Problem 12 (Ch. 2)

Elementary School offers 3 language classes
Spanish, French, German

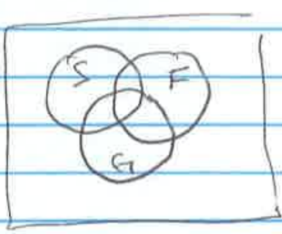
100 students in school (classes open to all)

There are 28 students in Spanish
" " 26 " " French
" " 16 " " German.
" " 12 " " S + F
" " 4 " " S + G
" " 6 " " F + G
" " 2 " " F + S + G

a) If a student is chosen randomly, what is the probability he/she is not in a language class?

$$P(S \cup F \cup G) = P(S) + P(F) + P(G) - P(SF) - P(SG) - P(FG) + P(SFG)$$

inclusion exclusion



$$= \frac{28}{100} + \frac{26}{100} + \frac{16}{100} - \frac{12}{100} - \frac{4}{100} - \frac{6}{100} + \frac{2}{100}$$

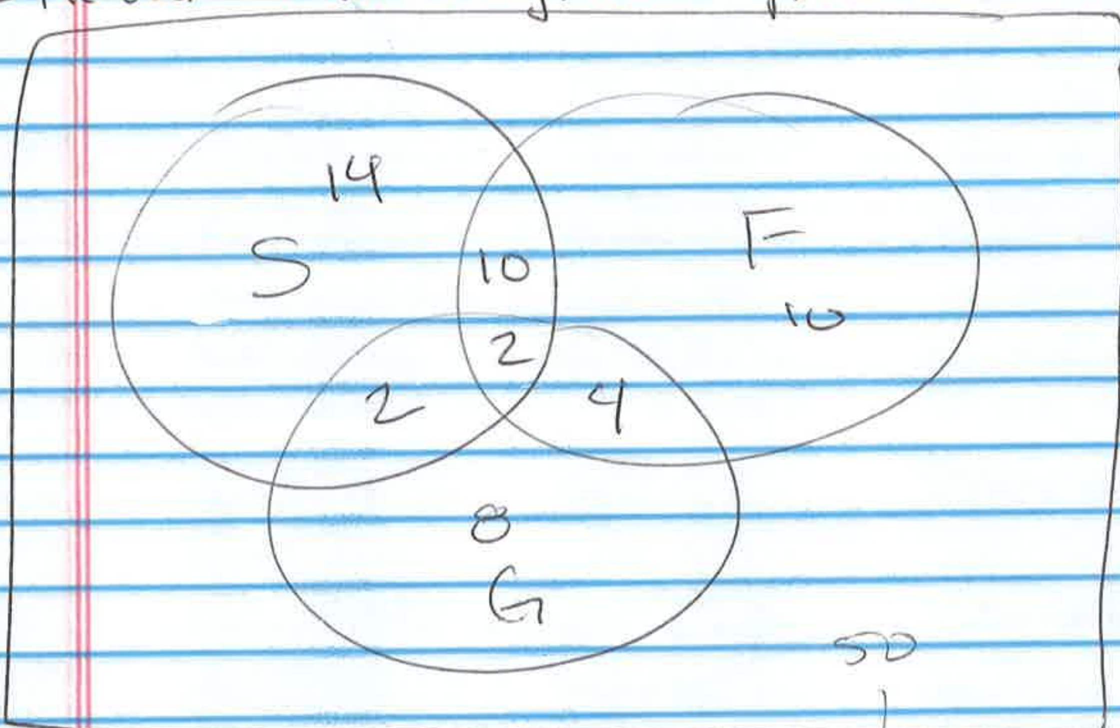
$$= \frac{50}{100}$$

So

$$P((S \cup F \cup G)^c) = 1 - \frac{50}{100} = \frac{50}{100} = \frac{1}{2}$$

41.1

Fill out Venn Diag. from given info.



not in any language since there are 100 total students.

b) If a student is chosen randomly, what is the probability he/she is taking exactly one language class?

$$P(\text{only } S) = P(S) - P(S \cap F) - P(S \cap G) + P(FSG)$$

$$P(\text{only } F) = P(F) - P(S \cap F) - P(G \cap F) + P(FSG)$$

$$P(\text{only } G) = P(G) - P(G \cap F) - P(G \cap S) + P(FSG)$$

sum $= \dfrac{28 + 26 + 16 - 2(12) - 2(4) - 2(6) + 3(2)}{100}$

$P(\text{only } S) + P(\text{only } F) + P(\text{only } G) = \dfrac{70 - 24 - 8 - 12 + 6}{100} = \dfrac{32}{100} = .32$

c) If 2 students are chosen randomly, what is the probability that at least one is taking a language class?

$$P(\text{neither is taking language}) = \frac{\#\text{ of outcomes with no language}}{\text{total }\#\text{ of outcomes}}$$

$$= \frac{\binom{50}{2}}{\binom{100}{2}}$$

$\binom{50}{2}$ ← choose 2 of 50 not in lang.

$\binom{100}{2}$ ← choose 2 of 100 students

assume we don't mean that we could pick the same student twice.

$$= \frac{50 \cdot 49}{100 \cdot 99} = \frac{49}{198}$$

→ $\boxed{\text{so } P(\text{at least one lang.}) = 1 - \frac{49}{198} = \frac{149}{198}}$

intuition?

Assumptions
- all days equally likely
- no birthdays on Feb. 29 (giving 365 possible days out of 366 actual days)

Example 5i (p.37)

If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year?

total with n different dates

$$P(\text{all birthdays differ}) = \frac{365 \cdot 364 \cdot \ldots \cdot (365-n+1)}{365 \cdot 365 \cdot \ldots \cdot 365}$$

total # of possibilities

$$= \frac{365!}{(365-n)!} \, \frac{1}{365^n}$$

n	P(different)	
10	0.883	
15	0.747	
20	0.589	
21	0.556	
22	0.524	← 47.6% chance of match
23	0.493	← 50.7% " "
25	0.431	
30	0.294	
40	0.109	← ~ 90% chance of a match
50	0.0296	
60	0.00588	
70	0.000840	99.91% chance of a match
80	0.000857	← ~~& 0.08% chance of~~
90	6.15×10^{-6}	
100	3.07×10^{-7}	
⋮		
$n \geq 365$	0	← pigeonhole principle!

intuition ?

Example 5j (p. 37)

A deck of 52 playing cards is shuffled. Cards turned up one at a time until the first ace appears. Is the card following the first ace more likely to be the ace of spades or the two of clubs?

• total # of orderings of 52 cards = 52!

want: # of Orderings resulting in As following first ace.

note: # of Ordering of the 51 other cards = 51!

note, getting As ~~first~~ first would not be a way As could follow first ace

~~each one of these card~~

each one of these 51! orderings has only one way to put the As after the first ace. So

$$P(\text{As after A\#1}) = \frac{51! \cdot 1}{52!} = \frac{1}{52}$$

The same argument would apply if we ordered 51 cards excluding the 2c and then inserting the 2c after first ace. So

$$P(\text{2c after A\#1}) = \frac{51!}{52!} = \frac{1}{52}$$

both events are equally-likely.

P(A spades after first ace)

$$= \frac{51! \cdot 1}{52!}$$

number of permutations of all cards

number of permutations of 51 cards

(A spades temporarily removed)

or any other card for that matter

# of ways the A spades can follow the first ace.

Note: if we didn't care where we put the Ace of spades the "1" would be 52 and the ratio would just be $\frac{52!}{52!} = 1$

i.e. the probability of getting any ordering of the 52 cards...

Ch. 2 solutions

Problem 1, 2, 4 (see Dan's notes)

Problem 8: Suppose A and B are mutually exclusive events.

$P(A) = 0.3$

$P(B) = 0.5$

What is the probability that...

a) either A or B occurs

$P(A \cup B) = P(A) + P(B) - P(AB)$

$P(A \cup B) = P(A) + P(B)$ since A, B are mutually exclusive

$= 0.3 + 0.5$

$\boxed{= 0.8}$

b) A occurs but B does not

$P(A) = 0.3$

note "B does not" is B^c

B^c and A are not mutually exclusive and so the question is simply what is the probability that A occurs.

$\boxed{0.3}$

c) both A and B occur.

$P(AB) = 0$ since A, B are mutually exclusive.

Problem 36 (p.51)

Two cards chosen at random from a deck of 52 cards. What is the probability that they

a) are both aces?

$$\underbrace{\frac{4}{52}}_{\text{first card}} \cdot \underbrace{\frac{3}{51}}_{\text{2nd card}} = \frac{12}{52 \cdot 51} = \boxed{\frac{1}{13} \cdot \frac{1}{17}}$$

b) have the same value

$$\frac{52}{52} \cdot \frac{3}{51} = \frac{3}{51}$$

any card works — must match first card

same.

or $\left(\frac{4}{52} \cdot \frac{3}{51}\right) \times 13 = \frac{3}{51}$

works for any of the 13 cards

Ch. 2 Problem 45

A woman has n keys, one of which opens her door.

a) If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her k^{th} try?

b) What if she does not discard previously tried keys?

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a) $k=1$: $P(k=1) = \frac{1}{n}$

$k=2$: $P(k=2) = \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}$

$k=3$ $P(k=3) = \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{1}{n-2} = \frac{1}{n}$

⋮

In general

$$P(k) = \underbrace{\overset{\text{try } 1}{\frac{n-1}{n}} \cdot \overset{2}{\frac{n-2}{n-1}} \cdots \overset{\text{try } k-1}{\frac{n-(k-1)}{n-(k-2)}}}_{\text{right key not chosen}} \cdot \underbrace{\frac{1}{n-(k-1)}}_{\text{right key chosen of the remaining } n-k+1 \text{ keys}}$$

$$\boxed{P(k) = \frac{1}{n}}$$

b) $P(k) = \underbrace{\left(\frac{n-1}{n}\right)^{k-1}}_{\text{wrong key } k-1 \text{ times}} \cdot \underbrace{\frac{1}{n}}_{\text{right key}}$

Note: $\sum_{k=1}^{\infty} P(k) = \sum_{k=1}^{\infty} \left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}$

$= \frac{1}{n}\left[\frac{1}{1-\left(\frac{n-1}{n}\right)}\right] = 1$ {Probability = 1 she eventually gets in door.}

Ch. 2 Problem 28 see also 2.35 (48.4)

$\boxed{2.28}$ Urn: 5 red
6 blue 19 total
8 green

Set of 3 balls randomly selected. What is the probability that each ball will be

a) of the same color? NO REPLACEMENT

3 red: $P_{3R} = \dfrac{\binom{5}{3}}{\binom{19}{3}} = \dfrac{10}{969} = $

$P_{3B} = \dfrac{\binom{6}{3}}{\binom{19}{3}} = \dfrac{20}{969}$

$P_{3G} = \dfrac{\binom{8}{3}}{\binom{19}{3}} = \dfrac{56}{969}$

so $P_{3R} + P_{3B} + P_{3G} = P_{SAME} = \dfrac{10+20+56}{969} = \boxed{\dfrac{86}{969}}$

$= 0.08875$

b) of different colors? NO REPLACEMENT

$P_{RGB} = \dfrac{\binom{5}{1}\binom{6}{1}\binom{8}{1}}{\binom{19}{3}} = \boxed{\dfrac{240}{969}} = 0.2477$

repeat ~~■~~ 2.28 with replacement

a) $P_{3R} = \left(\frac{5}{19}\right)^3$

$P_{3B} = \left(\frac{6}{19}\right)^3$

$P_{3G} = \left(\frac{8}{19}\right)^3$

$$P_3 = \left(\frac{5}{19}\right)^3 + \left(\frac{6}{19}\right)^3 + \left(\frac{8}{19}\right)^3$$

$$= \frac{125 + 216 + 512}{6859} = \frac{853}{6859}$$

$$= 0.1244$$

b) $P_{RGB} = \left(\frac{5}{19}\right)\left(\frac{6}{19}\right)\left(\frac{8}{19}\right)3! = \frac{240 \cdot 6}{6859} = \underline{0.2099}$

$$= \boxed{\frac{1440}{6859}}$$

this accounts for 6 possible orders of these 3 colors being drawn.

$\boxed{2.35}$ Seven balls randomly withdrawn from urn containing

12 red
16 blue   46 total
18 green

balls.

a) Find the probability that 3 red, 2 blue, and 2 green balls are withdrawn. (without replacement)

$$\frac{\binom{12}{3}\binom{16}{2}\binom{18}{2}}{\binom{46}{7}} = \frac{\frac{12!}{9!3!}\cdot\frac{16!}{14!2!}\cdot\frac{18!}{16!2!}}{\frac{46!}{39!7!}}$$

$$= \frac{\frac{39!}{9!14!16!}\cdot\frac{7!}{3!2!2!}}{\frac{46!}{12!16!18!}}$$

$$\frac{(\text{pick 3 red})(\text{pick 2 blue})(\text{pick 2 green})}{(\text{pick 7 total})}$$

∴ # of arrangements where 3 r, 2 b, 2 g can be in 7 selected spots and the other 9 red, 14 blue, 16 green can be in the other 39 spots

divided by # of arrangements of 46 items in 3 groups of 12, 16, 18.

b) At least two red balls withdrawn

$= 1 - P(\text{no red}) - P(\text{one red})$

$$\boxed{P\left(\begin{matrix}\text{at least}\\ 2R\end{matrix}\right) = 1 - \sum_{i=0}^{7} \frac{\binom{12}{0}\binom{16}{i}\binom{18}{7-i}}{\binom{46}{7}} - \sum_{i=0}^{6} \frac{\binom{12}{1}\binom{16}{i}\binom{18}{6-i}}{\binom{46}{7}}}$$

c) All withdrawn balls are the same color

$$P_{allR} = \frac{\binom{12}{7}\binom{16}{0}\binom{18}{0}}{\binom{46}{7}} = \frac{\binom{12}{7}}{\binom{46}{7}}$$

$$P_{allB} = \frac{\binom{12}{0}\binom{16}{7}\binom{18}{0}}{\binom{46}{7}} = \frac{\binom{16}{7}}{\binom{46}{7}}$$

$$P_{allG} = \frac{\binom{13}{0}\binom{16}{0}\binom{18}{7}}{\binom{46}{7}} \quad \frac{\binom{18}{7}}{\binom{46}{7}}$$

$$\boxed{P_{allsame} = P_{allR} + P_{allB} + P_{allG}}$$

d) either exactly 3 red or exactly 3 blue

$$P_{3R} = \sum_{i=0}^{4} \frac{\binom{12}{3}\binom{16}{i}\binom{18}{4-i}}{\binom{46}{7}}$$

$$P_{3B} = \sum_{i=0}^{4} \frac{\binom{12}{i}\binom{16}{3}\binom{18}{4-i}}{\binom{46}{7}}$$

Note that the $i=3$ case is counted twice so

$$\boxed{P = P_{3R} + P_{3B} - \frac{\binom{12}{3}\binom{16}{3}\binom{18}{1}}{\binom{46}{7}}}$$

More generally...

choose R red, B blue, G green balls from urn with N_R red balls, N_B blue balls, N_G green balls

N_T = total = $N_R + N_B + N_G$

- without replacement...

$$P_{RBG} = \frac{\binom{N_R}{R}\binom{N_B}{B}\binom{N_G}{G}}{\binom{N_R+N_B+N_G}{R+B+G}}$$

e.g. one of each $\dfrac{\binom{N_R}{1}\binom{N_B}{1}\binom{N_G}{1}}{\binom{N_R+N_B+N_G}{3}}$

- with replacement...

$$P_{RBG} = \left(\frac{N_R}{N_T}\right)^R \cdot \left(\frac{N_B}{N_T}\right)^B \cdot \left(\frac{N_G}{N_T}\right)^G \cdot \frac{(R+B+G)!}{R!\,B!\,G!}$$