5.5 Exponential Distribution (see also notes p. 152, 153)

A continuous random variable X, whose probability density function (pdf) is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

EXAMPLE IDEA!

X = waiting time until an event occurs when events occur at random rate $\lambda > 0$.

i.e. $P\{X < x\} = \int_{-\infty}^{x} f(x)\,dx \equiv F(x)$ = cumulative distribution function

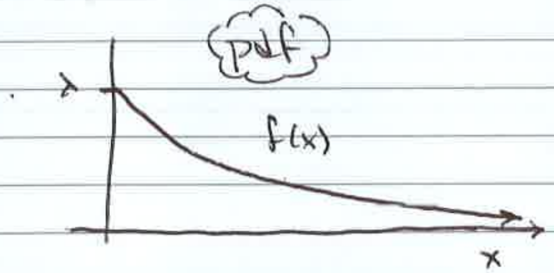
$= 0$ if $x \leq 0$

$$= \int_0^x \lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big|_0^x = -e^{-\lambda x} - (-1)$$

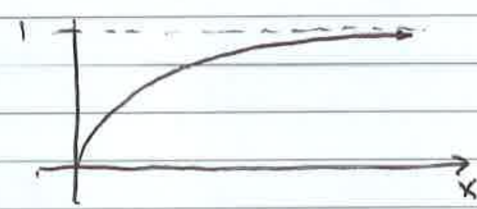
$$= 1 - e^{-\lambda x}$$

These could describe unscheduled things, or things that take unknown lengths of time to complete

pdf



cdf



we've shown $\boxed{E[X] = \frac{1}{\lambda}}$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \text{integrate by parts} = \frac{2}{\lambda^2}$$

(see text p. 198)

$$Var(X) = E[X]^2 - (E[X])^2 \qquad \boxed{Var(X) = \frac{1}{\lambda^2}}$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

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Problem $\boxed{5.32}$ (exponential)

The time in hours to repair a machine is an exponentially-distributed random variable with parameter $\lambda = 1/2$.

What is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a) the probability that the repair time exceeds 2hrs?

let X = # of hours for the repair

$$P\{X > 2\} = 1 - P\{X \leq 2\}$$

$$= 1 - \int_{-\infty}^{2} f(x)\,dx = 1 - \int_0^2 \lambda e^{-\lambda x}\,dx$$

$$= 1 + e^{-\lambda x}\Big|_0^2 = 1 + e^{-2\lambda} - 1 = e^{-2\lambda}$$

but $\lambda = 1/2$ so $\boxed{P\{X > 2\} = e^{-1}}$

b) the conditional probability that the repair takes at least 10 hrs, given that its duration exceeds 9 hrs.

$$P\{X \geq 10 \mid X > 9\} = \frac{P\{X \geq 10 \text{ AND } X > 9\}}{P\{X > 9\}}$$

$$= \frac{P\{X \geq 10\}}{P\{X > 9\}} = \frac{1 - P\{X < 10\}}{1 - P\{X \leq 9\}}$$

$$= \frac{1 - \int_0^{10} \lambda e^{-\lambda x}\,dx}{1 - \int_0^{9} \lambda e^{-\lambda x}\,dx} = \frac{1 + e^{-\lambda x}\Big|_0^{10}}{1 + e^{-\lambda x}\Big|_0^{9}} = \frac{e^{-10\lambda}}{e^{-9\lambda}}$$

$\lambda = 1/2$

$$= \frac{e^{-5}}{e^{-4.5}} = e^{-0.5} = \boxed{e^{-1/2}}$$

EXAMPLE (Problem $\boxed{5.34}$)

Number of miles (in ~~…~~ units of 1000) before a car needs to be junked is assumed to be an exponential random variable with parameter $\lambda = \frac{1}{20}$.

X = # of miles in ~~…~~ 1000's.

~~…~~ used car has 10,000 miles on it.

What is the probability you can get 20,000 more miles out of it?

$$P\{X > 30 \mid X > 10\} = \frac{P\{X > 30\}}{P\{X > 10\}}$$

$$= \frac{1 - P\{X \leq 30\}}{1 - P\{X \leq 10\}}$$

$$P\{X \leq K\} = \int_0^K \lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big|_0^K = 1 - e^{-K\lambda}$$

$\lambda = \frac{1}{20}$ $\quad P\{X \leq 30\} = 1 - e^{-\frac{30}{20}} = 1 - e^{-3/2}$

$$P\{X \leq 10\} = 1 - e^{-\frac{10}{20}} = 1 - e^{-1/2}$$

so $\quad P\{X > 30 \mid X > 10\} = \dfrac{e^{-3/2}}{e^{-1/2}} = e^{-1} = \boxed{\dfrac{1}{e}}$

Same.

$$P\{X > 20\} = 1 - P\{X \leq 20\} = 1 - (1 - e^{-20\lambda}) = 1 - 1 + e^{-\frac{20}{20}} = e^{-1}$$

$$f = \begin{cases} \frac{1}{40} & 0 < x < 40 \\ 0 & \text{otherwize} \end{cases}$$ *

If the lifetime of the car was instead uniformly distributed over $(0,40)$ (units of 1000), then

$$P\{X > 30 \mid X > 10\} = \frac{1 - P\{X \leq 30\}}{1 - P\{X \leq 10\}}$$

$$= \frac{1 - \frac{30}{40}}{1 - \frac{10}{40}} = \frac{10/40}{30/40} = \boxed{1/3}$$ a little less than $\frac{1}{e}$.

"Memoryless" property of exponential random variable X

Consider

$$P\{X > s+t \mid X > t\}$$

e.g. X lifetime exceeds $s+t$ given that it has exceeded s.

$$= \frac{P\{X > s+t \text{ AND } X > t\}}{P\{X > t\}}$$

← by defn of conditional probability

$$= \frac{P\{X > s+t\}}{P\{X > t\}}$$

for X an exponential R.V.

$$= \frac{\int_{s+t}^{\infty} \lambda e^{-\lambda x} dx}{\int_{t}^{\infty} \lambda e^{-\lambda x} dx}$$

$$= \frac{-e^{-\lambda x}\Big|_{s+t}^{\infty}}{-e^{-\lambda x}\Big|_{t}^{\infty}} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda(t)}} = e^{-\lambda s} = \int_s^{\infty} \lambda e^{-\lambda x} dx$$

$$= P\{X > s\}$$

see textbook p.199

so $P\{X > s+t \mid X > t\} = P\{X > s\}$

property of Memoryless R.V.

e.g. in Problem 5.34

$$\underbrace{P\{X > 20+10 \mid X > 10\}}_{= e^{-1}} = \underbrace{P\{X > 20\}}_{e^{-1}} \qquad \lambda = \tfrac{1}{20}$$

Given that the car has lasted 10,000 miles the probability of getting another 20,000 is the same as the unconditional probability of getting the new car to 20,000 miles.

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5.6.1 The Gamma Distribution

A random variable, X, has a gamma distribution with parameters (α, λ), $\lambda > 0$, $\alpha > 0$ if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\boxed{\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy \quad \text{is the gamma function}}$$

Note: • if $\alpha = 1$ $\quad f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$\Gamma(1) = 1$

then this is an exponential random variable.

• when $\alpha = n$ = positive integer

X = waiting time until n events occur when events always occur at random rate $\lambda > 0$.

• $\boxed{\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)}$ for $\alpha > 1$

follows from integration by parts. (see text, p. 204)

• for $n = 1, 2, 3, \ldots$ $\quad \boxed{\Gamma(n) = (n-1)!}$

lots of interesting properties

see Applied Math Talk – Nov. 2 2013 Srinivasan

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connecting Poisson R.V. to exponential R.V.

Recall the Discrete Poisson R.V.

$$p(i) = P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!} \qquad i = 0, 1, 2, \ldots$$

so

$$P\{X \le n\} = \sum_{i=0}^{n} e^{-\lambda}\frac{\lambda^i}{i!} = e^{-\lambda}\sum_{i=0}^{n}\frac{\lambda^i}{i!}$$

here $E[X] = \lambda$ (see p.137 text)

$Var(X) = \lambda$ (see p.138 text)

There we described conditions for a "Poisson Process"

see Text book p.144

and for the number of "events" occurring in a fixed time interval to be a Poisson Random Variable:

(1): The probability that exactly 1 event occurs in a given interval of length h is

$$\lambda h + o(h)$$

(2): The probability of 2 or more events occurring in an interval of length h is $o(h)$.

(3): For any integers $n, j_1, j_2, \ldots, j_n$ and any set of n non-overlapping intervals, if we define E_i to be the event that exactly j_i of the events occur in the i^{th} interval, then $E_1, E_2, \ldots, E_n$ are independent.

*

The arguments following this in the text (pp. 144-145) led to the result

see p. 145 Eq. (7.5)

$$\boxed{P\{N(t)=k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \qquad k=0,1,2,\ldots}$$

where $N(t)$ denotes the number of events occurring in the interval $[0,t]$.

In this context, where ①,②,③ hold, let

X = time until ~~first~~ first event occurs

Then $\underset{\uparrow}{X<t}$ if and only if $\underbrace{N(t)\geq 1}_{\text{at least one event in } [0,t]}$

first event occurs in $[0,t)$

so $P\{N(t)\geq 1\} = 1-P\{N(t)=0\} = 1 - e^{-\lambda t}\frac{(\lambda t)^0}{0!}$

$$= 1-e^{-\lambda t}$$

so $P\{X<t\} = P\{N(t)\geq 1\} = 1-e^{-\lambda t}$

$$= \int_0^t \lambda e^{-\lambda \bar{t}}\,d\bar{t}$$

$$= \int_{-\infty}^t f(\bar{t})\,d\bar{t}$$

so X is an exponential R.V.

where

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t\geq 0 \\ 0 & t<0 \end{cases}$$

*

$\underline{X}$ R.V. with

Connect Poisson R.V. to Gamma Distribution

Let T_n denote the time at which the n^{th} event occurs.

$$F_T(t) = P\{T_n \le t\} = P\{N(t) \ge n\} = \sum_{j=n}^{\infty} P\{N(t)=j\}$$

cumulative distribution function

at least n events happen in $[0,t]$

$$= \sum_{j=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

Then, the probability density function $f_T(t)$ is $\frac{d}{dt}F_T$

$$f_T(t) = \sum_{j=n}^{\infty} \frac{-\lambda e^{-\lambda t}(\lambda t)^j}{j!} + \frac{e^{-\lambda t}\, j(\lambda t)^{j-1}\cdot\lambda}{j!}$$

$$= \sum_{j=n}^{\infty} \lambda e^{-\lambda t}\left[\frac{(\lambda t)^{j-1}}{(j-1)!} - \frac{(\lambda t)^j}{j!}\right]$$

$$= \lambda e^{-\lambda t}\left\{\left[\frac{(\lambda t)^{n-1}}{(n-1)!} - \cancel{\frac{(\lambda t)^n}{n!}}\right] + \left[\cancel{\frac{(\lambda t)^n}{n!}} - \frac{(\lambda t)^{n+1}}{(n+1)!}\right] + \dots + \Big[\quad\Big]_{n\to\infty}\right.$$

collapsing and note $\frac{(\lambda t)^j}{j!} \to 0$ as $j\to\infty$

* $$f_T(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \qquad t \ge 0$$

* Note: This is the probability density function for a random variable X with gamma distribution $(\alpha = n)$

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \ge 0 \\ 0 & x < 0 \end{cases} \qquad \Gamma(n) = (n-1)!$$

$$P\{N(t)=k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Problem $\boxed{4.54}$

Suppose that the average number of cars abandoned weekly on a certain highway is 2.2

a) Approximate the probability that there will be no abandoned cars in the next week.

$$\underset{\text{with } t=1}{P\{N(t)=0\}} = e^{-\lambda t}\frac{(\lambda t)^0}{0!}\bigg|_{t=1} = e^{-\lambda}$$

↳ note: $\lambda h = 2.2$ with $h = 1$ week.

so $\lambda = 2.2$ ← i.e. $\frac{2.2 \text{ cars}}{\text{week}}$.

$$\boxed{P\{N(1)=0\} = e^{-2.2}}$$

b) Approximate the probability that there will be at least two abandoned cars in the next week

$$P\{N(t)\geq 2\} = 1 - P\{N(t)=0\} - P\{N(t)=1\}$$

where $t=1$.

$$= 1 - e^{-2.2}\frac{(2.2)^0}{0!} - e^{-2.2}\frac{(2.2)^1}{1!}$$

$$= 1 - e^{-2.2}(1+2.2)$$

$$= \boxed{1 - 3.2\,e^{-2.2}}$$

An equivalent way to think about this is to compute the probability of having to wait less than or equal to one week for 2 abandoned cars (using the Gamma Distribution) with $n=2$ events

$$P\{T' \leq 1\} = \int_0^1 \frac{\lambda e^{-\lambda x}(\lambda x)^{2-1}}{(2-1)!}\,dx$$

$$= \int_0^1 \lambda^2 \underbrace{e^{-\lambda x}}_{dv}\, \underbrace{x}_{u}\,dx$$

$$= \lambda^2 \left[x\left(-\frac{e^{-\lambda x}}{\lambda}\right)\Big|_0^1 - \int_0^1 \left(-\frac{e^{-\lambda x}}{\lambda}\right)dx \right]$$

$$= \lambda^2 \left[-\frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda}\int_0^1 e^{-\lambda x}\,dx \right]$$

$$= \lambda^2 \left[-\frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda}\,\frac{e^{-\lambda x}}{(-\lambda)}\Big|_0^1 \right]$$

$$= \lambda^2 \left[-\frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda^2}\left(1-e^{-\lambda}\right) \right]$$

$$= 1 - e^{-\lambda}(1+\lambda)$$

$$= 1 - 3.2e^{-2.2}$$ ⟵ same as p. 190.

use $P\{N(t)=k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$ or gamma distribution $f(x) = \begin{cases} \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} & t \geq 0 \\ 0 & t < 0 \end{cases}$

Problem $\boxed{4.63}$ (192)

People enter a casino at a rate of 1 every 2 minutes

$\lambda = \frac{1}{2}$ (time in minutes)

Assume a Poisson Process

a) What is the probability that no one enters between 12:00 and 12:05?

$$P\{N(5)=0\} = \frac{e^{-\frac{1}{2}(5)}\left(\frac{1}{2}\cdot 5\right)^0}{0!} = \boxed{e^{-5/2}}$$

b) What is the probability that at least 4 people enter the casino during this time?

$$P\{N(5)\geq 4\} = 1 - \sum_{i=0}^{3} P\{N(5)=i\}$$

$$= 1 - e^{-\lambda 5}\sum_{i=0}^{3}\frac{(\lambda 5)^i}{i!}$$

$$= 1 - e^{-5\lambda}\left(1 + 5\lambda + \frac{1}{2}(5\lambda)^2 + \frac{1}{3!}(5\lambda)^3\right)$$

$$= \boxed{1 - e^{-5/2}\left(1 + 5/2 + \frac{1}{2}(5/2)^2 + \frac{1}{3!}(5/2)^3\right)}$$

Alternatively, using the gamma distribution

a) $$P\{T_1 \geq 5\} = \int_5^\infty \frac{\frac{1}{2} e^{-\frac{1}{2}t} \left(\frac{1}{2}\lambda\right)^{1-1}}{(1-1)!} dt = \int_5^\infty \frac{1}{2} e^{-\frac{1}{2}t} dt$$

time for 1 event ≥ 5 (no people enter)

$$= -\frac{1}{2} 2 e^{-\frac{1}{2}t} \Big|_5^\infty = 0 - (-e^{-5/2})$$

$$= \boxed{e^{-5/2}}$$

b) $$P\{T_4 \leq 5\} = \int_0^5 \frac{\frac{1}{2} e^{-\frac{1}{2}t} \left(\frac{1}{2}t\right)^{4-1}}{(4-1)!} dt$$

$$= \frac{1}{3!} \frac{1}{2^4} \boxed{\int_0^5 t^3 e^{-\frac{1}{2}t} dt}$$

must integrate by parts to work this out..

should match

$$1 - e^{-5/2}\left[1 + \frac{5}{2} + \frac{1}{2}\left(\frac{5}{2}\right)^2 + \frac{1}{3!}\left(\frac{5}{2}\right)^3\right]$$