5.2 — continued...

5.3 Variance of a Continuous Random Variable

The variance of a continuous random variable X is defined by

$$Var(X) = E[(X-\mu)^2]$$
$$= E[X^2] - (E[X])^2$$

where $\mu = E[X]$

Comments

- this is the same definition that we saw for discrete random variables.

- Recall

$$E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx$$ where f is the prob. density function for X

- To compute $Var(X)$ we'll need to be able to compute the expected value of a function $g(X) = (X-\mu)^2$ (or $g(X) = X^2$)
  - we can do this by finding the prob. density function for $Y = (X-\mu)^2$ or use the following result...

Proposition 2.1 (p. 181, Ross)

If X is a continuous random variable with probability density function $f(x)$, then for any real valued function g,

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f(x)\,dx$$

A proof of Proposition 2.1 is given in the textbook (p. 181).

EXAMPLE (see previous example p. 154 – 155 notes)

- Let X have prob. density function $f_X(x) = \begin{cases} 0 & x<0 \\ e^{-x} & x \geq 0 \end{cases}$

(exponential with $\lambda = 1$)

- Let $Y = g(X) = X^2$
- Recall that we found $f_Y(y) = \begin{cases} 0 & y<0 \\ \frac{e^{-\sqrt{y}}}{2\sqrt{y}} & y>0 \end{cases}$

Find the expected value of Y $E[Y]$ (i.e. $E[X^2]$).

Method 1: use pdf f_Y

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \int_0^{\infty} y \frac{e^{-\sqrt{y}}}{2\sqrt{y}}\, dy$$

$$= \frac{1}{2}\int_0^{\infty} \sqrt{y}\, e^{-\sqrt{y}}\, dy$$

$$\begin{cases} \text{Let } s = +\sqrt{y} \\ ds = +\frac{1}{2}y^{-1/2}dy = +\frac{1}{2\sqrt{y}}dy \end{cases} \qquad dy = 2\sqrt{y}\, ds = 2s\, ds$$

$u = s^2$ $du = 2s\,ds$
$dv = e^{-s}ds$ $v = -e^{-s}$

$$= \int_0^{\infty} \underbrace{s^2}_{u}\, \underbrace{e^{-s}\, ds}_{dv}$$

integrate by parts

$$= -s^2e^{-s}\Big|_0^{\infty} + \int_0^{\infty} 2s e^{-s}\, ds$$

$u = s$ $du = ds$
$dv = e^{-s}ds$ $v = -e^{-s}$

$$= 0 + 2\int_0^{\infty} s e^{-s}\, ds$$

$$= 2\left[-se^{-s}\Big|_0^{\infty} + \int_0^{\infty} e^{-s}\, ds\right] = 2\left[0 - e^{-s}\Big|_0^{\infty}\right] = \boxed{2} = E[Y]$$

Method 2 use pdf f_X and Proposition 2.1

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \underbrace{\int_0^{\infty} x^2 e^{-x}\,dx}_{\text{we just did this integral! } \boxed{=2}}$$

so $\boxed{E[X^2] = 2} \;\boxed{= E[I]}$

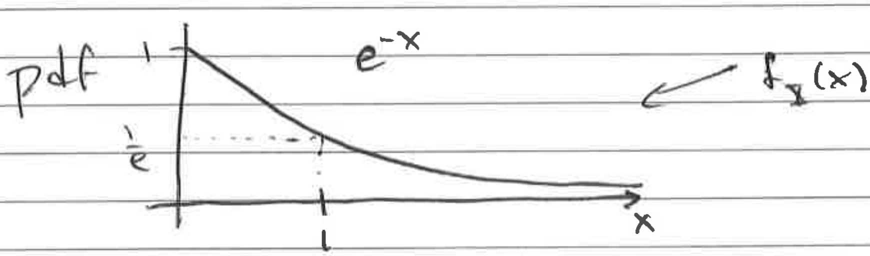
So for this random variable X with $f_X(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_0^{\infty} \underbrace{x}_{u}\, \underbrace{e^{-x}dx}_{dv}$$

$$= x(-e^{-x})\Big|_0^{\infty} - \int_0^{\infty} -e^{-x}\,dx$$

$$= 0 + -e^{-x}\Big|_0^{\infty} = 1 \qquad \text{so } E[X] = 1$$

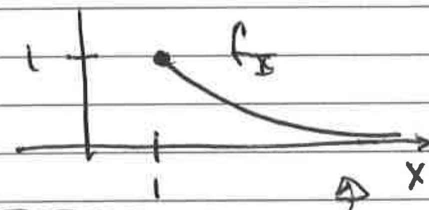
$$Var(X) = E[X^2] - (E[X])^2 = 2 - 1 = \boxed{1 = Var(X)}$$



EXAMPLE

Let X be a random variable with

$$f_X(x) = \begin{cases} 0 & x < 1 \\ \dfrac{1}{x^2} & x \geq 1 \end{cases}$$



However, Note:

$$E[X] = \int_{-\infty}^{+\infty} x \, f_X(x)\,dx$$

Note: $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{\infty} = 1$

$$= \int_1^{\infty} x \, \frac{1}{x^2}\,dx = \int_1^{\infty} \frac{1}{x}\,dx = \ln x \Big|_1^{\infty} \to +\infty$$

So the Expected Value of X does not exist.

Lemma 2.1 (Ross, p. 181)

For a nonnegative random variable Υ (range of $\Upsilon \geq 0$)

The result

$$\boxed{E[\Upsilon] = \int_0^\infty P\{\Upsilon > y\}\,dy}$$

cumulative dist. function

recall $P\{\Upsilon \leq y\} = \int_{-\infty}^{y} f_\Upsilon(t)\,dt$

to see ...

$$= \int_0^\infty \underbrace{\left[\int_y^\infty f_\Upsilon(t)\,dt\right]}_{u} \underbrace{dy}_{dv}$$

integrate by parts

$$\begin{cases} dv = dy & v = y \\ u = \int_y^\infty f_\Upsilon(t)\,dt & du = -f_\Upsilon(y) \end{cases}$$

$$= \left[y \int_y^\infty f_\Upsilon(t)\,dt \Big|_{y=0}^{\infty} - \int_0^\infty y\left(-f_\Upsilon(y)\right)dy \right]$$

0

$$= +\int_0^\infty y\, f_\Upsilon(y)\,dy = E[\Upsilon].$$

5.3 The Uniform Random Variable

The continuous random variable X is a uniform random variable on (a,b) if it has probability density function

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & \text{otherwise} \end{cases}$$

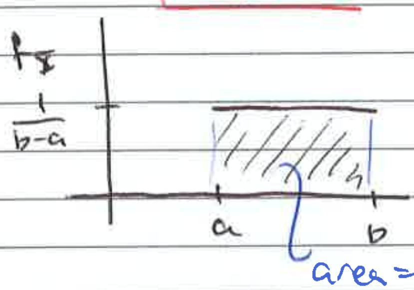
i.e. X is uniformly distributed on (a,b)

(equivalently, we can write $[a,b]$ and $a \le x \le b$ since the probability of one-point sets for continuous random variables is zero).

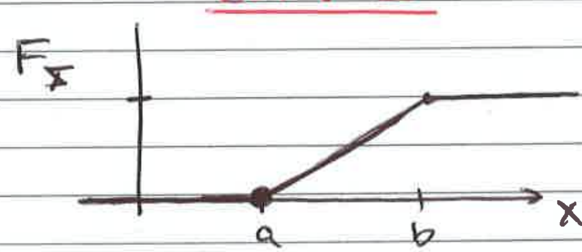
- Cumulative distribution function

$$F_X(x) = P\{X \le x\} = \int_{-\infty}^{x} f_X(x)\,dx = \begin{cases} 0 & x<a \\ \int_a^x \frac{1}{b-a}dx = \frac{x-a}{b-a} & a\le x\le b \\ 1 & x>b \end{cases}$$

pdf of X



cdf of X



- Expected Value

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_a^b \frac{x}{b-a}dx = \frac{\frac{1}{2}x^2}{(b-a)}\Big|_a^b = \frac{1}{2}\frac{b^2-a^2}{b-a} = \frac{1}{2}(b+a)$$

midpoint of segment (a,b)

- $$E[X^2] = \int_a^b \frac{x^2}{b-a}dx = \frac{1}{3}\frac{b^3-a^3}{b-a} = \frac{1}{3}\frac{(b-a)(b^2+ab+a^2)}{(b-a)} = \frac{b^2+ab+a^2}{3}$$

- Variance

$$Var(X) = E[X^2]-(E[X])^2 = \frac{1}{3}(b^2+ab+a^2) - \frac{1}{4}(b+a)^2 = \frac{4b^2+4ab+4a^2-3(b^2+2ab+a^2)}{12}$$

$$= \frac{1}{12}(b^2-2ab+a^2) = \frac{1}{12}(b-a)^2 = Var(X)$$

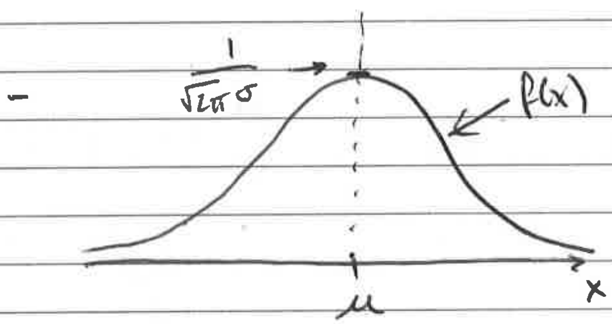
function of width.

5.4 Normal Random Variable

A random variable X is a <u>normal random variable</u> (or X is normally-distributed) with parameters μ and σ^2 if its probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

- recall, then $P\{a \le X \le b\} = \int_a^b f(x)\,dx$

- 

- "bell-shaped" curve

- $E[X] = \mu$
- $Var(X) = \sigma^2$

} see p. 169 in notes we'll check these in the context of standard normal then generalize...

* Let's first check some properties of the <u>standard normal random variable</u> X with

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad (\text{i.e. } \sigma = 1, \mu = 0)$$

$$-\infty < x < \infty$$

- First, let's confirm that

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

is equal to one (is it?)

Let $I = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

nifty trick...

Note:

$$I^2 = \left(\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right) \left(\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)/2} dx\, dy$$

convert to polar coordinates $\quad dx\,dy \rightarrow r\,dr\,d\theta$

$r^2 = x^2 + y^2$

$$I^2 = \frac{1}{2\pi} \int_0^{2\pi} \left[\int_0^{\infty} e^{-r^2/2} r\,dr \right] d\theta$$

$$= \frac{1}{2\pi} 2\pi \int_0^{\infty} r e^{-r^2/2} dr$$

$$= -e^{-r^2/2} \Big|_0^{\infty} = 0 - (-1) = 1 \quad \checkmark$$

- Next, let's compute the expected value $E[X]$.

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} x e^{-\frac{x^2}{2}} dx + \int_{0}^{\infty} x e^{-x^2/2} dx \right]$$

$$\int_{0}^{\infty} x e^{-\frac{x^2}{2}} dx = \int_{0}^{-\infty} -e^{+u} du = -e^{+u}\Big|_{0}^{-\infty} = +1$$

$u = -\frac{x^2}{2}$

$du = -x\,dx$

$$\int_{-\infty}^{0} x e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{0} -e^{u} du = -e^{u}\Big|_{-\infty}^{0} = -1$$

So both of these integrals are finite and it follows

$$\boxed{E[X] = 0}$$

• Next, let's compute the Variance (X = standard normal)

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_{-\infty}^{+\infty} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2}} dx$$

$u = x \qquad du = dx$

$dv = x e^{-x^2/2} dx \qquad v = -e^{-x^2/2}$

$$= \frac{1}{\sqrt{2\pi}} \left[\cancel{-x e^{-x^2/2} \Big|_{-\infty}^{\infty}}^{\,0} - \int_{-\infty}^{\infty} -e^{-x^2/2} dx \right]$$

$$= \underbrace{\boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx}}_{=1} = \int_{-\infty}^{\infty} f(x) dx = 1$$

So $Var(X) = 1 - 0 = 1$

∴ For standard normal distribution $\quad P\{X \in B\} = \int_B f(x)dx$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad -\infty < x < \infty$$

$E[X] = 0$

$Var(X) = 1$

Cumulative Distribution Function

common tabulated quantity – see p. 190

$$\boxed{F(x) = P\{X \le x\} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\,dt \quad \equiv \Phi(x) \text{ often}}$$

← X = standard normal distribution

Note:

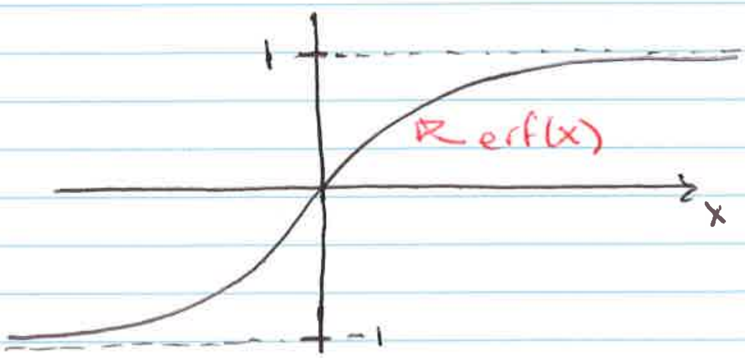
$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\,ds = \text{error function}$$

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-s^2}\,ds = \text{complementary error function}$$

$\mathrm{erf}(0) = 0$

$\mathrm{erf}(x \to \infty) = 1$

$\mathrm{erf}(x) + \mathrm{erfc}(x) = 1$



Let $t = -\sqrt{2}\,s$

$dt = -\sqrt{2}\,ds$

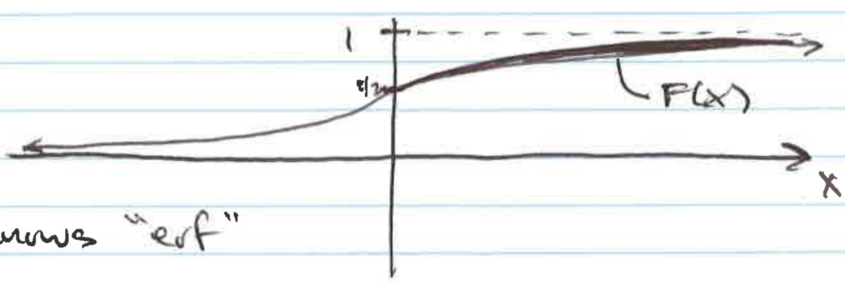
Then

$$F(x) = \int_{+\infty}^{-\frac{x}{\sqrt{2}}} -\frac{1}{\sqrt{2\pi}} e^{-s^2} \sqrt{2}\,ds = \frac{1}{\sqrt{\pi}} \int_{+\infty}^{-\frac{x}{\sqrt{2}}} -e^{-s^2}\,ds = \frac{1}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{2}}}^{\infty} e^{-s^2}\,ds$$

So

$$F(x) = \frac{1}{2}\,\mathrm{erfc}\left(-\frac{x}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(1 - \mathrm{erf}\left(-\frac{x}{\sqrt{2}}\right)\right) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$$



The table in book p. 190 shows values for this

Matlab knows "erf"

We can connect normally distributed X with standard normal random variable Y as follows.

- Let X be a normal random variable with parameters μ and σ^2. So

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Let $\boxed{Y = \frac{X-\mu}{\sigma}}$. Then

often called $Z = \frac{X-\mu}{\sigma}$

$$F_Y(y) = P\{Y \le y\} = P\left\{\frac{X-\mu}{\sigma} \le y\right\} = P\{X \le y\sigma+\mu\}$$

$$= \int_{-\infty}^{y\sigma+\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

but then

$$f_Y = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_{-\infty}^{y\sigma+\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\Bigg|_{x=y\sigma+\mu} \cdot \underbrace{\sigma}_{\frac{d}{dy}(y\sigma+\mu)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$\leftarrow$ standard normal prob. density function.

So if $E[Y] = 0$

$$Var(Y) = E[Y^2] - (E[Y])^2 = 1 - 0 = 1$$

then

$$E[X] = E[\mu + \sigma Y]$$

$$= \int_{-\infty}^{+\infty} (\mu + \sigma y) f_Y(y) dy$$

$$= \mu \underbrace{\int_{-\infty}^{+\infty} f_Y(y) dy}_{=1} + \sigma \underbrace{\int_{-\infty}^{+\infty} y f_Y(y) dy}_{\substack{=0 \\ =E[Y]}} = \mu \checkmark$$

and

$$E[X^2] = E[(\mu + \sigma Y)^2]$$

$$= E[\mu^2 + 2\mu\sigma Y + \sigma^2 Y^2]$$

$$= \int_{-\infty}^{+\infty} (\mu^2 + 2\mu\sigma y + \sigma^2 y^2) f_Y(y) dy$$

$$= \mu^2 \underbrace{\boxed{\int_{-\infty}^{+\infty} f_Y(y) dy}}_{=1} + 2\mu\sigma \underbrace{\boxed{\int_{-\infty}^{+\infty} y f_Y(y) dy}}_{=E[Y]=0} + \sigma^2 \underbrace{\boxed{\int_{-\infty}^{+\infty} y^2 f_Y(y) dy}}_{\substack{E[Y^2] \\ =1}}$$

$$= \mu^2 + \sigma^2$$

So

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$= \sigma^2 \checkmark$$

EX

Suppose X is a normal random variable with $\mu = -3$ and $\sigma^2 = 9$

Find $P\{-4 \le X \le 0\}$

write $Z = \dfrac{X-\mu}{\sigma}$

then $X = \mu + \sigma Z$

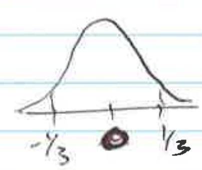
$$P\{-4 \le X \le 0\} = P\{-4 \le \mu + \sigma Z \le 0\}$$

$$= P\left\{ \frac{-4-\mu}{\sigma} \le Z \le \frac{0-\mu}{\sigma} \right\}$$

$$= P\left\{ \frac{-4-(-3)}{\sqrt{9}} \le Z \le \frac{0-(-3)}{\sqrt{9}} \right\}$$

$$= P\left\{ -\frac{1}{3} \le Z \le 1 \right\}$$

$$= P\{Z \le 1\} - \underbrace{P\{Z \le -1/3\}}_{\text{by symmetry } = P\{Z \ge \frac{1}{3}\}}$$



$$= P\{Z \le 1\} - P\{Z \ge 1/3\}$$

$$= P\{Z \le 1\} - (1 - P\{Z < 1/3\})$$

$$= F(1) - 1 + F(1/3)$$

where $F(x) = \dfrac{1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}{2}$ see p. 167

$$= \frac{1+\operatorname{erf}\left(\frac{1}{\sqrt{2}}\right)}{2} - 1 + \frac{1+\operatorname{erf}\left(\frac{1}{3\sqrt{2}}\right)}{2} = 0.8413 - 1 + 0.6306 = \boxed{0.4719}$$

Note:

$$P\{-\tfrac{1}{3} \le \Upsilon \le 1\}$$

$$= P\{\Upsilon \le 1\} - P\{\Upsilon \le -\tfrac{1}{3}\}$$

$$= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{1}{\sqrt{2}}\right)\right) - \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{-1/3}{\sqrt{2}}\right)\right)$$

$$= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{-1/3}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left[\operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{-1/3}{\sqrt{2}}\right)\right]$$

Note $\operatorname{erf}\left(\frac{-1/3}{\sqrt{2}}\right) = -\operatorname{erf}\left(\frac{1/3}{\sqrt{2}}\right)$

$$= \frac{1}{2}\left[\operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) + \operatorname{erf}\left(\frac{1/3}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{2}\left[0.6827 + 0.2611\right] = \underline{\underline{0.4719}}$$