Chapter 3 (Seq. and Series)

Section 3.1 Seq. and their limits

3.1.1 Definition A sequence of real numbers (or a sequence in $\mathbb{R}$) is a function defined on $\mathbb{N}$ whose range is contained in the set $\mathbb{R}$

$$X : \mathbb{N} \to \mathbb{R}, \quad \text{but we use } x_n := X(n).$$

and use the notations

$$X, \; (x_n), \; \{x_n\}, \; (x_n : n \in \mathbb{N}), \; \{x_n : n \in \mathbb{N}\}.$$

$Y = (y_n)$, $Z = (z_i)$

Example $X := \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$,

or $X := \{\frac{1}{n} : n \in \mathbb{N}\}$.

An important way to define sequences is via recursion (or inductively)

Example - $\{2n : n \in \mathbb{N}\}$ can be defined as $x_1 := 2$ and $x_{n+1} := x_n + 2$

- Fibonacci $f_1 := 1$, $f_2 := 1$, $f_{n+1} := f_{n-1} + f_n$

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

The Limit of a sequence

3.1.3 Definition A sequence $X := \{x_n\}$ is said to converge to $x \in \mathbb{R}$ if:

$$\forall \epsilon > 0, \; \exists K(\epsilon) \in \mathbb{N} : \forall n \geq K(\epsilon), \; |x_n - x| < \epsilon$$

We also say x is the limit of $\{x_n\}$.

- A sequence is said to be convergent if it has a limit.
- If a sequence has no limit then it is said to be divergent

We denote

$\lim X = x$, $\lim x_n = x$, $\lim_{n\to\infty} x_n = x$,

and $x_n \to x$ as $n \to \infty$.

3.1.4 Proposition (Uniqueness of limits)

A sequence in $\mathbb{R}$ can have at most 1 limit.

Proof: Suppose $\{x_n\}$ has two different limits $\tilde{x}$ and $\hat{x}$. Then $\forall \epsilon > 0$, $\exists$ $\tilde{K}(\epsilon), \hat{K}(\epsilon) \in \mathbb{N}$:

$$|\tilde{x} - x_n| < \epsilon \quad \text{and} \quad |\hat{x} - x_m| < \epsilon$$
$$n \geq \tilde{K}(\epsilon) \qquad\qquad m \geq \hat{K}(\epsilon).$$

Then for $n \geq \max(\tilde{K}(\epsilon), \hat{K}(\epsilon))$

$$|\tilde{x} - \hat{x}| = |\tilde{x} - x_n - (\hat{x} - x_n)|$$
$$\leq |\tilde{x} - x_n| + |\hat{x} - x_n|$$
$$\leq \epsilon + \epsilon = 2\epsilon$$

Since $\epsilon > 0$ is arbitrary $\tilde{x} - \hat{x} = 0$ ◻

Using the concept of ϵ-neighborhood of x as $V_\epsilon(x) = \{y \in \mathbb{R} : |x - y| < \epsilon\}$,

we have

3.1.5. Theorem Let $X = \{x_n\}$ be a seq. of real numbers, and let $x \in \mathbb{R}$. The following are equivalent.

(a) X converges to x

(b) The definition (3.1.3)

$\}$ Trivial.

(c) $\forall \epsilon > 0$, $\exists K = K(\epsilon) \in \mathbb{N}$: $\forall n \geq K$ we have

$$x - \epsilon < x_n < x + \epsilon$$

(d) $\forall$ $V_\epsilon(x)$ of x, $\exists K = K(\epsilon) \in \mathbb{N}$: $\forall n \geq K$

we observe

$$x_n \in V_\epsilon(x).$$

Examples - $\lim \frac{1}{n^2} = 0$.

$\forall \epsilon > 0$ by the A.P. $\exists K \in \mathbb{N}$:

$\frac{1}{\sqrt{\epsilon}} < K \Rightarrow \frac{1}{k^2} < \epsilon$

For $n \geq K \Rightarrow$

$$\left| \frac{1}{n^2} - 0 \right| = \frac{1}{n^2} \leq \frac{1}{k^2} < \epsilon$$

- $\lim \frac{(-1)^n}{n} = 0$.

$\forall \epsilon > 0$ by the A.P. $\exists k \in \mathbb{N}$:

$\frac{1}{\epsilon} < K$ or $\frac{1}{k} < \epsilon$.

For $n \geq K$

$$\left| \frac{(-1)^n}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{k} < \epsilon.$$

3.1.10 Theorem

Let $\{x_n\}$ be a sequence and $x \in \mathbb{R}$. If $\{a_n\}$ is a seq. of positive real numbers with $\lim a_n = 0$ and if for some $C > 0$ and some $m \in \mathbb{N}$ we have

$$|x_n - x| \leq C a_n \quad \forall n \geq m$$

then $\lim x_n = x$.

Proof: Since $a_n \to 0$, $\forall \epsilon > 0$, $\exists k \in \mathbb{N}$: if $n \geq K$ $\Rightarrow a_n < \epsilon$.

Choose $\epsilon = \frac{\tilde{\epsilon}}{C} \Rightarrow$

$$|x_n - x| \leq C a_n < \tilde{\epsilon} \quad \forall n \geq K.$$

Since $\tilde{\epsilon}$ is arbitrary, the result is proven ∎

Application $\lim \frac{1}{n^p} = 0 \quad p \in \mathbb{N}$.

$$\left| \frac{1}{n^p} - 0 \right| \leq \frac{1}{n}$$

Section 3.2 Limit Theorem

3.2.1 A sequence $X = \{x_n\}$ is said to be **bounded** if there exists a real number $M > 0$ such that

$$|x_n| \le M \qquad \forall n \in \mathbb{N}.$$

3.2.2 Theorem. A convergent sequence is bounded.

Proof: Let $\lim x_n = x \Rightarrow$ for $\epsilon = 1$, $\exists K \in \mathbb{N}$: $|x_n - x| < 1$ for $n \ge K$.

Then,

$$|x_n| = |x_n - x + x| \le |x_n - x| + |x|$$

$$\le 1 + |x| \qquad \forall n \ge K.$$

For $M = \max(|x_1|, |x_2|, \dots, |x_{K-1}|, 1 + |x|\}$

$$|x_n| \le M \qquad \forall n \ge K \qquad \blacksquare$$

Proof (2): Suppose $X = \{x_n\}$ is convergent and not bounded. Then

$$\forall m \in \mathbb{N};\ \exists n_m \in \mathbb{N}: \quad |x_{n_m}| > m$$

$y_m = x_{n_m}$. Suppose $\{y_m\}$ is convergent

$$\forall \epsilon > 0,\ \exists \tilde{K}(\epsilon): \ |y_m - y| < \epsilon, \ \forall m \ge \tilde{K}(\epsilon)$$

but

$$m - |y| < |y_m| - |y| \le |y_m - y| < \epsilon \qquad \forall m \ge \tilde{K}$$

$$m < \epsilon + |y| \qquad \forall m \ge \tilde{K}$$

$\Rightarrow \Leftarrow \quad \blacksquare$

Questions :

- Are bounded sequences convergent ?

3.2.3 Theorem (a) Let $\{x_n\}$ and $\{y_n\}$ be convergent seq. of real numbers (to x and y respectively) and let $c \in \mathbb{R}$. Then,

$\{x_n + y_n\}$, $\{x_n - y_n\}$, $\{x_n \cdot y_n\}$, and $\{c x_n\}$

Converge to

$x+y$, $x-y$, xy, cx,

respectively.

(b) If inaddition $y_n \neq 0$ $\forall n \in \mathbb{N}$ and $y \neq 0$ then $\left\{ \frac{x_n}{y_n} \right\}$

Converges to $\frac{x}{y}$.

3.2.4 Theorem If $X = \{x_n\}$ is a convergent sequence and if $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x = \lim x_n \geq 0$.

Proof: Let $x < 0$, then choose $\epsilon = -\frac{x}{2} > 0$, for $V_\epsilon(x)$ neighborhood $\exists$ K: for $n \geq K$, then $x_n \in V_\epsilon(x)$

So

$$|x_n - x| < -\frac{x}{2},$$

$$\frac{x}{2} < x_n - x < -\frac{x}{2}$$

$$\frac{3}{2}x < x_n < \frac{x}{2}$$

$\Rightarrow x_n < 0. \Rightarrow \Leftarrow$ ∎

3.2.5 Theorem Let $X = \{x_n\}$ and $Y = \{y_n\}$ be convergent sequences such that $x_n \leq y_n$ $\forall n \in \mathbb{N}$, then $\lim x_n \leq \lim y_n$.