Section 2.1 Algebraic and Order in $\mathbb{R}$.

2.1.1 Algebraic Properties of $\mathbb{R}$ On the set $\mathbb{R}$ of real numbers there are two binary operations, denoted by $+$ and $\cdot$ and called **addition** and **multiplication**, respectively. These operations satisfy the following properties:

(A1) $a + b = b + a$ for all a, b in $\mathbb{R}$ (*commutative property of addition*);

(A2) $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathbb{R}$ (*associative property of addition*);

(A3) there exists an element 0 in $\mathbb{R}$ such that $0 + a = a$ and $a + 0 = a$ for all a in $\mathbb{R}$ (*existence of a zero element*);

(A4) for each a in $\mathbb{R}$ there exists an element $-a$ in $\mathbb{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$ (*existence of negative elements*);

(M1) $a \cdot b = b \cdot a$ for all a, b in $\mathbb{R}$ (*commutative property of multiplication*);

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in $\mathbb{R}$ (*associative property of multiplication*);

(M3) there exists an element 1 in $\mathbb{R}$ *distinct from* 0 such that $1 \cdot a = a$ and $a \cdot 1 = a$ for all a in $\mathbb{R}$ (*existence of a unit element*);

(M4) for each $a \neq 0$ in $\mathbb{R}$ there exists an element $1/a$ in $\mathbb{R}$ such that $a \cdot (1/a) = 1$ and $(1/a) \cdot a = 1$ (*existence of reciprocals*);

(D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all a, b, c in $\mathbb{R}$ (*distributive property of multiplication over addition*).

A's are related to addition

M's " " multiplication

D combines bot.

If we interested in a lot of consequences of (AI) (MI) (D) take advanced linear algebra.

2.2 Absolute value and the Real line

2.2.1. Definition If $a \in \mathbb{R}$ then $|a|$ is defined as

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

2.2.2 Theorem (a) $|ab| = |a||b|$

(b) $|a|^2 = a^2$

(c) If $c \geq 0$, then $|a| \leq c$ iff $-c \leq a \leq c$

(d) $-|a| \leq a \leq |a|$

Proof. (a) Just consider all cases. - $a = 0$ (or) $b = 0$ is trivial

- $a > 0, b > 0 \Rightarrow ab > 0 \Rightarrow |ab| = ab$ and $|a||b| = ab$ ✓
- $a < 0, b < 0 \Rightarrow ab > 0 \Rightarrow |ab| = ab$ and $|a||b| = (-a)(-b) = ab$ ✓
- $a < 0, b > 0 \Rightarrow ab < 0 \Rightarrow |ab| = -ab$ and $|a||b| = (-a)b = -ab$ ✓
- $a > 0, b < 0$ identical.

(b) $a^2 = |a^2| = |a \cdot a| = |a||a| = |a|^2$

(c)(I) $|a| \le c \Rightarrow \begin{cases} a \le c & \text{if } a > 0 \\ 0 \le c & a = 0 \\ -a \le c & \text{if } a < 0 \end{cases}$

then $a \le c$ and $-a \le c$ in any case. since $c \ge 0$.

then $-c \le a \le c$.

(II) $-c \le a \le c \Rightarrow a \le c$ and $-a \le c$

$\Rightarrow |a| \le c$. ∎

(d) $c = |a|$.

2.2.3 $\triangle$ ineq.

If $a, b \in \mathbb{R}$ then

$$|a+b| \le |a| + |b|.$$

Proof: $\left.\begin{array}{l} -|b| \le b \le |b| \\ -|a| \le a \le |a| \end{array}\right\} -(|b|+|a|) \le a+b \le |a|+|b|$

$\Rightarrow |a+b| \le |a| + |b|$. ∎

2.2.4 Corollary

(a) $\big||a| - |b|\big| \le |a-b|$

(b) $|a-b| \le |a| + |b|$.

Examples (a) Determine the set A of $x \in \mathbb{R}$: $|2x+3| < 7$.

By definition. ~~...~~ and 2.2.2.(c)

$-7 < 2x+3 < 7 \Rightarrow -10 < 2x < 4$

$\Rightarrow -5 < x < 2$.

$A = \{x \in \mathbb{R} : -5 < x < 2\}$.

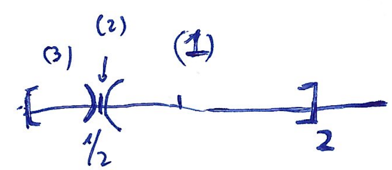
(c) Solve $|2x - 1| \le x + 1$

Find all x's satisfying the inequality.

By definition $\begin{cases} 2x-1 \le x+1 & \text{if } 2x-1 > 0 \\ 0 \le x+1 & \text{if } 2x-1 = 0. \\ -(2x-1) \le x+1 & \text{if } 2x-1 < 0. \end{cases}$

(1) $x > \frac{1}{2}$ and $x \le 2$ (2) $0 \le \frac{1}{2} + 1$

(3) $x < \frac{1}{2}$ and $0 \le 3x$



$= \{x \in \mathbb{R} : 0 \le x \le 2\}$.

The real line

The distance between two numbers $a, b \in \mathbb{R}$ is $|a-b|$. Then.

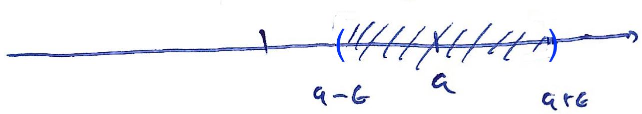
2.2.7 Definition Let $a \in \mathbb{R}$ and $\epsilon > 0$. Then the ϵ-neighborhood of a is the set $V_\epsilon(a) = \{x \in \mathbb{R} : |x-a| < \epsilon\}$.

(Set of all points x : the distance to a is less to ϵ)

By 2.2.2-(c) $\Rightarrow$ $-\epsilon < x-a < \epsilon$

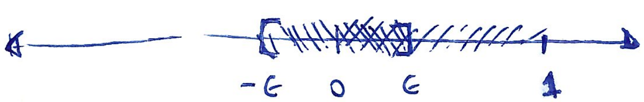
$\Rightarrow$ $-\epsilon + a < x < \epsilon + a$

$V_\epsilon(a)$.



Example 1 $I = \{x : 0 \leq x \leq 1\}$

then $\forall \epsilon > 0$ $V_\epsilon(0)$ contains points not in I.



Choose $-\epsilon/2 \in V_\epsilon(0) = \{x : |x| < \epsilon\}$.

Section 2.3 The completeness property of $\mathbb{R}$

Suprema and Infima:

2.3.1. Definition Let S be a non-empty subset of $\mathbb{R}$.

(a) The set S is said to be bounded above if $\exists\ u \in \mathbb{R}$: $s \leq u$ $\forall s \in S$. Each such u is called an upper bound of S.

(b) Analogously we define bounded below and lower bounds

(c) A set is said to be bounded if it is both bounded above and bounded below.

A set is said to be unbounded if it is not bounded

Example – $\{x \in \mathbb{R} : x > 2\}$ is unbounded

– $\underbrace{\{x \in \mathbb{R} : x = \frac{1}{n} \text{ for some } n \in \mathbb{N}\}}_{A :=}$ is bounded

2.3.2 Definition

(a) If S is bounded above, then a number u is said to be a supremum (or least upper bound) of S if it satisfies the conditions.

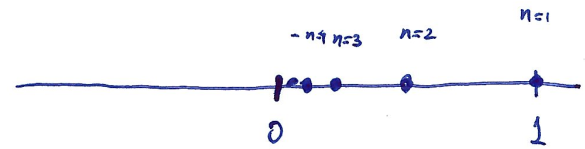
(1) u is an upper bound of S

(2) If v is an upper bound of S then $u \leq v$.

(b) Analogously we define infimum (or greatest lower bound).

We denote $\text{Sup } S$ and $\text{Inf } S$, respectively provided they exist.

Example $A = \{x \in \mathbb{R} : x = \frac{1}{n} \text{ for some } n \in \mathbb{N}\}$



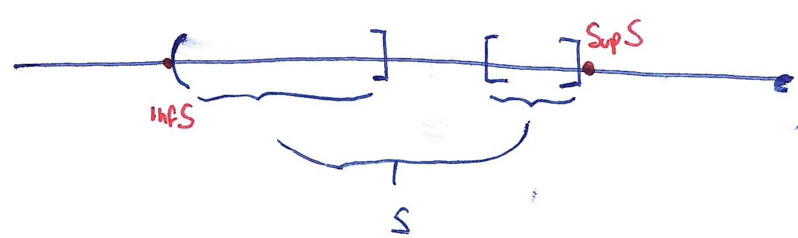
$\boxed{\text{Sup } A = 1}$ + $\frac{1}{n} \leq 1$ so 1 is an upper bound.

+ Let v be another upper bound: $v < 1$
but for $n = 1$ $v < \frac{1}{n}$ then v can't be an upper bound.

$\boxed{\text{Inf } A = 0}$ + Clearly $0 \leq \frac{1}{n}$ $\forall n \in \mathbb{N}$
So 0 is a lower bound.

+ let v be a lower bound and $0 < v$
then $0 < v \leq \frac{1}{n}$ $\Rightarrow$ taking $\lim_{n \to \infty}$ $\Rightarrow$ $0 < v \leq 0$
$(\Rightarrow\Leftarrow)$ it can't be a lower bound.

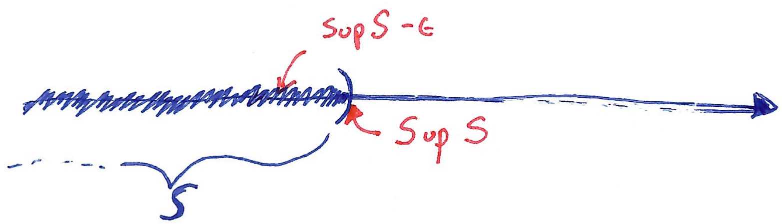
Graphically is



2.3.4. Lemma An upper bound u of a non-empty set S in $\mathbb{R}$ is the supremum of S

if and only if

$$\forall \epsilon > 0, \ \exists \ s_\epsilon \in S : u - \epsilon < s_\epsilon . \quad (*)$$



Proof: Let u be an upper bound and suppose $(*)$ holds.

Suppose that v is an upper bound and $v < u$.

Let $\epsilon := u - v \Rightarrow \exists \ s_\epsilon \in S$

$: u - (u - v) < s_\epsilon \Rightarrow v < s_\epsilon \ (\Rightarrow\Leftarrow)$

then $\operatorname{Sup} S = u$.

Let u be an upper bound and further a supremum.

Let $\epsilon > 0$. Since $u - \epsilon < u$ then $u - \epsilon$ is not an upper bound of S. That is, $\exists \ s_\epsilon \in S$ $: u - \epsilon < s_\epsilon$ ☒

The completeness property of $\mathbb{R}$

2.3.6 The Comp. prop. of $\mathbb{R}$ Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.

2.4 Applications to the Supremum property.

Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$

Proposition Let S be a nonempty subset of $\mathbb{R}$ that is bounded above and let $a \in \mathbb{R}$ be arbitrary. Then, for $a+S := \{ a+s : s \in S \}$, we have

$$\sup (a+S) = a + \sup S$$

Proof: $\boxed{\sup(a+S) \leq a + \sup S}$

Since $x \leq \sup S \quad \forall x \in S \Rightarrow x + a \leq \sup S + a$ $\forall x \in S \Rightarrow \sup S + a$ is an upper bound, then

$$\sup(a+S) \leq a + \sup S.$$

$$\boxed{a + \sup S \leq \sup(a+S)}$$

Let ν be any upper bound of $a+S$, then $a + x \leq \nu \quad \forall x \in S$. Then $x \leq \nu - a \quad \forall x \in S$ $\Rightarrow \nu - a$ is an upper bound to S. Hence $\sup S \leq \nu - a$ or $\sup S + a \leq \nu$. Since ν is an arbitrary upper bound, then chose $\nu = \sup(a+S)$
$\Rightarrow a + \sup S \leq \sup(a+S)$. ∎

Proposition: Let A and B be non-empty subsets of $\mathbb{R}$, with the property.

$$a \leq b \quad \forall a \in A, \ \forall b \in B.$$

Then
$$\sup A \leq \inf B.$$

Proof: ~~[illegible]~~ For a given $b \in B$
$a \leq b \quad \forall a \in A \Rightarrow b$ is an upper bound to $A \Rightarrow \sup A \leq b$. Since b was arbitrary, $\sup A$ is a lower bound to $B \Rightarrow$

$$\sup A \leq \inf B. \quad ∎$$

Functions

Recall that for $f: D \to \mathbb{R}$,
$f(D) = \{ f(x) : x \in D \}$.

- We say that f is bounded above (below) if $f(D)$ is bounded above (below).

- If f is bounded above and below then $\exists M \in \mathbb{R} : |f(x)| \leq M \quad \forall x \in D$.

Proposition: Suppose $f: D \to \mathbb{R}$, $g: D \to \mathbb{R}$ are functions.

(a) If $f(x) \leq g(x)$ $\forall x \in D$ then

$$\sup f(D) \leq \sup g(D)$$

[also written $\sup_{x\in D} f(x) \leq \sup_{x \in D} g(x)$]

(b) If $f(x) \leq g(y)$ $\forall x \in D, \forall y \in D$

then $\sup f(D) \leq \inf g(D)$

[also written $\sup_{x\in D} f(x) \leq \sup_{x\in D} g(x)$]

Proof: (a) Since $f(x) \leq g(x) \leq \sup g(D)$ then $\sup g(D)$ is an upper bound of $f(D)$ then $\sup f(D) \leq \sup g(D)$

(b) This follows from the previous

Proposition ∎

Compare this to the "cheating" proof using limits from previously.

Archimedean Property (2.4.3)

$\boxed{\text{If } x \in \mathbb{R}, \text{ then there exists } n_x \in \mathbb{N} \text{ such that } x \leq n_x}$

Proof: Suppose the statement is false, i.e., $n \leq x$ $\forall n \in \mathbb{N}$. Hence x is an upper bound to $\mathbb{N}$. Then by the supremum property $\sup(\mathbb{N}) \in \mathbb{R}$. Hence, $\sup(\mathbb{N})-1$ is not an upper bound to $\mathbb{N}$, then $\exists m \in \mathbb{N}$: $\sup(\mathbb{N})-1 \leq m \Rightarrow \sup(\mathbb{N}) \leq m+1 \in \mathbb{N}$.

$\Rightarrow \Leftarrow$ ∎

2.4.4 Corollary If $S := \{\frac{1}{n} : n \in \mathbb{N}\}$ then $\inf S = 0$

Proof: $S \neq \emptyset$ and 0 is a lower bound then $0 \leq \inf S$. For each $\epsilon > 0$, $\exists n \in \mathbb{N}$: $\frac{1}{\epsilon} < n$ (by the Archimedean property) $\Rightarrow$ $\frac{1}{n} < \epsilon$. Hence $0 \leq w \leq \frac{1}{n} < \epsilon$ and since $\epsilon > 0$ is arbitrary $\Rightarrow w = 0$. ∎

Two more important results

2.4.5 Corollary If $t>0$ then there exists $n_t \in \mathbb{N}$ such that $0 < \frac{1}{n_t} < t$.

2.4.6 Corollary If $y>0$, $\exists\ n_y \in \mathbb{N}$ such that $n_y - 1 \leq y < n_y$.

Density of Rational Numbers

2.4.8 the Density Theorem If x and y are real numbers with $x<y$, then there exists a rational number $r \in \mathbb{Q}$ such that

$$x < r < y.$$

Proof: Suppose $x>0$ (Otherwise add M to x and y so $x+M>0$)

Since $0 < y-x \overset{\text{corollary } 2.4.5}{\Longrightarrow} \frac{1}{n} < y-x \Rightarrow$

$nx + 1 < ny$. ⊛

Apply 2.4.6 to $nx > 0 \Rightarrow \exists\ m$:

$$m-1 \leq nx < m$$

Then by ⊛ and $+1$ $\rightarrow$ $m \leq nx+1 < ny$

$$nx < m < ny$$

$$\Rightarrow \quad x < \frac{m}{n} < y$$

If $x<0 \Rightarrow -x>0 \Rightarrow$ by the A.P. $\exists\ n$: $-x \leq n$.
$< \underbrace{n+1}_{=:\tilde{n}}$

$$\Rightarrow \quad 0 < \tilde{n} + x < \tilde{n} + y$$

by (2.4.8) $\Rightarrow$ $\tilde{n} + x < r < \tilde{n} + y$

$$x < \frac{p}{q} - \tilde{n} < y$$

$$x < \frac{p - q\tilde{n}}{q} < y$$

Section 2.5 (Intervals)

$(a,b) := \{x \in \mathbb{R} : a < x < b\}$

$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$

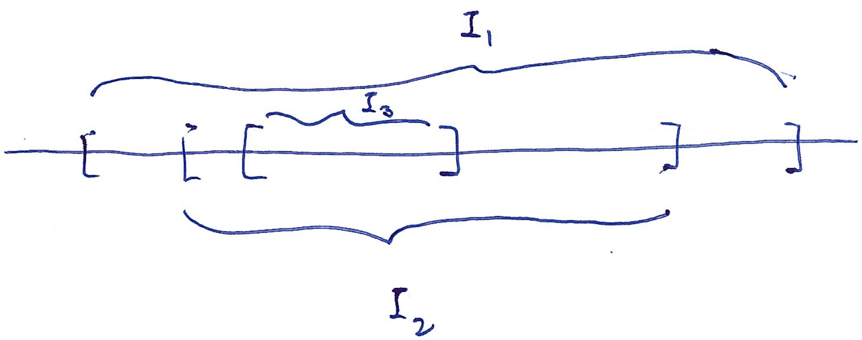
$(a,+\infty)$, $[a,+\infty)$, $(-\infty,b)$, $(-\infty,b]$

$(-\infty,+\infty) = \mathbb{R}$.

Nested Intervals

I_n, $n \in \mathbb{N}$ is nested if

$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$



2.5.2 Nested Intervals property

If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence ~~with~~ of closed and bounded intervals $\Rightarrow$ $\exists\, \xi \in \mathbb{R} : \xi \in I_n \ \forall n \in \mathbb{N}$

Proof: – Since $I_n \subseteq I_1 \Rightarrow a_n \leq b_1 \ \forall n \in \mathbb{N}$

Hence $\{a_n : n \in \mathbb{N}\}$ is bounded above, and we define ξ to be the $\sup(\{a_n : n \in \mathbb{N}\})$, so $a_n \leq \xi$.

– We prove that $\xi \leq b_n$, by proving that b_n is always an upper bound to $\{a_n : n \in \mathbb{N}\}$. Let $k \in \mathbb{N}$ be arbitrary

$$a_k \leq \begin{cases} a_k \leq b_k \leq b_n & n \leq k \\ a_k \leq a_n \leq b_n & n > k \end{cases}$$

$\Rightarrow$ b_n is an upper bound of $\{a_k : k \in \mathbb{N}\}$

$\Rightarrow$ $\xi \leq b_n \ \forall n \in \mathbb{N}$.

$\Rightarrow$ $a_n \leq \xi \leq b_n \quad \forall n \in \mathbb{N}$. ∎

The uncountability on $\mathbb{R}$

Remember $\mathbb{N}$ is countable (denumerable)

$\mathbb{Q}$ is also

but $P(\mathbb{N})$ and $P(\mathbb{Q})$ are not.

Theorem (2.5.5) The unit interval $[0,1] := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ is ~~...~~ not countable.

Proof: Note that every $x \in [0,1]$ has a decimal representation

$$x = \alpha_1 \alpha_2 \alpha_3 \alpha_4 \cdots\cdots$$

Suppose there is a ~~...~~ surjection

$$f : \mathbb{N} \to [0,1]$$

Then.

$$f(1) = 0.a_1 a_2 a_3 a_4 \cdots\cdots$$
$$f(2) = 0.b_1 b_2 b_3 b_4 \cdots$$
$$f(3) = 0.c_1 c_2 c_3 c_4 \cdots$$
$$f(4) = 0.d_1 d_2 d_3 d_4 \cdots$$
$$\vdots \qquad \vdots$$

Define the number $y \in [0,1]$ as.

$$y = 0.\beta_1 \beta_2 \beta_3 \beta_4 \cdots\cdots$$

with $\beta_1 \neq a_1$, $\beta_2 \neq b_2$, $\beta_3 \neq c_3$, $\beta_4 \neq d_4$, ……

then there is no $m \in \mathbb{N} : f(m) = y$

∎