Practice Final Answer Key

Practice Final a

1. True.

   Proof. Fix $x, y \in \mathbb{R}$ and put $k = |x - y|$. $\square$

2. False.

   Proof. Put $x = 1$ and $y = 2$. Let $z > 0$ and observe that $x - y = -1 \leq z$. $\square$

3. False.

   Proof. Let $z = 0$ and $w \in \mathbb{C}$. We have $zw = 0 \neq 1$. $\square$

4. True.

   Proof. Suppose that $a, a' \in A$ satisfy $(g \circ f)(a) = (g \circ f)(a')$. As g is injective, and as $g(f(a)) = g(f(a'))$, we deduce that $f(a) = f(a')$. As f is injective, we conclude that $a = a'$. $\square$

5. True.

   Proof. Fix $a, b, c \in A$.

   Since $f(a) \sim f(a)$, it follows that aRa and we deduce that R is reflexive.

   Now suppose that aRb. Thus, $f(a) \sim f(b)$, from which follows $f(b) \sim f(a)$, and hence bRa. Consequently, R is symmetric.

   If aRb and bRc, then $f(a) \sim f(b)$ and $f(b) \sim f(c)$, whence $f(a) \sim f(c)$, and thus aRc. Therefore, R is transitive. $\square$

6. True.

   Proof. Let $k, \ell \in \mathbb{Z}$ and observe that

   $$\begin{aligned} \phi(k + \ell) &= 2(k + \ell) \\ &= 2k + 2\ell \\ &= \phi(k) + \phi(\ell) \end{aligned}$$

   and

   $$\phi(-k) = 2(-k) = -2k = -\phi(k).$$

   $\square$

7. True.

   Proof. Fix $\varepsilon > 0$, choose $\delta > 0$ so that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$, and choose $N \in \mathbb{N}$ so that $d_X(x_i, x) < \delta$ whenever $n > N$. It follows that $d_Y(f(x_i), f(x)) < \varepsilon$ for all $n > N$. $\square$

8. False.

   Proof. Suppose to the contrary that S is the set of all sets. From $\mathcal{P}(S) \subseteq S$ it follows that $|\mathcal{P}(S)| \leq |S|$, while Cantor's theorem asserts $|S| < |\mathcal{P}(S)|$. This provides the desired contradiction. $\square$

Practice Final b

1. True.

   Proof. Fix $x \in \mathbb{R}$. Let $y = \lfloor x \rfloor$ be the greatest integer less than or equal to x. We have $|x - y| = x - y \leq 1$. $\square$

2. False.

   Proof. Let $k \in \mathbb{Z}$ and put $\ell = k + 1$. It follows that $(k - \ell)^2 = 1 \neq 0$. $\square$

3. False.

   Proof. Put $z = 1$ and $r = 2$. Let $k \in \mathbb{N}$. We have $|z|^k = 1 \leq r$. $\square$

4. False.

   Proof. Consider the bijection

   $$\begin{aligned} f : \mathbb{Z} &\to \mathbb{Z} \\ k &\mapsto k + 1 \end{aligned}$$

   and put $a = 0$. Let $k \geq 1$ and observe that $f^k(0) = k \neq 0$. $\square$

5. False.

   Proof. Consider the power set $\mathcal{P}(\{0, 1\})$. Observe that neither $\{0\} \subseteq \{1\}$ nor $\{1\} \subseteq \{0\}$. $\square$

6. True.

   Proof. First observe that $1_G \in \ker \phi$ ensures $\ker \phi \neq \varnothing$.
   Let $g, h \in \ker \phi$ and observe that

   $$\phi(g^{-1}) = \phi(g)^{-1} = 1_H$$

   and

   $$\phi(gh) = \phi(g)\phi(h) = 1_H.$$

   $\square$

7. False.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x = -1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $(f \circ f)(x) = 0$. In particular $f \circ f$ is continuous. $\square$

8. True.

Proof. Suppose to the contrary that there is a set A and a bijection $f : A \to \mathcal{P}(A)$. Define the set

$$B = \{a \in A \mid a \notin f(a)\}.$$

Choose $b \in A$ with $f(b) = B$. If $b \notin f(b)$, then it follows by the construction of B that $b \in B = f(b)$. However, if $b \in f(b)$, then $b \notin B = f(b)$. This yields the desired contradiction. $\square$

Practice Final c

1. False.

Proof. Let $x = 1$ and $y = 0$. Let $z \in \mathbb{R}$ and observe that $z \leq x$ or $z \geq y$. $\square$

2. True.

Proof. Let $k = 0$ and choose $x, y \in \mathbb{R}$. It follows that $xyk = k$. $\square$

3. False.

Proof. Put $z = 1$ and $w = 0$. We have $z - w \in \mathbb{R}$ and $z \neq \bar{w}$. $\square$

4. False.

Proof. Let $f : \{0\} \to \mathbb{R}$ and $g : \mathbb{R} \to \{0\}$ be given by

$$f(0) = 0$$

and

$$g(x) = 0.$$

Observe that f and g are each nonbijective, while $g \circ f : \{0\} \to \{0\}$ is bijective. $\square$

5. False.

Proof. Let $A = \{a, a'\}$ and $B = \{b\}$, equip B with the partial order given by $b \leq b$, and let $f : A \to B$ be given by $f(a) = f(a') = b$. From $b \leq b$, it follows that aRa', and we conclude that R is not antisymmetric. $\square$

6. True.

Proof. Let $r, s \in S \cap S'$. From $r, s \in S$ we have

$$-r, r+s, rs, 1_R \in S.$$

Similarly, from $r, s \in S'$ we obtain

$$-r, r+s, rs, 1_R \in S'.$$

We conclude that

$$-r, r+s, rs, 1_R \in S \cap S'.$$

□

7. True.

Proof. Suppose not. It follows that $d(x, y) > 0$. Choose $N \in \mathbb{N}$ so that $d(x_n, x) < \frac{1}{3}d(x, y)$ and $d(x_n, y) < \frac{1}{3}d(x, y)$ for all $n \geq N$. It follows that

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{1}{3}d(x, y) + \frac{1}{3}d(x, y) < d(x, y).$$

This yields the desired contradiction. □

8. True.

Proof. First observe that from $1 = 1^2$ it follows that the claim is true when $n = 1$.

Now fix $n \geq 1$ and suppose that

$$1 + 3 + 5 + (2n - 1) = n^2.$$

Adding $2n + 1$ to each side yields

$$1 + 3 + 5 + (2n - 1) + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2,$$

which establishes the claim for $n + 1$. □