Practice Midterm 2 Answer Key

Practice Midterm 2a

1. $11 - 3i$

2. $13 + i$

3. 4

4. $-i$

5. i

6. True.

   Proof. Let $x \in X$. From xIx it follows that I is reflexive.

   If $x, y, z \in X$ satisfy xIy and xIz, then from $x = y$ and $y = z$ we obtain $x = z$, so that xIz. Therefore, I is transitive.

   Given $x, y \in X$ such that xIy and yIx, it immediately follows that $x = y$, whence I is antisymmetric. $\square$

7. True.

   Proof. Symmetry and reflexivity are clear.

   Suppose that $(m, n) \sim (m', n')$ and $(m', n') \sim (m'', n'')$. Adding the equalities

   $$m + n' = m' + n \qquad \text{and} \qquad m' + n'' = m'' + n'$$

   and subtracting $m' + n'$ yields

   $$m + n'' = m'' + n.$$

   We conclude that $\sim$ is transitive. $\square$

8. *Answers will vary.*

Practice Midterm 2b

1. $-4 + 3i$

2. $-5 + 3i$

3. $-11 - 60i$

4. i

5. -1

6. True.

   Proof. Let $(p, q) \in \mathbb{Z}^2$. From $pq = pq$ it follows that $(p, q) \sim (p, q)$, whence $\sim$ is reflexive.

   Suppose that $(p, q) \sim (p', q')$ and $(p', q') \sim (p'', q'')$. If $p' = 0$. Thus,

   $$pq' = p'q \qquad \text{and} \qquad p'q'' = p''q'.$$

   If $p' = 0$, then it immediately follows that $p = p'' = 0$ and consequently that $(p, q) \sim (p'', q'')$. Hence suppose that $p \neq 0$. Multiplying the preceding equalities together provides

   $$pp'q'q'' = p'p''qq'.$$

   and dividing through by $p'q'$ yields $pq'' = p''q$, so that $(p, q) \sim (p'', q'')$. Therefore, $\sim$ is transitive.

   Finally, the symmetry of $\sim$ follows directly from that of the relation $=$ on $\mathbb{Z}$. □

7. True.

   Proof. Let $m \in \mathbb{N}$. Since $m^1 = m$, it follows that R is reflexive.

   Suppose that $m, n, p \in \mathbb{N}$ such that mRn and nRp. Thus, there are $k, \ell \in \mathbb{N}$ with $m^k = n$ and $n^\ell = p$. Raising each side of the first equality to the power of ℓ provides

   $$m^{k\ell} = n^\ell = p$$

   from which follows mRp. Thus, R is transitive.

   Now suppose that $m, n \in \mathbb{N}$ satisfy mRn and nRm and let $k, \ell \in \mathbb{N}$ such that $m^k = n$ and $n^\ell = m$. If either k or ℓ is equal to 0, then $m = n = 1$ and we are done. Otherwise, $m^k = n$ yields $m \leq n$, while $n^\ell = m$ provides $n \leq m$, and we conclude that $m = n$. Therefore, R is antisymmetric. □

8. *Answers will vary.*

Practice Midterm 2c

1. $-12 + 18i$
2. $-\dfrac{1}{5} + \dfrac{2}{5}i$
3. $-3 - i$
4. $\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$
5. $\dfrac{3}{10} + \dfrac{i}{10}$
6. True.

   Proof. Let $m \in \mathbb{N}$. As $m \leq m$ we deduce that $\leq$ is reflexive.

   Suppose that $m, n, p \in \mathbb{N}$ with $m \leq n$ and $n \leq p$. As $m \leq p$ it follows that $\leq$ is transitive.

   Let $m, n \in \mathbb{N}$. If $m \leq n$ and $n \leq m$, then $m = n$ and we conclude that $\leq$ is antisymmetric. □

7. True.

   Proof. Let $A \subseteq X$. Since $A \subseteq A$ it follows that $\subseteq$ is reflexive.

   Suppose that $A, B, C \subseteq X$ with $A \subseteq B$ and $B \subseteq C$. It follows that $A \subseteq C$ and we deduce that $\subseteq$ is transitive.

   If $A, B \subseteq X$ with $A \subseteq B$ and $B \subseteq A$, then $A = B$ and we conclude that $\subseteq$ is antisymmetric. □

8. *Answers will vary.*