Practice Midterm 1 Answer Key

Practice Midterm 1a

1. i. $\forall k \in \mathbb{Z} : \exists \ell \in \mathbb{N} : \ell \leq k$
   ii. $\exists k \in \mathbb{Z} : \forall \ell \in \mathbb{N} : \ell > k$
   iii. false
   iv. Let $k = -1$ and choose $\ell \in \mathbb{N}$. We have $\ell > k$.

2. i. $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : (x < y \implies \exists z \in \mathbb{R} : x < z < y)$
   ii. $\exists x \in \mathbb{R} : \forall y \in \mathbb{R} : (x < y \ \wedge \ \forall z \in \mathbb{R} : (x \geq z \ \vee \ z \geq y))$
   iii. true
   iv. Let $x \in \mathbb{R}$ and let $y = x - 1$. This completes the proof as it is not the case that $x < y$.

3. i. $\forall$ sets $A, B, C : A \cap B \subseteq C \implies (A \subseteq C \ \vee \ B \subseteq C)$
   ii. $\exists$ sets $A, B, C : A \cap B \subseteq C \ \wedge \ A \not\subseteq C \ \wedge \ B \not\subseteq C$
   iii. false
   iv. Let $A = \{1\}$, $B = \{2\}$, and $C = \{3\}$. It follows that $A \cap B = \varnothing \subseteq C$, while $A, B \not\subseteq C$.

4. i. $\forall$ nonsurjections $f : A \to B,\ g : B \to C : g \circ f$ nonsurjective
   ii. $\exists$ nonsurjections $f : A \to B,\ g : B \to C : g \circ f$ surjective
   iii. true
   iv. Choose $c \in C$ so that $g(b) \neq c$ for all $b \in B$ and observe that $g(f(a)) \neq c$ for all $a \in A$.

5. i. $\forall$ bijections $f : A \to B : \exists$ function $g : B \to A : g \circ f = \mathrm{id}_A \ \wedge \ f \circ g = \mathrm{id}_B$
   ii. $\exists$ bijection $f : A \to B : \forall$ functions $g : B \to A : g \circ f \neq \mathrm{id}_A \ \vee \ f \circ g \neq \mathrm{id}_B$
   iii. true
   iv. For each $b \in B$, the surjectivity of f provides an element $g(b) \in A$ with $f(g(b)) = b$, while the injectivity of f ensures this value is unique. In particular, for $a \in A$, we have $f(g(f(a))) = f(a)$, from which follows $g(f(a)) = a$ by the injectivity of f.

6. Fix $\varepsilon > 0$, choose $\delta = \sqrt{\varepsilon}$, and let $x \in \mathbb{R}$ with $|x| < \delta$. It follows that $|x^2| < \delta^2 = \varepsilon$.

Practice Midterm 1b

1. i. $\forall m, n \in \mathbb{Z} : m \leq n \implies m^2 \leq n^2$
   ii. $\exists m, n \in \mathbb{Z} : m \leq n \wedge m^2 > n^2$
   iii. false
   iv. If $m = -1$ and $n = 0$, then $m^2 = 1 > 0 = n^2$.

2. i. $\exists k \in \mathbb{Z} : \forall x \in \mathbb{R} : kx \notin \mathbb{Z}$
   ii. $\forall k \in \mathbb{Z} : \exists x \in \mathbb{R} : kx \in \mathbb{Z}$
   iii. false
   iv. Let $k \in \mathbb{Z}$ and $x = 0$. We have $kx = 0 \in \mathbb{Z}$.

3. i. $\forall$ sets $A, B, C : A \subseteq B \subseteq C \implies (A \cap C \neq \varnothing \implies A \cap B \neq \varnothing)$
   ii. $\exists$ sets $A, B, C : A \subseteq B \subseteq C \;\wedge\; A \cap C \neq \varnothing \;\wedge\; A \cap B = \varnothing$
   iii. true
   iv. Choose $x \in A \cap C$. It follows that $x \in A$ and, as $A \subseteq B$, that $x \in B$. Thus, $x \in A \cap B$.

4. i. $\forall$ noninjections $f : A \to B$, $g : B \to C : g \circ f$ noninjective
   ii. $\exists$ noninjections $f : A \to B$, $g : B \to C : g \circ f$ injective
   iii. true
   iv. Choose distinct $a, a' \in A$ with $f(a) = f(a')$ and observe that $g(f(a)) = g(f(a'))$.

5. i. $\forall$ functions $f : A \to B$, $g : B \to C : (g \circ f = \mathrm{id}_A \;\wedge\; f \circ g = \mathrm{id}_A) \implies (f$ bijective $\wedge$ g bijective$)$
   ii. $\exists$ functions $f : A \to B$, $g : B \to C : (g \circ f = \mathrm{id}_A \wedge f \circ g = \mathrm{id}_A) \;\wedge\; (f$ nonbijective $\vee$ g nonbijective$)$
   iii. true
   iv. Let $a, a' \in A$ with $f(a) = f(a')$. From $a = g(f(a)) = g(f(a')) = a'$, we deduce that f is injective. Surjectivity follows as $f(g(b)) = b$ for all $b \in B$. The proof for g is analogous.

6. Fix $\varepsilon > 0$ and choose $N > 0$ so that $f(y) > \frac{1}{\varepsilon}$ for all $y > N$. Thus, $x > N$ implies $\frac{1}{f(x)} < \varepsilon$.

Practice Midterm 1c

1. i. $\forall m, n \in \mathbb{Z} : m \leq n \implies m^2 \leq n^2$
   ii. $\exists m, n \in \mathbb{Z} : m \leq n \wedge m^2 > n^2$
   iii. false
   iv. If $m = -1$ and $n = 0$, then $m^2 = 1 > 0 = n^2$.

2. i. $\forall m, n \in \mathbb{Z} : (m < n \implies \exists x \in \mathbb{R} : xm > n)$
   ii. $\exists m, n \in \mathbb{Z} : (m < n \;\wedge\; \forall x \in \mathbb{R} : xm \leq n)$
   iii. false
   iv. If $m = 0$ and $n = 1$, then $xm = 0 \leq n$ for all $x \in \mathbb{R}$.

3. i. $\forall$ sets $A, B : (\exists x \in A \cup B) \implies (A \neq \varnothing \;\vee\; B \neq \varnothing)$
   ii. $\exists$ sets $A, B : (\exists x \in A \cup B) \;\wedge\; A = \varnothing \;\wedge\; B = \varnothing$
   iii. true
   iv. By the definition of union, we have $x \in A$ or $x \in B$. In the first case, $A \neq \varnothing$; in the second, $B \neq \varnothing$.

4. i. $\forall$ sets $A_1, \ldots, A_n : (A_1 \cap \cdots \cap A_n = \varnothing) \implies (\exists i, j \in [0, n] : A_i \cap A_j = \varnothing)$
   ii. $\exists$ sets $A_1, \ldots, A_n : (A_1 \cap \cdots \cap A_n = \varnothing) \;\wedge\; (\forall i, j \in [0, n] : A_i \cap A_j \neq \varnothing)$
   iii. false
   iv. Let $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, and $A_3 = \{1, 3\}$. It follows that $A_1 \cap A_2 \cap A_3 = \varnothing$ while $A_1 \cap A_2$, $A_2 \cap A_3$, and $A_1 \cap A_3$ are each nonempty.

5. i. $\forall$ surjections $f\colon A \to B : \forall S \subseteq A : \forall b \in B : \exists s \in S : f(s) = b$
   ii. $\exists$ surjection $f\colon A \to B : \exists S \subseteq A : \exists b \in B : \forall s \in S : f(s) \neq b$
   iii. false
   iv. Let $f : \{1\} \to \{1\}$ be the identity function, let $S = \varnothing$, and let $b = 1$. It is vacuously true that $f(s) \neq 1$ for all $s \in \varnothing$.

6. Fix $M > 0$, choose $\delta = \frac{1}{\sqrt{M}}$, and let $x \in \mathbb{R}$ with $|x| < \delta$. From $|x| < \frac{1}{\sqrt{M}}$ we conclude that $\frac{1}{x^2} = \frac{1}{|x|^2} > M$.