

Disperzija (**varianca**)

D
(
X
)
=
E
(
(
X
−
E
(
X
)

)

2

)
=
E
(

X

2

)
−
(
E
(
X
)

)

2

{\displaystyle D(X)=E((X-E(X))^{2})=E(X^{2})-(E(X))^{2}}

Lastnosti:

- D*(*X*) ≥ 0
- D*(*X*) = 0 ⇐⇒ *P*(*X* = *E*(*X*)) = 1
- D*(*aX*) = *a*²*D*(*X*)

Standardna diviacija/odklon:

σ
(
X
)
=

D
(
X
)

{\displaystyle \sigma (X)={\sqrt {D(X)}}}

zanjo velja *σ*(*aX*) = |*a*|*σ*(*X*).

Nekoreliranost

Sl. sprem. *X* in *Y* sta nekorelirani, če velja:

E
(
X
Y
)
=
E
(
X
)
E
(
Y
)

{\displaystyle E(XY)=E(X)E(Y)}

X,*Y* neodvisni ⇒ *X*,*Y* nekorelirani

Če imata *X* in *Y*, je nekoreliranost ekvivalentna zvezi:

D
(
X
+
Y
)
=
D
(
X
)
+
D
(
Y
)

{\displaystyle D(X+Y)=D(X)+D(Y)}

Kovarianca

K
(
X
,
Y
)
=
E
(
(
X
−
E
(
X
)
)
(
Y
−
E
(
Y
)
)
)
=
E
(
X
Y
)
−
E
(
X
)
E
(
Y
)

{\displaystyle K(X,Y)=E((X-E(X))(Y-E(Y)))=E(XY)-E(X)E(Y)}

- K*(*X*,*X*) = *D*(*X*)
- K*(*X*,*Y*) = 0 ⇐⇒ *X*,*Y*nekorelirani
- K*(*aX*,*bY*,*Z*) = *aK*(*X*,*Z*) + *bK*(*Y*,*Z*)

- K*(*X*,*Y*) = *K*(*Y*,*X*)

- K*(*aX* + *b*, *cY* + *d*) = *acK*(*X*,*Y*)

- |*K*(*X*,*Y*)| ≤ √*D*(*X*)*D*(*Y*)

- D*(*X* + *Y*) = *D*(*X*) + *D*(*Y*) + 2*K*(*X*,*Y*)

- D*(*X*₁ + ... + *X*_{*n*}) = *D*(*X*₁) + ... + *D*(*X*_{*n*}) + 2∑_{*i*=1}^{*n*-1} ∑_{*j*=*i*+1}^{*n*} *K*(*X*_{*i*},*X*_{*j*})

Standardizacija

X

S

=

X
−
E
(
X
)

σ
(
X
)

{\displaystyle X_{S}={\frac {X-E(X)}{\sigma (X)}}}

Korelacijski koeficient

r
(
X
,
Y
)
=

K
(
X
,
Y
)

σ
(
X
)
σ
(
Y
)

=
E
(

X

S

,

Y

S

)

{\displaystyle r(X,Y)={\frac {K(X,Y)}{\sigma (X)\sigma (Y)}}=E(X_{S},Y_{S})}

Lastnosti:

- r*(*X*,*Y*) = 0 ⇐⇒ *X*,*Y*nekorelirani

- −1 ≤ *r*(*X*,*Y*) ≤ 1

- r*(*X*,*Y*) = 1 ⇐⇒ *P*(*X*_{*S*} = *Y*_{*S*}) = 1

- r*(*X*,*Y*) = −1 ⇐⇒ *P*(*X*_{*S*} = −*Y*_{*S*}) = 1

- r*(*aX* + *b*, *cY* + *d*) = *r*(*X*,*Y*)

Pogojne porazdelitve

Pogojna porazdelitev sl. sprem. *X* glede na dogodek *B*:

X

|
B

∼
⎡

P

(
X

=

a

1

|
B
)

P

(
X

=

a

2

|
B
)

⋯

⎣

{\displaystyle X|B\sim \left(P(X=a_{1}|B)\quad P(X=a_{2}|B)\quad \cdots \right)}

Pogojna porazdelitvena funkcija

sl. sprem. *X* glede na dogodek *B*:

F

X
|
B

(
x
)
=

F

X

(
x
|
B
)
=
P
(
X
≤
x
|
B
)
=

P
(
(
X
≤
x
)
∩
B
)
P
(
B
)

{\displaystyle F_{X|B}(x)=F_{X}(x|B)=P(X\leq x|B)={\frac {P((X\leq x)\cap B)}{P(B)}}}

Če je pogojna porazdelitev zvezna, objstaja tudi **pogojna porazdelitvena gostota**:

p

X
|
B

(
x
)
=

F

′

X
|
B

(
x
)

{\displaystyle p_{X|B}(x)=F'_{X|B}(x)}

Pogojna gostota

p

X

(
x
|
Y
=
y
)
≡

p

X

(
x
|
y
)
=

p
(
X
,
Y
)
(
x
,
y
)

p

Y

(
y
)

{\displaystyle p_{X}(x|Y=y)\equiv p_{X}(x|y)={\frac {p_{X,Y}(x,y)}{p_{Y}(y)}}}

Pogojno matematično upanje

E
(
h
(
X
)
|
B
)
=

∑

x

h
(
x
)
P
(
X
=
x
|
B
)

{\displaystyle E(h(X)|B)=\sum _{x}h(x)P(X=x|B)}

E
(
X
|
Y
=
y
)
=

∫

−
∞

∞

x

p

(
X
|
Y
)
(
x
|
y
)
d
x

=

1

p

Y

(
y
)

∫

−
∞

∞

x

p

(
X
,
Y
)
(
x
,
y
)
d
x

{\displaystyle E(X|Y=y)=\int _{-\infty }^{\infty }xp_{(X|Y)(x|y)}dx={\frac {1}{p_{Y}(y)}}\int _{-\infty }^{\infty }xp_{(X,Y)(x,y)}dx}

E
(
h
(
X
,
Y
)
|
Y
=
y
)
=
E
(
h
(
X
,
y
)
|
Y
=
y
)

{\displaystyle E(h(X,Y)|Y=y)=E(h(X,y)|Y=y)}

E
(
h
(
X
,
Y
)
|
Y
)
=

∑

x

h
(
x
,
Y
)
P
(
X
=
x
|
Y
)

{\displaystyle E(h(X,Y)|Y)=\sum _{x}h(x,Y)P(X=x|Y)}

E
(
h
(
X
,
Y
)
|
Y
)
=

∫

−
∞

∞

h
(
x
,
Y
)

P

X
|
Y

(
x
|
Y
)
d
x

{\displaystyle E(h(X,Y)|Y)=\int _{-\infty }^{\infty }h(x,Y)P_{X|Y}(x|Y)dx}

Za vsako slučajno spremenlivko *X* in dogodek *B* veleja:

E
(
X
|
B
)
=

E
(
X
Z
)
P
(
B
)

=

E
(
X
Z
)
E
(
Z
)

{\displaystyle E(X|B)={\frac {E(XZ)}{P(B)}}={\frac {E(XZ)}{E(Z)}}}

kjer je sl. sprem. *Z* indikator dogodka *B*.

Za vsako sl. sprem. *X* z mat. up. in popoln sistem dogodkov *H*₁,*H*₂,... velja **izrek o polni pričakovani vrednosti**

E
(
X
)
=
P
(

H

1

)
E
(
X
|

H

1

)
+
P
(

H

2

)
E
(
X
|

H

2

)
+
.
.
.

{\displaystyle E(X)=P(H_{1})E(X|H_{1})+P(H_{2})E(X|H_{2})+...}

Regresijska funkcija

ϕ
(
y
)
=
E
(
X
|
Y
=
y
)

{\displaystyle \phi (y)=E(X|Y=y)}

Za vsako sl. sprem. *X* z mat. up. in diskretno sl. sprem. *Y* velja:

E
(
X
g
(
Y
)
|
Y
)
=
E
(
X
|
Y
)
g
(
Y
)

{\displaystyle E(Xg(Y)|Y)=E(X|Y)g(Y)}

E
(
X
g
(
Y
)
)
=
E
(
E
(
X
|
Y
)
g
(
Y
)
)

{\displaystyle E(Xg(Y))=E(E(X|Y)g(Y))}

E
(
X
)
=
E
(
E
(
X
|
Y
)
)

{\displaystyle E(X)=E(E(X|Y))}

Za vsak dododek *A* in vsako sl. sprem *Y* velja:

E
(
P
(
A
|
Y
)
)
=
P
(
A
)

{\displaystyle E(P(A|Y))=P(A)}

Momenti

Moment reda *k* glede na točko *a* je

m

k

(
a
)
=
E
(
(
X
−
a

)

k

)

{\displaystyle m_{k}(a)=E((X-a)^{k})}

 če obstaja

- Začetni moment** *z*_{*k*} := *m*_{*k*}(0) = *E*(*X*^{*k*})

- Centralni moment** *m*_{*k*} := *m*_{*k*}(*E*(*X*)) = *E*((*X* − *E*(*x*))^{*k*})

- Faktorski moment** reda *r*: *E*(*X*(*X* − 1) ... (*X* − *r* + 1)) = *G*_{*X*}^(*r*)(1)

z

1

=
E
(
X
)

m

2

=
D
(
X
)

{\displaystyle z_{1}=E(X)\qquad m_{2}=D(X)}

Če obstaja *m*_{*n*}(*a*), obstaja tudi *m*_{*k*}(*a*) za ∀*k* < *n*.

Če obstaja *z*_{*n*}, obstaja tudi *m*_{*n*}(*a*) za ∀*a* ∈ ℝ

Centralne momente lahko izračunamo iz začetnih:

m

n

=

∑

k
=
0

n

n

k

(
−
1

)

n
−
k

z

1

n
−
k

z

k

{\displaystyle m_{n}=\sum _{k=0}^{n}{n \choose k}(-1)^{n-k}z_{1}^{n-k}z_{k}}

Asimetrija

A
(
X
)
=
E
(

X

S

3

)
=
E
⎡
⎡

X
−
E
(
X
)

σ
(
X
)

3

⎣
=

m

3

m

2

3

2

{\displaystyle A(X)=E(X_{S}^{3})=E\left(\left({\frac {X-E(X)}{\sigma (X)}}\right)^{3}\right)={\frac {m_{3}}{m_{2}^{3/2}}}}

∀λ > 0 : *A*(λ*X*) = *A*(*X*)

Sploščenost (kurtozis)

K
(
X
)
=
E
⎡
⎡

X
−
E
(
X
)

σ
(
X
)

4

⎣
=

m

4

m

2

2

{\displaystyle K(X)=E\left[\left({\frac {X-E(X)}{\sqrt {D(X)}}}\right)^{4}\right]={\frac {m_{4}}{m_{2}^{2}}}}

Presežna sploščenost:

K
∗
(
X
)
=
K
(
X
)
−
3

{\displaystyle K^{*}(X)=K(X)-3}

Vrstilne karakteristike

Kvantil reda *p*

je vsaka vrednost *x*_{*p*}, za katero velja:

P
(
X
≤

x

p

)
≥
p

{\displaystyle P(X\leq x_{p})\geq p}

 in

P
(
X
≥

x

p

)
=
1
−
p

{\displaystyle P(X\geq x_{p})=1-p}

oz. *F*(*x*_{*p*}−) ≤ *p* ≤ *F*(*x*_{*p*})

- Mediana: *x*

1
2

{\displaystyle x_{\frac {1}{2}}}
- Kvartili: *x*

1
4

,

x

2
4

,

x

3
4

{\displaystyle x_{\frac {1}{4}},x_{\frac {2}{4}},x_{\frac {3}{4}}}
- (Per)centili: *x*

1
100

,
.
.
.
,

x

99
100

{\displaystyle x_{\frac {1}{100}},\ldots ,x_{\frac {99}{100}}}

Semi interkvartilni razmik

s
=

1
2

⎡

x

3
4

−

x

1
4

⎣

{\displaystyle s={\frac {1}{2}}\left(x_{\frac {3}{4}}-x_{\frac {1}{4}}\right)}

Rodovne funkcije

Naj bo *X* sl. sprem. z zalogo vrednosti ℕ ∪ {0}:

p

k

=
P
(
X
=
k
)

k
=
0
,
1
,
2
,
.
.
.

{\displaystyle p_{k}=P(X=k)\qquad k=0,1,2,\ldots }

Rodovna funkcija sl. sprem. *X*:

G

X

(
s
)
=

p

0

+

p

1

s
+

p

2

s

2

+
.
.
.
=

∑

k
=
0

∞

p

k

s

k

{\displaystyle G_{X}(s)=p_{0}+p_{1}s+p_{2}s^{2}+\cdots =\sum _{k=0}^{\infty }p_{k}s^{k}}

Obstaja za vse |*s*| ≤ 1.

P
(
X
=
k
)
=

G

X

(
k
)
(
0
)

k
!

{\displaystyle P(X=k)={\frac {G_{X}^{(k)}(0)}{k!}}}

G

X

(
0
)
=

p

0

G

X

(
1
)
=
1

G

X

(
s
)
=
E
(

s

X

)

{\displaystyle G_{X}(0)=p_{0}\qquad G_{X}(1)=1\qquad G_{X}(s)=E(s^{X})}

Izrek o enoličnosti:

∀*s* ∈ [−1, 1] : *G*_{*X*}(*s*) = *G*_{*Y*}(*s*)

⇐⇒ *P*(*X* = *k*) = *P*(*Y* = *k*) ∀*k* = 0, 1, 2, ...

lim

s
↑
1

G

X

′
(
s
)
=
lim

s
↑
1

∑

k
=
1

∞

k

p

k

s

k
−
1

=

∑

k
=
1

∞

lim

s
↑
1

k

p

k

s

k
−
1

E
(
X
)

{\displaystyle \lim _{s\uparrow 1}G'_{X}(s)=\lim _{s\uparrow 1}\sum _{k=1}^{\infty }kp_{k}s^{k-1}=\sum _{k=1}^{\infty }\lim _{s\uparrow 1}kp_{k}s^{k-1}E(X)}

Naj bo *X* sl. sprem. z rodovno funkcijo *G*_{*X*}, potem je:

G

X

(

n

)
(
1
−
)
=
E
(
X
)
(
X
−
1
)
(
X
−
2
)
.
.
.
(
X
−
n
+
1
)

{\displaystyle G_{X}^{(n)}(1-)=E(X)(X-1)(X-2)\ldots (X-n+1)}

Naj bosta *X*₁,...,*X*_{*n*} nedovisne sl. sprem. z rodovnimi funkcijami *G*_{*X*₁},...*G*_{*X*_{*n*}}:

G

X

1
+
.
.
.
+

X

n

=
G
(

X

1

)
.
.
.
G
(

X

n

)

∀
s
∈
[
−
1
,
1
]

{\displaystyle G_{X_{1}+\cdots +X_{n}}=G(X_{1})\ldots G(X_{n})\qquad \forall s\in [-1,1]}

Naj bodo ∀*n* ∈ ℕ sl. sprem *N*,*X*₁,...,*X*_{*n*} neodvisne. Naj ima *N* rodovno funkcijo *G*_{*N*} in *X*_{*i*} rodovno funkcijo *G*_{*X*}(*X*₁,...,*X*_{*n*} so enako porazdeljene). Naj bo *S* = *X*₁ + ... + *X*_{*n*}. Potem je:

G

S

=
G

N

(

G

X

(
s
)
)

∀
s
∈
[
−
1
,
1
]

{\displaystyle G_{S}=G_{N}(G_{X}(s))\qquad \forall s\in [-1,1]}

Velja tudi *E*(*S*) = *E*(*N*)*E*(*X*).

G

2
X

(
s
)
=

G

X

(

s

2

)

{\displaystyle G_{2X}(s)=G_{X}(s^{2})}

Znane rodovne funkcije

∑

n
=
0

∞

q

n

=

1

1
−
q

∑

n
=
0

b

q

n

=

1
−

q

b
+
1

1
−
q

{\displaystyle \sum _{n=0}^{\infty }q^{n}={\frac {1}{1-q}}\qquad \sum _{n=0}^{b}q^{n}={\frac {1-q^{b+1}}{1-q}}}

∑

n
=
a

∞

q

n

=

q

a

1
−
q

∑

n
=
a

b

q

n

=

q

a

−

q

b
+
1

1
−
q

{\displaystyle \sum _{n=a}^{\infty }q^{n}={\frac {q^{a}}{1-q}}\qquad \sum _{n=a}^{b}q^{n}={\frac {q^{a}-q^{b+1}}{1-q}}}

a

n

−

b

n

=
(
a
−
b
)
(

a

n
−
1

+

a

n
−
2

b
+
...
+

a

b

n
−
2

+

b

n
−
1

)

{\displaystyle a^{n}-b^{n}=(a-b)(a^{n-1}+a^{n-2}b+...+ab^{n-2}+b^{n-1})}

a

0

+
.
.
.
+

a

k
−
1

x

k
−
1

1
−

x

k

=

a

0

+
.
.
.
+

a

k
−
1

x

k
−
1

+

a

0

k

+
.
.
.
+

a

k
−
1

x

2
k
−
1

+
.
.
.

{\displaystyle {\frac {a_{0}+...+a_{k-1}x^{k-1}}{1-x^{k}}}=a_{0}+...+a_{k-1}x^{k-1}+a_{0}^{k}+...+a_{k-1}x^{2k-1}+...}

(
x
+
y

)

n

=

∑

k
=
0

n

n

k

x

n
−
k

y

k

{\displaystyle (x+y)^{n}=\sum _{k=0}^{n}{n \choose k}x^{n-k}y^{k}}

1

(
1
−
x

)

n

=

∑

k
=
0

n

n
+
k
−
1

k

x

k

{\displaystyle {\frac {1}{(1-x)^{n}}}=\sum _{k=0}^{n}{n+k-1 \choose k}x^{k}}

B

λ

(
x
)
=

∑

n

λ

n

x

n

=
(
1
+
x

)

λ

;

⎡

λ

n

⎣
=

λ

n
!

{\displaystyle B_{\lambda }(x)=\sum _{n}{\lambda \choose n}x^{n}=(1+x)^{\lambda };~\quad {\binom {\lambda }{n}}={\frac {\lambda ^{n}}{n!}}}

Momentno rodovna funkcija

M

X

(
t
)
=
E
(

e

t
X

)

∀
t
∈
R

če obstaja

{\displaystyle M_{X}(t)=E(e^{tX})\qquad \forall t\in R\quad \!{\it {če obstaja}}}

=
1
+

z

1

t
+

z

2

2
!

t

2

+

z

3

3
!

t

3

+
.
.
.

{\displaystyle =1+z_{1}t+{\frac {z_{2}}{2!}}t^{2}+{\frac {z_{3}}{3!}}t^{3}+\ldots }

V primeru, ko ima *X* zalogo vrednosti v ℕ ∪ {0}, je

M

X

(
t
)
=
E
(

e

t
X

)
=

G

X

(

e

t

)

{\displaystyle M_{X}(t)=E(e^{tX})=G_{X}(e^{t})}

Za zvezno porazdeljeno sl. sprem. *X* velja:

M

X

(
t
)
=

∫

