Sets and quantifiers

Contents

1 Introduction to set notation

Informal Definition. A *set* is a collection of objects.

Conventions.

- Sets are frequently denoted by uppercase letters (e.g. A, B, C).
- If x is in A, then we say that x is an *element* of A or that A *contains* x, and we write $x \in A$.
- Otherwise, we write $x \notin A$.
- If the elements of A are precisely $a_1, \ldots, a_n$, then we write $A = \{a_1, \ldots, a_n\}$.

Examples.

i. natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$[1]

ii. integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

iii. rational numbers $\mathbb{Q}$

iv. real numbers $\mathbb{R}$

v. complex numbers $\mathbb{C}$

Convention. If the elements of A are precisely those of B that satisfy a condition P, then we write

$$A = \{x \in B \,|\, x \text{ satisfies the condition } P\}.$$

Examples.

i. $\mathbb{N} = \{n \in \mathbb{Z} \,|\, n \geq 0\}$

ii. $\mathbb{Q} = \left\{ \frac{n}{m} \,\middle|\, n, m \in \mathbb{Z}, m \neq 0 \right\}$

Definition. The *empty set* $\varnothing$ is the set that contains no elements.

That is, $\varnothing = \{\}$.

[1] There is an alternative convention that $\mathbb{N} = \{1, 2, 3, \ldots\}$.

2 Quantifiers

Informal Definition. If there is an element $x \in A$ that satisfies the condition P, then we write

$$\exists\, x \in A : x \text{ satisfies the condition } P.$$

The symbol $\exists$ is called the *existential quantifier.*

Convention. There are a few ways this can be read. Examples include,

- "There exists an x in A such that x satisfies P."
- "There is an x in A such that..."
- "There is an x in A that satisfies the condition P."

Examples. The following statements are true:

i. $\exists\, n \in \mathbb{Z} : n$ is even

ii. $\exists\, x \in \mathbb{R} : x > 3$

iii. $\exists\, n \in \mathbb{N} : n > 3$ and n is even

The following are false:

iv. $\exists\, n \in \mathbb{N} : n < 0$

v. $\exists\, n \in \mathbb{Z} : n > 3$ and $n < 1$

vi. $\exists\, x \in \mathbb{R} : x^2 = -1$

Informal Definition. If every $x \in A$ satisfies the condition P, then we write

$$\forall\, x \in A : x \text{ satisfies the condition } P.$$

The symbol $\forall$ is called the *universal quantifier.*

Convention. This may be read as, for example,

- "For all/every/any x in A, x satisfies the condition P"
- "All x in A satisfy..."
- "Every/Any x in A satisfies..."

Examples. True statements:

i. $\forall n \in \mathbb{N} : n \geq 0$

ii. $\forall k \in \mathbb{N} : k \in \mathbb{Z}$

iii. $\forall x \in \mathbb{R} : x^2 \geq 0$

False statements:

iv. $\forall x \in \mathbb{R} : x \in \mathbb{N}$

v. $\forall m \in \mathbb{Z} : m$ is even

vi. $\forall n \in \mathbb{N} : \sqrt{n} \in \mathbb{N}$

Remark. Note that

$$\text{If } x \in A, \text{ then } P(x)$$

may also be formalized as

$$\forall x \in A : P(x).$$

Examples. Quantifiers can be strung together:

i. $\forall m \in \mathbb{Z} : \exists n \in \mathbb{N} : m < n$

ii. $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x - y = 2$

iii. $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : \forall z \in \mathbb{R} : (x - y)z = 0$

Convention. When introducing new variables of the same type, it is convenient to do so alphabetically (e.g. a, b, c, or x, y, z).

3 Proofs with quantifiers

To prove a claim of the form

$$\exists x \in A : P(x),$$

we have simply to exhibit an $x \in A$ that satisfies the condition P.

Consider the following example:

Claim. *There is a $k \in \mathbb{Z}$ such that $k^2 = k$.*

Proof. We have $0 \in \mathbb{Z}$ and $0^2 = 0$. □

Remark. We could have just as well chosen $k = 1$. Only a single $k \in \mathbb{Z}$ satisfying $k^2 = k$ is required to prove the claim.

Convention. A proof should consist of grammatically correct English sentences. It is considered undesirable to begin a sentence with a mathematical symbol. To adhere to this rule, it is often convenient to preface an otherwise-bare mathematical formula with a brief phrase such as

- "We have..."
- "Observe that..."
- "Note that..."

To prove a claim of the form

$$\forall x \in A : P(x),$$

there are two steps:

1. Introduce an arbitrary $x \in A$.
2. Show that x satisfies P.

The first step is accomplished by means of a statement such as

- "Let $x \in A$."
- "Fix $x \in A$."
- "Suppose that $x \in A$."

Claim. *If $q \in \mathbb{Q}$, then $\frac{q}{2} \in \mathbb{Q}$.*

Proof. Fix $q \in \mathbb{Q}$. By the definition of $\mathbb{Q}$, there are $m, n \in \mathbb{Z}$ with $n \neq 0$ such that $q = \frac{m}{n}$. Thus,

$$\frac{q}{2} = \frac{m}{2n} \in \mathbb{Q}.$$

$\square$

Convention. Common prefaces to a conclusion include

- "Thus,"
- "Hence,"
- "Therefore,"
- "It follows that,"

Claim. *For every $m \in \mathbb{Z}$, there is an $n \in \mathbb{Z}$ with $m < n$.*

Proof. Fix $m \in \mathbb{Z}$ and let $n = m + 1$. It follows that $m < n$. $\square$

Convention. The following phrases have similar meanings:

- "such that"
- "with"
- "subject to the condition that"
- "satisfying"
- "for which"

4 Negation

Informal Definition. The *negation* of a statement S is the statement that *it is not the case that* S, written $\neg S$.

Convention. The symbol $\neg$ is read "not".

Examples.

i. The negation of
$$\exists\, x \in \mathbb{R} : x^2 = -1$$
is
$$\neg\exists\, x \in \mathbb{R} : x^2 = -1,$$
which states that *it is not the case that* there is a real number that squares to -1.

ii. The negation of
$$\forall n \in \mathbb{Z} : n \geq 0$$
is
$$\neg\forall n \in \mathbb{Z} : n \geq 0,$$
which asserts that *it is not the case that* every integer is positive.

Informal Definition. To *disprove* a statement S is to prove that S is false. This is equivalent to proving $\neg S$.

It is useful to note that

$$\neg\forall x \in A : P(x) \qquad \text{is equivalent to} \qquad \exists x \in A : \neg P(x)$$

and

$$\neg\exists x \in A : P(x) \qquad \text{is equivalent to} \qquad \forall x \in A : \neg P(x).$$

Examples. i. The negation of

$$\exists x \in \mathbb{R} : \forall y \in \mathbb{R} : x = y$$

is

$$\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x \neq y$$

ii. The negation of

$$\forall m \in \mathbb{Z} : \exists n \in \mathbb{N} : m + n < 0$$

is

$$\exists m \in \mathbb{Z} : \forall n \in \mathbb{N} : m + n \geq 0$$

Proof of i. Fix $x \in \mathbb{R}$. If $y = x + 1$, then $x \neq y$. □

Proof of ii. Put $m = 0$ and let $n \in \mathbb{N}$. Since $n \geq 0$, it follows that $m + n \geq 0$. □