Worksheet 7

1. Let X be a mathematical object (e.g. a set, group, vector space, ring, or poset). Which of the following term necessarily describes $\{f : X \to X \mid f \text{ is an isomorphism}\}$?

   magma, (commutative) semigroup, (commutative) monoid, (abelian) group

2. Let k be a field and $k[X]$ the polynomial ring in k over X. Fix $a \in k$ and define the *evaluation map*

$$\begin{aligned} \varepsilon_a : k[X] &\to k \\ p(X) &\mapsto p(a), \end{aligned}$$

   where if

$$p(X) = c_0 + c_1 X + \cdots c_n X^n$$

   then

$$p(a) = c_0 + c_1 a + \cdots c_n a^n.$$

   Is ε_a a ring homomorphism? Prove or disprove.

3. Let $G = \{1, i, -1, -i\}$ be a group with operation complex multiplication, and let $H = \{0, 1, 2, 3\}$ be a group with operation modular addition. Are G and H isomorphic groups? Prove or disprove.[1]

4. The *trivial group* is the group $G = \{e\}$ with operation given by $e \cdot e = e$. Is G isomorphic to $(\mathbb{Z}_2, +)$? Prove or dispove.

5. Is $(\mathbb{Z}, \leq)$ isomorphic to $(\mathbb{N}, \leq)$? Prove or dispove.

6. Prove that the automorphism group of $(\mathbb{N}, \leq)$ is trivial.

7. Prove that the automorphism group of $(\mathbb{Z}, \leq)$ is isomorphic to $(\mathbb{Z}, +)$.

8. Is the function

$$\begin{aligned} f : \mathbb{Z} &\to \mathbb{Z} \\ k &\mapsto k^2 \end{aligned}$$

   a homomorphism from $(\mathbb{Z}, +)$ to $(\mathbb{Z}, +)$? Prove or disprove.

9. Let $(A, \leq)$ and $(B, \preccurlyeq)$ be partially ordered sets. A function $f : A \to B$ is called *antitone* if

$$\forall a, a' \in A : a \leq a' \implies f(a') \preccurlyeq f(a).$$

   Prove or disprove that if a function $g : A \to B$ is both monotone and antitone, then g is necessarily constant.

[1] Here and throughout this worksheet, you may use the fact that a group homomorphism is an isomorphism precisely when it is bijective.

10. Define the *direct product* of two groups G and H to comprise the Cartesian product

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

together with the product

$$(g, h) \cdot (g', h') = (g \cdot g', h \cdot h').$$

Is $(\mathbb{Z}_2, +) \times (\mathbb{Z}_2, +)$ isomorphic to $(\mathbb{Z}_4, +)$? Prove or disprove.