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diff --git a/main.tex b/new.tex
index cabd02c..f9d852f 100644
--- a/main.tex
+++ b/new.tex
@@ -28,14 +28,16 @@
\documentclass{ams}
\usepackage{amsmath,amsthm,amssymb,amsfonts}
\usepackage{tikz}
\usepackage{tikzpicture}
\begin{document}
\begin{theorem}
\textit{Sard's Theorem.} \textit{Let }  $f : M \rightarrow N$  \textit{ be a smooth map, where }  $M$  \textit{ and }  $N$  \textit{ are manifolds of dimension }  $m$  \textit{ and }  $n$  \textit{ respectively, with }  $m < n$ . \textit{ Then the set of critical values of }  $f$  \textit{ has Lebesgue measure zero.}
\end{theorem}
\begin{proof}
\textit{In general, it is too much to hope that the set of critical values of a smooth map be finite. But this set will be ``small,''} in the sense indicated by the next theorem, which was proved by A. Sard in 1942 following earlier work by A.P. Morse, (References [30], [24].)
\textit{In general, it is too much to hope that the set of critical values of a smooth map be finite. But this set will be ``small,''} in the sense indicated by the next theorem, which was proved by A. Sard in 1942 following earlier work by A.P. Morse, (References [30], [24].)
\end{proof}
\begin{theorem}
\textit{Let }  $f : U \rightarrow \mathbb{R}^n$  \textit{ be a smooth map, defined on an open set }  $U \subset \mathbb{R}^m$ , \textit{ and let }  $S = \{x \in U \mid \text{rank}(df_x) < n\}$ . \textit{ Then the image }  $f(S)$  \textit{ has Lebesgue measure zero.}
\end{theorem}
\begin{proof}
\textit{In other words, given any }  $\epsilon > 0$ , \textit{ it is possible to cover }  $f(S)$  \textit{ by a sequence of cubes in }  $\mathbb{R}^n$  \textit{ having total }  $n$ -dimensional volume less than }  $\epsilon$ .
\end{proof}
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\textit{Since a set of measure zero cannot contain any nonvacuous open set, it follows that the complement }  $\mathbb{R}^n - f(S)$  \textit{ must be everywhere dense.} (Proved by Arthur B. Brown in 1935. This result was rediscovered by Dubovickil in 1953 and by Thorn in 1954. (References [5], [8], [36].)) in  $\mathbb{R}^n$ .
\textit{The proof will be given in [3]. It is essential for the proof that [3] should [43, 72 + 45, 87].}
\textit{We will be mainly interested in the case }  $m \geq n$ . \textit{If }  $m < n$ , \textit{ then clearly }  $S = U$ ; \textit{hence the theorem says simply that }  $f(U)$  \textit{ has measure zero.}
\end{proof}
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\textit{More generally consider a smooth map }  $f : M \rightarrow N$ , \textit{ from a manifold }  $M$  \textit{ to a manifold }  $N$  \textit{ of dimension }  $m$  \textit{ to a manifold of dimension }  $n$ . \textit{Let }  $S$  \textit{ be the set of all }  $x \in M$  \textit{ such that }  $\text{rank}(df_x) < n$ . \textit{Then }  $S$  \textit{ is called the set of critical points, }  $f(S)$  \textit{ the set of critical values, and the complement }  $N - f(S)$  \textit{ the set of regular values of }  $f$ . (This agrees with our previous definitions in the case }  $m = n$ .) \textit{Since }  $M$  \textit{ can be covered by a countable collection of neighborhoods each diffeomorphic to an open subset of }  $\mathbb{R}^m$ , \textit{ we have:}
\end{proof}
\begin{corollary}
\textit{A.B. Brown. The set of regular values of a smooth map }  $f : M \rightarrow N$  \textit{ is everywhere dense in }  $N$ .
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\textit{In order to exploit this corollary we will need the following:}
\begin{lemma}
\textit{If }  $f : M \rightarrow N$  \textit{ is a smooth map between manifolds of dimension }  $m \geq n$ , \textit{ and if }  $y \in N$  \textit{ is a regular value, then the set }  $f^{-1}(y) \subset M$  \textit{ is a smooth manifold of dimension }  $m - n$ .
\textit{If }  $f : M \rightarrow N$  \textit{ is a smooth map between manifolds of dimension }  $m \geq n$ , \textit{ and if }  $y \in N$  \textit{ is a regular value, then the set }  $f^{-1}(y) \subset M$  \textit{ is a smooth manifold of dimension }  $m - n$ .
\end{lemma}
\begin{proof}
\textit{Let }  $x \in f^{-1}(y)$ . \textit{Since }  $y$  \textit{ is a regular value, the derivative }  $df_x : T_x M \rightarrow T_y N$  \textit{ is surjective. The null space }  $\mathcal{N}_x = \ker(df_x) \subset T_x M$  \textit{ of }  $df_x$  \textit{ will therefore be an }  $(m - n)$ -dimensional vector space.}
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\textit{If }  $M \subset \mathbb{R}^k$  \textit{ choose a linear map }  $L : \mathbb{R}^m \rightarrow \mathbb{R}^{m-n}$  \textit{ that is nonsingular on this subspace }  $\mathcal{N}_x \subset T_x M$ . \textit{Now define }  $F : M \rightarrow \mathbb{R}^m \times \mathbb{R}^{m-n}$  \textit{ by }  $F(x) = (f(x), L(x))$ . \textit{The derivative }  $dF_x$  \textit{ is clearly given by the formula }  $dF_x(v) = (df_x(v), L(v))$ . \textit{Thus }  $dF_x$  \textit{ is nonsingular. Hence }  $F$  \textit{ maps some neighborhood }  $U$  \textit{ of }  $x$  \textit{ diffeomorphically onto a neighborhood }  $V$  \textit{ of }  $(y, L(x))$ . \textit{Note that }  $f^{-1}(y) \cap U$  \textit{ corresponds, under }  $F$ , \textit{ to hyperplane }  $S = \mathbb{R}^{m-n} \times \{0\}$ . \textit{This proves that }  $f^{-1}(y)$  \textit{ is a smooth manifold of dimension }  $m - n$ .}
\textit{ }  $dF_x(v) = (df_x(v), L(v))$ .
\textit{Thus }  $dF_x$  \textit{ is nonsingular. Hence }  $F$  \textit{ maps some neighborhood }  $U$  \textit{ of }  $x$  \textit{ diffeomorphically onto a neighborhood }  $V$  \textit{ of }  $(y, L(x))$ . \textit{Note that }  $f^{-1}(y) \cap U$  \textit{ corresponds, under }  $F$ , \textit{ to hyperplane }  $S = \mathbb{R}^{m-n} \times \{0\}$ . \textit{This proves that }  $f^{-1}(y)$  \textit{ is a smooth manifold of dimension }  $m - n$ .}
\textit{As an example we can give an easy proof that the unit sphere }  $S^{n-1}$  \textit{ is a smooth manifold. Consider the function }  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  \textit{ defined by }  $f(x) = x_1^2 + x_2^2 + \dots + x_m^2$ .
\textit{Any }  $y \neq 0$  \textit{ is a regular value, and the smooth manifold }  $f^{-1}(y)$  \textit{ is the unit sphere.}
\textit{If }  $M$  \textit{ is a manifold which is contained in }  $\mathbb{R}^m$ , \textit{ it has already been noted that }  $T_x M$  \textit{ for }  $x \in M$  \textit{ is a subspace of }  $T_x \mathbb{R}^m$ . \textit{The orthogonal complement of }  $T_x M$  \textit{ in }  $T_x \mathbb{R}^m$  \textit{ is then a vector space of dimension }  $m - m$  \textit{ called the } \textit{space of normal vectors to }  $M$  \textit{ at }  $x$ .}
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\textit{In particular let }  $M = f^{-1}(y)$  \textit{ for a regular value }  $y$  \textit{ of }  $f : M \rightarrow \mathbb{R}$ .
\textit{The null space of }  $df_x : T_x M \rightarrow T_y \mathbb{R}$  \textit{ is precisely equal to the tangent space }  $T_x M \subset T_x \mathbb{R}^m$  \textit{ of the submanifold }  $M = f^{-1}(y)$ . \textit{Hence }  $df_x$  \textit{ maps the orthogonal complement of }  $T_x M$  \textit{ isomorphically onto }  $T_y \mathbb{R}$ .}
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\end{lemma}
\begin{proof}
\textit{From the diagram}
\begin{tikzpicture}
\draw[thick, ->] (4,4) -- (4,0) node[anchor=north west] {y};
\draw[thick, ->] (4,0) -- (8,0) node[anchor=north west] {N};
\draw[thick, ->] (4,4) -- (8,4) node[anchor=north west] {M};
\draw[thick, ->] (8,4) -- (8,0);
\end{tikzpicture}
\textit{we see that }  $df_x$  \textit{ maps the subspace }  $T_x M \subset T_x \mathbb{R}^m$  \textit{ to zero. Counting dimensions we see that }  $df_x$  \textit{ maps the space of normal vectors to }  $M$  \textit{ isomorphically onto }  $T_y \mathbb{R}$ .}
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\end{proof}
\section{Manifolds with Boundary}
\textit{The lemmas above can be sharpened so as to apply to a map defined on a smooth "manifold with boundary." Consider first the closed half-space }  $H^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}$ .
\textit{ }  $H^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}$ .
\textit{The boundary }  $\partial H^m$  \textit{ is defined to be the hyperplane }  $\mathbb{R}^{m-1} \times \{0\} \subset \mathbb{R}^m$ .}
\begin{definition}
\textit{A subset }  $X \subset \mathbb{R}^m$  \textit{ is called a } \textit{smooth }  $m$ -manifold with boundary \textit{ if each }  $x \in X$  \textit{ has a neighborhood }  $U \subset \mathbb{R}^m$  \textit{ diffeomorphic to an open subset }  $V \subset H^m$  \textit{ of }  $H^m$ . \textit{The } \textit{boundary} \textit{ of }  $X$  \textit{ is the set of all points in }  $X$  \textit{ which correspond to points of }  $\partial H^m$  \textit{ under such a diffeomorphism.}
\textit{A subset }  $X \subset \mathbb{R}^m$  \textit{ is called a } \textit{smooth }  $m$ -manifold with boundary \textit{ if each }  $x \in X$  \textit{ has a neighborhood }  $U \subset \mathbb{R}^m$  \textit{ diffeomorphic to an open subset }  $V \subset H^m$  \textit{ of }  $H^m$ . \textit{The } \textit{boundary} \textit{ of }  $X$  \textit{ is the set of all points in }  $X$  \textit{ which correspond to points of }  $\partial H^m$  \textit{ under such a diffeomorphism.}
\end{definition}
\textit{It is not hard to show that }  $\partial X$  \textit{ is a well-defined smooth manifold of dimension }  $m - 1$ . \textit{The } \textit{interior} \textit{ of }  $X$  \textit{ is a smooth manifold of dimension }  $m$ . \textit{The tangent space }  $T_x X$  \textit{ is defined just as in }  $\mathbb{R}^m$ , \textit{ so that }  $T_x X$  \textit{ is a full }  $m$ -dimensional vector space, even if }  $x \in \partial X$  \textit{ is a boundary point.}
\textit{Here is one method for generating examples. Let }  $M$  \textit{ be a manifold without boundary and let }  $g : M \rightarrow \mathbb{R}$  \textit{ have }  $0$  \textit{ as regular value.}
\begin{lemma}
\textit{The set of }  $x \in M$  \textit{ with }  $g(x) \geq 0$  \textit{ is a smooth manifold, with boundary equal to }  $g^{-1}(0)$ .
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\end{lemma}
\textit{The proof is just like the proof of Lemma 1.}
\end{document}
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