4.7 The Poisson Random Variable

Def: A random variable X that takes on one of the values $0, 1, 2, \ldots$ is said to be a Poisson Random Variable with parameter λ if, for some $\lambda > 0$

$$\boxed{p(i) = P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!} \quad \text{for } i = 0, 1, 2, \ldots}$$

Note: $$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!} = e^{-\lambda}\underbrace{\left(\sum_{i=0}^{\infty}\frac{\lambda^i}{i!}\right)}_{=e^{\lambda}} = 1.$$

(Taylor Expansion of e^{λ})

Comments:

- Comparison to Binomial Random Variable

$$P_{BIN}\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}$$

- consider the situation where
  - n is large
  - p is small
  - i is not too large (moderate)

Let $\lambda = np$ (so λ is moderate in size)

Then $$P_{BIN}\{X = i\} = \frac{n!}{i!(n-i)!}\left(\frac{\lambda}{n}\right)^i \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$$

$$= \underbrace{\frac{n(n-1)(n-2)\cdot\ldots\cdot(n-i+1)}{n^i}}_{\approx 1} \frac{\lambda^i}{i!} \frac{\left(1-\frac{\lambda}{n}\right)^n \; \}\approx e^{-\frac{\lambda}{n}n} = e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^i \; \}\approx 1}$$

$\left(\frac{\lambda}{n} \ll 1 \text{ and } i \text{ not big}\right)$

$$\approx \frac{\lambda^i}{i!}e^{-\lambda} = P_{POIS.}\{X = i\}$$

X = Poisson Random Variable

* Note that

$$P_{Pois.}\{X = i+1\} = e^{-\lambda}\frac{\lambda^{i+1}}{(i+1)!}$$

$$= \frac{e^{-\lambda}\lambda^{i}}{i!}\cdot\frac{\lambda}{i+1}$$

$$= P_{Pois.}\{X = i\}\cdot\frac{\lambda}{i+1}$$

see section 4.7.1 (p. 146)

* $$E[X] = \sum_{i=0}^{\infty} i\, e^{-\lambda}\frac{\lambda^{i}}{i!} = \sum_{i=1}^{\infty} i\, e^{-\lambda}\frac{\lambda^{i}}{i!}$$

$$= \lambda\sum_{i=1}^{\infty} e^{-\lambda}\frac{\lambda^{i-1}}{(i-1)!}$$

let $j = i-1$

$$= \lambda\,\boxed{\sum_{j=0}^{\infty} e^{-\lambda}\frac{\lambda^{j}}{j!}} = 1$$

$$E[X] = \lambda$$

* $$E[X^2] = \sum_{i=0}^{\infty} i^2 e^{-\lambda}\frac{\lambda^{i}}{i!} = \sum_{i=1}^{\infty} i^2 e^{-\lambda}\frac{\lambda^{i}}{i!}$$

$$= \sum_{i=1}^{\infty}(i^2 - i)e^{-\lambda}\frac{\lambda^{i}}{i!} + \boxed{\sum_{i=1}^{\infty} i\, e^{-\lambda}\frac{\lambda^{i}}{i!}}$$

$= E[X] = \lambda$

$$= \sum_{i=2}^{\infty} i(i-1)e^{-\lambda}\frac{\lambda^{i}}{i!} + \lambda$$

⊛

$$E[X^2] = \sum_{i=2}^{\infty} e^{-\lambda} \frac{\lambda^i}{(i-2)!} + \lambda$$

$$= \lambda^2 \underbrace{\boxed{\sum_{i=2}^{\infty} e^{-\lambda} \frac{\lambda^{i-2}}{(i-2)!}}}_{=1} + \lambda = \lambda^2 + \lambda$$

so

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\left.\begin{array}{l} \boxed{Var(X) = \lambda} \\ \text{and } \boxed{E[X] = \lambda} \end{array}\right\} \text{for Poisson Random Variable}$$

EXAMPLE (7a, p. 137)

Suppose the number of typographical errors on a single page of our text book has a Poisson distribution (i.e. $p(i) = e^{-\lambda}\frac{\lambda^i}{i!}$) with parameter $\lambda = 1/2$.

Calculate the probability that there is at least one error on p. 137.

$$P\{X \geq 1\} = 1 - P\{X = 0\}$$
$$= 1 - e^{-\lambda}\frac{\lambda^0}{0!}$$
$$= 1 - e^{-1/2} = 1 - \frac{1}{\sqrt{e}} \approx \underline{\underline{0.393}}$$

EXAMPLE (Problem $\boxed{4.51}$)

The expected number of typographic errors on a magazine page is 0.2.

What is the probability that the next page you read contains

a) 0 typographical errors?

b) 2 or more typographical errors?

Explain.

- Assume the # of typographical errors on a page is a Poisson Random Variable.

$$P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}$$

- Assume each page is independent and has n letters on the page.

- Further assume that each letter is wrong with the same probability p.

$$E[X] = 0.2 = \lambda = np$$

So on the next page

a) $P\{X=0\} = e^{-0.2}\,\dfrac{(0.2)^0}{0!} = e^{-0.2} = 0.8187$

b) $P\{X \geq 2\} = 1 - P\{X=0\} - P\{X=1\}$

$$= 1 - e^{-0.2} - \frac{0.2}{1} e^{-0.2}$$

$$= 1 - 1.2 e^{-0.2}$$

$$\approx 0.0175$$

EXAMPLE (7e - p. 146 - earthquakes)

Suppose that erthquakes occur in western U.S. at the rate of 2 per week.

Assuming:

see p. 144 in Ross, ed. #9

(1) The probability that exactly one "event" occurs in a given interval of length h is equal to $\lambda h + o(h)$

"little o" of h

where $o(h)$ stands for any function f with

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$ i.e. f goes to zero faster than h.

(2) The probability that 2 or more "events" occur in an interval of length h is $o(h)$ (i.e. negligible)

(3) An event in one interval occurs independently from events in other intervals.

With these assumptions, the number of events in an interval of length t is a Poisson Random Variable with parameter λt. Our original form was

$$P\{X = i\} = e^{-\lambda} \frac{\lambda^i}{i!} \qquad i = 0, 1, 2, \ldots$$

$$E[X] = \lambda$$

So the adjusted form to describe the probability of $N=k$ events happening in an interval of length t is

$$\boxed{P\{N(t)=k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \qquad k=0,1,2,\ldots} \quad *$$

Here $E[N] = \lambda t$

$Var(N) = \lambda t$

So back to the question.

Earthquakes occur at a rate of 2 per week

Let the time unit be a week. Then,

$\lambda = 2$

(e.g. if $t = 1$ week $E[N] = 2$

if $t = 2$ weeks $E[N] = 4$, etc.)

a) Find the probability that at least 3 earthquakes occur during the next 2 weeks.

$$P\{N(2)\geq 3\} = 1 - P\{N(2)=0\} - P\{N(2)=1\} - P\{N(2)=2\}$$

$\lambda = 2$

$$= 1 - e^{-\lambda \cdot 2}\frac{(\lambda 2)^0}{0!} - e^{-\lambda 2}\frac{(\lambda 2)^1}{1!} - e^{-\lambda 2}\frac{(\lambda 2)^2}{2!}$$

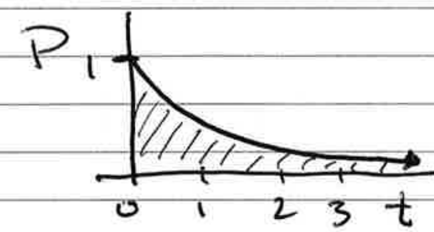
$$= 1 - e^{-4}\left[1 + 4 + \frac{16}{2}\right] = \boxed{1-13e^{-4}}$$

b) Find the probability distribution of the time, starting from now, until the next earthquake.

Let X be the amount of time (in weeks) until the next earthquake.

$$P\{X > t\} = \underbrace{P\{N(t) = 0\}}_{\substack{\text{probability that}\\ \text{no earthquakes}\\ \text{occur in the}\\ \text{interval of length}\\ t \text{ (starting now)}}} = e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$= e^{-2t}$$



e.g. Probability that no earthquakes in the next 3 weeks $= e^{-3\cdot 2} = e^{-6}$ (i.e. small number!)

The probability distribution function is

$$F(t) = P\{X \le t\} = 1 - P\{X > t\}$$

$$= 1 - e^{-\lambda t}$$

For $\lambda = 2$

$$F(t) = 1 - e^{-2t}$$

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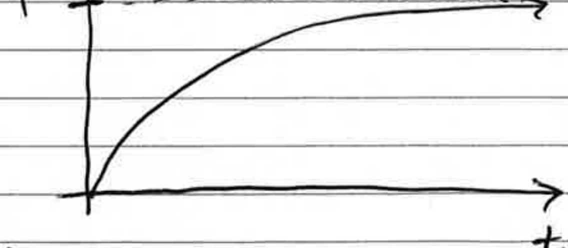
probability that an earthquake occurs before time t (in weeks)



See Ch. 4.7 and pp. 136–144 for other examples of phenomena described by Poisson Random Variables.