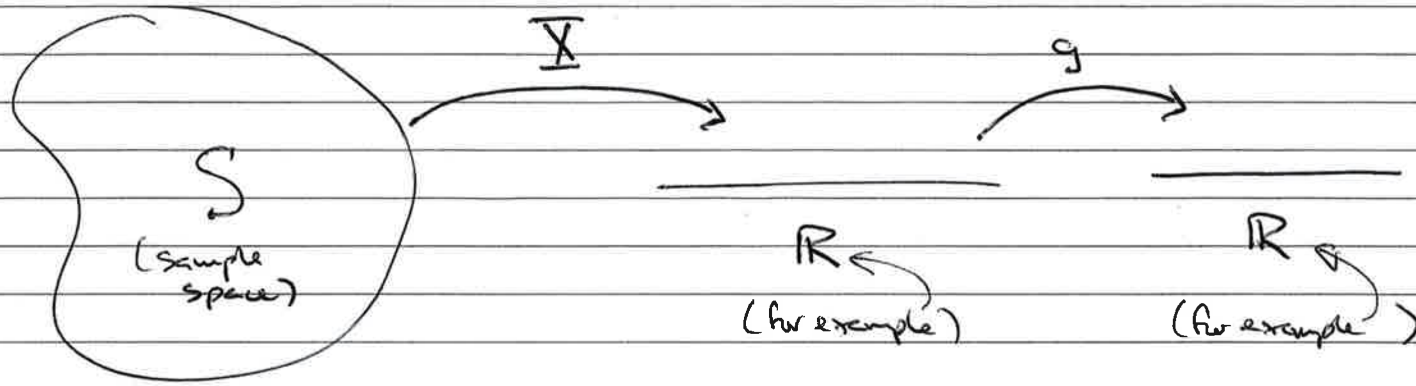
4.4 Functions of a Random Variable

Consider



So $g(X)$ is a function from S to $\mathbb{R}$ (really, a composite function). It is a random variable (i.e. it is a function from a sample space of a random experiment) to a set T (e.g. real numbers).

EX Flip a fair coin 3 times.

Let X = # of heads − # tails in the 3 flips

$i = -3$ (no heads, 3 tails)

$P\{X=-3\} = \left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3\binom{3}{0} = \frac{1}{8}$

$i = -1$ (1 heads, 2 tails)

$P\{X=-1\} = \left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2\cdot\binom{3}{1} = \frac{3}{8}$

$i = +1$ (2 heads, 1 tails)

$P\{X=1\} = \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)\cdot\binom{3}{2} = \frac{3}{8}$

$i = 3$

$P\{X=3\} = \left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0\cdot\binom{3}{3} = \frac{1}{8}$

note $P\{X=0\}=0$ $P\{X=2\}=0$ $P\{X=-2\}=0$

← computing probability mass function for X.

Let $p_X(i) = P\{X=i\}$

the expected value of X is

$$E[X] = (-3)\left(\frac{1}{8}\right) + (-1)\left(\frac{3}{8}\right) + (+1)\left(\frac{3}{8}\right) + (+3)\left(\frac{1}{8}\right)$$

$$\boxed{E[X] = 0}$$

Now let $Y = |X|$ = difference between heads and tails (absolute value)

(e.g. $g(x) = |x|$ (function of random variable)

so $P\{Y=3\} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

$P\{Y=1\} = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$

compute new probability mass function for Y

(e.g. $p_Y(i) = P\{Y=i\}$

The expected value of Y is

$$E[Y] = 3 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \boxed{\frac{3}{2}} = E[Y]$$

Question: Can we obtain the expected value for $Y = g(X)$ using the probability mass function for X (that we had already computed)?

p_X denotes prob. mass function for X

Note:

$$\underbrace{|-3|\left(\tfrac{1}{8}\right) + |-1|\left(\tfrac{3}{8}\right) + |1|\left(\tfrac{3}{8}\right) + |3|\left(\tfrac{1}{8}\right)} =$$

$$= \sum_i g(x_i)\, p_X(x_i)$$

here $x_i = i$ for the relevant i values

$$= 3\left(\frac{1}{8} + \frac{1}{8}\right) + 1\left(\frac{3}{8} + \frac{3}{8}\right) = 3\left(\sum_{i:\, g(x_i)=3} p_X(x_i)\right) + 1 \cdot \left(\sum_{i:\, g(x_i)=1} p_X(x_i)\right)$$

$$= \frac{6}{8} + \frac{6}{8} = \frac{12}{8} = \boxed{3/2}.$$

Here we are grouping the numbers in the range of X
(i.e. -3 groups with 3 and -1 groups with 1)
that get sent to the same element of the range of g.

So

$$\sum_i g(x_i)\, p_x(x_i) = 3\left(\sum_{i:g(x_i)=3} p_x(x_i)\right) + 1\cdot\left(\sum_{i:g(x_i)=1} p_x(x_i)\right)$$

if $y_1 \equiv 1$
$y_2 \equiv 3$

$$= \sum_{j=1}^{2} y_j \boxed{\sum_{i:g(x_i)=y_j} p_x(x_i)}$$

$$= \sum_{j=1}^{2} y_j\, P\{g(X)=y_j\}$$

$$= E[g(X)]$$

So it looks like

$$E[g(X)] = \sum_i g(x_i)\, p_x(x_i)$$

where p_x is the probability mass function for the random variable X.

Let's work this out more carefully + more generally...

Suppose X is a discrete random variable that takes on the values x_i, $i \geq 1$ with associated probabilities $p(x_i)$. Suppose g is a real-valued function.

$$\sum_i g(x_i)p(x_i) = \sum_j \left(\underbrace{\sum_{i:g(x_i)=y_j} g(x_i)p(x_i)}_{\text{group values of } x_i \text{ with the same value } g(x_i) \text{ (forming } j \text{ such groups)}} \right)$$

e.g. $\underbrace{\underbrace{g(x_1)p(x_1) + g(x_2)p(x_2)}_{\sum_{i:g(x_i)=y_1} g(x_i)p(x_i)}}_{j=1} + \underbrace{\underbrace{g(x_3)p(x_3) + g(x_4)p(x_4)}_{\sum_{i:g(x_i)=y_2} g(x_i)p(x_i)}}_{j=2}$

$$= \sum_j \left(\sum_{i:g(x_i)=y_j} y_j p(x_i) \right)$$

each y_j is indep. of i.

$$= \sum_j y_j \left(\sum_{i:g(x_i)=y_j} p(x_i) \right)$$

$$= \sum_j y_j \left(P\{g(X)=y_j\} \right)$$

$$= E\left[g(X) \right]$$ by definition of expected value.

(112.1)

$$\sum_i g(x_i)\,p(x_i)$$

$$\overbrace{g(x_1)}^{y_1}p(x_1) + \overbrace{g(x_2)}^{y_2}p(x_2) + \overbrace{g(x_3)}^{y_2}p(x_3) + \overbrace{g(x_4)}^{y_1}p(x_4) + \overbrace{g(x_5)}^{y_3}p(x_5)$$

$$\sum_j \left(\sum_{i:g(x_i)=y_i} g(x_i)\,p(x_i) \right)$$

$$= \sum_j \left(\sum_{i:g(x_i)=y_i} y_j\,p(x_i) \right)$$

$$= y_1 \left(\sum_{i:g(x_i)=y_1} p(x_i) \right) + y_2 \left(\sum_{i:g(x_i)=y_2} p(x_i) \right) + y_3 \left(\sum_{i:g(x_i)=y_3} p(x_i) \right)$$

$$= \sum_j y_j \left(\sum_{i:g(x_i)=y_j} p(x_i) \right)$$

$$= y_1 \left(P\{g(X)=y_1\} \right) + y_2 \left(P\{g(X)=y_2\} \right) + y_3 \left(P\{g(X)=y_3\} \right)$$

$$= \sum_j y_j \left(P\{g(X)=y_j\} \right)$$

def. of Expected Value

$= \sum$(values of g) × (Prob of g = value)

$$= E[g(X)] \qquad = E[g(X)]$$

so $$E[g(X)] = \sum_i g(x_i)\,p(x_i)$$

(i.e. stuff on previous page)

This is the proof of Proposition 4.1

X = discrete random variable

$p(x_i)$ = probability mass function for X

g = real valued function

$$\Rightarrow \quad E[g(X)] = \sum_i g(x_i)p(x_i)$$

Corollary 4.1

If a and b are constants then

$$E[aX + b] = aE[X] + b$$

Proof: follows directly from Proposition 4.1 with $g(x) = ax+b$

Comments

- $E[X]$ = expected value of X is also called the mean, or first moment of X, often denoted by μ.

- $E[X^n]$ = n^{th} moment of X

$$= \sum_{x : p(x) > 0} x^n p(x)$$

- next up... variance...

4.5 Variance

Def: If X is a random variable with mean μ (i.e. expected value $E[X]=\mu$), then the variance of X, denoted by $Var(X)$, is defined by

$$Var(X) = E\left[(X-\mu)^2\right]$$

Equivalently, using results from Section 4.4 (e.g. $g(X) = (X-\mu)^2$)
$g(x) = (x-\mu)^2$

$$Var(X) = \sum_{x:p(x)>0} (x-\mu)^2 p(x)$$

$$= \sum (x^2 - 2\cdot\mu x + \mu^2)\, p(x)$$

$$= \underbrace{\sum x^2 p(x)}_{E[X^2]} - 2\mu \underbrace{\sum x p(x)}_{=E[X]=\mu} + \mu^2 \underbrace{\sum p(x)}_{=1}$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2$$

$$Var(X) = E[X^2] - \left(E[X]\right)^2$$

Comment:

- the standard deviation of X is $\sqrt{Var(X)}$

$$\boxed{SD(X) = \sqrt{Var(X)}}$$

EX Urn w/ 3 red balls
2 ~~green~~ blue balls
1 green ball

see p. 97 notes

3 balls drawn, X = # of reds

without replacement

$p(0) = \frac{1}{20}$

$p(1) = 9/20$

$p(2) = 9/20$

$p(3) = 1/20$

$E[X] = 1.5$ (see p. 106 calc.)

$= 3/2$

$Var(X) = E[X^2] - (E[X])^2$

$E[X^2] = 0^2 \cdot \frac{1}{20} + 1^2 \cdot \frac{9}{20} + 2^2 \cdot \frac{9}{20} + 3^2 \cdot \frac{1}{20}$

$= \frac{9}{20} + \frac{36}{20} + \frac{9}{20}$

$= \frac{54}{20} = 2.7$

$Var(X) = \frac{54}{20} - \frac{9}{4}$

$= \frac{54}{20} - \frac{45}{20} = \frac{9}{20} = .45$

with replacement

$p(0) = \frac{27}{6^3}$

$p(1) = \frac{81}{6^3}$

$p(2) = \frac{81}{6^3}$

$p(3) = \frac{27}{6^3}$

note symmetry about $X = 1.5$

$E[X] = 1.5$ (see p. 106 calc.)

$Var(X) = E[X^2] - (E[X])^2$

$E[X^2] = 0^2 \cdot \frac{27}{6^3} + 1^2 \frac{81}{6^3} + 2^2 \cdot \frac{81}{6^3} + 3^2 \cdot \frac{27}{6^3}$

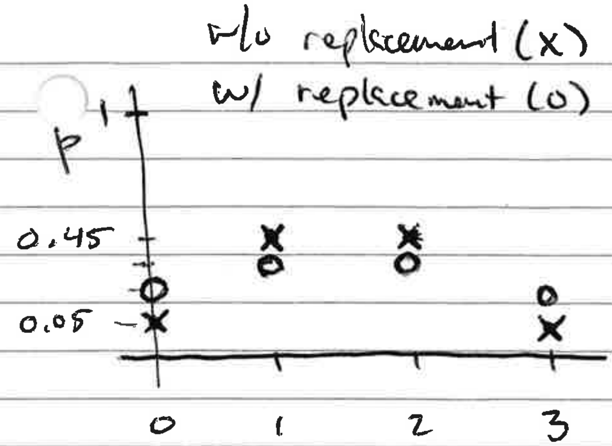
$= \frac{81 + 4 \cdot 81 + 9 \cdot 27}{6^3}$

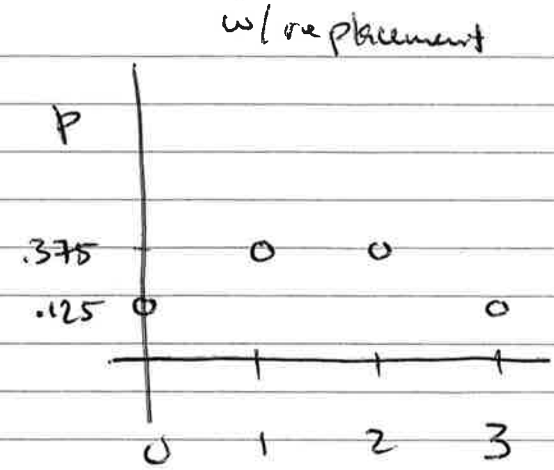
$= \frac{81 + 324 + 243}{216}$

$= \frac{648}{216} = 3$

$Var(X) = 3 - \frac{9}{4} = \frac{12}{4} - \frac{9}{4} = \frac{3}{4}$

$= .75$





smaller variance

$Var(X) = 0.45$

$E[X] = 1.5$

larger variance

$Var(X) = 0.75$

$E[X] = 1.5$

Again, note symmetry about $X = 1.5$

EX Flip a fair coin 3 times.

X = # of heads that appear.

From before $p(0) = 1/8$

$p(1) = 3/8$

$p(2) = 3/8$

$p(3) = 1/8$

$$E[X] = 0 \cdot \frac{1}{8} + 1 \cdot 3/8 + 2 \cdot 3/8 + 3 \cdot \frac{1}{8} = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{3}{2} = \mu$$

$$Var(X) = E\left[(X - \mu)^2\right]$$

Method 1: $Y = (X-\mu)^2$

$$\left(0 - 3/2\right)^2 = \left(3 - \frac{3}{2}\right)^2 = \frac{9}{4}$$

$$\left(1 - 3/2\right)^2 = \left(2 - 3/2\right)^2 = \frac{1}{4}$$

$$P\{Y = 9/4\} = \frac{1}{8} + \frac{1}{8}$$

$$P\{Y = 1/4\} = 3/8 + 3/8$$

so $E[Y] = Var(X) = \frac{1}{4} \cdot \frac{9}{4} + \frac{6}{8} \cdot \frac{1}{4} = \frac{12}{16} = \frac{3}{4}$ use p from Y

or... Method 2 $E\left[(X-\mu)^2\right] = (0-3/2)^2 \cdot \frac{1}{8} + (1-3/2)^2 \cdot 3/8 + (2-3/2)^2\, 3/8 + (3-3/2)^2 \frac{1}{8}$

$$= \frac{9}{4} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{3}{8} + \frac{1}{4} \cdot \frac{3}{8} + \frac{9}{4} \cdot \frac{1}{8} = 3/4$$

use p from X.