbb{R}^{n_f \times n_u}$ and $f \in \mathbb{R}^{n_f}$ are known.

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The goal of this paper is to design a controller that, at each iteration j, solves the following infinite horizon robust optimal control problem

$$\begin{aligned} V^{j,\star}(x_S) := & \\ \min_{u_0^j, u_1^j(\cdot), \ldots} \quad & \sum_{t \geq 0} \ell\left(x_t^j(\mathbf{0}), u_t^j\left(x_t^j(\mathbf{0})\right)\right) \\ \text{s.t.} \quad & x_{t+1}^j = Ax_t^j + Bu_t^j(x_t^j) + Ew_t^j, \\ & Fx_t^j + Gu_t^j \leq f, \ \forall w_t^j \in \mathbb{W}, \\ & x_0^j = x_S, \ t = 0, 1, \ldots, \end{aligned} \tag{3}$$

where x_t^j and u_t^j denote the system state and the control input at time t, respectively. Moreover, $\ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}_+$ is a positive definite stage cost, and $x_t^j(\mathbf{0})$ denotes the disturbance-free, also known as "nominal" and "certainty-equivalent", state[1]. Notice that (3) minimizes the nominal cost. While other types of cost functions such as worst-case or expected-value cost are also considered in the literature, this paper focuses on certainty-equivalent cost due to its conceptual simplicity and because it has been shown to deliver good performance in practical applications [3], [18]. We point out that, as system (1) is uncertain, the optimal control problem (3) consists of finding feedback policies $[u_0^j, u_1^j(\cdot), u_2^j(\cdot), \ldots]$, where $u_t^j : \mathbb{R}^{n_x} \ni x_t^j \mapsto u_t^j = u_t^j(x_t^j) \in \mathbb{R}^{n_u}$ are state feedback policies. The first input u_0^j is static since x_0^j is assumed fixed and equal to x_S. A discussion on the benefits of "closed-loop" (i.e., feedback) policies over "open-loop" policy can be found in [13], [19], [20].

The optimal control problem (3) is generally difficult to solve since (i) the optimization is performed over the infinite-dimensional space of all feedback policies, (ii) the control horizon is infinite. In this paper, we exploit the fact that in iterative tasks the same problem (3) has to be solved over and over again, and previous data can be exploited to improve the performance at the next iteration.

III. Technical Background

In this section, we present the technical background that lays the foundation for the proposed Robust Learning MPC algorithm.

A. Control Policy Approximation

Recall that (3) is difficult to solve because (i) it requires the optimization over all possible feedback policies and (ii) the control horizon is infinite. A common approach to address (i) is to consider affine state feedback policies of the form [14], [21]

$$u_t^j(x_t) = K\left(x_t^j - x_t^j(\mathbf{0})\right) + v_t^j, \tag{4}$$

where $K \in \mathbb{R}^{n_u \times n_x}$ is a fixed feedback gain same for all iterations j. $x_t^j(\mathbf{0})$ is the nominal state and v_t^j is an auxiliary control input ("offset"). It is well-known that parametrization (4) allows us to decouple the state dynamics (1) into a nominal state $s_t^j \equiv x_t^j(\mathbf{0})$ and an error state $e_t^j := x_t^j - s_t^j$, whose respective dynamics are given by

$$s_{t+1}^j = As_t^j + Bv_t^j, \quad s_0^j = x_S, \quad j = 0, 1, \ldots \tag{5a}$$

$$e_{t+1}^j = \Psi e_t^j + Ew_t, \quad e_0^j = 0, \quad j = 0, 1, \ldots, \tag{5b}$$

where $\Psi := (A + BK)$. Above, the state s_t^j is disturbance-free, and the disturbance affects in e_t^j only.

The original problem (3) is approximated by the feedback policy (4) where the offset is computed by solving the following optimization problem

$$\begin{aligned} \hat{V}^{j,\star}(x_S) := & \min_{v_0^j, v_1^j, \ldots} \sum_{t \geq 0} \ell\left(s_t^j, v_t^j\right) \\ \text{s.t.} & \\ & \begin{bmatrix} s_{k+1}^j \\ e_{k+1}^j \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & \Psi \end{bmatrix} \begin{bmatrix} s_k^j \\ e_k^j \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v_k^j + \begin{bmatrix} 0 \\ E \end{bmatrix} w_t^j, \\ & Fs_t^j + Gv_t^j + (F + GK)e_k^j \leq f, \ \forall w_t^j \in \mathbb{W}, \\ & s_0^j = x_S, \ t = 0, 1, \ldots \end{aligned} \tag{6}$$

It follows from (4) that $\hat{V}^{j,\star}(x_S) \geq V^{j,\star}(x_S)$.

In the following we introduce the notion of safe set and iteration cost, which we borrow from Learning MPC for deterministic systems [11], [22], and extend hereby to the uncertain system (1).

B. Convex Safe Set

At iteration j, let the vectors

$$\mathbf{v}^j = [v_0^j, \ v_1^j, \ \ldots, \ v_t^j, \ \ldots], \tag{7a}$$

$$\mathbf{s}^j = [s_0^j, \ s_1^j, \ \ldots, \ s_t^j, \ \ldots], \tag{7b}$$

denote the nominal input and state of system (5a). To exploit the iterative nature of the control design, we define the *sampled Safe Set* $\mathcal{SS}^j$ at iteration j as

$$\mathcal{SS}^j = \left\{ \bigcup_{i \in M^j} \bigcup_{t=0}^{\infty} s_t^i \right\}, \tag{8}$$

where $M^j \subseteq \{0, \ldots, j\}$ is the set of indices associated with successful iterations, i.e.,

$$M^j = \left\{ k \in [0, j] : \lim_{t \to \infty} s_t^k = 0 \right\}.$$

In other words, $\mathcal{SS}^j$ is the collection of all nominal state trajectories up to iteration j that have converged to the origin. We define the *convex Safe Set* as

$$\mathcal{CS}^j = \text{conv}(\mathcal{SS}^j). \tag{9}$$

Notice that since the system (5a) is linear and the constraints are convex, we have that for every element of $\mathcal{CS}^j$ there exists an input sequence that steers the system to the origin [19]. Therefore, $\mathcal{CS}^j$ is a control invariant set, see [19, Section 11] for a precise definition thereof. As we will see later on, $\mathcal{CS}^j$ will be used as the terminal set in our Robust Learning MPC algorithm.

[1] The notation is motivated by the fact that, given a fixed control policy and initial state x_S, the state x_t^j depends on the previous uncertainty $\mathbf{w}_{t-1}^j := [w_0^j, \ldots, w_{t-1}^j]$ only, i.e., $x_t^j = x_t^j(\mathbf{w}_{t-1}^j)$.

C. Terminal Cost

At time t of the j-th iteration, the cost-to-go associated with the closed loop trajectory (7b) and input sequence (7a) is defined as

$$V^j_{t\to\infty}(s^j_t) := \sum_{k=0}^{\infty} \ell(s^j_{t+k}, v^j_{t+k}),$$

where $\ell(\cdot,\cdot)$ is the stage cost of problem (6). Note that $V^j_{0\to\infty}(s^j_0)$ quantifies the controller performance at the j-th iteration.

We are now in place to define the barycentric function [23]

$$P^j(s) := \begin{cases} p^{j,*}(s) & \text{if } s \in \mathcal{CS}^j \\ +\infty & \text{else} \end{cases}, \tag{10}$$

where

$$\begin{aligned} p^{j,*}(s) := \min_{\lambda_t \geq 0, \forall t\in[0,\infty)} & \sum_{k=0}^{j}\sum_{t=0}^{\infty} \lambda^k_t V^k_{t\to\infty}(s^k_t) \\ \text{s.t.} \quad & \sum_{k=0}^{j}\sum_{t=0}^{\infty} \lambda^k_t = 1, \ \sum_{k=0}^{j}\sum_{t=0}^{\infty} \lambda^k_t s^k_t = s, \end{aligned}$$

where s^k_t is the nominal state at time t of the k-th iteration, as defined in (7b). Intuitively, the function $P^j(\cdot)$ assigns to every point in $\mathcal{CS}^j$ the minimum cost-to-go along the trajectories in $\mathcal{CS}^j$.

IV. Robust Learning MPC (RLMPC)

In this section, we present the proposed Robust Learning MPC (RLMPC) algorithm for the linear systems with additive uncertainty (1).

A. The RLMPC Algorithm

The *Robust Learning MPC (RLMPC)* problem solves at each time step t of iteration j the following finite horizon optimal control problem

$$\begin{aligned} & V^{\text{RLMPC},j}_{t\to t+N}(s^j_t) := \\ & \min_{v^j_{t|t},\dots,v^j_{t+N-1|t}} \quad \sum_{k=t}^{t+N-1} \ell\left(s^j_{k|t}, v^j_{k|t}\right) + P^{j-1}(s^j_{t+N|t}) \\ & \text{s.t.} \quad s^j_{k+1|t} = A s^j_{k|t} + B v^j_{k|t}, \\ & \qquad F s^j_{k|t} + G v^j_{k|t} \leq f - \Phi e_t, \ \forall e_t \in \mathcal{E}, \\ & \qquad s^j_{t|t} = s^j_t, \ s^j_{t+N|t} \in \mathcal{CS}^{j-1}, \\ & \qquad k = t, \dots, t+N-1, \end{aligned} \tag{12}$$

where $\Phi := (F + GK)$. Notice that (12) forces the terminal state into the control invariant set $\mathcal{CS}^{j-1}$ defined in (9) and that the terminal cost $P^{j-1}(\cdot)$ is the barycentric function in (10). Furthermore, $\mathcal{E}$ in (12) is assumed to be a robust positive invariant set for the error in (5b). Upon solving (12), the controller applies

$$u^j_t(x_t) := K(x_t - s^j_t) + v^{j,\star}_{t|t} \tag{13}$$

to the system (1), where $v^{j,\star}_{t|t}$ is the first input from the optimal input sequence of (12). The following assumptions on (12) are used throughout the remainder of the paper.

Assumption 1: The following two conditions hold:

(i) The set $\mathcal{E} \subset \mathbb{R}^{n_x}$ is a robust positive invariant set for the error dynamics (5b), i.e., for all $e_0 \in \mathcal{E}$ it holds $e_t(\mathbf{w}_{t-1}) \in \mathcal{E}$ for all $\mathbf{w}_{t-1} = [w_0, \dots, w_{t-1}] \in \mathbb{W}^t$ and all $t \geq 1$.

(ii) We are given an initial state and input sequence $(\mathbf{s}^0, \mathbf{v}^0)$ that satisfies the constraints (5a) and $Fs^0_t + Gv^0_t \leq f - \max_{e\in\mathcal{E}}\{\Phi e\}$ for all $t \geq 0$, where $\mathcal{E}$ is a robust positive invariant set for (5b). □

Part (ii) in Assumption 1 is not restrictive in many practical applications. For linear systems with additive uncertainty, it is possible to compute robust positive invariant sets and tighten the constraints accordingly [21]. This would allow us to formulate a conservative optimal control problem to find a feasible trajectory which can be used to initialize the RLMPC.

Remark 1: One difficulty in solving (12) is that the state and input constraints must be satisfied robustly for all $e_t \in \mathcal{E}$. While ensuring this condition can be hard for general sets $\mathcal{E}$, it turns out that this can be done efficiently for most convex sets $\mathcal{E}$ of practical interest [24]. Therefore, from a practical point of view, the set $\mathcal{E}$ is often chosen to be a polytope or an ellipsoid.

B. Feasibility, Stability and Convergence of RLMPC

In this section, we analyze the control law generated by the RLMPC algorithm (12) and (13) when it is applied in closed loop to the uncertain system (1).

Theorem 1: Consider system (1) in closed loop with the RLMPC (12) and (13). Let Assumption 1 hold. Then RLMPC (12), (13) is feasible for all times $t \geq 0$ and all iterations $j \geq 1$. Moreover, the closed-loop system (1), (13) converges asymptotically to the set $\mathcal{E}$ for every iteration $j \geq 1$ and all $w \in \mathbb{W}$. □

Proof: Note that the system dynamics in (5a) and (12) are identical. Moreover, the optimization problem (12) is deterministic. Therefore recursive feasibility of RLMPC (12), (13) follows from [22, Theorem 1]. Moreover from [22, Theorem 1] we have that $\lim_{t\to\infty} s^j_t = 0, \ \forall j \geq 1$. From the definition of $e_t = x^j_t - s^j_t$ we have $\lim_{t\to\infty} x^j_t = \lim_{t\to\infty}(s^j_t + e^j_t) = \lim_{t\to\infty} e^j_t \in \mathcal{E}, \ \forall j \geq 1$ where the last inclusion holds by virtue of Assumption 1. ■

Before stating the convergence properties we introduce the following definitions which will be used in the upcoming *Theorem 2*.

Definition 1 (one-step successor set from the set $\mathcal{S}$): For the system (5a) we denote the *one-step successor set from the set $\mathcal{S}$* as

$$\mathcal{R}_1(\mathcal{S}) = \text{Succ}(\mathcal{S}) \cap \mathcal{X}. \tag{14}$$

where

$$\begin{aligned} \text{Succ}(\mathcal{S}) \triangleq \{s \in \mathbb{R}^{n_x} \ : \ & \exists \bar{s} \in \mathcal{S}, \exists v \in \mathbb{R}^{n_u} \\ & \text{s.t. } A\bar{s} + Bv = s\}. \end{aligned} \tag{15}$$

Definition 2 (one-step controllable set to the set $\mathcal{S}$): For the system (5a) we denote the *one-step controllable set to the set* $\mathcal{S}$ as

$$\mathcal{K}_1(\mathcal{S}) = \text{Pre}(\mathcal{S}) \cap \mathcal{X}. \tag{16}$$

where

$$\text{Pre}(\mathcal{S}) \triangleq \{s \in \mathbb{R}^n \ : \ \exists v \in \mathbb{R}^{n_u} \text{ s.t. } As + Bv \in \mathcal{S}\}. \tag{17}$$

Theorem 2: Consider RLMPC (12) and (13) in closed loop with system (1). Let Assumption 1 hold. Then the iteration cost $V^j_{0\to\infty}(x_S)$ is non-increasing in the iteration j. Furthermore, if the RLMPC (12), (13) converges to a steady state solution $(\mathbf{s}^\infty, \mathbf{v}^\infty)$ for which $s^\infty_k \in \text{Int}(\text{Pre}(s^\infty_{k+1}))$ and $s^\infty_{k+1} \in \text{Int}(\text{Succ}(s^\infty_k))$ for all $k \geq 0$, then $(\mathbf{s}^\infty, \mathbf{v}^\infty)$ is the optimal solution to the following finite-horizon robust optimal control problem

$$\begin{aligned} V^\star_{T,\mathcal{E}}(x_S) :=& \\ \min_{v_0, v_1, \dots} \quad & \sum_{t=0}^{T-1} \ell\left(s_t, v_t\right) \\ \text{s.t.} \quad & s_{t+1} = As_t + Bv_t, \ s_0 = x_S, \\ & Fs_t + Gv_t \leq f - \Phi e_t, \ \forall e_t \in \mathcal{E} \\ & t = 0, \dots, T \\ & s_T = s^\infty_T \end{aligned} \tag{18}$$

for all $T \geq 0$, where e_t is confined to the robust positive invariant set $\mathcal{E}$. □

Proof: Follows from (12) being deterministic and [22, Theorems 2-3]. ■

Theorems 1-2 ensure that, whenever a feasible initial trajectory $(\mathbf{s}^0, \mathbf{v}^0)$ is given, then the RLMPC algorithm generates a sequence of robustly feasible policies $\mathbf{u}^j(\cdot) := [u^j_0, u^j_1(\cdot), u^j_2(\cdot), \dots]$ with non-deteriorating performance (in terms of objective value). The following corollary establishes the relationship between the proposed RLMPC algorithm and our original optimal control problem (3).

Corollary 1: Let Assumption 1 hold. Then, for each iteration $j \geq 1$, the control policy $\mathbf{u}^j(\cdot)$ obtained from the RLMPC algorithm (12) and (13) is a feasible solution to the original infinite-horizon optimal control problem (3). □

Proof: It follows from (5) that constraint (2) is satisfied if and only if $Fs_t + Gv_t + \Phi e_t(\mathbf{w}_{t-1}) \leq f$ for all $\mathbf{w}_{t-1} \in \mathbb{W}^t$. By virtue of Assumption 1, $e_t(\mathbf{w}_{t-1}) \in \mathcal{E}$ for all $\mathbf{w}_{t-1}$ and all $t \geq 0$. Therefore, any (s_t, v_t) satisfying $Fs_t + Gv_t \leq f - \max_{e_t \in \mathcal{E}}\{\Phi e_t\}$ satisfies $Fs_t + Gv_t \leq f - \max_{\mathbf{w}_{t-1}\in\mathbb{W}^t}\{\Phi e_t(\mathbf{w}_{t-1})\}$. ■

V. Illustrative Example: L_1-Regulator

In this section, we apply the proposed Robust LMPC algorithm to compute feasible solutions to the following iterative infinite horizon optimal control problem

$$\begin{aligned} V^{j,\star}(x_S) =& \\ \min_{u^j_0, u^j_1(\cdot), \dots} \quad & \sum_{t\geq 0} \left\| x^j_t\left(\mathbf{0}\right)\right\|_1 + 10 \left\| u^j_t\left(x^j_t\left(\mathbf{0}\right)\right)\right\|_1 \\ \text{s.t.} \quad & x^j_{t+1} = \begin{bmatrix} 1.2 & 1.5 \\ 0 & 1.3 \end{bmatrix} x^j_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u^j_t(x^j_t) + w^j_t, \\ & \begin{bmatrix} -10 \\ -10 \\ -1 \end{bmatrix} \leq \begin{bmatrix} x^j_k \\ u^j_k \end{bmatrix} \leq \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix}, \ \forall w^j_t \in \mathbb{W}, \\ & x^j_0 = x_S, \ t = 0, 1, \dots, \end{aligned} \tag{19}$$

where $w_t \in \mathbb{W} = \{w \in \mathbb{R}^2 : ||w||_\infty \leq 0.1\}$. In the following we solve the above problem for four arbitrarily chosen initial states $x_S \in \{[-6.7, 1.4], [4.6, 0.8], [-2.3, -0.3], [5.6, -0.9]\}$ to test the consistency of the proposed algorithm.

The RLMPC in (12), (13) is implemented with a control horizon of $N = 3$, and the feedback gain K in (13) is chosen to be the optimal LQR gain for system (5a) with parameters $Q = I$ and $R = 10$. The robust positive invariant set $\mathcal{E}$ in (12) is chosen to be the minimal robust positive invariant set which we computed using the approximation scheme presented in [21]. A feasible initial trajectory to initialize $\mathcal{CS}^0$ and $P^0(\cdot)$ was found by applying the rigid tube MPC method of [21]. The linear programs (LP) arising in the RLMPC algorithm are solved in Matlab with the solver GUROBI [25] using the Yalmip interface [26]. For each iteration j, the RLMPC schemes was terminated at step t whenever $V^{\text{RLMPC},j}_{t\to t+N}(s^j_t) \leq 10^{-8}$.

A. Number of Iterations

Table I lists the iteration cost $V^j_{0\to\infty}(x_S)$ for $j = 0, 1, \dots, 9$ and four different initial conditions. In accordance with Theorem 1, we see that the iteration costs are non-increasing from iteration to iteration. The numerical simulations indicate that, empirically, the RLMPC algorithm converges after $j \leq 9$ iterations for all four initial states. For each initial condition, the bold number in their respective column indicate the iteration at which

$$\frac{|V^j_{0\to\infty}\left(x^i_S\right) - V^\star_{T,\mathcal{E}}(x^i_S)|}{V^\star_{T,\mathcal{E}}(x^i_S)} \leq 10^{-4},$$

where $V^\star_{T,\mathcal{E}}(x^i_S)$ is the optimal solution of the finite horizon optimal control problem (18)[2]. Finally, we remark that for each iteration j and each initial condition, no more than 15 LPs needed be solved, with an average solution time of 3ms for each LP on a MacBook Pro that is equipped with an i7 2.5GHz processor with 16GB of RAM.

B. Growth of Safe Set

In this section, we examine the growth of the convex sampled safe sets $\mathcal{CS}^j = \text{conv}(\mathcal{SS}^j)$ as a function of the iteration j. For the purpose of illustration, Figure 2 depicts

[2]For this numerical example, the finite horizon solution was obtained by solving (18) with $T = 20$, a horizon long enough such that s^∞_T has already converged to the origin for all initial conditions.

TABLE I

ITERATION COST $V^j_{0\to\infty}(x^i_S)$ FOR FOUR INITIAL CONDITIONS x^1_S, x^2_S, x^3_S AND x^4_S.

Iteration	Iteration cost $V^j_{0\to\infty}(x^i_S)$ $i=1$	$i=2$	$i=3$	$i=4$
$j=0$	94.03853	84.96294	83.15883	95.45890
$j=1$	74.15600	67.40028	69.31017	78.49602
$j=2$	71.69370	66.13863	67.46463	77.78788
$j=3$	70.80008	65.96067	66.83440	77.25786
$j=4$	70.68970	65.94979	66.81273	77.15558
$j=5$	70.68759	65.94899	66.81003	77.13188
$j=6$	70.68756	65.94893	66.80986	**77.13184**
$j=7$	70.68755	65.94892	66.80986	77.13184
$j=8$	**70.68755**	**65.94892**	66.80985	77.13184
$j=9$	70.68755	65.94892	**66.80985**	77.13184
$V^\star_{\mathcal{E}}(x^i_S)$	70.68751	65.94888	66.80985	77.13178

the evolution of $\mathcal{SS}^j$ for x^1_S only, where the first sampled set $\mathcal{SS}^0$ was constructed from the solution of the rigid tube MPC method presented in [21]. We see from Figure 2 that $\mathcal{SS}^j$ grows as the iteration number increases, until the RLMPC algorithm has converged at the 8-th iteration, after which also the safe set has converged, see also column two in Table I. Notice from Figure 2 that the safe sets $\mathcal{SS}^j$ do not grow uniformly in all directions, but rather only in those that are "relevant" in improving our cost function. In fact, it turns out in practice that $\mathcal{SS}^\infty$ is often much smaller than the maximal control (or positive) invariant set used in tradition (robust) MPC for approximating the terminal set. We will come back to this in the next section. We finish this subsection by pointing out that, it is the growth of $\mathcal{SS}^j$ that allows RLMPC to iteratively improve its cost $V^j_{0\to\infty}(\cdot)$ as shown in Table I.

C. Control Performance

It is well-known that conservativeness and numerical efficiency of control methods are problem dependent. Moreover there exists a considerable body of literature on effective MPC schemes for uncertain systems, see e.g. [14], [21], [27] and the references therein. Nevertheless we believe that the following analysis of the previous example will provide insights into our RLMPC algorithm.

Figure 1 compares the initial state trajectory $\mathbf{s}^0$ with the steady state solution $\mathbf{s}^\infty$ of RLMPC, together with ten random realizations of $\mathbf{x}^\infty(\mathbf{w})$. We notice that $\mathbf{s}^\infty$ clearly deviates from $\mathbf{s}^0$, explaining the decrease in cost from 94.0 to 70.7 (column 2, Table I). Figure 1 also depicts (solid black line) the robust positive invariant set which is the set of all feasible initial condition for the rigid tube MPC method [21]. We see that, for this particular example, the RLMPC method outperforms the rigid tube MPC method since RLMPC converges to a solution that is deemed infeasible by the rigid tube MPC algorithm. Furthermore, we see in Figure 1 that the (disturbed) states $\mathbf{x}^\infty(\mathbf{w})$ indeed converge to the set $\mathcal{E}$, empirically validating Theorem 1.

Finally, we examine the conservatism of the proposed RLMPC algorithm with respect to (6), the infinite horizon robust optimal control problem with linear state feedback

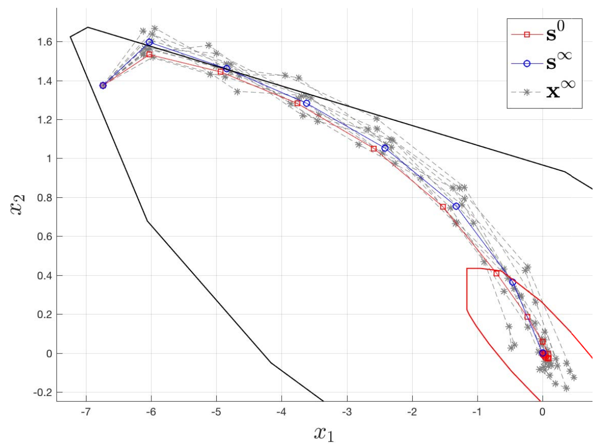

Fig. 1. Comparison between the first feasible trajectory $\mathbf{s}^0$ (red squares) and the steady state solution to the RLMPC $\mathbf{s}^\infty$ (blue circles). Solid black line depicts feasible region obtained from the rigid tube MPC method [21].

policy. Recall that to guarantee robust constraint satisfaction and persistent feasibility in our RLMPC scheme, we introduced the robust positive invariant set $\mathcal{E}$ in (12) to tighten the constraints. Since $\mathcal{E}$ is typically larger than what the exact propagated error set is, conservatism is introduced. To estimate the induced conservatism, we compare the solution of RLMPC to that of (6). For our small example, (6) can be solved[3] and the associated optimal input trajectory. Numerical simulations indicate that the optimal value of (6) is 66.45, which is lower than the solution obtained by our RLMPC algorithm of $V^\infty_{0\to\infty}(x^1_S) = V^\star_{\mathcal{E}}(x^1_S) = 70.69$. As expected, the robust positive invariant set $\mathcal{E}$ indeed induces conservatism into the control design.

VI. CONCLUSIONS

In this paper, we have extended the standard Learning MPC (LMPC) algorithm to uncertain systems that are subject to external disturbances and whose constraints must be satisfied robustly for all uncertainty realizations. Given the knowledge of a feasible trajectory, the proposed Robust LMPC (RLMPC) algorithm has been shown to iteratively improve the control performance while ensuring persistent feasibility and robust constraint satisfaction. Compared to most existing iterative Robust MPC methods, the proposed RLMPC algorithm is reference-free and learns from past iterations to improve its control performance. Our numerical example has demonstrated that the performance of the presented RLMPC is constrained through the introduction of a robust positive invariant set for the error dynamics. Future work concentrates on finding better approximations by exploiting knowledge from past iterations, and by co-optimizing the size of the uncertainty set in the optimization problems using the techniques of [28].

[3]Similar as before, the infinite horizon solution was obtained with $N = 20$, a horizon long enough to drive s_t to the origin.

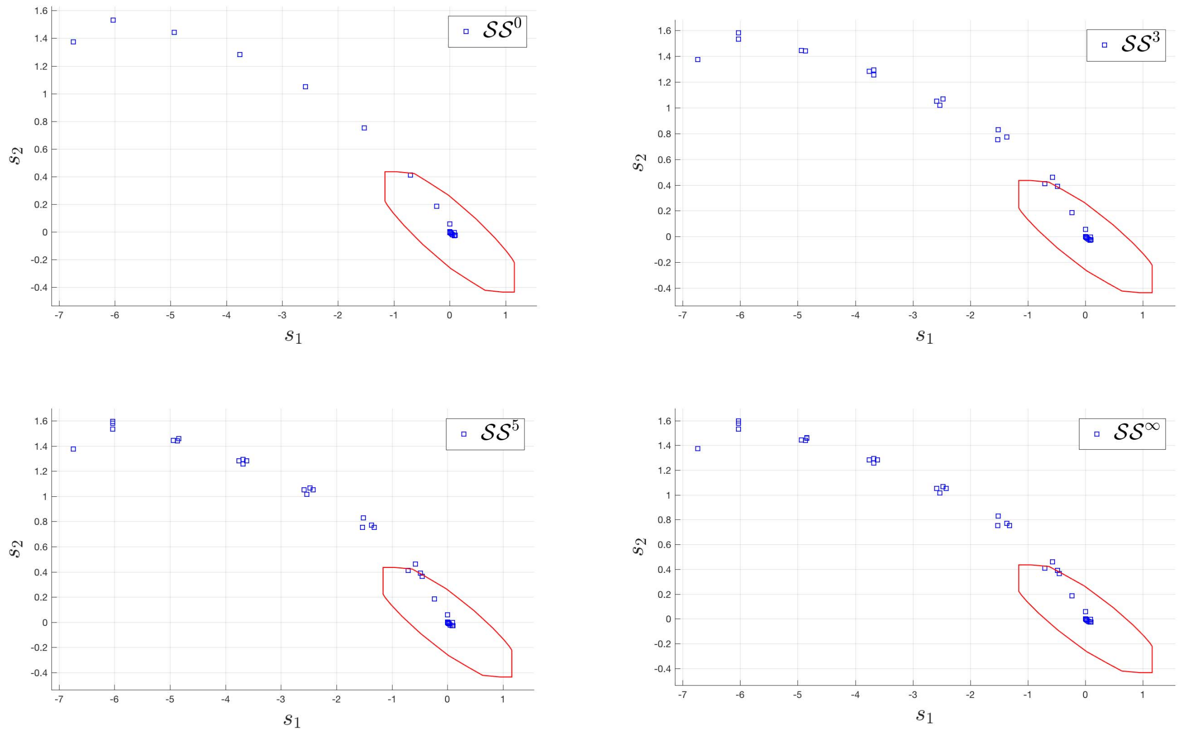

Fig. 2. Evolution of sampled safe $\mathcal{SS}^{j-1}$ set used to construct $\mathcal{CS}^{j-1} = \text{conv}(\mathcal{SS}^{j-1})$ for $j = 1, 3, 5, 9$ (converged) and $x_S = [-6.7, 1.3]$. $\mathcal{SS}^0$ (top left) was obtained from the rigid tube MPC method [21], while the others were generated from the RLMPC algorithm. The (red) set containing the origin shows $\mathcal{E}$.

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