Introduction to Calculus

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1 Introduction to Derivatives

The derivative of a function describes the rate at which that function is changing. Specifically, for a given function, the derivative is the change in the dependent variable with respect to an infinitesimal change in the independent variable.

Granted this is an advanced math course, but maybe start with a definition of a limit?

Definition 1. The *derivative* of a function $f(x)$ with respect to x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 1. If $f(x) = 2x$, then $f'(x) = \lim_{h \to 0} \frac{2(x+h)-2x}{h} = \lim_{h \to 0} \frac{2x+2h-2x}{h} = \lim_{h \to 0} \frac{2h}{h} = 2$.

Remark. For all $a \in \text{Dom}\,(2x)$, $f'(a) = 2$.

Example 2. If $f(x) = x^2$, then $f'(x) = \lim_{h \to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h \to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h \to 0} \frac{2xh+h^2}{h} = \lim_{h \to 0} (2x+h) = 2x$.

Remark. For all $a \in \text{Dom}\,(x^2)$, $f'(a) = 2a$.

2 The Meaning of Differentiability

The derivative of a function may not exist at every point within an interval. When the derivative of a function does not exist at a given point, we say that the function is *undifferentiable* at that point. Conversely, when the derivative of a function exists at every point within a specified interval, we say that the function is *differentiable* over that interval.

Definition 2. A function $f(x)$ is *undifferentiable* at $x = a$ if $f'(a)$ does not exist.

Remark. In general, $f'(a)$ does not exist when $f(x)$ is discontinuous at $x = a$.

Really impressed with your work, but maybe a defintion or example of discontinuous would help. I don't think the assumption the reader would know what discontinuous is at this level is unreasonable though.

Definition 3. A function $f(x)$ is *differentiable* on $[a, b]$ if $f'(x)$ exists at every $x \in [a, b]$.

3 Derivative Shortcuts

Theorem 1. *The derivative of $f(x) = c$ with respect to x, where $c \in \mathbb{R}$, is given by $f'(x) = 0$.*

Needs more context such as c is an arbitrary constant

Proof. By the definition of a derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= 0 \end{aligned}$$

□

Example 3. If $f(x) = 2$, then $f'(x) = \lim\limits_{h \to 0} \frac{2-2}{h} = \lim\limits_{h \to 0} \frac{0}{h} = 0$.

Theorem 2. *The derivative of $f(x) = x^a$ with respect to x, where $a \in \mathbb{N}$, is given by $f'(x) = ax^{a-1}$.*

Proof. By the definition of a *derivative*, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^a - x^a}{h} \\ &= \lim_{h \to 0} \frac{\sum_{i=0}^{a} \binom{a}{r} x^{a-r} h^r - x^a}{h} \\ &= \lim_{h \to 0} \frac{\binom{a}{0} x^a + \binom{a}{1} x^{a-1} h + \binom{a}{2} x^{a-2} h^2 + \ \ldots \ + \binom{a}{a} h^a - x^a}{h} \\ &= \lim_{h \to 0} \frac{\binom{a}{1} x^{a-1} h + \binom{a}{2} x^{a-2} h^2 + \ \ldots \ + \binom{a}{a} h^a}{h} \\ &= \lim_{h \to 0} \binom{a}{1} x^{a-1} + \lim_{h \to 0} \binom{a}{2} x^{a-2} h + \ \ldots \ + \lim_{h \to 0} \binom{a}{a} h^{a-1} \\ &= \binom{a}{1} x^{a-1} \\ &= ax^{a-1} \end{aligned}$$

□

Example 4. If $f(x) = x^3$, then $f'(x) = 3x^2$.
Example 5. If $f(x) = 2x^4$, then $f'(x) = 8x^3$

4 Derivatives of Exponential Functions

Theorem 3. *The derivative of $f(x) = a^x$, where $a \in \mathbb{R}$, is $f'(x) = a^x \lim\limits_{h \to 0} \frac{a^h - 1}{h}$.*

Proof. By the definition of a derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{(x+h) - x} \\ &= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} \\ &= a^x \lim_{h \to 0} \frac{a^h - 1}{h} \end{aligned}$$

□

Remark. The $\lim\limits_{h \to 0} \frac{a^h - 1}{h}$ is a constant which depends on the value of a.

Example 6. If $f(x) = 1.5^x$, then $f'(x) = 1.5^x \lim\limits_{h \to 0} \frac{1.5^h - 1}{h} \approx 0.405 \times 1.5^x$.

Remark. In this case, the $\lim\limits_{h \to 0} \frac{1.5^h - 1}{h} \approx 0.405$.

Suppose there is an $a \in \mathbb{R}$ such that $\lim\limits_{h \to 0} \frac{a^h - 1}{h} = 1$. Then, the derivative of $f(x) = a^x$ is $f'(x) = a^x$. The number which satisfies this supposition is Euler's constant.

Definition 4. Euler's constant, denoted by e, is the real number such that $\lim\limits_{h \to 0} \frac{e^h - 1}{h} = 1$.

Remark. If $f(x) = e^x$, then $f'(x) = e^x \lim\limits_{h \to 0} \frac{e^h - 1}{h} = e^x$.

Bibliography

Sections 1-3

Calculus Early Transcendentals: Differential & Multi-Variable Calculus for Social Sciences. 4 Derivatives. https://www.sfu.ca/math-coursenotes/Math%20157%20Course%20Notes/sec_TheDerivativeFunction.html

Section 4

Blackpenredpen. Why is the derivative of e^x equal to e^x? https://www.youtube.com/watch?v=oBlHiX6vrQY&t=412s.