Ch. 3 Conditional Probability and Independence

EX - toss two dice

die #2

die #1	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- Probabilities of sums of the two dice

$\boxed{\text{36 outcomes}}$

$$P(\text{sum}=2) = \frac{1}{36} = P(\text{sum}=12)$$
$$P(\text{sum}=3) = \frac{2}{36} = P(\text{sum}=11)$$
$$P(\text{sum}=4) = \frac{3}{36} = P(\text{sum}=10)$$
$$P(\text{sum}=5) = \frac{4}{36} = P(\text{sum}=9)$$
$$P(\text{sum}=6) = \frac{5}{36} = P(\text{sum}=8)$$
$$P(\text{sum}=7) = \frac{6}{36}$$

note: showing the "first" die or the "second" die is equivalent and leads to the same 6 possible outcomes shown below

- Suppose I roll two dice, show you that one of the dice shows a 1, and ask you to give me the probabilities of these various sums (e.g. before showing you the second die)

6 Possible outcomes $\boxed{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)}$
(given that first die is a 1)

So given this piece of information

$P(\text{sum}=2) = \frac{1}{6}$ $\quad P(\text{sum}=5) = \frac{1}{6}$ $\quad P(\text{sum}=8) = 0$

$P(\text{sum}=3) = \frac{1}{6}$ $\quad P(\text{sum}=6) = \frac{1}{6}$ $\quad P(\text{sum}=9) = 0$

$P(\text{sum}=4) = \frac{1}{6}$ $\quad P(\text{sum}=7) = \frac{1}{6}$ $\quad P(\text{sum}=10) = 0$

$P(\text{sum}=11) = 0$

$P(\text{sum}=12) = 0$

These new probabilities are called <u>conditional probabilities</u>.

Def: Suppose E and F represent events. The conditional probability that E occurs given that F has occurred is denoted by

$$P(E|F)$$

Further, if $P(F) > 0$ then

$$P(E|F) = \frac{P(EF)}{P(F)}$$

$P(EF)$ ← probability of both E and F

$P(F)$ ← probability of event F

- Note: the denominator represents that our probabilities are measured relative to a new, reduced, sample space

- In our previous example denote

F = event that first die # is a 1

so $P(F) = \frac{1}{6}$

E = event that the sum of the two dice is x

(where $x = 2,3,4,5,6,7,8,9,10,11,12$)

x	2	3	4	5	6	7	8	9	10	11	12
$P(EF)$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0	0	0	0	0

so

x	2	3	4	5	6	7	8	9	10	11	12
$P(E\|F) = \frac{P(EF)}{P(F)}$	$\frac{1/36}{1/6} = \frac{1}{6}$	$\frac{1}{6} = \frac{1/36}{1/6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	0	0

(50.5)

one idea for computing conditional probabilities is that with the given information you can reduce the sample space so that

$$P = \frac{(\text{\# of ways an event can happen given info})}{(\text{total \# of outcomes of the restricted space – i.e. given info rules things out})}$$

conditional probability - determined by working with a reduced sample space

Example 2c (p.58)

Card Game of Bridge : (4 players)

52 cards dealt out - 13 to each player called N, S, E, W.

Suppose we are told that N and S have a total of 8 spades.

What is the probability that E holds {3 of the / 4 of the " / 5 of the "} remaining 5 spades?

Sol : work with reduced sample space

E and W hold 26 of the ~~remaining~~ cards.

→ conditional probability that East holds ~~______~~ 3 of remaining 5 spades $= \dfrac{\underbrace{\binom{5}{3}}_{\text{choose 3 of 5 spades}}\underbrace{\binom{21}{10}}_{\text{choose 10 other non-spades}}}{\underbrace{\binom{26}{13}}_{\text{reduced space}}} \approx \underline{\underline{0.339}}$

$\binom{26}{13}$ ↖ total # of outcomes for East's hand

$$= \frac{(10)(352{,}716)}{10{,}400{,}600}$$

→ conditional probability that East holds 4 of the remaining 5 spades $= \dfrac{\binom{5}{4}\binom{21}{9}}{\binom{26}{13}}$

$$= \frac{5 \cdot 293{,}930}{10{,}400{,}600} \approx \underline{\underline{0.141}}$$

→ conditional probability that East holds 5 of the remaining 5 spades $= \dfrac{\binom{5}{5}\binom{21}{8}}{\binom{26}{13}} = \dfrac{1 \cdot 203490}{10400600}$

$$= \underline{\underline{0.01956}}$$

Note:

- The conditional probability $P(E|F) = \dfrac{P(EF)}{P(F)}$

can also be written as a formula for the probability of both E and F occurring. That is,

$$\boxed{P(EF) = P(E|F)\cdot P(F)} = P(F|E)\cdot P(E)$$

more generally the "multiplication rule" for probability of the intersection of an arbitrary number of events is

~~P(EFG) = P(E) P~~

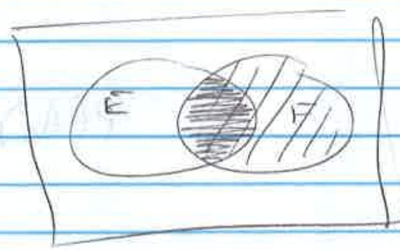
$$\boxed{\begin{aligned} P(E_1E_2\ldots E_n) = \underbrace{P(E_1)\cdot P(E_2|E_1)}_{P(E_1E_2)}\cdot P(E_3|E_1E_2)\cdots \\ \cdot P(E_n|E_1E_2\cdots E_{n-1}) \end{aligned}}$$

e.g. $P(E_1E_2E_3) = \underbrace{P(E_1)\cdot P(E_2|E_1)}_{P(E_1E_2)}\cdot P(E_3|E_1E_2)$

$= P(E_3)\cdot P(E_3|E_2)\cdot P(E_1|E_2E_3)$ } different varieties.

$\vdots$

- $P(E/F) = \dfrac{P(EF)}{P(F)} = \dfrac{\#\text{ in } EF}{\#\text{ in } F}$



e.g. start by considering

$$P(E_1E_2\cdots E_n) = P(E_n|E_1E_2\cdots E_{n-1})\cdot P(E_1E_2\cdots E_{n-1})$$

$$\boxed{P(E_{n-1}|E_1E_2\cdots E_{n-2})\cdot \underbrace{P(E_1\cdots E_{n-2})}}$$

etc.

Problem 3.11 (p. 98)

Two cards are randomly chosen without replacement from a standard deck of 52 cards.

Let B = event that both cards are aces.

Let A_s = event that the Ace of Spades is chosen. (is one of the two cards)

Let A = event that at least one ace is chosen.

a) Find $P(B|A_s)$ i.e. the probability that both cards are aces given that one of the cards is the Ace of Spades.

$$= \frac{P(BA_s)}{P(A_s)}$$

First,

- $P(A_s) = \frac{1 \cdot 51}{\binom{52}{2}}$ $\left(\binom{1}{1}\binom{51}{1}\right)$ ← 51 hands out of $\binom{52}{2}$ hands have A_s

$= \frac{2}{52}$

- $P(BA_s) = \frac{1 \cdot 3}{\binom{52}{2}}$ $\left(\binom{1}{1}\binom{3}{1}\right)$ ← 3 hands out of $\binom{52}{2}$ hands have both aces including A_s.

so

$$P(B|A_s) = \frac{3/\binom{52}{2}}{51/\binom{52}{2}} = \frac{3}{51} = \boxed{\frac{1}{17} = P(B|A_s)}$$

$\approx 0.0588\ldots$

b) $P(B|A) = \frac{P(BA)}{P(A)}$

- $P(A) = \frac{\binom{4}{1}\binom{48}{1}}{\binom{52}{2}} + \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{4 \cdot 48 + 6}{\binom{52}{2}} = \frac{198}{\binom{52}{2}}$

one ace   two aces

Note that $P(A)$ can also be computed as

$$P(A) = \frac{\binom{52}{2} - \binom{48}{2}}{\binom{52}{2}}$$ ← ways to get no aces

$$= \frac{\frac{52\cdot 51}{2} - \frac{48\cdot 47}{2}}{\binom{52}{2}} = \frac{1326 - 1128}{1326} = \frac{198}{1326}$$

$$= \frac{33}{221}$$

$$\approx 0.14932$$

Also

$$P(BA) = P(B) \quad \text{since} \quad B \subset A \text{ and } BA = B$$

↑ intersection

$$= \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{6}{\binom{52}{2}}$$

Then

$$P(B|A) = \frac{\frac{6}{\binom{52}{2}}}{\frac{198}{\binom{52}{2}}} = \frac{6}{198} = \boxed{\frac{1}{33} = P(B|A)}$$

$\approx 0.03030\ldots$

= the probability that both cards are aces given that ~~one of them~~ at least one of the cards is an ace.

Example 2d (p. 58)

Celine needs to decide on one course to take ~~this~~ next semester: French or Chemistry.

- She estimates her probability of getting an A in French is $1/2$ and of getting an A in chemistry is $2/3$.
- Celine likes probability too and so will let her decision be based on the flip of a fair coin.

Q: What is the probability that she gets an A in chemistry?

A: Note: the event of getting an A in chemistry means that she had to take chemistry and that she had to get an A in it.

Let C = event she takes chemistry

A = event she gets an A in whatever course she takes.

$C \cap A$ = so CA = event she gets A in chemistry

$P(CA)$ = probability she takes Chem. and gets an A in it.

$= P(A|C)P(C)$

$= \frac{2}{3} \cdot \frac{1}{2} = \boxed{\frac{1}{3} = P(CA)}$

Some other variations / computations:

Let F = event she takes French

Note: A^c = event she does not get an A in whatever course she takes.

- $P(FA) = P(A|F)P(F)$ ← probability of A in French

$$= \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4} = P(FA)}$$

- $P(FA^c) = P(A^c|F)P(F)$

$$= \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4} = P(FA^c)}$$

- $P(CA^c) = P(A^c|C) \cdot P(C) =$

$$\frac{1}{3} \cdot \frac{1}{2} = \boxed{\frac{1}{6} = P(CA^c)}$$

Notes
- $P(CA) + P(CA^c) + P(FA) + P(FA^c) = 1$
- $C = CA \cup CA^c$ with CA and CA^c mutually exclusive events

  $P(C) = P(CA) + P(CA^c)$
- $F = FA \cup FA^c$ with FA and FA^c mutually exclusive events

  $P(F) = P(FA) + P(FA^c)$
- $P(C) + P(F) = 1$
- $C \cup F = S$ = entire class sample space

Based on these computations, what is the probability that Celine ~~got~~ will get an A ~~this~~ next semester?

$$P(A) = P(CA) + P(FA)$$ = sum of probabilities of A in Chem. or A in French

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

We can think of this another way, start with A, use axiom 3, ...

$A = CA \cup FA$ where CA and FA are mutually exclusive events

So

$P(A) = P(CA) + P(FA)$ by Axiom 3

$$= P(A|C)P(C) + P(A|F)P(F)$$

$$= \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{2}{6} + \frac{1}{4} = \frac{7}{12}$$

Comment: This is the idea of Bayes's Formula, which we'll introduce shortly... (note here $C^c = F$, $F^c = C$)

See notes p. 67

EX

Celine has to choose 1 of 6 classes, C_i, $i = 1,2,3,4,5,6$

Her Probability of getting an A in C_1 is p_1
" " " " C_2 is p_2
$\vdots$
C_6 " p_6

She'll roll a fair die to decide which class to take?

Q: What is the probability that she gets an A in C_i.

$$P(C_i A) = P(A|C_i) \cdot P(C_i)$$
$$= p_i \cdot \frac{1}{6}$$

Q: what is the probability that she gets an A ~~this~~ next semester?

$$A = \bigcup_{i=1}^{6} C_i A \quad \text{where } C_iA \text{ and } C_jA \text{ mutually exclusive for } i \neq j$$

$$P(A) = \sum_{i=1}^{6} P(C_i A)$$

$$= \sum_{i=1}^{6} P(A|C_i) \cdot P(C_i) = \boxed{\sum_{i=1}^{6} p_i \cdot \frac{1}{6} = P(A)}$$