Chapter 1

Section 1.1

Some notation

- If A is a subset of B we write
  $A \subseteq B$ or $B \supseteq A$.

- If A is a proper subset of B
  $A \subset B$.

- A and B are equal iff
  $A \subseteq B$ and $B \subseteq A$.

- $\mathbb{N} = \{1, 2, 3, \cdots\}$
- $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$
- $\mathbb{Q} = \{m/n : m, n \in \mathbb{Z}$ and $n \neq 0\}$
- $\mathbb{R}$ = real numbers ...

- For A and B sets, union is denoted as $A \cup B$ and intersection as $A \cap B$.

- $A \setminus B := \{x : x \in A$ and $x \notin B\}$.

- Let $\{A_1, A_2, \dots\}$ be an infite collection of sets

- $\bigcup_{n=1}^{\infty} A_n := \{x : x \in A_n$ for some $n \in \mathbb{N}\}$

- $\bigcap_{n=1}^{\infty} A_n := \{x : x \in A_n$ for all $n \in \mathbb{N}\}$

Functions:

The cartesian product of A and B nonempty sets is defined as

$$A \times B := \{(a, b) : a \in A, b \in B\}$$

Definition (function) A function f from a set A into a set B is a rule of correspondance that assigns to each element $x \in A$ a unique element $f(x) \in B$

- We distinguish between the function and function values.

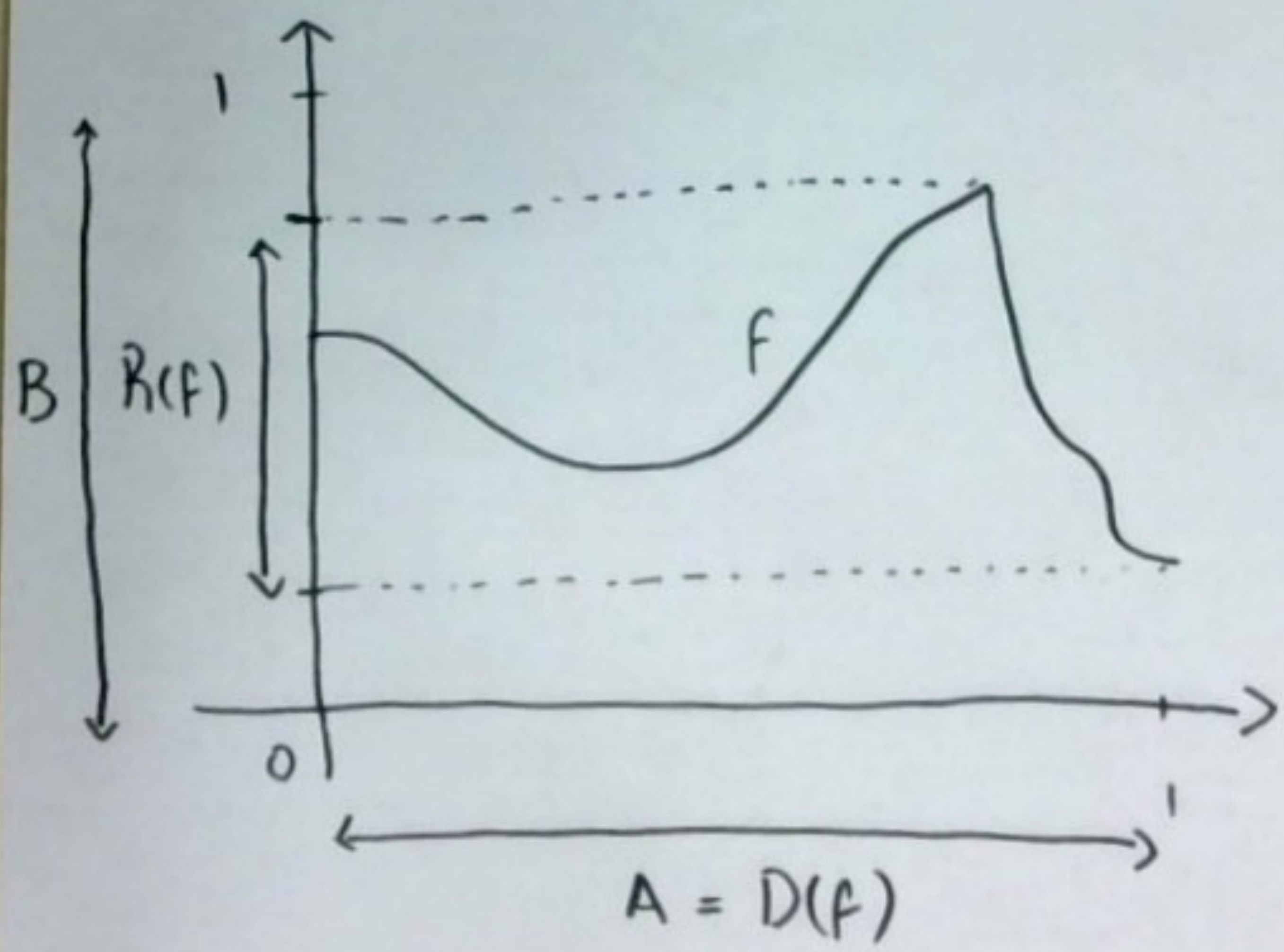
Definition (function) A function f from set A into set B is a set of ordered pairs in $A \times B$ such that for each $a \in A$ there is a unique $b \in B$ with $(a, b) \in f$.

Consequences

- if $(a,b) \in f$, $(a,b') \in f \Rightarrow b = b'$.

Notation:
- $f: A \to B$
- $a \mapsto f(a)$

- f is a (map, mapping) of A into B
- A is called the domain $D(f) := A$
- The set of all second elements is the range $R(f) \subseteq B$.



$f: [0,1] \to [0,1]$

- Let $f: A \to B$
- If $E \subseteq A$ then the image or direct image of E under f is

$$f(E) := \{ f(x) : x \in E \}$$

- If $H \subseteq B$ the the pre-image or inverse image of H under f is

$$f^{-1}(H) := \{ x \in A : f(x) \in H \}.$$

Example $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

- If $E = \{x : 0 \leq x \leq 2\} \Rightarrow f(E) = \{y : 0 \leq y \leq 4\}$
- If $G = \{y : 0 \leq y \leq 4\} \Rightarrow f^{-1}(G) = \{x : -2 \leq x \leq 2\}$

$\Rightarrow f^{-1}(f(E)) \neq E$

- $f(f^{-1}(G)) = G$

- If $H = \{y : -1 \leq y \leq 1\}$ then

$f^{-1}(H) = \{x : -1 \leq x \leq 1\}$ and

$[0,1] = f(f^{-1}(H)) \neq H.$

Special classes of functions

Definition Let $f: A \to B$

(a) f is said to be injective (one-to-one) if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$

(b) f is said to be surjective (onto) if $F(A) = B$ or $R(f) = B$.

(c) injection + surjection = bijection.

Example $f(x) = x^2$

$f: \mathbb{R} \to \mathbb{R}$ not bijective

$f: [0, +\infty) \to [0, +\infty)$ bijective

Inverse functions

If $f: A \to B$ is a bijection

then

$$g := \{ (b, a) \in B \times A : (a, b) \in f \}$$

is called the inverse function of and denoted by f^{-1}

Consequences if f is a bijection.

$$D(f) = R(f^{-1})$$

$$R(f) = D(f^{-1}).$$

Example $f(x) = \dfrac{2x}{x-1}$

Let $A = \{ x \in \mathbb{R} : x \neq 1 \}$ - (a) f is injective

assume $f(x_1) = f(x_2) \Rightarrow x_1(x_2 - 1) = x_2(x_1 - 1)$

$\Rightarrow x_1 = x_2.$

(b) Identify where it is surjective $y = \dfrac{2x}{x-1}$

$\Rightarrow x = y/(y-2) \Rightarrow y \neq 2$

then $B = \{ y \in \mathbb{R} : y \neq 2 \}$

$f: A \to B$ is bijective.

Definition If $f: A \to B$, $g: B \to C$ then the composite fuction $g \circ f$ is defined as

$$(g \circ f)(x) := g(f(x)) \quad x \in A.$$

Section 1.2

Math induction ✓

One of its forms is

1.2.2. Principle of Math Induction

Let S be a subset of $\mathbb{N}$ such that

(1) $1 \in S$

(2) $\forall k \in \mathbb{N}$, if $k \in S \Rightarrow k+1 \in S$

Therefore $S = \mathbb{N}$.

$\Downarrow$

For each $n \in \mathbb{N}$, let $P(n)$ be a statement about n. Suppose that

(1') $P(1)$ is true

(2') $\forall k \in \mathbb{N}$, if $P(k)$ is true $\Rightarrow P(k+1)$ is true.

Therefore

$P(n)$ is true for all $n \in \mathbb{N}$.

Section 1.3 Finite and Infinite sets

Example $r \in \mathbb{R}$, $r \neq 1$, $n \in \mathbb{N}$, then

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

(1') $n = 1$ $\quad 1 + r = \dfrac{1 - r^2}{1 - r} = \dfrac{\cancel{(1-r)}(1+r)}{\cancel{(1-r)}}$

(2') Let $1 + r + r^2 + \cdots + r^k = \dfrac{1 - r^{k+1}}{1 - r}$

add r^{k+1}

$$1 + r + r^2 + \cdots + r^k + r^{k+1} = \frac{1 - r^{k+1}}{1 - r} + r^{k+1} = \frac{1 \cancel{- r^{k+1}} \cancel{+ r^{k+1}} - r^{k+2}}{1 - r}$$

✓

Section 1.3

How to rigourously count in math?

1.3.1 Definition

(a) The empty set is said to have 0 elements

(b) A set S is said to have n elements ($n \in \mathbb{N}$) if there exists a bijection from $\mathbb{N}_n := \{1, 2, \dots, n\}$ onto S

(c) A set S is said to be finite if it is either empty or has n elements ($n \in \mathbb{N}$)

(d) A set S is said to be infinite if it is not finite.

Since inverse of a bijection is also a bijection (see homework), the bijection can be considered from S onto $\mathbb{N}_n$

1.3.2 Uniqueness Theorem

If S is a finite set, the number of elements in S is a unique number

Proof: Suppose the opposite, i.e., S has n and m elements with $n \neq m$, $n, m \in \mathbb{N}$. Then, there exists two bijections f, g

$$f : \mathbb{N}_n \to S \quad \text{and} \quad g : S \to \mathbb{N}_m$$

Then $f \circ g : \mathbb{N}_n \to \mathbb{N}_m$ is a bijection (see homework) which is a contradiction. ∎

1.3.3 Theorem

The set of natural numbers $\mathbb{N}$ is an infinite set

Proof: Suppose the opposite, i.e., $\mathbb{N}$ has n-elements. Then $\exists\, f : \mathbb{N} \to \mathbb{N}_n$ which is a bijection with some $n \in \mathbb{N}$ which is a contradiction. ∎

We can use: If $n \neq m \Rightarrow$ there is no bijection between $\mathbb{N}_n$ and $\mathbb{N}_m$.

- There is no injection from $\mathbb{N}$ and $\mathbb{N}_n$, $n \in \mathbb{N}$.

1.3.4. Theorem

(a) If A is a set with m elements and B " " " n elements, and $A \cap B = \emptyset$, then $A \cup B$ has $m+n$ elements

(b) If A is a set with m elements and $C \subseteq A$ is a set with 1 element, then $A \setminus C$ is a set with $m-1$ elements.

(c) If C is an infinite set and B is a finite set, then $C \setminus B$ is an infinite set.

Proof: Let $f : \mathbb{N}_m \to A$ and $g : \mathbb{N}_m \to B$ be the bijections.

Define

$$h(i) = \begin{cases} f(i) & i = 1, 2, \dots, m \\ g(i-m) & i = m+1, m+2, \dots, m+n \end{cases}$$

Injectivity of $h : \mathbb{N}_{n+m} \to A \cup B$ follows from $A \cap B = \emptyset$.

Surjectivity follows since f and g are surjective

(b) Hint: construct the bijection $f : \mathbb{N}_{m-1} \to A \setminus C$

(c) Hint: use (a) to get a contradiction ■

Subsets of finite sets are finite and supsets of infite sets are infinite.

1.3.5 Theorem

Suppose that T and S are sets and $T \subseteq S$

(a) If S is a finite set $\Rightarrow$ T is a finite set

(b) If T is an infinite set $\Rightarrow$ S is an infinite set

Proof: (a) suppose the oposite, i.e, T is infinite. Let $f : S \to \mathbb{N}_n$ be the bijection since S has n-elements.

The restriction of f to T is injective $\underbrace{\phantom{f \text{ to } T}}_{\text{defined as } f_T}$

$$f_T : T \longrightarrow \mathbb{N}_n$$

by definition, and $R(f_T) \subseteq \mathbb{N}_n \Rightarrow$

$$f_T : T \longrightarrow R(f_T)$$

is bijective(*), and $R(f_T)$ has $\tilde{n} \leq n$ elements and $\exists \quad g : R(f_T) \to \mathbb{N}_{\tilde{n}}$

a bijection. Then $f_T \circ g : T \to \mathbb{N}_{\tilde{n}}$

is a bijection, a contradiction. ∎

Note that $P_1 \Rightarrow P_2$

is equivalent to $\neg P_2 \Rightarrow \neg P_1$

which shows (b).

(*) Check homework.

Countable sets

We want to identify and characterize certain sets that are infinite.

1.3.6. Definition

(a) A set S is said to be <u>denumerable</u> (or countably infinite) if there is a bijection of $\mathbb{N}$ to S.

(b) S is <u>countable</u> if it's finite or denumerable

(c) S is <u>uncountable</u> if it's not countable

<u>Examples</u> - Odd numbers are denumerable
- even " " "

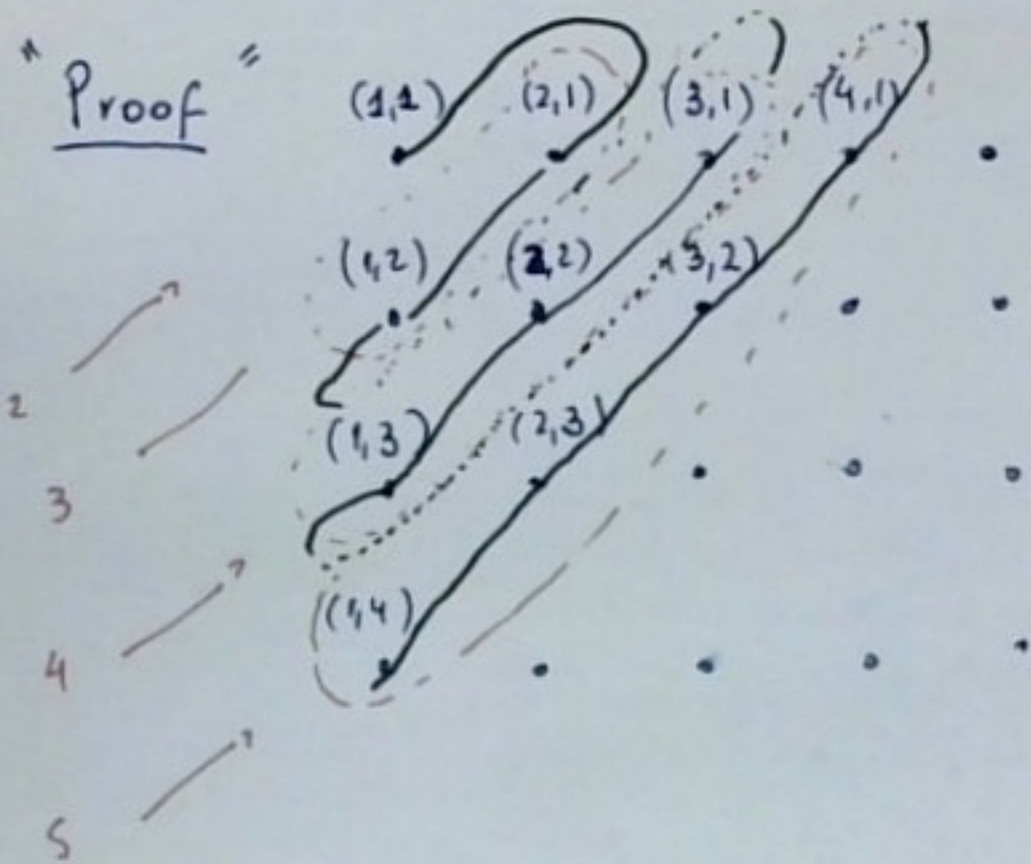
$f(n) = 2n$, $g(n) = 2n-1$ are the bijections

- $\mathbb{Z}$ is denumerable.

- <u>Important</u>: If A and B are denumerable (and disjoint) $\Rightarrow$ $A \cup B$ is denumerable.

1.38. Theorem the set $\mathbb{N} \times \mathbb{N}$ is denumerable

"**Proof**"



$(1,1), (2,1), (1,2), (3,1), (2,2), (1,3), \ldots$

∎

1.3.9 - Theorem Suppose $T \subseteq S$

(a) If S is countable $\Rightarrow$ T is countable.

(b) If T is uncountable $\Rightarrow$ S is uncountable.

Proof:

(a) if S is finite we did it already.

Suppose S is denumerable.

Then $\exists\ f: S \to \mathbb{N}$ is a bijection.

The restriction $f_T: T \to \mathbb{N}$ is injective and bijective if $f_T: T \to R(f_T)$

Since T is not finite and $R(f_T) \subseteq \mathbb{N}$

then $\exists\ g: R(f_T) \to \mathbb{N}$ a bijection

$\Rightarrow \quad g \circ f_T: T \to \mathbb{N}$ is a bijection, a contradiction ∎

We use that

B.4 Theorem If $A \subseteq \mathbb{N}$, then A is countable

1.3.9 Theorem The following are equivalent

(a) S is countable

(b) $\exists$ a surjection of $\mathbb{N}$ onto S

(c) $\exists$ an injection of S into $\mathbb{N}$.

Proof:

(9)

1.3.11 Theorem $\mathbb{Q}$ is denumerable.

Proof: - There is a bijection $f: \mathbb{N} \to \mathbb{N}\times\mathbb{N}$.

- Build $g(n,m) = \frac{n}{m}$ so that, $g: \mathbb{N}\times\mathbb{N} \to \mathbb{Q}^+$ is surjective.

~~Then~~ $g \circ f : \mathbb{N} \longrightarrow \mathbb{Q}^+$ is surjective. [By (b) 1.3.9 Theorem $\mathbb{Q}^+$ is denumerable.

- We can do the same for $\mathbb{Q}^-$, the negative rationals

- $Q^+ \cup Q^-$ is denumerable [see notes]

- $Q^+ \cup Q^- \cup \{0\} = \mathbb{Q}$ is denumerable ∎

1.3.12 Theorem A_m countable for $m \in \mathbb{N}$

$\Rightarrow \bigcup_{m=1}^{\infty} A_m = A$ is countable.

Proof diagonal argument ∎

1.13.13 Cantor's Theorem

If A is a set $\mathcal{P}(A)$ is the set of all subsets.

Example

$A = \{a, b\}$, $\mathcal{P}(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

Theorem

If A is any set, then there is no surjection of A onto $\mathcal{P}(A)$

Proof: Suppose the opposite, and let

$A \longmapsto \varphi(A)$ be the surjection.

Since $\varphi(a) \subseteq A \Rightarrow \begin{cases} a \in \varphi(a) & \text{or} \\ a \notin \varphi(a) \end{cases}$

Let

$$D := \{a \in A : a \notin \varphi(a)\}$$

Since φ is a surjection and $D \subseteq A$

$\exists\, a_0 \in A : \quad \varphi(a_0) = D$.

$a_0 \notin D$.

two cases: - $a_0 \in D \Rightarrow a_0 \in \varphi(a_0) \Rightarrow \Leftarrow$

- $a_0 \notin D \Rightarrow a_0 \notin \varphi(a_0) \Rightarrow a_0 \in D \Rightarrow \Leftarrow$

Then NO SURJECTION $\blacksquare$