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Worksheet 3

Let A, B, *and* C be sets. Prove or disprove the following statements.

1. If $A \cap B = \emptyset$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$

   Proof. Let $A = \{a\}$ and $C = \{a, c\}$ and $B = \{b\}$. Observe $\{a\} \cap \{b\} = \emptyset$ and $\{b\} \cap \{a, c\} = \emptyset$ while $\{a\} \cap \{a, c\} = \{a\}$ □

2. If $A \not\subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$

   Proof. Let $A = \{a\}$ and $C = \{a, c\}$ and $B = \{b\}$ Observe $\{a\} \not\subseteq \{b\}$ and $\{b\} \not\subseteq \{a, c\}$, while $\{a\} \subset \{a, c\}$ □

3. If $A \subseteq \emptyset$, then $a = \emptyset$

   Proof. Assume the negation $A \subseteq \emptyset$ and $A \neq \emptyset$. If $A \neq \emptyset$ then $A \not\subseteq \emptyset$ by definition of $\emptyset$ □

4. If $A \subseteq C$ and $B \subseteq C$, then $A \cap B \subseteq C$

   Proof. Fix $x \in A \cap B$ by defintion of intersection $x \in A$ and $x \in B$. From the inclusion $A \subseteq C$ it follows that $x \in C$. □

5. If $f : A \to B$ is injective and $g : B \to C$ is injective, then $g \circ f : A \to C$ is injective.

   Proof. Fix $x, y \in A$ and suppose $g(f(x)) = g(f(y))$. By injectivity of g we have $f(x) = f(y)$ and by injectivity of f we conclude that $x = y$. □

6. If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is surjective

   Proof. Fix $c \in C$. The surjectivity of g implies the existence of $b \in B$ with $g(b) = c$, while that of f yields an $a \in A$ with $f(a) = b$. We have, $g(f(a)) = g(b) = c$. □

7. Give an example of a function $f : A \to A$ that is injective but not surjective.

   Proof. Fix $b \in \mathbb{Z}$. $g : b \mapsto 2b$ maps to only the even co-domain. □

8. Give an example of a function $g : A \to A$ that is surjective but not injective.

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be given by

$$\begin{cases} k-1 & k \geq 1 \\ 0 & k = 0 \end{cases}$$

$\square$

9. Let $f : A \to B$ and $g : B \to A$. If $g \circ f = id_a$, then both f and g are bijections.

   Proof. Put $f : \{0\} \mapsto 1$ and $g : \mathbb{N} \to \{0\}$. Then

   $$g \circ f : \{0\} \to \{0\}$$

   $0 \mapsto 0$ Observe that $g \circ f$ is a bijection while g is not a bijection.

   $\square$

10. If $f : A \to A$ is surjective, and if A is a finite set, then f is injective.

    Proof. By definition of surjective $\forall a \in A : \exists b \in A : f(b) = a$. By definition of a function no parameter may map to more than one value. Hence, if the domain and co-domain are both a finite set and the function is surjective then the function must be injective. $\square$

11. If $f : A \to A$ satisfies the property that $f \circ f = id_a$ then f is a bijection.

    Proof. Let $x, y \in A$ with $f(x) = f(y)$. By applying f to f, we have

    $$x = f \circ f(x) = f \circ f(y) = y$$

    whence f is injective. Fix $b \in A$. From the identity $f(f(b)) = b$ we conclude that f is surjective. $\square$