An exponent tells us how many times to multiply a number by itself, e.g. $3^2 = 3 * 3 = 9$. An exponential function is a function where we let the independent variable be the exponent, e.g. $f(x) = y = 3^x$. Logarithmic functions are the inverse of exponential functions. In essence, logarithms tell us what exponent a value is raised to to equal another value. The formal definition of a logarithm is as follows:

Definition: Let x and b be positive numbers, and $b \neq 1$. Then, the *logarithm* of x to the base b, $\log_b x$, is the power to which b must be raised to equal x. This can be written as:

$$\begin{aligned} \text{if} \quad & y = \log_b x, \quad \text{then} \\ & x = b^y \end{aligned}$$

$y = log_b x$ is written in logarithmic form, whereas $x = b^y$ is written in exponential form. These are equivalent statements presented in different forms. For a visual trick on how to convert logarithms into exponents, click here. Some written examples:

(1) $\log_4 x = 2$, then

$$\begin{aligned} x &= 4^2 \\ x &= 16 \end{aligned}$$

(2) $\log_2 \frac{y}{3} = 4$, then

$$\begin{aligned} \frac{y}{3} &= 2^4 = 16 \\ y &= 16 * 3 = 48 \end{aligned}$$

1. Properties of Logarithms

Logarithms have some important properties that follow from their definition. Some of the common, useful properties are:

$$\log_b uv = \log_b u + \log_b v \tag{1}$$

$$\log_b \frac{u}{v} = \log_b u - \log_b v \tag{2}$$

$$\log_b u^a = a \log_b u \tag{3}$$

$$\log_b u = \log_b v \quad \text{if and only if} \quad u = v \tag{4}$$

$$\log_b u = \frac{\log_a u}{\log_a b} \tag{5}$$

Property (5) is known as the Change of Base Formula and is very helpful in several applications. Let's see some of these properties in a simple example:

$$\begin{aligned} \log_2(4 * 8) &= \log_2 4 + \log_2 8 && \text{Use property (1) to write as a sum} \\ &= \log_2 2^2 + \log_2 2^3 && \text{Re-write 4 as } 2^2 \text{ and 8 as } 2^3 \\ &= 2\log_2 2 + 3\log_2 2 && \text{Use property (3) to move the exponent to the front} \\ &= 2 + 3 && \text{Use the fact that } \log_2 2 = 1 \\ &= 5 \end{aligned}$$

[1] Portions of this section were adapted from this webpage.

2. The Natural Logarithm

The natural logarithm, written as ln, is a type of logarithm where $\ln x = \log_e x$. e is an irrational number approximately equal to 2.718. The natural logarithm is the frequently seen and used in statistical applications. Somewhat confusingly, you will often see the *natural logarithm* written as log. All of the properties of logarithms hold for the natural logarithm.

3. Exercises

Here are some practice exercises based on the definition and properties of the general and natural logarithms.

(1) Re-write the exponential equation $x^y = z$ as an equivalent logarithmic equation.
(2) Solve for x: $\log_x(81) = 4$
(3) Solve for x: $\log_3 x = \log_3 7 + \log_3 3$
(4) The streptococci bacteria population N at time t (in months) is given by $N = N_0 e^2 t$ where N_0 is the initial population. If the initial population was 100, how long does it take for the population to reach one million?