Monday March 3, 2025

Bayes's Summary

Recall <u>3.3</u> Bayes's Formula

E, F are events

$$P(F|E) = \frac{P(EF)}{P(E)} \qquad EF = E \cap F$$

$$P(E|F) = \frac{P(EF)}{P(F)}$$

$E = EF \cup EF^c$ & mutually exclusive

$$P(E) = P(EF) + P(EF^c)$$

$$= P(E|F)P(F) + P(E|F^c)P(F^c)$$

so

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)}$$

More generally, for $S = \bigcup_{i=1}^{N} F_i$ $\quad$ F_i mutually exclusive events

$$P(F_j|E) = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^{N} P(E|F_i)P(F_i)}$$

$E = EF_1 \cup EF_2 \cup \ldots \cup EF_n$

$\boxed{\text{Example 2g (p. 61)}}$

An ordinary deck of cards is randomly divided into 4 piles of 13 cards each.

What is the probability that each pile has exactly one ace?

Method 1

Recall, the number of ways to distribute 52 cards into ~~3~~ 4 piles of 13 is

$$\frac{52!}{(13!)(13!)(13!)(13!)}$$

If we want to count ~~separately~~ the number of ways for a particular distribution we need

$$\underbrace{\frac{4!}{1!\,1!\,1!\,1!}}_{\substack{\text{ways to distribute}\\ \text{4 aces into}\\ \text{4 piles}}} \cdot \underbrace{\frac{48!}{12!\,12!\,12!\,12!}}_{\substack{\text{distribute the other}\\ \text{48 cards into 4 piles}\\ \text{of 12.}}}$$

multiply since any ace can go with any pile of 12.

So $P\left(\begin{matrix}\text{exactly one ace}\\ \text{per pile}\end{matrix}\right) = \dfrac{4!\,\dfrac{48!}{(12!)^4}}{\dfrac{52!}{(13!)^4}} = \dfrac{4!\,48!}{52!}(13)^4 = \dfrac{4!\,(13)^4}{52\cdot 51\cdot 50\cdot 49}$

$\boxed{\approx 0.105}$

9.20-2018 (62)

Note that this calculation is equivalent to

$$\boxed{\frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} \cdot \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}} \cdot \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}} \cdot \frac{\binom{1}{1}\binom{12}{12}}{\binom{13}{13}}}$$

= probability of each hand containing one of the four aces and 12 other cards.

<u>Method 2</u> – use conditional probabilities (not Bayes's yet... just conditional prob.)

Let E_1 = event that $\underbrace{\text{ace of spades}}_{A_1}$ (~~Ace~~ Ace #1) is in any one of the piles

E_2 = event that ace of spades + ace of hearts are in different piles (~~e.g.~~ A_1 and A_2 are in ~~different piles~~)

E_3 = event that ace of spades, ace of hearts, ace of diamonds are in different piles (e.g. A_1, A_2, A_3 are in different piles)

E_4 = event that all four aces are in different piles.

Use multiplication rule

$$P(E_4) = P(E_1E_2E_3E_4) = P(E_4|E_1E_2E_3)\,P(E_1E_2E_3)$$

$$= P(E_4|E_1E_2E_3)\,\boxed{P(E_3|E_1E_2)\,P(E_1E_2)}$$

$$= P(E_4|E_1E_2E_3)\,P(E_3|E_1E_2)\,\boxed{P(E_2|E_1)\,P(E_1)}$$

Let's compute the probabilities in this expression

$$P(E_1) = 1$$

12 spots in pile 1

51 cards to choose from...

$P(E_2|E_1) = 1 -$ probability that A_2 is in pile with A_1

1+2 in separate piles

$$= 1 - \frac{12}{51} = \frac{39}{51} \quad (> 3/4)$$

$P(E_3|E_1E_2) = 1 -$ probability that A_3 is in one of first two piles

24 spots in two piles

$$= 1 - \frac{24}{50}$$

50 possible cards

2 aces already in place

$$= \frac{26}{50} \quad (> 1/2)$$

$P(E_4|E_1E_2E_3) = 1 -$ probability that A_4 is in one of first 3 piles

$$= 1 - \frac{36}{49} = \frac{13}{49} \quad (> 1/4)$$

So

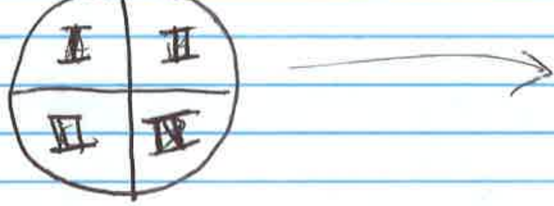
$$P(E_1E_2E_3E_4) = \frac{13}{49} \cdot \frac{26}{50} \cdot \frac{39}{51} \cdot 1 \approx \boxed{0.105}$$

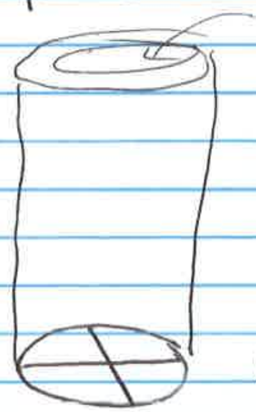
*

Intuition?

EX

Consider a related problem: Drop 4 coins down a ~~[illegible]~~ deep wishing well with the following grid pattern on the bottom





Assume the probability that a coin lands in Quad i is $1/4$ (each area is $1/4$ total area)

What is the probability that the four coins land in different quadrants (assume ~~[illegible]~~ each coin drop is an independent event).

$P(A_i)$ = probability that ~~[illegible]~~ coin i lands in an "unoccupied" quadrant (given that $i-1$ quadrants are occupied)

First coin: $P(A_1) = 1$

2nd ~~coin~~: $P(A_2) = \frac{3}{4}$ (one occupied)

$P(A_3) = 1/2$ (two occupied)

$P(A_4) = 1/4$

$$P(A_1)\cdot P(A_2) P(A_3)\cdot P(A_4) = \frac{3}{4\cdot 4\cdot 2} = \frac{3}{32}$$

$\boxed{= 0.09375}$

ie this is also

$$P = \frac{4!}{4^4}$$

← # of ways for the four coins to be separate

← total number of 4 coin position outcomes

* Why is this less likely than the four aces in 4 piles problem?

intuition?

EX

Consider another related problem.

Consider a nonstandard deck of cards with $4N$ cards including four aces. None of the other cards are aces. Divide this deck into 4 piles of N cards. What is the probability that ~~three~~ each pile has an ace?

Let's use the same definitions of E_1, E_2, E_3, E_4 (see notes, p. 62).

$$P(E_1E_2E_3E_4) = P(E_4|E_1E_2E_3) \cdot P(E_3|E_2E_1)P(E_2|E_1)P(E_1)$$

$$P(E_1) = 1$$

$$P(E_2|E_1) = 1 - \underbrace{\frac{N-1}{4N-1}}_{\text{probability that } A_2 \text{ is in pile 1 given } A_1 \text{ is in pile 1.}}$$

$N-1$: # of spots left in pile 1

$4N-1$: # of cards that could go in pile

$$P(E_3|E_1E_2) = 1 - \frac{2(N-1)}{4N-2} = 1 - \text{prob. that } A_3 \text{ is in piles 1 or 2 given } A_1 \text{ in pile 1, } A_2 \text{ in pile 2}$$

$$P(E_4|E_1E_2E_3) = 1 - \frac{3(N-1)}{4N-3} = 1 - \text{prob. } A_4 \text{ is in piles 1, 2, 3 given } A_1 \text{ in pile 1, } A_2 \text{ in pile 2, } A_3 \text{ in pile 3}$$

So

$$P(E_1E_2E_3E_4) = \left(1 - \frac{3(N-1)}{4N-3}\right)\left(1 - \frac{2(N-1)}{4N-2}\right)\left(1 - \frac{N-1}{4N-1}\right) \cdot 1$$

$$= \frac{N}{(4N-3)} \cdot \frac{2N}{(4N-2)} \cdot \frac{3N}{(4N-1)}$$

	N	P(all aces separate)
(4 cards)	1	1
(8 cards)	2	$\frac{2}{5} \cdot \frac{4}{6} \cdot \frac{6}{7} = \frac{8}{35} \approx \underline{0.229}$
(12 cards)	3	$\frac{3}{9} \cdot \frac{6}{10} \cdot \frac{9}{11} = \frac{18}{110} = \frac{9}{55} \approx \underline{0.164}$
⋮		
(52 cards)	13	$\frac{13}{49} \cdot \frac{26}{50} \cdot \frac{39}{51} \approx 0.105498...$
(104 cards)	26	$\frac{26}{101} \cdot \frac{52}{102} \cdot \frac{78}{103} \approx 0.09938$
↓ ∞	$N \to \infty$	↓ $\frac{1}{4} \cdot \frac{2}{4} \cdot \frac{3}{4} = \frac{3}{32} = 0.09375$

using Multinomial...

$$P = \frac{4! \dfrac{(4N-4)!}{((N-1)!)^4}}{\dfrac{(4N)!}{(N!)^4}} \longleftarrow \frac{\binom{\text{choose}}{\text{aces}}\binom{\text{choose remaining}}{\text{cards}}}{\text{total \# of ways to divide } 4N \text{ cards into 4 piles of } N \text{ cards}}$$

(66.2)

$$P = 4! \frac{(4N-4)!}{(4N)!} \left(\frac{N!}{(N-1)!}\right)^4$$

$$= 4! \left[\frac{1}{4N \cdot (4N-1) \cdot (4N-2) \cdot (4N-3)}\right] \left[\frac{N}{1}\right]^4$$

$$= \frac{\cancel{4} \cdot 3 \cdot 2 \cdot 1 \cdot N^{\cancel{4}\,3}}{\cancel{4N} \cdot (4N-1)(4N-2)(4N-3)}$$

$$\boxed{P = \frac{3N}{4N-1} \cdot \frac{2N}{4N-2} \cdot \frac{N}{4N-3}}$$

$\leftarrow$ Matches result on p. (66)