Metric spaces

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Math 300

Section 1

Definition and examples

Let X be a set.

Definition

A *metric* on X is a function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

satisfying

i. $d(x, y) = 0$ if and only if $x = y$,
ii. $d(x, y) = d(y, x)$,
iii. $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

The pair (X, d) is called a *metric space*.

Is it a metric?

X any set

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Yes! The *discrete metric* on X

Is it a metric?

X any set

$$d(x, y) = 0$$

No!

Is it a metric?

$$X = \mathbb{R}^n$$

$$d\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Yes! The *Euclidean metric*

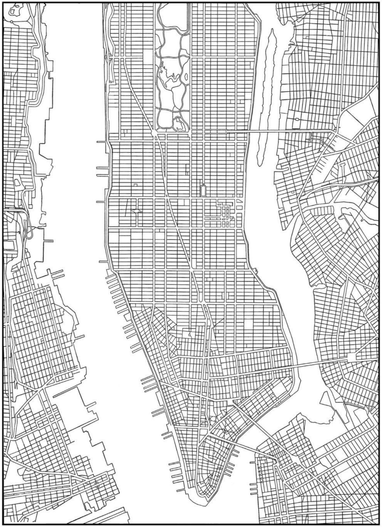
Is it a metric?

$$X = \mathbb{R}^n$$

$$d\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = |x_1 - y_1| + \cdots + |x_n - y_n|$$

Yes! The *taxicab metric*

Taxicab metric



MAP OF LOWER MANHATTAN, AN ISLAND IN NEW YORK BAY BOUNDED BY THE HUDSON, EAST RIVER (A STRAIT), AND HARLEM RIVER, AND COMPRISING THE BOROUGH OF MANHATTAN OF GREATER NEW YORK. NEW YORK IS THE MOST IMPORTANT PORT ON THE ATLANTIC SEABOARD, AND IS BORDERED BY GREAT PIERS

Is it a metric?

$$X = \mathbb{R}^n$$

$$d\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) = \sqrt[p]{|x_1 - y_1|^p + \cdots + |x_n - y_n|^p}$$

Yes! The *ℓ^p-metric*

Is it a metric?

$$X = \mathbb{R}^n$$

$$d\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = \max_{0 \le i \le n} |x_i - y_i|$$

Yes! The ℓ^∞-*metric*

Is it a metric?

$$X = C_0(\mathbb{R})$$

$$d(f,g) = \int_{-\infty}^{\infty} |f(x) - g(x)| \,\mathrm{d}x$$

Yes! The L^1*-metric*

Is it a metric?

$$X = C_0(\mathbb{R})$$

$$d(f, g) = \left(\int_{-\infty}^{\infty} \left(f(x) - g(x)\right)^p \, \mathrm{d}x \right)^{1/p}$$

Yes! The L^p*-metric*

Is it a metric?

$$X = C_0(\mathbb{R})$$

$$d(f,g) = \max_{x \in \mathbb{R}} |f(x) - g(x)|$$

Yes! The *L^∞-metric*

Section 2

Normed vector spaces

Let V be an $\mathbb{R}$-vector space.

Definition

A *norm* on V is a function

$$\|\cdot\| : V \to \mathbb{R}_{\geq 0}$$

such that

i. $\|v\| = 0$ if and only if $v = 0$,

ii. $\|sv\| = |s|\,\|v\|$,

iii. $\|u + v\| \leq \|u\| + \|v\|$ (*triangle inequality*).

The pair $(V, \|\cdot\|)$ is called a *normed vector space*.

Proposition

If $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ is a norm on V, then

$$\begin{aligned} d : V \times V &\to \mathbb{R} \\ (u, v) &\mapsto \|u - v\| \end{aligned}$$

is a metric on V.

Proof. (Condition i.)

Let $u, v \in V$. First observe that

$$\begin{aligned} d(u, v) = 0 &\iff \|u - v\| = 0 \\ &\iff u - v = 0 \\ &\iff u = v. \end{aligned}$$

$\square$

Proof. (Conditions ii. and iii.)

Moreover, d is symmetric as

$$\begin{aligned} d(u,v) &= \|u-v\| \\ &= \|(-1)\cdot(u-v)\| \\ &= \|v-u\| \\ &= d(v,u). \end{aligned}$$

Finally, given $u, v, w \in V$, we have

$$\begin{aligned} d(u,w) &= \|u-w\| \\ &= \|(u-v)+(v-w)\| \\ &\leq \|u-v\| + \|v-w\| \\ &= d(u,v) + d(v,w). \end{aligned}$$

□

Is it a norm?

$$(\mathbb{R}, |\cdot|)$$

Yes!

Is it a norm?

$$X = \mathbb{R}^n$$

$$\left\|(x_1, \ldots, x_n)\right\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$$

Yes! The *Euclidean norm*

Is it a norm?

$$X = \mathbb{R}^n$$

$$\left\|(x_1, \ldots, x_n)\right\|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$$

Yes! The ℓ^p-*norm*

Is it a norm?

$$X = \mathbb{R}^n$$

$$\left\|(x_1, \ldots, x_n)\right\|_1 = |x_1| + \cdots + |x_n|$$

Yes! The ℓ^1-*norm*

Is it a norm?

$$X = \mathbb{R}^n$$

$$\left\|(x_1, \ldots, x_n)\right\|_\infty = \max_{1 \le i \le n} |x_i|$$

Yes! The ℓ^∞*-norm*

Is it a norm?

$$X = C_0(\mathbb{R})$$

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x$$

Yes! The L^1*-norm*

Is it a norm?

$$X = C_0(\mathbb{R})$$

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p \, \mathrm{d}x \right)^{1/p}$$

Yes! The *L^p-norm*

Is it a norm?

$$X = C_0(\mathbb{R})$$

$$\|f\|_\infty = \max_{x \in \mathbb{R}} |f(x)|$$

Yes! The L^∞*-norm*

Section 3

Convergence

Definition

Let $(x_i)_i$ be a sequence in X and fix $x \in X$. We say that $(x_i)_i$ *converges* to x if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, x) \leq \varepsilon.$$

In this case, we write $x_i \to x$ or $\lim_{i \to \infty} x_i = x$ and we say that x is the *limit* of $(x_i)_i$.

Definition

If the sequence $(x_i)_i$ does not converge to any point $x \in X$, then $(x_i)_i$ is said to *diverge*.

Proposition

If $(x_i)_i$ is a constant sequence with value $x \in X$, then $x_i \to x$.

Proof.

Let $\varepsilon > 0$.For all $n \geq 1$, we have $d(x_n, x) = 0 \leq \varepsilon$. □

Proposition

If $x_i \to x$ and $x_i \to y$, then $x = y$.

Proof.

Suppose not. Then there is an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(x_n, x) \leq \frac{1}{3} d(x, y) \qquad \text{and} \qquad d(x_n, y) \leq \frac{1}{3} d(x, y).$$

Consequently,

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \leq \frac{2}{3} d(x, y).$$

This yields the desired contradiction. □

Example

Consider the sequence of functions $(f_i)_i$ given by

$$f_i(x) = \begin{cases} i - i^3|x| & \text{if } |x| < \frac{1}{i^2} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $f_i \to 0$ with respect to the L^1-metric, and that f_i diverges with respect to the L^∞-metric.

Image credits

- https://maps-manhattan.com/manhattan-grid-map