Midterm 1 Answer Key

1. i. $\forall k \in \mathbb{Z} : \exists \ell \in \mathbb{Z} : k\ell = 1$
   ii. $\exists k \in \mathbb{Z} : \forall \ell \in \mathbb{Z} : k\ell \neq 1$
   iii. false
   iv. Put $k = 0$ and let $\ell \in \mathbb{Z}$. It follows that $k\ell = 0 \neq 1$.

2. i. $\exists x \in \mathbb{R} : \forall y \in \mathbb{R} : x < y^2$
   ii. $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x \geq y^2$
   iii. true
   iv. Let $x = -1$ and choose $y \in \mathbb{R}$. We have $x < y^2$.

3. i. $\forall$ sets $A, B, C : (A \subseteq B \subseteq C) \implies (A \cap B) \subseteq (A \cap C)$
   ii. $\exists$ sets $A, B, C : (A \subseteq B \subseteq C) \ \wedge \ (A \cap B) \nsubseteq (A \cap C)$
   iii. true
   iv. Let $x \in A \cap B$. It follows that $x \in A$ and $x \in B$. From $B \subseteq C$, it follows that $x \in C$. We conclude that $x \in A \cap C$.

4. i. $\forall$ injective $f : A \to B, g : B \to C : g \circ f$ injective
   ii. $\exists$ injective $f : A \to B, g : B \to C : g \circ f$ noninjective
   iii. true
   iv. Suppose that $a, a' \in A$ with $g(f(a)) = g(f(a'))$. By the injectivity of g we have $f(a) = f(a')$, and by the injectivity of f we conclude that $a = a'$.

5. i. $\forall$ surjective $f : A \to B : \forall$ injective $g : B \to C : g \circ f$ bijective
   ii. $\exists$ surjective $f : A \to B : \exists$ injective $g : B \to C : g \circ f$ nonbijective
   iii. false
   iv. Let $f : \{1, 2\} \to \{1\}$ and $g : \{1\} \to \{1\}$ be the constant functions with value 1. Observe that the composition $g \circ f$ is noninjective and thus nonbijective.

6. Fix $M > 0$, choose $N = \sqrt{M}$, and let $x > N$. From $x > \sqrt{M}$, it follows that $x^2 > M$.