Parsing Binomials & Multinomials in Probability

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Introduction

The application of the Binomial and Multinomial theorems in probability can often lack clarity. However, the relations between polynomials and probability can provide insight into how models work in probability. This article aims to clarify the connection between expansions of multinomials and binomials, combinations, total probability, and probability mass functions.

1 Binomial Theorem

The binomial theorem expresses the expansion of two monomial terms x and y such that $(x + y)^2 = x^2 + 2xy + y^2$. The later is useful in algebra and other fields of mathematics, but how binomial expansion relates to probability is not intuitvely obvious. The solution is to treat x and y as two independent events. The resulting binomial expansion can express the sample space or total probability of all combinations of two independent events.

Example 1. Let an unfair coin be flipped twice with $P(Tails) = 0.3$ and $P(Heads) = 0.7$
We know the probability must sum to 1. In two flips then, $(T + H)^2 = T^2 + 2TH + H^2$. This aligns with the outcomes of $T^2 = TT$, $2TH = TH + HT$, and $H^2 = HH$ for 2 flips. Additionally, substituting in the probabilities we have $0.3^2 + 2 * 0.3 * 0.7 + 0.7^2 = 1$.

Theorem 1. *The Binomial Theorem can express the sum of all probabilities of two independent events.*

Proof. Let A and B be independent events with probability $A = \frac{1}{a}$ and $B = (1 - \frac{1}{a})$. By the binomial theorem we have

$$(\frac{1}{a} + (1 - \frac{1}{a}))^2$$

$$(\frac{1}{a})^2 + 2\frac{1}{a}(1 - \frac{1}{a}) + (1 - \frac{1}{a})^2$$

$$\frac{1}{a^2} + \frac{2}{a}\frac{a - 1}{a} + \frac{(a - 1)^2}{a^2}$$

$$\frac{1}{a^2} + \frac{2(a - 1)}{a^2} + \frac{a^2 - 2a + 1}{a^2}$$

$$\frac{a^2 + 2 - 2 + 2a - 2a}{a^2}$$

$$\frac{a^2}{a^2} = 1$$

$\square$

To compute the expansion of two monomials the theorem uses:

Definition 1 (Factorial n!). Count everyway to permute a set of n distinct objects

$$n! = \prod_{i=1}^{n} i$$

where $0! = 1$ and $n \geq 0$.

Building on factorials, the theorem uses the

Definition 2 (Binomial Coefficient). Count everyway to combine a set of n objects of k size

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Note that the binomial coeffecient can be used to determine the number of combinations of a particular size the binomial theorem produces.

Example 2. 3 flips of a coin yields:

$$(T + H)^3 = TTT + TTH + THT + HTT + HHT + HTH + THH + HHH$$
$$= T^3 + 3T^2H + 3H^2T + H^3$$
$$\binom{3}{3} = 1 \text{ hence } T^3 = TTT \text{ or } H^3 = HHH$$
$$\binom{3}{2} = 3 \text{ hence } 3T^2H = TTH + THT + HTT \text{ or } 3H^2T = HHT + HTH + THH$$

Remark 1. The total number of combinations of the binomial is 2^n.
$2^3 = T^3 + 3T^2H + 3H^2T + H^3 = 1 + 3 + 3 + 1 = 8$

Many combinations can be simplified by the use of Pascal's identity:

Pascal's Identity.

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$
$$= (n-1)![\frac{n-k}{k!(n-k)!} + \frac{k}{k(n-k)!}]$$
$$= (n-1)!\frac{n}{k!(n-k)!}$$
$$= \frac{n!}{k!(n-k)!}$$
$$= \binom{n}{k}$$

$\square$

Binomial Theorem. Assume that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ and by the the definition of the binomial

coeffecient $n \geq 0$. For the case $(n = 0) \Rightarrow (a + b)^0 = 1$. For the case $n \geq 0$.

$$(a + b)^{n+1} = (a + b)(a + b)^n = (a + b) \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} nn - k + \sum_{k=o}^{n} \binom{n}{k} a^k b^{n-k+1}$$

$$m = k + 1$$

$$= \sum_{m=1}^{n+1} \binom{n}{m-1} a^m b^{n-m+1} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

$$= b^{n+1} + \sum_{k=1}^{n} [\binom{n}{k} + \binom{n}{k-1}] a^k b^{n-k+1} + a^{n+1}$$

$$= b^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n-k+1} + a^{n+1}$$

$$= \sum_{k}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}$$

$\square$

2 Multinomial Theorem

While the binomial theorem applies to two independent events, the multinomial theorem generalizes to any number of groups or events. The theorem uses the multinomial coeffecient.

Definition 3 (Multinomial Coeffecient).

$$\binom{N}{n_1..n_r} = \frac{N!}{n_1!..n_r!}$$

Where n_1 to n_r are different group sizes.

Example 3. For 13 items we want to know how many combinations of 5, 5, and 3 can be made

$$\binom{13}{5,5,3} = \binom{13}{5}\binom{8}{5}\binom{3}{3}$$

$$= \frac{13!}{5!(13-5)!} \frac{8!}{5!(8-5)!} \frac{3!}{3!(3-3)!}$$

$$= \frac{13!}{5!5!3!}$$

We now combine the multinomial coeffecents with monomials raised to a power to get:

Definition 4 (multinomial theorem).

$$(x_1 + ... + x_r)^n = \sum_{(n1,...,nr)} \binom{n}{n1, ..., nr} x_1^{n1}...x_r^{nr}$$

where $n1 + ... + nr = n$

Multinomial Theorem. Fix $r = 1$ and observe that $(x_1)^n = \sum_{(n1=1)} \binom{n}{n1=1} x_1^{n1=1} = nx_1$

Fix $m = r + 1$ and $(x_r + x_{r+1})^n = \sum_{(r,...,r+1)} \binom{n}{x_1,...,x_{r+1}} x_1^r x_{r+1}^{r+1}$ and observe that this is provable using the binomial theorem. $\square$