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CONTROL OF HIGHWAY VEHICLES FOR MINIMUM FUEL CONSUMPTION OVER VARYING TERRAIN†

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Abstract—In this paper we develop a feedback algorithm for driving a highway vehicle for minimum fuel consumption. The algorithm allows the driver to choose an arbitrary steady state velocity for a level road, but modifies the speed for optimal fuel consumption on various grades. The algorithm is derived from the Pontryagin maximum principle to provide a mathematically optimal performance subject to the driver's choice of a trip time limit. Computer simulation is used to illustrate the implications of this analysis for gear shifts, braking and throttle settings as the road grade changes.

1. INTRODUCTION

Recent shortages of gasoline and other petroleum fuels have renewed interest in factors affecting the fuel economy of highway vehicles. Various proposals for reducing fuel consumption have been made, ranging from simple modifications such as the replacement of bias-ply tires with radial tires, to extensive design changes such as the development of alternative fuel sources and power plants. A noticeable trend toward smaller and more efficient automobiles has appeared in new car sales, and fuel economy has become a selling point in automotive advertisement. Most of the proposals for conserving fuel resources have involved vehicle design changes. One exception is the national reduction of highway speed limits to 55 m.p.h. This is an energy management program which affects existing vehicles as well as future designs. This speed reduction has led conscientious drivers to try to conserve fuel by holding their speed as near as possible to the 55 m.p.h. limit under all road conditions. Other research has suggested that a better conservation policy might be to maintain a constant throttle setting, rather than constant speed. This paper examines the problem of defining an optimal program for conserving fuel under varied road conditions.

In this paper we approach the problem of energy efficient driving as a question of mathematical optimization. We develop a mathematical model for automobile fuel consumption and obtain the optimal control algorithm for this model. We further analyze its performance in a numerical simulation of a hypothetical compact automobile to determine the general implications for an energy conscious driver. These results confirm the procedures of professional economy drivers and suggest a strategy which is a compromise between the constant speed and constant throttle policies mentioned above. Although this conclusion appears self evident in retrospect, one can readily convince most laymen that either the constant speed or constant throttle strategy suggested above is optimal.

No single fuel economy measure in driving is likely to have a major effect on the overall energy consumption in the United States, but a continued attention to automobile fuel conservation measures may carry over into other areas of energy use and multiply the effect. The primary objective of this study has been the mathematical verification of various empirical driving tips for economical driving. A possible practical consequence of these observations is the posting of lower suggested speeds on upslopes and higher ones on the downgrades of highways. The results of our simulation suggest that a 15% variation from the level road speed limit would be appropriate for a 10% grade, not an excessive speed variation. The full implementation of the optimal fuel economy algorithm developed below would involve sensing impending changes in road grade before they occur, and would be technologically difficult. On the other hand, a near optimal algorithm which modifies the throttle setting to its optimal value for the road grade and vehicle speed observed at the time could be implemented easily in an electronic micro-processor unit for present day automobiles. Although such a device would be most appropriate for open roads, its use in controlling vehicle acceleration and gear shifting in urban driving might also prove beneficial.

2. VEHICLE DYNAMICS AND FUEL CONSUMPTION MODEL

Fuel consumption

The quantity of fuel used by a highway vehicle in a given period of time is the product of the power consumed

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times the rate of fuel consumption per unit of power under the operating conditions of the vehicle. The rate of fuel consumption for internal combustion engines depends on the engine speed and on the fraction of available power (load) being used. The typical form of this relationship is indicated in Figs. 1 and 2. As indicated by these graphs, automobile engines are most efficient when operating at about half their rated speed, and at about 80%

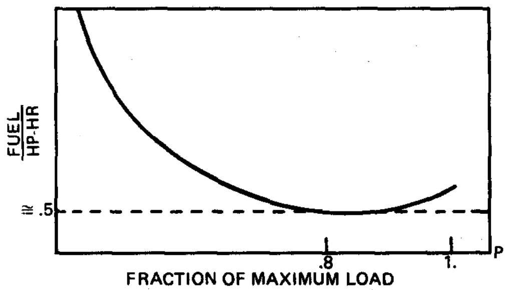

Fig. 1. Typical fuel consumption rate per unit of power for an internal combustion engine expressed as a function of engine load p (fraction of available power).

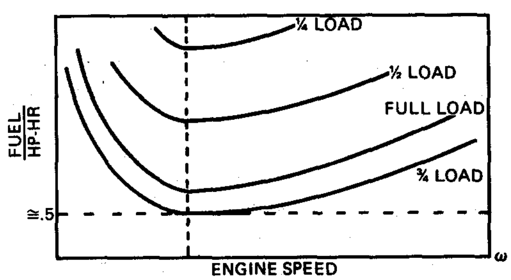

Fig. 2. Typical fuel consumption rate per unit of power for an internal combustion engine expressed as a function of engine speed ω for various loads p.

load. The engine speed and load are determined by the vehicle power demand and gear ratios, however, so that this optimal relationship cannot be maintained in actual practice. Overdrives operate to improve economy by allowing the automobile engines to be run at moderate speed and higher loads without altering the speed of the vehicle. According to Figs. 1 and 2, an approximate model for the rate of consumption per unit of power of an internal combustion engine is

$$\frac{R(p)\,Q(\omega)}{p\omega} \tag{2.1}$$

where ω is engine speed, p is the fraction of available power used, and where R and Q are second order polynomials

$$\begin{aligned} R(p) &= A_2p^2 + A_1p + A_0 \\ Q(\omega) &= B_2\omega^2 + B_1\omega + B_0, \end{aligned} \tag{2.2}$$

whose coefficients depend on the engine fuel consumption curves (Figs. 1 and 2).

Internal combustion engines are capable of differing horsepowers at different engine speeds. Typical automotive engines develop maximum power at about 4600 r.p.m. as indicated in Fig. 3. The torque H produced by an automobile engine is also dependent on the engine speed:

$$\text{horse power} = \frac{2\pi H(\omega)\,\omega}{550}$$

where ω is given in revolutions per second. Typical torque curves are parabolic and have maximum value around 2600 r.p.m. as indicated in Fig. 4. We are thus led to a

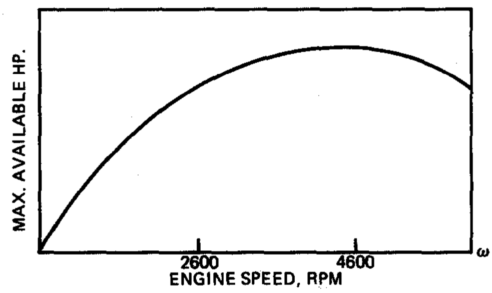

Fig. 3. Typical maximum power output for an automotive internal combustion engine as a function of engine speed ω.

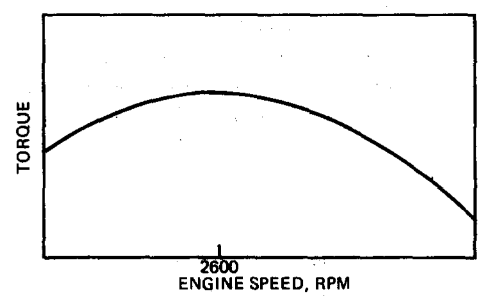

Fig. 4. Typical maximum torque output for an automotive internal combustion engine as a function of engine speed ω.

model for engine output in which the maximum torque H at engine speed ω is given by

$$H(\omega) = K_2\omega^2 + K_1\omega + K_0 \tag{2.3}$$

and consequently the available engine power P is given by

$$P(\omega) = \frac{2\pi}{550} H(\omega)\,\omega. \tag{2.4}$$

The requirement that the maximum torque and horsepower occur at specified speeds ω_H and ω_P, respectively, imposes the conditions

$$\begin{aligned} 0 &= 3\,K_2\omega_P^2 + 2\,K_1\omega_P + K_0 \\ 0 &= 2\,K_2\omega_H + K_1, \end{aligned}$$

and consequently

$$\begin{aligned} K_0 &= \omega_P(4\,\omega_H - 3\,\omega_P)\,K_2 \\ K_1 &= -2\,\omega_H K_2. \end{aligned} \tag{2.5}$$

Using (2.4) as the model for the available engine power we can express the fuel consumption per unit of time as

$$\text{consumption rate} = \frac{R(p)\,Q(\omega)\,p\,P(\omega)}{p\omega} = R(p)\,Q(\omega)\frac{2\pi}{550}H(\omega).$$

Thus the total fuel used in a trip of duration T is given by integral

$$\text{fuel} = \int_0^T \frac{2\pi}{550} R[p(t)]\,Q[\omega(t)]\,H[\omega(t)]\,\mathrm{d}t. \tag{2.6}$$

Vehicle dynamics

The differential equation describing the velocity of a highway vehicle is

$$M\dot{v} = -MgC_r - \beta v - \tfrac{1}{2}\rho C_d A v^2 - Mg\sin[\theta(s)] + F \tag{2.7}$$

where: M, mass of the vehicle; v, velocity of the vehicle; g, acceleration of gravity; C_r, coefficient of rolling friction (0.015 for bias-belted tires); β, braking force ($=0$ unless the brakes are applied); ρ, ambient air density; C_d, drag coefficient (approximately 0.5 for most passenger cars); A, frontal area (20 to 25 square feet for most passenger cars); $\theta(s)$, angle of inclination of the road surface at a distance s from the origin; F, propulsive force. The force F is determined by the throttle setting and gear ratio at a given engine speed through the relationship

$$\text{power} = \frac{\pi D F \omega}{550\,\sigma\,\delta} = \frac{2\pi}{550} H(\omega)\,\omega \tag{2.8}$$

where: D, vehicle tire diameter; σ, transmission reduction ratio; δ, differential reduction ratio. Thus for a given engine power output $pP(\omega)$ we can compute the propulsive force

$$F = \frac{p\,\sigma\,\delta\,2H(\omega)}{D} \tag{2.9}$$

in (2.7).

Optimal control model

We simplify our model somewhat by using the relation

$$v = \frac{\pi D \omega}{\sigma \delta}$$

to eliminate ω from (2.6) and (2.7), and absorbing all constants into general coefficients. Thus we pose the optimal control problem: minimize the cost

$$J = \int_0^T K\,r[p(t)]q[\sigma(t)v(t)]h[\sigma(t)v(t)]\,\mathrm{d}t \tag{2.10}$$

subject to

$$\dot{v} = -c_0 - \beta v - c_2 v^2 - g\sin(\theta(s)) + p\sigma h(\sigma v) \tag{2.11}$$
$$\dot{s} = v. \tag{2.12}$$

The functions r, q, and h are all second order polynomials

$$r(z) = a_2 z^2 + a_1 z + a_0$$
$$q(z) = b_2 z^2 + b_1 z + b_0$$
$$h(z) = d_2 z^2 + d_1 z + d_0$$

and K is a constant. The system (2.10)–(2.12) is a standard nonlinear optimal control system with control variables p, β and σ with values

$$p\in[0,1];\quad \beta\in[0,B];\quad \text{and}\quad \sigma\in\{\sigma_1,\sigma_2,\ldots,\sigma_k\}$$

where the numbers $\sigma_1, \sigma_2, \ldots, \sigma_k$ are the transmission reduction ratios for the different gears. Boundary conditions are given by

$$\begin{aligned} s(0) &= 0, & s(T) &\geq S_0 \\ v(0) &= 0, & v(T) &= 0. \end{aligned} \tag{2.13}$$

These boundary conditions are appropriate for seeking the most economical control program for a specified trip. This is not really what we want, however. The optimal steady state speed for most cars is between 15 and 35 miles per hour, a speed currently considered too slow for highway traffic. What we really seek is a program for traversing the road without going "too slow". We will thus specify that the trip not take longer than a specified time:

$$T \leq T_0. \tag{2.14}$$

The problem (2.10)–(2.14) is our optimal control problem.

3. SYSTEM OPTIMIZATION

Finite time optimal program

We approach the problem of minimizing fuel consumption as a problem in nonlinear optimal control which can be solved using the Pontryagin maximum principle (Pontryagin *et al.*, 1962). In applying this principle we form the function

$$\begin{aligned} \mathscr{H}(t,v,s,p,\sigma,\beta,\lambda) = &-\lambda_0 K\,r(p)\,q(\sigma v)\,h(\sigma v) \\ &+ \lambda_1\{-c_0 - \beta v - c_2 v^2 - g\sin[\theta(s)] \\ &+ p\,\sigma\,h(\sigma v)\} + \lambda_2 v. \end{aligned} \tag{3.1}$$

The maximum principle says that the optimal solution to the fuel economy problem, if one exists, must satisfy the condition

$$\begin{aligned} \bar{\mathscr{H}}(t) &\equiv \mathscr{H}(t,\bar{v}(t),\bar{s}(t),\bar{p}(t),\bar{\sigma}(t),\bar{\beta}(t),\lambda(t)) \\ &= \max_{\sigma,p,\beta} \mathscr{H}(t,\bar{v}(t),\bar{s}(t),p,\sigma,\beta,\lambda(t)) \end{aligned} \tag{3.2}$$

where the overbar above indicates functions that are evaluated along the optimal program. As a consequence of (3.2) we see that if $\lambda_1 > 0$, then $\beta = 0$ and if $\lambda_1 < 0$, $\beta = B$. Furthermore, if $\bar{p} \in (0,1)$ then the optimal throttle setting $\bar{p}$ may be found by setting the partial derivative $\partial_p \mathscr{H}$ equal to zero. Doing this we obtain

$$\begin{aligned} \lambda_1(t) &= \lambda_0 K\,r'(\bar{p}(t))\,q[\bar{\sigma}(t)\bar{v}(t)]/\bar{\sigma}(t) \\ &= \lambda_0 K\,[2a_2\bar{p}(t) + a_1]q[\bar{\sigma}(t)]\bar{v}(t)]/\bar{s}(t). \end{aligned} \tag{3.3}$$

The maximum principle also says that the multipliers λ_0,

λ_1 and λ_2 are not all zero at any one point of $[0, T]$ and λ_0 can be chosen to be either 0 or 1. The remaining parts of the maximum principle give us the equations

$$\dot{\lambda}_1(t) = \lambda_0\{K\, r[\bar{p}(t)\ \bar{q}'(\bar{\sigma}(t)\bar{v}(t))\ h(\bar{\sigma}(t)\ \bar{v}(t)) + K\, r[\bar{p}(t)] \times q[\bar{\sigma}(t)\bar{v}(t)]\ h'[\bar{\sigma}(t)\bar{v}(t)]\}\bar{\sigma}(t) - \lambda_1(t)\{-2c_2\bar{v}(t) - \beta + \bar{p}(t)_{\sigma}^{-2}(t)\ h'[\bar{\sigma}(t)\bar{v}(t)]\} - \lambda_2(t). \tag{3.4}$$

$$\dot{\lambda}_2(t) = \lambda_1(t)\ g\ \cos[\theta(\bar{s}(t)]\ \theta'[\bar{s}(t)] \tag{3.5}$$

$$\dot{\bar{\mathscr{H}}}(t) = 0 \tag{3.6}$$

and the end conditions

$$\lambda_2(T) \equiv \nu \geq 0, \qquad \bar{\mathscr{H}}(T) = \mu \geq 0. \tag{3.7}$$

Careful analysis of this system would show that the multiplier λ_0 cannot be zero, and consequently we could solve (3.3) for $\bar{p}$ and use the differential equations(2.11) and (3.4) to find a differential equation for $\bar{p}(t)$ in any fixed gear. Although this is a solution to our problem, it is not very satisfactory. In the first place, this procedure gives us an open loop solution for the problem, rather than a closed loop solution. We have a set of instructions to be carried out at specified times, rather than a response to observed road conditions.

The second drawback to this method of solution is that the resulting system is highly unstable as a function of the parameters (in particular, as a function of $\lambda_2(T) \equiv \nu$). Our ultimate desire is to devise an algorithm for controlling a vehicle on long trips. For this purpose we are interested in the limiting behavior of our vehicle when the trip length S_0 is increased indefinitely. The value assigned to $\lambda_2(T)$ in a numerical simulation determines the trip length S_0. The instability of the system makes it impossible to specify this value with enough precision to prevent divergence of the numerical algorithms in attempts to simulate longer trips.

Our alternate approach is to observe that (3.6) and (3.7) together imply that $\bar{\mathscr{H}}(t)$ must have the constant value μ. For $p \in (0,1)$, we see from (3.3) that $\lambda_1 > 0$ so $\beta = 0$. Inserting (3.3) into (3.1) then gives a quadratic equation for $\bar{p}(t)$ in terms of the other variables:

$$0 = -K\,(a_2\bar{p}^2 + a_1\bar{p} + a_0)\,q(\bar{\sigma}\bar{v})\ h(\bar{\sigma}\bar{v}) + K\,(2a_2\bar{p} + a_1)\,\frac{q(\bar{\sigma}\bar{v})}{\bar{\sigma}}\{-c_0 - c_2\bar{v}^2 - g\sin[\theta(\bar{s})] + \bar{p}\,\bar{\sigma}\,h(\bar{\sigma}\bar{v})\} + \lambda_2\bar{v} - \mu.$$

Here we have suppressed the independent variable t for simplicity of notation. The solution of this equation gives $\bar{p}$ as a function of the variables v, s and σ. Thus if $p \in (0,1)$

$$\bar{p}(v,s,\sigma) = \frac{c_0 + c_2v^2 + g\sin[\theta(s)]}{\sigma\,h(\sigma v)} \pm \frac{1}{\sigma h(\sigma v)}\left\{[c_0 + c_2v^2 + g\sin(\theta(s))]^2 + \frac{\sigma h(\sigma v)}{a_2}\,a_1[c_0 + c_2v^2 + g\sin(\theta(s))] + \frac{h(\sigma v)\sigma^2(\mu - \lambda_2 v)}{a_2\,K\,q(\sigma v)} + (\sigma h(\sigma v))^2\frac{a_0}{a_2}\right\}^{1/2}. \tag{3.8}$$

Substitution of (3.8) into (2.11) shows that the sign of the radical above is the same as that of $\dot{v}$. A more precise determination of this sign is the major difficulty in the implementation of (3.8) as a practical algorithm. In most highway situations the correct sign will be easy to compute, however. Formula (3.8) gives the throttle setting in a closed loop form provided the optimal answer is between 0 and 1. The complete formula is obtained by truncating (3.8) so that $\bar{p}$ is set to 0 if (3.8) gives a value less than or equal to 0 and $\bar{p}$ is set to 1 if the output is as large as 1.

Formula (3.8) gives the program for operating the vehicle provided it has only one gear, or is operating in a range of velocities where no gear shift is called for. If there are multiple gears, then we must develop an algorithm for determining the shift velocities (switching points). For this we make use of the fact that (3.4) implies that λ_1 must be continuous, and shifting can occur only when (3.3) would give the same value for both gear ratios. Applying the maximum principle at $v = 0$, and noting that λ_1 must be positive for the vehicle to accelerate at all, we see that the initial gear reduction ratio must be as large as possible (i.e. the gear is the lowest available). Shifting then occurs sequentially as speed increases. A shift from a gear with reduction ratio σ_i to one with reduction ratio σ_j occurs when

$$\left(2a_2\bar{p}(\bar{v},\bar{s},\sigma_i) + a_1\right)\frac{q(\sigma_i\bar{v})}{\sigma_i} = \left(2a_2\bar{p}(\bar{v},\bar{s},\sigma_j) + a_1\right)\frac{q(\sigma_j\bar{v})}{\sigma_j}. \tag{3.9}$$

Steady state analysis

Formulas (3.8) and (3.9) give a closed loop solution to the fuel minimization problem. The problem of instability of the optimal solution still remains, however. Our goal is to use the algorithm developed above to obtain an approximation to the optimal system for very long trips which is also stable enough to be practically implemented. We first restrict our attention to a vehicle with a single gear ratio operating on a road with constant grade θ. Such a vehicle accelerates from zero velocity and continues to accelerate until the sign of $\dot{\bar{v}}$, and hence the sign of the radical in (3.8), changes. Since λ_1 is continuous, $\bar{p}$ must also be continuous, and this change of sign must occur when the radical in (3.8) is zero. Simulation shows that the longer the trip, the larger the maximum velocity, so we choose λ_2 to be the smallest value such that this radical becomes zero during the trip. This computation corresponds to setting the system parameters to obtain the maximum velocity for any finite trip. (Note that λ_2 appears in (3.8) with a negative sign, so large values tend to make the radical zero at smaller velocities.) The velocity at which the radical becomes zero with this minimum choice of λ_2 is approximately the steady state velocity for a very long trip. We may thus approximate the system behavior for such a trip, by choosing the sign of the radical in (3.8) to be positive until the vehicle begins to slow down at the end of the trip. Using the above procedure we determine a value of λ_2 for any grade angle θ. This is the appropriate value of λ_2 on any long grade of constant slope.

Relation (3.5) gives the change in λ_2 as the road grade changes. In order to understand the implications of this equation it is convenient to assume that at a given distance S_1 from the origin of the trip, the angle changes abruptly from a value θ_1 to the value θ_2. This assumption is outside the hypotheses of the standard maximum principle, so we consider it as the limiting case of (3.5) when the time (i.e. distance) required to make a smooth change from grade angle θ_1 to θ_2 goes to zero. Since λ_1 and $\bar{v}$ both have bounded derivatives, we assume that they are approximately constant on the interval $(t_0, t_0+\Delta t)$ on which the grade angle changes. Integrating (3.5) over this interval we have

$$\lambda_2(t_0+\Delta t)-\lambda_2(t_0) = \int_{t_0}^{t_0+t} \frac{\lambda_1(\tau)}{v(\tau)} g \cos[\theta(\bar{s}(\tau))\, \dot{\bar{s}}(\tau)]\, \mathrm{d}\tau$$

$$\cong \frac{\lambda_1(t_0)}{v(t_0)} g\{\sin(\theta_2)-\sin(\theta_1)\}.$$

That is to say that the change in λ_2 when the road grade changes from θ_1 to θ_2 is

$$\Delta\lambda_2(t_0) = \frac{\lambda_0 K(2a_2\bar{p}(t_0,\bar{v},\bar{\sigma})+a_1)\, q(\bar{\sigma}\bar{v})}{\bar{v}\bar{\sigma}} \cdot g\{\sin(\theta_2)-\sin(\theta_1)\}. \qquad (3.10)$$

If the hill with new grade θ_2 is of substantial length then the difference determined in (3.10) must agree with the change of values of λ_2 determined by the steady state behavior of the vehicle as in the paragraph above. This means that at the instant t_0 the optimal velocity $\bar{v}(t_0)$ must be such as to make the two values coincide. We will see in the simulation at the end of this paper that this implies that the optimal velocity must increase at the base of a hill, and then drop off as the vehicle starts to climb. A reverse behavior holds at the crest of a downslope.

4. SIMULATION RESULTS

Although the previous sections have specified the optimal algorithm for minimizing fuel consumption, the qualitative implication of this analysis is not clear. In this section we illustrate the behavior of a vehicle operating under this optimal program through a computer simulation. The vehicle we model is a small passenger sedan with moderate power and three forward gears. Some of the parameters had to be estimated (e.g. the coefficients of R and Q), so the magnitude of some of the results below is suspect, but the general pattern illustrates what one would expect from the actual vehicle. We assume the following parameters in (2.7) and (2.8).

power = 90 h.p. at 4600 r.p.m.
weight = 2300 lb ($M = 71.5$ sl)
$A = 20\,\mathrm{ft}^2$
$C_d = 0.5$
$C_r = 0.015$
$D = 2\,\mathrm{ft}$
$\sigma = 3.7$
$\sigma_1 = 3.337$ (first gear)
$\sigma_2 = 1.653$ (second gear)
$\sigma_3 = 1.000$ third gear)

We begin by examining the steady state behavior of the vehicle. Figures 5 and 6 show the asymptotic behavior of the throttle setting and velocity as the car accelerates in third gear. Figure 5 shows the results for the optimal steady state velocity, 30 f/sec, and Fig. 6 shows the same curves where the parameter μ is chosen so that the terminal velocity on level ground is 60 f/sec. Similar curves can be plotted for each value of μ and each angle of slope θ. These curves are plotted with the vehicle in third gear. Figure 7 indicates the optimal throttle behavior and shift points for acceleration to 60 f/sec using all three gears. Shifting occurs at low speeds where engine power is low, producing only moderate acceleration in spite of the high engine load. This suggests that modification of automatic transmissions to produce gear shifting at lower speeds might prove economical for urban driving.

In the remaining Figs. 8–11 we indicate the optimal behavior on hills and downslopes. The illustrations are for

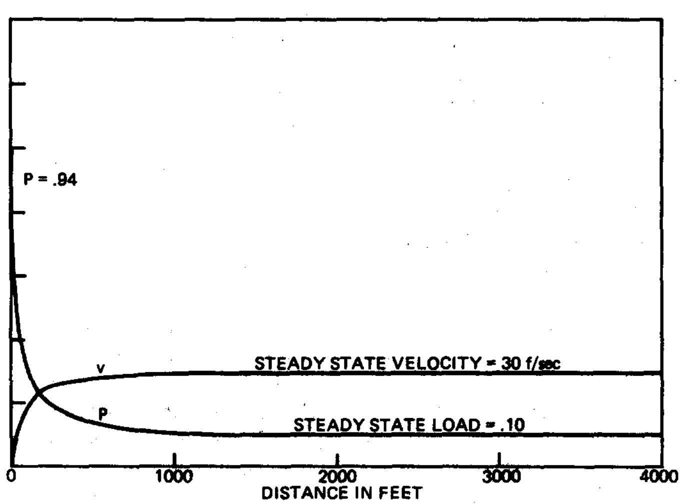

Fig. 5. Optimal initial velocity v and engine load p curves for acceleration to the most economical level road speed.

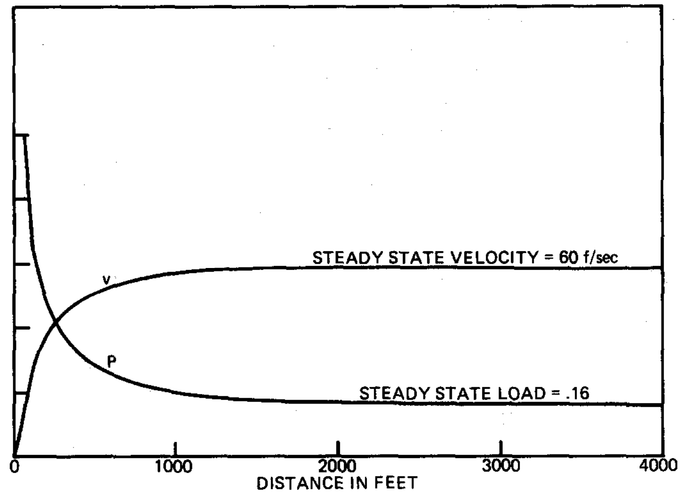

Fig. 6. Optimal initial velocity v and engine load p curves for acceleration to a level road speed of 60 f/sec.

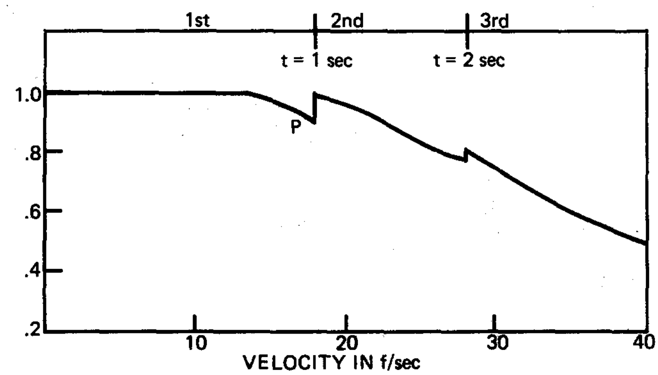

Fig. 7. Optimal gear change points and engine load p for acceleration to 60 f/sec using three gears on a level road.

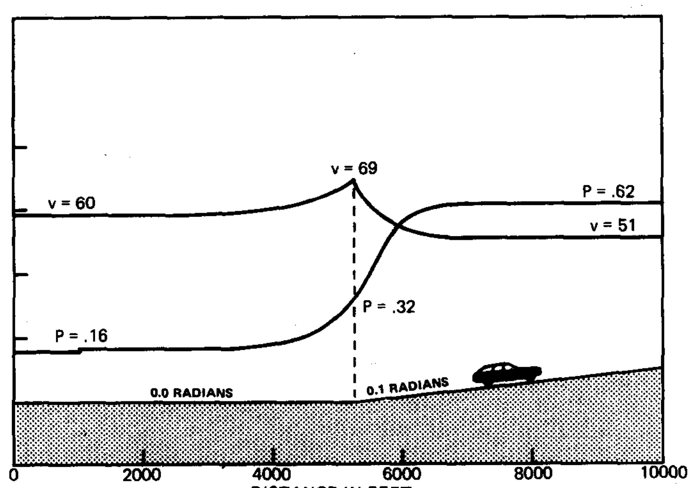

Fig. 8. Optimal velocity v and engine load p when approaching a 10% upslope (0.1 radian) where the steady state velocity on a level road is to be 60 f/sec.

parameters chosen to give a 60 f/sec terminal velocity on a road with zero grade, rather than the optimal velocity. These conditions are more nearly appropriate to highway speeds than the slower optimal speeds of 20 m.p.h.

Figures 8 and 9 show that the optimal behavior on an upslope is to increase speed while approaching the base of the hill, and then allow the speed to drop off while climbing the hill. The appropriate steady state velocity on the upslope is slower than that on level ground, but the throttle setting is greater. Just prior to reaching the crest

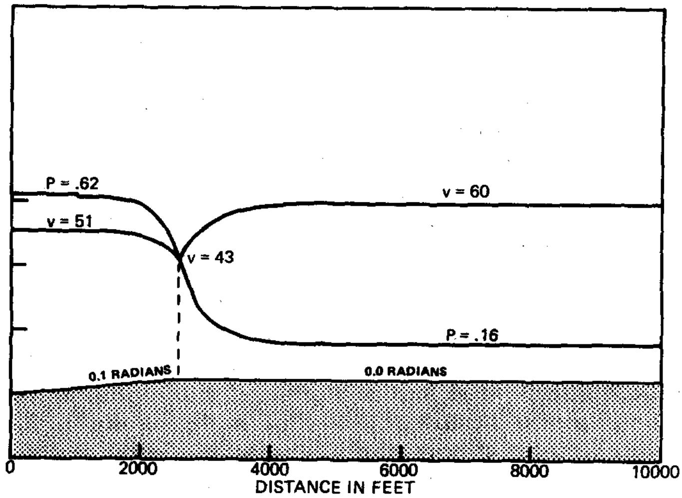

Fig. 9. Optimal velocity v and engine load p when approaching the crest of a 10% upslope (0.1 radian) where the steady state velocity on a level road is to be 60 f/sec.

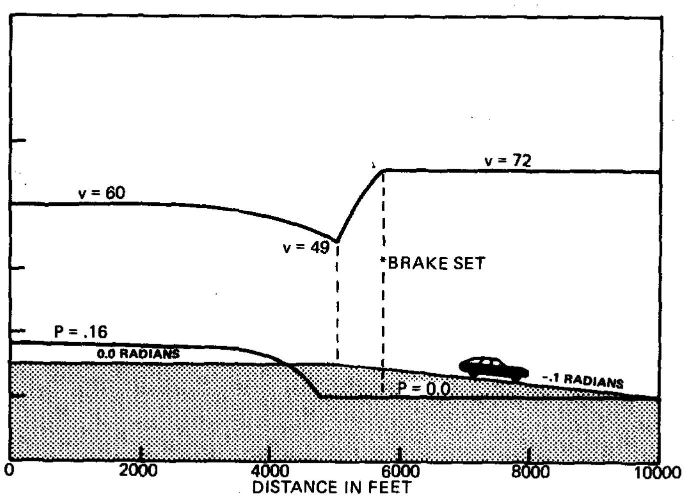

Fig. 10. Optimal velocity v, engine load p and braking behavior when approaching a 10% downslope (− 0.1 radian) where the steady state velocity on a level road is to be 60 f/sec.

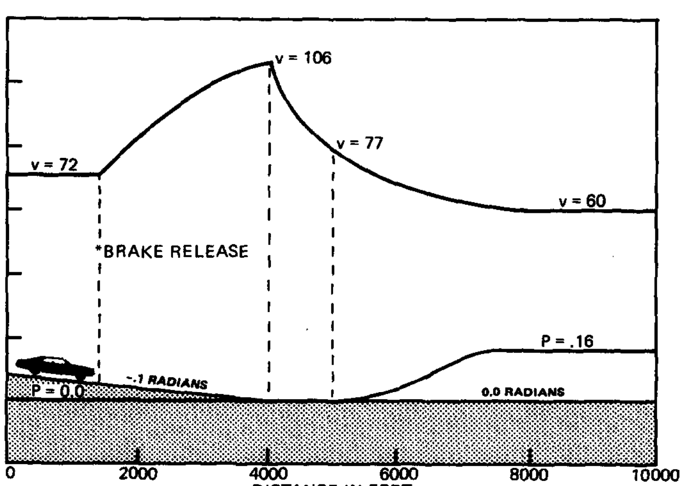

Fig. 11. Optimal velocity v and engine load p when approaching the base of a 10% downslope (− 0.1 radian) with brakes used on the downslope to maintain optimal velocity and with a steady state velocity on a level road of 60 f/sec.

of the hill, the speed drops off slightly and then picks up again on the level road. A reverse behavior is appropriate for a downslope. In this analysis we have assumed that the vehicle remains in gear throughout the trip. Thus when p is zero the motor produces no propulsive force, but the engine will turn over at a speed determined by the vehicle velocity and will use more fuel at higher vehicle speeds. Now recall that if λ_1 is positive then the maximum principle indicated that no braking force is to be applied, and when λ_1 is negative the brakes are to be applied with full force. If $\lambda_1 = 0$, however, the braking force is undetermined by the maximum principle. The dependence of λ_1 on v indicates that if λ_1 should reach zero, then the brakes should be applied with exactly enough force to maintain that velocity. We conclude that if brakes are to be used on a downslope, then λ_2 must be chosen so that the minimum value of λ_1 on the downslope is zero. The optimal program is then indicated in Figs. 10 and 11. The vehicle is allowed to pick up speed until $\lambda_1 = 0$, where its acceleration is checked. Near the base of the downslope the brakes are released and the vehicle is allowed to accelerate again.

The algorithm described above is fairly complex, in that it requires sensing the approach of a change in grade before it occurs. However, a near optimal, non-anticipating algorithm can be obtained by allowing the vehicle behavior to be governed by the algorithm (3.8), where λ_2 is determined for the grade angle at the vehicle location, without insisting that (3.10) hold. Thus we cancel the effect of anticipating grade changes, and simply respond to the changes.

One additional comment is in order. Some existing research has suggested that a constant throttle setting should prove optimal for fuel economy. Indeed if we were to run two identical vehicles with controls set so that they had identical zero grade steady state velocities, the one operating under constant throttle would use less fuel climbing a hill than would the one run under the proposed algorithm. This is not a fair comparison, however, since one should choose settings so that an entire trip is completed in the same time under both algorithms. In such a test the proposed algorithm would effect a saving by producing a more economical velocity on the level grades. An additional advantage of the optimal algorithm is that, although it does not produce a constant velocity, the variations of velocity on differing grades are not as extreme as with a fixed throttle, and should not disturb the flow of highway traffic as badly.

REFERENCES

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