3.4 Independent Events

Def: Two events E and F are said to be <u>independent</u> if

$$P(EF) = P(E)\cdot P(F)$$

we've been using this idea already ...

Note: • If E and F are independent

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(E)P(F)}{P(F)} = P(E)$$

so.. knowledge of F does not change the probability of E happening. That is, $P(E|F) = P(E)$

• Similarly, for E and F independent events

$$P(F|E) = \frac{P(EF)}{P(E)} = \frac{P(E)P(F)}{P(E)} = P(F)$$

EXAMPLE 4c (p.75)

Toss two fair dice.

Let E_1 = event that sum of two dice is 6.

Let F = event that first die is a 4.

Are events E_1 and F independent?

$P(E_1) = \frac{5}{36}$ $\quad E_1 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$

$P(F) = \frac{1}{6}$

$P(EF) = P(\{(4,2)\}) = \frac{1}{36}$

but $P(E_1) \cdot P(F) = \frac{5}{36} \cdot \frac{1}{6} \neq P(EF) = \frac{1}{36}$

So E_1 and F are not independent events (makes sense since ~~___~~ the chances of getting a 6 clearly depends on the value of the first die).

Related Question

- Let E_2 = event that sum of two dice is 7
  Let F = event that first die is a 4.

Are events E_2 and F independent?

$P(E_2) = \frac{6}{36}$ $\quad E_2 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$P(F) = \frac{1}{6}$

$P(E_2F) = P(\{(4,3)\}) = \frac{1}{36}$

Note:

$P(E_2) \cdot P(F) = \frac{6}{36} \cdot \frac{1}{6} = \frac{1}{36}$

same. So in this scenario, E_2 and F are independent events.

(note, no matter the first die, the chances of getting a sum = 7 is the same. Also, vice versa.

Def: Independence of 3 events

Events E, F, and G are said to be independent if

$P(EFG) = P(E) \cdot P(F) \cdot P(G)$ } three way independence

and

$P(EF) = P(E)P(F)$
$P(EG) = P(E)P(G)$
$P(FG) = P(F)P(G)$ } pairwise independence

- see also general n event case, P. 77

n - independent events

That is, given events $E_1, E_2, \ldots, E_n$

These events are <u>independent</u> if for every subset

$$E_{1'}, E_{2'}, \ldots, E_{r'} \quad \text{of } r \text{ events}$$

with $r \le n$ we have

$$P(E_{1'} E_{2'} \cdots E_{r'}) = P(E_{1'}) \cdot P(E_{2'}) \cdots P(E_{r'})$$

<u>Independent Trials</u> (p. 77 bottom)

Suppose an experiment consists of a sequence of subexperiments. We say that the subexperiments are independent if $E_1, E_2, \ldots, E_n$ is an independent sequence of events whenever E_i is an event completely determined by the outcome of the i^{th} sub-experiment (i.e. its not influenced by the outcome of any other experiment).

If each subexperiment has the same set of possible outcomes, then the subexperiments are often called <u>trials</u>.

EXAMPLE 4h (p. 79)

Independent trials of rolling fair dice.

What is the probability that an outcome (= sum of dice) of a 5 occurs before an outcome of a 7 occurs?

Sol 1

E_n = no 5 or 7 on first $n-1$ trials, then a 5 on n^{th} trial

$P(\text{sum}=5) = \frac{4}{36}$

$P(\text{sum}=7) = \frac{6}{36}$

$$P(E_n) = \left(1-\frac{10}{36}\right)^{n-1} \cdot \left(\frac{4}{36}\right)$$

So

$$P = \sum_{n=1}^{\infty} P(E_n) = \frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{26}{36}\right)^{n-1} = \frac{1}{9}\frac{1}{1-\frac{26}{36}} = \frac{1}{9}\frac{36}{10} = \frac{4}{10} = \boxed{\frac{2}{5}}$$

we've done this one before

Sol 2

- use conditional probabilities

E = event 5 occurs before 7 (so we want $P(E)$)

all possible outcomes on first trial:

F = event that first trial results in a 5

G = " " " " " " 7

H = " " " " " " neither a 5 or 7.

$$P(E) = P(E|F)P(F) + P(E|G)P(G) + P(E|H)P(H)$$

$$= (1)\frac{4}{36} + (0)\cdot\frac{6}{36} + (P(E))\frac{26}{36}$$

* if first roll ≠ 5 or 7 we are back to the same problem

So

$$P(E) = \frac{4}{36} + \frac{26}{36} P(E)$$

$$\left(1 - \frac{26}{36}\right) P(E) = \frac{4}{36}$$

$$P(E) = \frac{\frac{4}{36}}{\frac{10}{36}} = \frac{4}{10} = \frac{2}{5}$$

Ch. 3 Problem 3.59 p. 102

Independent flips of a coin that land on heads with probability p are made. What is the probability that the first four outcomes are

a) H, H, H, H?

$$P(HHHH) = p^4$$

b) T, H, H, H?

$$P(THHH) = (1-p)p^3$$

c) What is the probability that the pattern

THHH

occurs before the pattern

HHHH ?

Sol: If a T appears anywhere, the pattern THHH necessarily appears before HHHH appears, since HHH must appear in order for HHHH to appear. The only way HHHH can appear before THHH is if the first four flips are all heads. Therefore,

$P(\text{THHH appears before HHHH}) = 1 - P(HHHH) = 1 - p^4$.

Ch.3 Problem 58

p. 102

Suppose we have a coin that lands on heads with probability p. (p may or may not $= 1/2$). Can we somehow use this coin to generate the outcome of the flip of a fair coin?

Consider the procedure:

1. Flip the coin
2. Flip the coin again
3. If both flips land on heads or both land on tails return to step 1.
4. Let the result of the last flip be the result of the experiment.

a) Show that the result is equally-likely to be heads or tails.

Denote:

E = event that (T,H) occurs on flips $2i-1$ and $2i$ for some i with (T,H) or (H,T) not occurring on any previous pair of flips (so P(E) = outcome probability for heads)

all possibilities
F = outcome of rolls 1 and 2 is (T,H)
G = " " " " " " (H,T)
H = " " " " " " is the same HH or TT

$$P(E) = P(E|F)P(F) + P(E|G)P(G) + P(E|H)P(H)$$
$$(1)(1-p)(p) + (0)p(1-p) + P(E)\cdot\left(p^2 + (1-p)^2\right)$$

* try simulating this with a die where you interpret $H = 6$, $T = 1,2,3,4,5$ so $p(H) = \frac{1}{6} = p$, $p(T) = \frac{5}{6} = 1-p$

Solve for $P(E)$

$$P(E) = (1-p)(p) + P(E)\left[p^2 + 1 - 2p + p^2\right]$$

$$P(E)\left[\cancel{1} - (2p^2 + \cancel{1} - 2p)\right] = (1-p)p$$

$$P(E) = \frac{(1-p)p}{2p - 2p^2} = \frac{(1-p)p}{2p(1-p)} = 1/2$$

so $\boxed{P(E) = 1/2}$ ← probability of heads as the outcome.

(i.e. this procedure is equally-likely to "predict" heads or tails.

b) Could we use a simpler procedure that continues to flip the coin until the last two flips are different and then lets the result be the outcome of the final flip?

$\cancel{P(H_k) = (1-p)^{k-1} p}$

$\cancel{P(H) = \sum_{k=1}^{\infty} (1-p)^{k-1} p = \frac{p}{1-(1-p)}}$

Note here that the ~~position~~ final flip outcome will be the opposite of the first flip. That is, ways to get heads

(TH), (TTH), $(TTTH)$, ... $P(H_{last}) = P(T_{first}) = 1-p$

similarly

$P(T_{last}) = P(H_{first}) = p$

and we know p is not necessarily $= 1/2$.

We could write this as a conditional probability

E = event that H follows T without TH or HT occurring previously

$\{$ ~~F = first flip = T~~

G = first flip = H

$$P(E) = P(E|F)\,P(F) + P(E|G)\,P(G)$$ condition on first flip

but $P(E|F) = 1$

$P(E|G) = 0$

and $P(F) = (1-p)$

~~$P(H) = p$~~

So

$$P(E) = 1\cdot(1-p) + 0\cdot(p) = 1-p$$ as previously discovered.