Ch. 2 Axioms of Probability

2.1 Introduction

2.2 Sample Spaces and Events

Consider experiments such as...

- flip a coin
- flip a coin twice
- draw a card from a deck
- roll a die

⋮

These are experiments where the outcome is not known in advance but the set of all possible outcomes is known in advance

Def: The set of all outcomes is called the sample space.

EX
Sample space of flipping a coin is $S = \{H, T\}$

EX
Sample space of flipping a coin twice is

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

EX
Sample space of drawing a card from a deck is

$$S = \{2,3,4,5,\ldots,J,Q,K,A \text{ (Hearts)}, \; 2,3,4,\ldots \text{ (Diamonds)}, \; 2,3,\ldots K,A, \text{ (Clubs)}, \; 2,3,\ldots,K,A \text{ (Spades)}\}$$

EX

Sample space of rolling one die is

$$S = \{1,2,3,4,5,6\}$$

EX

Turn on a light bulb and measure the ~~[illegible]~~ # of hours until it burns out

$$S = \{x \in \mathbb{R} : x \geq 0\}$$

Def: An event is a subspace of the sample space.

Notes

- That is, an event consists of possible outcomes of the experiment. Not necessarily just one outcome and not necessarily all outcomes.

EX (Deck of Cards)

Let $E = \{A\heartsuit, A\diamondsuit, A\clubsuit, A\spadesuit\}$

= event that an ace is drawn from a deck of cards

EX (One die)

Let $E = \{4,5,6\}$ = event of rolling a 4 or higher.

EX (Flip a coin twice)

Let $E = \{(H,H)\}$ = event of flipping heads two times in a row.

Further Notes

- If E and F are events in the same sample space S (so $E \subset S$, $F \subset S$)

union • Then $E \cup F$ is an event in sample space S

intersection and $EF = E \cap F$ is an event " " " "

- $E^c = S \setminus E$ = complement of E in S is an event in S.

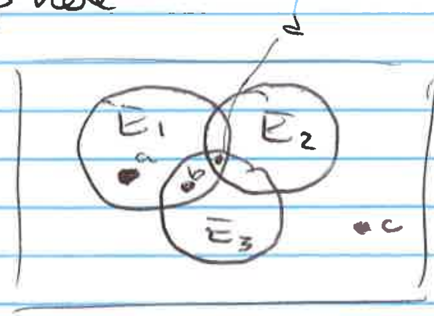
(i.e. $E \cup E^c = S$)

- If $E_1, E_2, E_3, \ldots$ are events in S then the union of these events $\bigcup_{n=1}^{\infty} E_n$ is the set of outcomes that are in E_n for at least one n.

Venn Diagram may help here

$a, b, d \in \bigcup_{n=1}^{3} E_n$

$c \notin \bigcup_{n=1}^{3} E_n$



- $\bigcap_{n=1}^{\infty} E_n$ is the set of outcomes in all of $E_1, E_2, E_3, \ldots$

$d \in \bigcap_{n=1}^{3} E_n$

- Review Basic Set Theory (see Ross, pp. 24-25)

Commutativity: $E \cup F = F \cup E$

$EF = FE$

Associativity: $(E \cup F) \cup G = E \cup (F \cup G)$

$(EF)G = E(FG)$

Distributivity: $(E \cup F)G = EG \cup FG$

$EF \cup G = (E \cup G)(F \cup G)$

DeMorgan's Laws

$(E \cup F)^c = E^c F^c$

$(EF)^c = E^c \cup F^c$

Mutually Exclusive Events

If $EF = \emptyset$ ($\emptyset$ = event with no outcomes = empty set)

then events E and F are said to be

<u>mutually exclusive</u>

<u>EX</u> (rolling die) $E = \{1,2,3\}$, $F = \{4,5,6\}$

$EF = \emptyset$ the event of rolling ≤ 3 and ≥ 4 has no outcomes

One possible way to define the probability of an event E is to ~~consider~~ define it as the limit

$$\boxed{P(E) = \lim_{n \to \infty} \frac{n(E)}{n}}$$

where n = # of repetitions of an experiment

and $n(E)$ is the number of times in the first n repitions that Event E occurs.

EX "loaded" dice.

(Blue) Die #1:

1	2	3	4	5	6
62	49	57	53	39	68

$n = 328$

e.g. $\frac{n(6)}{n} = \frac{68}{328} \approx 0.207 > \frac{1}{6} = 0.166\ldots$

Is this die "loaded" or is this just "random chance"

(Red) Die #2:

1	2	3	4	5	6
49	43	50	48	35	44

$n = 269$

e.g. $\frac{n(6)}{n} = \frac{44}{269} \approx 0.164 < \frac{1}{6} = 0.166\ldots$

For now we'll consider an approach to probability based on a set of axioms...

49 43 50 48 35 44 62 49 57 53 39 68

28.2

R						B					
1	2	3	4	5	6	1	2	3	4	5	6

2.3 Axioms of Probability

For each event E of a sample space S associated with some experiment, we assume that the probability of the event E, denoted by $P(E)$, is defined and satisfies the following 3 axioms:

(1): $0 \le P(E) \le 1$

(2): $P(S) = 1$

(3): For any sequence of mutually exclusive events $E_1, E_2, E_3, \ldots$ (i.e. events for which $E_iE_j = \phi$ when $i \ne j$)

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Notes:

- $\boxed{P(\phi) = 0}$ — i.e. ϕ = an event that can't happen

  : Follows from taking

  $E_1 = S$

  $E_i = \phi$ for $i = 2, 3, 4, \ldots$

  Then

  $$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

  $$\underbrace{P(S \cup \phi \cup \phi \ldots)}_{P(S)} = P(S) + \sum_{i=2}^{\infty} P(\phi)$$

  $$0 = 0 + \sum_{i=2}^{\infty} P(\phi)$$

  $\Rightarrow \boxed{P(\phi) = 0}$ since $P(\phi) \ge 0$

- For any finite sequence of mutually exclusive events

$E_1, E_2, \ldots, E_n$

define $E_i = \emptyset$ for $i > n$. Then

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i) \quad \leftarrow \text{infinite sum}$$

$$\boxed{P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)} + 0 + 0 + \ldots \quad \leftarrow \text{finite sum}$$

- Further, note if $\bigcup_{i=1}^{n} E_i = S$ then

if E_i's are mutually exclusive

$$P\left(\bigcup_{i=1}^{n} E_i\right) = P(S) = \boxed{1 = \sum_{i=1}^{n} P(E_i)}$$

EX

→ Suppose we roll a "fair" die (all six sides equally-likely)

Let ~~eg~~ $E_1 = \{1\}, E_2 = \{2\}, \ldots, E_6 = \{6\}$ ← events

~~[illegible]~~ Since these events are equally likely

$P(E_1) = P(E_2) = \ldots = P(E_6)$. Then

$$1 = \sum_{i=1}^{6} P(E_i) = 6P(E_1) \longrightarrow \boxed{P(E_1) = \ldots = P(E_6) = \frac{1}{6}}$$

as expected.

- If $F = \{5, 6\}$ $\leftarrow$ event of rolling a 5 or 6

by Axiom 3...

$$P(F) = P(E_5 \cup E_6) = P(E_5) + P(E_6)$$
$$= \frac{1}{6} + \frac{1}{6}$$
$$= \frac{1}{3} \text{ as expected.}$$

- Proposition 4.1: $\boxed{P(E^c) = 1 - P(E)}$

Note: E and E^c are mutually exclusive events

and $E \cup E^c = S$

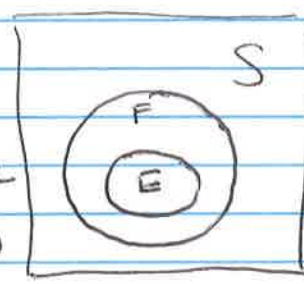
so $P(S) = 1 = P(E \cup E^c) = P(E) + P(E^c)$ ✓.

- Proposition 4.2: $\boxed{\text{If } E \subset F \text{ then } P(E) \leq P(F)}$

Note: $F = E \cup (E^c F)$

by Axiom 3, since E and $E^c F$ are mutually exclusive

$$P(F) = P(E \cup (E^c F)) = P(E) + P(E^c F)$$
$$\geq P(E) \text{ since } 0 \leq P(E^c F) \leq 1$$



- Proposition 4.3: $P(E \cup F) = P(E) + P(F) - P(EF)$

Proof: observe that

$$E \cup F = E \cup E^cF$$ where E and E^cF are mutually exclusive

and

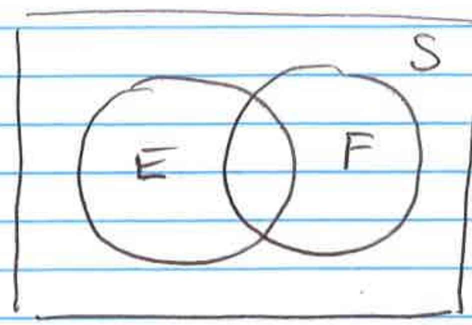
$$F = E^cF \cup EF$$ where E^cF and EF are mutually exclusive

Then

$$P(E \cup F) = P(E) + P(E^cF)$$

and

$$P(F) = P(E^cF) + P(EF)$$



So

$$P(E \cup F) = P(E) + P(F) - P(EF)$$ by algebra...

- by applying Proposition 4.3 to $P(E \cup F \cup G)$ ...

$$\begin{aligned}
P(E \cup F \cup G) &= P(E) + P(F \cup G) - P(E(F \cup G)) \\
&= P(E) + [P(F) + P(G) - P(FG)] - P(EF \cup EG) \\
&= P(E) + P(F) + P(G) - P(FG) \\
&\quad - [P(EF) + P(EG) - P(EFEG)]
\end{aligned}$$

$$\boxed{P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(FG) - P(EF) - P(EG) + P(EFG)}$$

See also the more general Proposition 4.4 (p.30) which is also known as the inclusion - exclusion identity.

2.5 Sample Spaces with equally likely outcomes.

Suppose S, the sample space, has N equally-likely outcomes. Denote these by

$$S = \{1, 2, \ldots, N\}$$

and also $E_i = i$ for $i = 1, \ldots, N$. So $P(E_1) = P(E_2) = \ldots = P(E_N)$

Assuming these events are mutually exclusive,

since $S = \bigcup_{i=1}^{N} E_i$

$$1 = P(S) = P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i) = \underbrace{N \cdot P(E_i)}_{\text{true for any } i = 1, \ldots, N}$$

Then

$$P(E_i) = \frac{1}{N}$$

If we denote by E any event (e.g. $E = \bigcup_{\substack{i=1 \\ \text{odd}}}^{N} E_i$)

$$\boxed{P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}}$$

Examples

EX Poker Probabilities

~~Recall~~ 5 cards (of 52) are dealt.

What is the probability ~~that~~ of

a) Royal Flush? 10, J, Q, K, A of same suit

there are only 4 such hands!

$$\text{So} \quad P(\text{royal flush}) = \frac{4}{\binom{52}{5}} = \frac{4}{2{,}598{,}960} = \boxed{1.539 \times 10^{-6}}$$

recall $\binom{52}{5}$ = # of different 5 card hands
(order is not important)

b) Straight Flush? a straight all in same suit

e.g. 4, 5, 6, 7, 8 of hearts

A, 2, 3, 4, 5 of clubs

A can be low or high

- in each suit there are 10 possible starting cards for the low card

$$\underline{10} \cdot \underbrace{\underline{1} \cdot \underline{1} \cdot \underline{1} \cdot \underline{1}}_{\text{after the first there is only one choice}}$$

A
2
3
4
5
6
7
8
9
10

So $\overset{\text{\# of suits}}{4} \cdot 10$ possible straight flush hands.

If we do not include the Royal flush cases we have $40 - 4 = \boxed{36 \text{ straight flushes that are not Royal flushes}}$

So $P(\text{straight flush}) = \dfrac{36}{\binom{52}{5}} = 1.385 \times 10^{-5}$

c) Four of a kind?

Total #: $\underbrace{13}_{\substack{\text{pick the denomination} \\ \text{e.g. } \binom{13}{1}}} \cdot \underbrace{\binom{4}{4}}_{\substack{\text{choose all 4 of} \\ \text{that card}}} \cdot \underbrace{48}_{\substack{\text{pick any} \\ \text{other card}}} = 624$

$P(4 \text{ of a kind}) = \dfrac{624}{\binom{52}{5}} = 2.401 \times 10^{-4}$

d) Full House? (a, a, a, b, b) a's have same denom. b's have same but different denomination than a.

Total # $\underbrace{13}_{\substack{\text{\# of denominations} \\ \text{of a}}} \cdot \underbrace{\binom{4}{3}}_{\substack{\text{choose} \\ \text{3 of} \\ \text{4 a's}}} \cdot \underbrace{12}_{\substack{\text{\# of} \\ \text{denom.} \\ \text{of b}}} \underbrace{\binom{4}{2}}_{\substack{\text{choose 2} \\ \text{of 4 b's}}} = 3744$

555 44 different from 444 55

or $\underbrace{\binom{13}{2}}_{\text{pick \#'s}} \times \underbrace{2}_{\substack{\text{which one is 3 s.} \\ \text{which one is 2.}}} \times \binom{4}{3}\binom{4}{2}$

$P(\text{full house}) = \dfrac{3744}{\binom{52}{5}} = 1.441 \times 10^{-3}$

NOTE :

(35.i)

Two Pair (aabbc)

$$\binom{13}{2} \quad \binom{4}{2}\binom{4}{2} \quad \binom{11}{1} \quad \binom{4}{1} = 123552$$

pick the two denominations

choose suits for the two pairs

pick remaining card

suit of remaining card

Note: this is not the same as

$$\underbrace{13\binom{4}{2}\cdot 12\binom{4}{2}}_{\text{this counts}} \cdot 11 \cdot \binom{4}{1} = 2\times(123552)$$

this counts
AAJJ 5
and JJAA 5
as two different hands (but they are not different)

Full House (aaabb)

$$13\binom{4}{3}\cdot 12\binom{4}{2} = 3744$$

pick the triple

pick the pair

Note AAAJJ
AAJJJ
are two different full houses.

or, equivalently $\binom{13}{2}\times \underset{\text{order matters}}{2} \times \binom{4}{3}\binom{4}{2} = 3744$

alternatively $S = \{$ ~~[illegible]~~ all possible 5-card poker hands dealt from standard 52 card deck $\}$

Sample Space of Poker Hands

$S = \{$ Royal Flush, Straight Flush, Four of a Kind, Full House, Flush, Straight, Three of a Kind, Two Pair, One Pair, High Card $\}$

where we interpret "Flush" as a flush that is not a better flush like a straight flush or Royal Flush.

$= \{ H_1, H_2, H_3, \ldots, H_{10} \}$

high card

Note: "high card" is a hand that meets none of the requirements of the "better" hands — no pairs, etc. Denote $E_i = \{H_i\}$ = event of getting hand type H_i

$$1 = P(S) = P\left(\bigcup_{i=1}^{10} E_i\right) = P(E_1) + \ldots + P(E_{10}).$$

E_i are mutually exclusive

After completing a few extra calculations, you should be able to show that

see practice problems.

$$P(E_{10}) = 1 - \sum_{i=1}^{9} P(E_i) = 0.5012$$

So there is slightly better than 50% chance of getting no pairs, straights, full house, etc.

Problem 25 [Ch.2]

A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a five occurs first.

Hint: Let E_n denote the event that a 5 occurs on the n^{th} roll and no 5 or 7 occurs on the first $n-1$ rolls.

Recall

second roll

first roll

	1	2	3	4	5	6
1	(1,1)	(1,2)		(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)		(2,5)	(2,6)
3		(3,2)		(3,4)		(3,6)
4	(4,1)		(4,3)			(4,6)
5		(5,2)	(5,3)		. .	(5,6)
6	(6,1)	(6,2)	...		(6,5)	(6,6)

36 possibilites

4 outcomes with sum = 5
6 outcomes with sum = 7

- Probability of no 5 or 7 on a roll $= \frac{26}{36}$
- probability of a 5 $= \frac{4}{36}$

So $P(E_n) = \left(\frac{26}{36}\right)^{n-1} \cdot \left(\frac{4}{36}\right)$

no 5 or 7 on first $n-1$ rolls

5 on n^{th} roll.

$$P(5 \text{ occurs first}) = \sum_{n=1}^{\infty} \left(\frac{26}{36}\right)^{n-1}\left(\frac{4}{36}\right)$$

$$= \frac{1}{9} \underbrace{\sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^{n-1}}_{\left[1 + \frac{13}{18} + \left(\frac{13}{18}\right)^2 + \dots\right]}$$

Recall $S_N \equiv \sum_{n=1}^{N} r^{n-1} = 1 + r + r^2 + \dots + r^{N-1}$

$$r S_N = \quad r + r^2 + \dots + r^N$$

$$S_N - r S_N = 1 - r^N$$

$$S_N = \frac{1 - r^N}{1 - r} \qquad \lim_{N \to \infty} S_N = \frac{1}{1-r}$$

So

$$P(5 \text{ occurs first}) = \frac{1}{9}\left(\frac{1}{1 - \frac{13}{18}}\right) = \frac{1}{9}\frac{1}{\left(5/18\right)} = \frac{18}{9 \cdot 5} = \boxed{\frac{2}{5}}$$

Poker ~ "without replacement"

Poker Dice ~ "with replacement"

Problem 16 (Ch.2)

EX (Poker Dice) Roll 5 dice simultaneously

(like Yatzee!

Let S be the set of all sequences $(n_1, n_2, n_3, n_4, n_5)$

where $n_i \in \{1,2,3,4,5,6\}$ for each i

S has $6^5 = 7776$ elements (contrast $\binom{52}{5}$ in cards)

a) $P(\text{no two alike}) = \dfrac{6\cdot5\cdot4\cdot3\cdot2}{6^5} = \dfrac{720}{7776} \approx 0.0926$

b) $P(\text{one pair}) = \dfrac{\binom{6}{1}\binom{5}{2}\cdot5\cdot4\cdot3}{6^5}$

pick which # to pair; of the 5 dice 2 must be pair; can't be paired #; can't be paired # or second #; can't be anything already picked

or $\binom{6}{1}\binom{5}{2}\left[\binom{5}{3}3!\right]$ choose other three #'s

$5\cdot4\cdot3$

$= \dfrac{6\cdot10\cdot5\cdot4\cdot3}{6^5} = \dfrac{3600}{6^5} \approx .4630$

c) $P(\text{two pair}) = \dfrac{\binom{6}{2}\binom{5}{2}\binom{3}{2}\cdot4}{6^5}$

pick the paired #'s; first pair from 5; second pair from 3; 4 options for last die

$= \dfrac{15\cdot20\cdot3\cdot4}{6^5} = \dfrac{1800}{6^5} \approx 0.2315$

which one alike

choose 3 of those

pick other numbers

40

d) $P(\text{three alike}) = \dfrac{\binom{6}{1}\binom{5}{3}\,5\cdot 4}{6^5} = \dfrac{6\cdot 10\cdot 5\cdot 4}{6^5}$

$= \dfrac{1200}{7776} \approx 0.1543$

the two #'s can be 3 group or 2 group

pick the two numbers

e) $P(\text{full house}) = \dfrac{2\cdot\binom{6}{2}\binom{5}{3}\binom{2}{2}}{7776} = \dfrac{300}{7776}$

pick 3 from 5

pick 2 from 2

≈ 0.0386

pick the #

f) $P(\text{four alike}) = \dfrac{\binom{6}{1}\cdot\binom{5}{4}\cdot 5}{7776} = \dfrac{150}{7776} \approx 0.0193$

pick 4 from 5

g) $P(\text{five alike}) = \dfrac{\binom{6}{1}\binom{5}{5}}{7776} = \dfrac{6}{7776} \approx 0.00077$

Note: $720 + 3600 + 1800 + 1200 + 300 + 150 + 6 = 7776$

The cases a) – g) account for all possibilities

note: "none alike" includes the possibility of a "straight"

(40.1)

d) Think about our 6x6 grid for two dice. once we have identified the # that will have 3 alike we pick the other two #'s that don't match this one and don't pair up

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
~~(4,1)~~	~~(4,2)~~	~~(4,3)~~	~~(4,4)~~	~~(4,5)~~	~~(4,6)~~
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6.5)	(6,6)

say 444 $\underbrace{\_\ \_}$

20 options = 5.4

Problem 12 (Ch. 2)

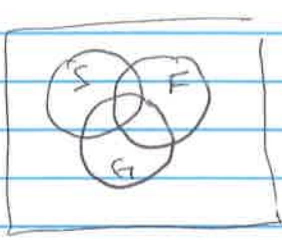
Elementary School offers 3 language classes
Spanish, French, German

100 students in school (classes open to all)

There are 28 students in Spanish
" " 26 " " French
" " 16 " " German.
" " 12 " " S + F
" " 4 " " S + G
" " 6 " " F + G
" " 2 " " F + S + G

a) If a student is chosen randomly, what is the probability he/she is not in a language class?

$$\left.\begin{aligned} P(S \cup F \cup G) = P(S) + P(F) + P(G) \\ - P(SF) - P(SG) - P(FG) \\ + P(SFG) \end{aligned}\right\} \text{inclusion exclusion}$$



$$= \frac{28}{100} + \frac{26}{100} + \frac{16}{100} - \frac{12}{100} - \frac{4}{100} - \frac{6}{100} + \frac{2}{100}$$

$$= \frac{50}{100}$$

So

$$P((S \cup F \cup G)^c) = 1 - \frac{50}{100} = \frac{50}{100} = \frac{1}{2}$$

b) If a student is chosen randomly, what is the probability he/she is taking exactly one language class?

$$P(\text{only } S) = P(S) - P(S \cap F) - P(S \cap G) + P(FSG)$$

$$P(\text{only } F) = P(F) - P(S \cap F) - P(G \cap F) + P(FSG)$$

$$P(\text{only } G) = P(G) - P(G \cap F) - P(G \cap S) + P(FSG)$$

$$\text{sum} = \frac{28 + 26 + 16 - 2(12) - 2(4) - 2(6) + 3(2)}{100}$$

$$P(\text{only } S) + P(\text{only } F) + P(\text{only } G) = \frac{70 - 24 - 8 - 12 + 6}{100} = \frac{32}{100} = .32$$

c) If 2 students are chosen randomly, what is the probability that at least one is taking a language class?

$$P(\text{neither is taking language}) = \frac{\#\text{ of outcomes with no language}}{\text{total } \#\text{ of outcomes}}$$

$$= \frac{\binom{50}{2}}{\binom{100}{2}} \quad \begin{matrix} \leftarrow \text{choose 2 of 50 not in lang.} \\ \leftarrow \text{choose 2 of 100 students} \end{matrix}$$

assume we don't mean that we could pick the same student twice.

$$= \frac{50 \cdot 49}{100 \cdot 99} = \frac{49}{198} \rightarrow \boxed{\text{so } P(\text{at least one lang.}) = 1 - \frac{49}{198} = \frac{149}{198}}$$

strategy?

lecture 6
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Problem 27 Ch. 2

An urn contains 3 red balls and 7 black balls. Player A and Player B withdraw balls from the urn consecutively until a red ball is selected. Player A draws first. There is no replacement of balls.

What is the probability that Player A selects the red ball?

Sol

Let E_n denote the event that Player A draws red on n^{th} draw

$P(E_1) = \frac{3}{10}$, $P(E_2) = 0$ (2^{nd} draw is Player B)

$$P(E_3) = \left(\underset{B}{\frac{7}{10}} \cdot \underset{B}{\frac{6}{9}}\right) \cdot \underset{R}{\frac{3}{8}}, \quad P(E_4) = 0$$

$$P(E_5) = \left(\underset{B}{\frac{7}{10}} \cdot \underset{B}{\frac{6}{9}}\right)\left(\underset{B}{\frac{5}{8}} \cdot \underset{B}{\frac{4}{7}}\right) \underset{R}{\frac{3}{6}}, \quad P(E_6) = 0$$

$$P(E_7) = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5} \cdot \frac{3}{4}, \quad P(E_8) = 0$$

$P(E_9) = 0$ since 8 draws had to happen before that and there are only 7 black balls.

So $P(\text{Player } A) = P(E_1) + P(E_3) + P(E_5) + P(E_7)$

$$= \frac{3}{10} + \frac{7\cdot 6\cdot 3}{10\cdot 9\cdot 8} + \frac{\cancel{7}\cdot\cancel{6}\cdot 5\cdot 4}{10\cdot 9\cdot 8\cdot\cancel{7}}\cdot\frac{3}{\cancel{6}}$$

$$+ \frac{\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot 3\cdot 2}{10\cdot 9\cdot 8\cdot\cancel{7}\cdot\cancel{6}\cdot\cancel{5}}\cdot\frac{3}{\cancel{4}}$$

$$= \frac{3\cdot 9\cdot 8 + 7\cdot 6\cdot 3 + 5\cdot 4\cdot 3 + 3\cdot 2\cdot 3}{10\cdot 9\cdot 8} = \frac{216+126+60+18}{720} = \frac{420}{720}$$

$$= \frac{7}{12}$$

So Player A has a better chance of "winning" (if picking first = win) than Player B.

How does this change with # of balls, etc.

(what is a good strategy to win if a player gets to pick who goes first (maybe flips coin to give player A or B the choice to go first or not).

intuition?

Assumptions
- all days equally likely
- no birthdays on Feb. 29 (giving 365 possible days out of 366 actual days)

Example 5i (p. 37)

If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year?

$$P(\text{all birthdays differ}) = \frac{\overbrace{365 \cdot 364 \cdot \ldots \cdot (365-n+1)}^{\text{total with } n \text{ different dates}}}{\underbrace{365 \cdot 365 \cdot \ldots \cdot 365}_{\text{total \# of possibilities}}}$$

$$= \frac{365!}{(365-n)!} \frac{1}{365^n}$$

n	P(different)	
10	0.883	
15	0.747	
20	0.589	
21	0.556	
22	0.524	← 47.6% chance of a match
23	0.493	← 50.7% " " "
25	0.431	
30	0.294	
40	0.109	← ~ 90% chance of a match
50	0.0296	
60	0.00588	
70	0.000840	99.91% chance of a match
80	0.000857	← ~~[illegible]~~
90	6.15×10^{-6}	
100	3.07×10^{-7}	
⋮		
$n \geq 365$	0	← pigeonhole principle!

intuition?

Example 5j (p.37)

A deck of 52 playing cards is shuffled. Cards turned up one at a time until the first ace appears. Is the card following the first ace more likely to be the ace of spades or the two of clubs?

- total # of orderings of 52 cards = 52!

want: # of Orderings resulting in As following first ace.

note: # of Ordering of the 51 other cards = 51!

~~only one of these card~~

note, getting As first would not be a way As could follow first ace

each one of these 51! orderings has only one way to put the As after the first ace. So

$$P(\text{As after A\#1}) = \frac{51! \cdot 1}{52!} = \frac{1}{52}$$

The same argument would apply if we ordered 51 cards excluding the 2c and then inserting the 2c after first ace. So

$$P(\text{2c after A\#1}) = \frac{51!}{52!} = \frac{1}{52}$$

both events are equally-likely.