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7.4 Covariance, Variance of Sums, and Correlations

Recall: For a single R.V.

Discrete: $E[X] = \sum_x x\,p(x)$

$$Var(X) = E\left[(X-\mu)^2\right] = E[X^2] - (E[X])^2$$

$$\mu = E[X]$$

Continuous

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \mu$$

$$Var(X) = E\left[(X-\mu)^2\right] = E[X^2] - (E[X])^2$$

Def: Covariance between X and Y

The covariance between X and Y, denoted by $Cov(X,Y)$ is

$$Cov(X,Y) = E\left[(X-E[X])(Y-E[Y])\right]$$

Note:

- $Cov(X,Y) = E\left[XY - E[X]Y - E[Y]X + E[X]E[Y]\right]$

$$= E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y]$$

$$\boxed{Cov(X,Y) = E[XY] - E[X]E[Y]}$$

- If X and Y are independent (say for continuous R.V. case)

$$E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f(x,y)\, dx\, dy$$

but $f(x,y) = f_X(x)\, f_Y(y)$ if X and Y are indep.

so

$$E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty} x f_X(x)\, dx \cdot \int_{-\infty}^{+\infty} y f_Y(y)\, dy$$

$$= E[X] \cdot E[Y].$$

- So, if X and Y are independent R.V.

$$Cov(X,Y) = E[XY] - E[X]\cdot E[Y]$$

$$= E[X]\cdot E[Y] - E[X]\cdot E[Y]$$

$$\boxed{Cov(X,Y) = 0}\,, \quad \leftarrow \text{true for continuous } X, Y \text{ and for discrete } X, Y.$$

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Proposition 4.1 (p 304, textbook)

If X and Y are independent, then for any functions h and g

$$E\left[g(X)\,h(Y)\right] = E\left[g(X)\right]\cdot E\left[h(Y)\right]$$

Comments
- true for discrete X, Y
- true for continuous X, Y
- proof follows the one outlined on p. 240 - see also textbook p. 305.

Proposition 4.2 (Properties of Covariance of X and Y)

a) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$ ✓

b) $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

c) $\mathrm{Cov}(aX,Y) = a\,\mathrm{Cov}(X,Y)$

$\mathrm{Cov}(X,Y)$
$= E[(X-E[X])(Y-E[$
$= E[XY]-E[X]E[Y]$

e) if X and Y are independent, $\mathrm{Cov}(X,Y)=0$

(but note $\mathrm{Cov}(X,Y)=0 \not\Rightarrow X, Y$ indep.)

d) $$\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}(X_i, Y_j)$$

proof of this - see book p. 306

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Proof of

$$\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}(X_i, Y_j)$$

$$\mathrm{Cov}\left(\underbrace{\sum_{i=1}^{n} X_i}_{U}, \underbrace{\sum_{j=1}^{m} Y_j}_{V}\right) = E\left[(U - E[U])(V - E[V])\right]$$

Note:

$$U = \sum_{i=1}^{n} X_i \qquad \text{let } E[X_i] = \mu_i$$

$$E[U] = E\left[\sum X_i\right] = \sum E[X_i] = \sum_{i=1}^{n} \mu_i$$

similarly

$$V = \sum_{j=1}^{m} Y_j \qquad \text{let } E[Y_j] = \nu_j$$

$$E[V] = E\left[\sum Y_j\right] = \sum E[Y_j] = \sum_{j=1}^{m} \nu_j$$

So

$$E\left[(U - E[U])(V - E[V])\right]$$

$$= E\left[\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right)\left(\sum_{j=1}^{m} Y_j - \sum_{j=1}^{m} \nu_j\right)\right]$$

$$= E\left[\left(\sum_{i=1}^{n} (X_i - \mu_i)\right)\cdot\left(\sum_{j=1}^{m} (Y_j - \nu_j)\right)\right]$$

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$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{m}(X_i-\mu_i)(Y_j-\nu_j)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\underbrace{E\left[(X_i-\mu_i)(Y_j-\nu_j)\right]}_{Cov(X_i,\,Y_j)}$$

So $= \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$ ✓

So...

$$Cov\left(\sum_{i=1}^{n} X_i,\ \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$$

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What can we say about

$$Var\left(\sum_{i=1}^{n} X_i\right) = Var\left(X_1 + X_2 + X_3 + \dots + X_n\right) ?$$

$$Var\left(\sum_{i=1}^{n} X_i\right) = Cov\left(\sum_{i=1}^{n} X_i , \sum_{i=1}^{n} X_i\right) \quad (b)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(X_i, X_j) \quad (d)$$

$$= \sum_{i=1}^{n} \left[Cov(X_i, X_i) + \sum_{j \neq i} Cov(X_i, X_j) \right]$$

$$= \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n} \sum_{j \neq i} Cov(X_i, X_j)$$

Equivalently,

$$\boxed{Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{i<j}\sum Cov(X_i, X_j)}$$

(note $Cov(X_i, X_j) = Cov(X_j, X_i)$)

In the case where the X_i's are pairwise independent, then

$$\boxed{Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)}$$

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<u>EXAMPLE</u> (4a, but see also Example 2c, notes p. 234)

Suppose $X_1, X_2, \ldots, X_n$ are independent and identically-distributed R.V.'s & <u>each</u> having cumulative distribution function F, expected value $E[X_i] = \mu$ for $i = 1, 2, \ldots, n$ and variance $Var(X_i) = \sigma^2$.

random variable → $$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \text{sample mean}$$

$$X_i - \bar{X} = \text{deviation (of } X_i) \quad \text{for } i = 1, 2, \ldots, n$$

random variable → $$S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1} \quad \text{is called the } \underline{\text{sample variance}}$$

a) Find $Var(\bar{X})$

b) $E[S^2]$

recall $E[\bar{X}] = \mu$

(see pp. 234 - 235)

$$Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$E[\bar{X}^2] - (E[\bar{X}])^2$

$$= \left(\frac{1}{n}\right)^2 Var\left(\sum_{i=1}^{n} X_i\right)$$

using * from previous page

$$= \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} Var(X_i)$$

$$= \left(\frac{1}{n}\right)^2 n \cdot \sigma^2 = \boxed{\frac{\sigma^2}{n} = Var(\bar{X})}$$

*

b) $$E[S^2] = E\left[\sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1}\right]$$

$$= E\left[\frac{1}{(n-1)} \sum_{i=1}^{n} \left(X_i - \mu + \mu - \bar{X}\right)^2\right]$$

$$= E\left[\frac{1}{(n-1)} \sum_{i=1}^{n} (X_i - \mu)^2 + 2(\mu - \bar{X})(X_i - \mu) + (\mu - \bar{X})^2\right]$$

$$= E\left[\frac{1}{(n-1)} \left(\sum_{i=1}^{n} (X_i - \mu)^2 + 2(\mu - \bar{X}) \sum_{i=1}^{n} (X_i - \mu) + \sum_{i=1}^{n} (\mu - \bar{X})^2\right)\right]$$

$$= E\left[\frac{1}{n-1} \left(\sum_{i=1}^{n} (X_i - \mu)^2 + 2(\mu - \bar{X})(n\bar{X} - n\mu) + n(\mu - \bar{X})^2\right)\right]$$

$$= E\left[\frac{1}{n-1} \left(\sum_{i=1}^{n} (X_i - \mu)^2 + 2n(\mu - \bar{X})(\bar{X} - \mu) + n(\mu - \bar{X})^2\right)\right]$$

$$= E\left[\frac{1}{n-1} \left(\sum_{i=1}^{n} (X_i - \mu)^2 - n(\bar{X} - \mu)^2\right)\right]$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} E(X_i - \mu)^2 - \frac{n}{n-1} E\left[(\bar{X} - \mu)^2\right]$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} Var(X_i) - \frac{n}{n-1} Var(\bar{X})$$

$$= \frac{1}{n-1}(n \cdot \sigma^2) - \frac{n}{n-1}\left(\frac{\sigma^2}{n}\right) = \frac{n-1}{n-1}\sigma^2 = \boxed{\sigma^2}$$

$$E[S^2] = \boxed{\sigma^2}$$

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EXAMPLE (4b, Textbook, p. 308)

Variance of Binomial R.V. (see also notes, p. 235)

Let X be a binomial R.V. with parameters n and p.

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i} \quad i = 0, 1, \ldots, n$$
$$= P\{X = i\}$$

i = # of successes in n total trials

p = prob. of success

Let $X = X_1 + X_2 + \ldots + X_n$ see also p. 235

$$X_i = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ trial is a success} \\ 0 & \text{otherwise } (i^{\text{th}} \text{ trial is failure}) \end{cases}$$

Note: the X_i's are independent Bernoulli R.V.'s
$E[X_i] = 1 \cdot p + 0 \cdot (1-p) = p$

$$Var(X) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) = n \cdot Var(X_i)$$

for any i.

$$Var(X_i) = E[X_i^2] - (E[X_i])^2$$

note: $X_i^2 = X_i$ see notes pp. 123-125

so

$$Var(X_i) = E[X_i] - p^2$$
$$= p - p^2$$

see Textbook, p. 132 Section 4.6.1

So $\boxed{Var(X) = n \cdot (p - p^2) = np(1-p)}$ ← as we've seen earlier

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The <u>correlation</u> of two random variables

X and Y, is

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

as long as $Var(X)$ $Var(Y)$ is positive

- ρ is a measure of linearity between X and Y

  $\rho = \pm 1$ (high degree of linearity)

  $\rho = 0$ means X and Y are uncorrelated

<u>Ex</u> Suppose $Y = aX + b$ for constants a and b

$$Var(Y) = E[Y^2] - (E[Y])^2$$

$$= E[(aX+b)^2] - (E[aX+b])^2$$

$$= E[a^2X^2 + 2abX + b^2] - (aE[X]+b)^2$$

$$= a^2E[X^2] + 2abE[X] + b^2 - a^2(E[X])^2 - 2abE[X] - b^2$$

$$= a^2(E[X^2] - (E[X])^2) = a^2 Var(X)$$

*

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$= E[X(aX+b)] - E[X]E[aX+b]$$

$$= aE[X^2] + bE[X] - a(E[X])^2 - bE[X]$$

$$= a\left(E[X^2] - (E[X])^2\right)$$

$$= a\,\mathrm{Var}(X)$$

So

$$\rho(X,Y) = \frac{a\,\mathrm{Var}(X)}{\sqrt{a^2\,\mathrm{Var}(X)\cdot \mathrm{Var}(X)}}$$

Note: $\mathrm{Var}(X) = E[(X-\mu)^2] \geq 0$ (>0 except if $X=\mu$ with probability 1)

So

$$\rho(X,Y) = \frac{a}{\sqrt{a^2}}\,\frac{\mathrm{Var}(X)}{\mathrm{Var}(X)}$$

$$= \frac{a}{|a|}$$

so

$$\boxed{\begin{array}{ll} \rho(X,Y) = 1 & \text{if } a>0 \\ \rho(X,Y) = -1 & \text{if } a<0 \end{array} \quad \text{when } Y = aX+b}$$