Parsing Binomials & Multinomials in Probability

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Introduction

The application of the Binomial and Multinomial theorems in probability can often lack clarity. However, the relations between polynomials and probability can provide insight into how models work in probability. We intend to clarify the connection between expansions of multinomials and binomials, combinations, total probability, and probability mass functions through examples and proofs of theorems.

1 Binomial Theorem

The Binomial Theorem expresses the expansion of two monomial terms x and y such that $(x + y)^2 = x^2 + 2xy + y^2$. The later is useful in algebra and other fields of mathematics, but how a binomial expansion relates to probability is not intuitvely obvious. The solution is to treat x and y as the two possible outcomes of independent event(s). The resulting binomial expansion can express the sample space or sum of probabilities for all combinations of the two outcomes for n independent event(s).

Example 1. Let an unfair coin be flipped twice with $P(Tails) = 0.3$ and $P(Heads) = 0.7$
The probability must sum to 1. In two flips then, $(T+H)^2 = T^2+2TH+H^2$. This aligns with the outcomes of $T^2 = TT$, $2TH = TH + HT$, and $H^2 = HH$ for 2 flips. Additionally, substituting in the probabilities we have $0.3^2 + 2 * 0.3 * 0.7 + 0.7^2 = 1$.

Theorem 1. *The Binomial Theorem can express the sum of all possible outcomes of n independent events.*

Proof. Let A and B be the two outcomes of n independent events with probability $A = \frac{1}{a}$, $B = (1 - \frac{1}{a})$, and $n = 2$. By the binomial theorem we have

$$
\begin{aligned}
&(\frac{1}{a} + (1 - \frac{1}{a}))^2 \\
&(\frac{1}{a})^2 + 2\frac{1}{a}(1 - \frac{1}{a}) + (1 - \frac{1}{a})^2 \\
&\frac{1}{a^2} + \frac{2}{a}\frac{a-1}{a} + \frac{(a-1)^2}{a^2} \\
&\frac{1}{a^2} + \frac{2(a-1)}{a^2} + \frac{a^2 - 2a + 1}{a^2} \\
&\frac{a^2 + 2 - 2 + 2a - 2a}{a^2} \\
&\frac{a^2}{a^2} = 1.
\end{aligned}
$$

□

The uses of the Binomial Theorem are obvious especially in calculating large numbers of events. To compute the expansion of two monomials the theorem uses factorials and binomial coefficients:

Definition 1 (Factorial n!)**.** The factorial of a nonnegative integer n is given by

$$n! = \prod_{i=1}^{n} i$$

where $0! = 1$.

Definition 2 (Binomial Coefficient)**.** Count everyway to combine a set of n objects of size k.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Remark 1. Note that the binomial coefficient can be used to determine the number of combinations for n independent events.

Example 2. 3 flips of a coin yields:

$$\begin{aligned}
&(T+H)^3 = TTT + TTH + THT + HTT + HHT + HTH + THH + HHH \\
&= T^3 + 3T^2H + 3H^2T + H^3 \\
&\binom{3}{3} = 1 \text{ hence } T^3 = TTT \text{ or } H^3 = HHH \\
&\binom{3}{2} = 3 \text{ hence } 3T^2H = TTH + THT + HTT \text{ or } 3H^2T = HHT + HTH + THH.
\end{aligned}$$

Remark 2. The total number of combinations of the binomial is 2^n.
$2^3 = T^3 + 3T^2H + 3H^2T + H^3 = 1 + 3 + 3 + 1 = 8$

Lemma 1. *Given two binomial coeffecients* $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$.

Pascal's Identity.

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= (n-1)![\frac{n-k}{k!(n-k)!} + \frac{k}{k(n-k)!}] \\
&= (n-1)!\frac{n}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

□

Remark 3. Pascal's Identity can be used to simplify multiple binomial coefficients into a single coefficient.

We are now ready to prove the Binomial Theorem by use of Pascal's Identity, Binomial Coefficients, and Factorials.

Proof. Assume that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ and by the the definition of the binomial coeffecient $n \geq 0$. For the case $(n = 0) \Rightarrow (a+b)^0 = 1$. For the case $n \geq 0$.

$$
\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}nn-k + \sum_{k=o}^{n}\binom{n}{k}a^k b^{n-k+1} \\
m &= k+1 \\
&= \sum_{m=1}^{n+1}\binom{n}{m-1}a^m b^{n-m+1} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k+1} \\
&= b^{n+1} + \sum_{k=1}^{n}[\binom{n}{k} + \binom{n}{k-1}]a^k b^{n-k+1} + a^{n+1} \\
&= b^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}a^k b^{n-k+1} + a^{n+1} \\
&= \sum_{k}^{n+1}\binom{n+1}{k}a^k b^{n+1-k}.
\end{aligned}
$$

□

2 Multinomial Theorem

While the binomial coefficient works with 2 outcomes many cases in probability have more than 2 outcomes. The multinomial coefficient generalizes to handle more than two outcomes.

Definition 3 (Multinomial Coefficient)**.**

$$\binom{N}{n_1..n_r} = \frac{N!}{n_1!..n_r!}$$

Where n_1 to n_r are different group sizes.

Example 3. For 13 items we want to know how many combinations of 5, 5, and 3 can be made

$$
\begin{aligned}
\binom{13}{5,5,3} &= \binom{13}{5}\binom{8}{5}\binom{3}{3} \\
&= \frac{13!}{5!(13-5)!}\frac{8!}{5!(8-5)!}\frac{3!}{3!(3-3)!} \\
&= \frac{13!}{5!5!3!}.
\end{aligned}
$$

Like the multinomial coefficient, the multionomial theorem generalizes to any number of outcomes and events. Like the binomial theorem the probabilities of the outcomes sum to 1 and the theorem provides an accurate count of the combinations found in the sample space.

Definition 4 (multinomial theorem)**.**

$$
\begin{aligned}
(x_1 + ... + x_r)^n = \sum_{(n1,...,nr)} \binom{n}{n1,...,nr} x_1^{n1}...x_r^{nr} \\
\text{where } n1 + ... + nr = n
\end{aligned}
$$

Multinomial Theorem. Fix $r = 1$ and observe that $(x_1)^n = \sum_{(n1=1)} \binom{n}{n1=1} x_1^{n1=1} = nx_1$
Fix $m = r + 1$ and $(x_r + x_{r+1})^n = \sum_{(r,\ldots,r+1)} \binom{n}{x_1,\ldots,x_{r+1}} x_1^r x_{r+1}^{r+1}$ and by the binomial theorem. $\square$