Worksheet 8

Name: ________________________________

Let (X, d_X) and (Y, d_Y) be metric spaces.

1. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Prove that $f + g : \mathbb{R} \to \mathbb{R}$ is continuous.
2. If $f : X \to Y$ is a continuous bijection, does it necessarily follow that $f^{-1} : Y \to X$ is continuous?
3. If $(x_i)_i \subseteq X$ is a divergent sequence, and if $f : X \to Y$ is continuous, does it necessarily follow that $\big(f(x_i)\big)_i \subseteq Y$ is divergent? That is, do continuous functions preserve *di*vergence? Prove or provide a counterexample.
4. An *isometric embedding* is a function $f : X \to Y$ that satisfies
$$\forall x, y \in X : d_X(x, y) = d_Y(f(x), f(y)).$$
Prove that every isometric embedding is continuous.
5. A *Cauchy sequence* is a sequence $(x_i)_i \subseteq X$ that satisfies
$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall m, n \geq N : d_X(x_m, x_n) < \varepsilon.$$
Informally, a Cauchy sequence is a sequence the terms of which become and remain infinitesimally close to each other. Prove that every convergent sequence is a Cauchy sequence.
6. Give an example of a Cauchy sequence $(x_i)_i \subseteq X$ that does not converge to any $x \in X$.
7. A function $f : X \to Y$ is said to be *uniformly convergent* when
$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon.$$
Prove that if $f : X \to Y$ is uniformly continuous then f is continuous.
8. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous.
9. Prove that $f(x) = x^2$ is *not* uniformly continuous.
10. A function $f : X \to Y$ satisfying
$$\exists c \in [0, 1) : \forall x, y \in X : d_Y(f(x), f(y)) \leq c d_X(x, y).$$
is called a *contraction*. Prove that if $f : X \to Y$ is a contraction, then f is uniformly continuous.