Induction

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Math 300

1. Weak induction

2. Strong induction

3. Proof or spoof?

Section 1

Weak induction

(Weak) induction

Suppose that $\{P(n)\}$ is a collection of assertions indexed by $n \in \mathbb{N}$.

Goal: For all $n \geq n_0$, prove $P(n)$.

In its simplest manifestation, the approach of *(weak) induction* is twofold:

1. Establish the *base case* $P(n_0)$,
2. Prove the *inductive hypothesis*:

$$\forall n \geq n_0 : P(n) \implies P(n+1).$$

Remark

Sometimes it is necessary to prove multiple base cases.

Proposition

For all $n \geq 1$, we have $1 + \cdots + n = \frac{n(n+1)}{2}$.

Proof.

First observe that the formula holds for $n = 0$.
Now suppose that $n \in \mathbb{N}$ satisfies $0 + \cdots + n = \frac{n(n+1)}{2}$, and observe that

$$\begin{aligned} 0 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

$\square$

Proposition

If $(a_i)_i \subseteq \mathbb{R}$ is the sequence with $a_0 = 1$ and $a_i = 2a_{i-1}$ for all $i \geq 1$, then $a_n = 2^n$ for all n.

Proof.

As $a_0 = 2^0$, the claim is true for $n = 0$.
Now let $n \geq 0$ and suppose that $a_n = 2^n$. We have

$$\begin{aligned} a_{n+1} &= 2a_n \\ &= 2 \cdot 2^n \\ &= 2^{n+1}. \end{aligned}$$

$\square$

Section 2

Strong induction

Strong induction

Suppose again that $\{P(n)\}_{n\in\mathbb{N}}$ is a collection of assertions.

Goal: For each $n \geq n_0$, prove $P(n)$.

The technique of *strong induction* is:

1. Establish the *base case* $P(n_0)$,
2. Prove the *inductive hypothesis*:

$$\forall n \geq n_0 : (\forall m \leq n : P(m)) \implies P(n+1).$$

Proposition

Every integer $n \geq 2$ is the product of prime numbers, which are unique up to rearrangement.

Proof.

First observe that $n = 2$ is the product of primes.
Now let $n > 2$ and suppose that every integer $m < n$ is the product of primes $m = p_1 \cdots p_k$. If n is prime then we are done. Thus suppose that $n = m \cdot m'$. By assumption,

$$m = p_1 \cdots p_k \qquad \text{and} \qquad m' = p'_1 \cdots p'_{k'}$$

for some primes $p_1, \ldots, p_k$ and $p'_1, \ldots, p'_k$. It follows that

$$n = mm' = p_1 \cdots p_k p'_1 \cdots p'_{k'}$$

is a product of primes. □

Section 3

Proof or spoof?

Let $\{\ell_i\}_i$ be a finite collection of lines in $\mathbb{R}^2$, and let $\mathcal{C}$ be the resulting collection of regions in $\mathbb{R}^2$.

Claim

The members of $\mathcal{C}$ can be colored red and blue in such a way that no two adjacent regions have the same color.

Proof?

If $\{\ell_i\}_i$ is empty then $\mathcal{C} = \{\mathbb{R}^2\}$. Coloring $\mathbb{R}^2$ either red or blue satisfies the claim.
Fix $n \in \mathbb{N}$, suppose the claim is true for collections of n lines, and let $\{\ell_1, \ldots, \ell_{n+1}\}$. By hypothesis, the regions $\mathcal{C}'$ bounded by $\{\ell_1, \ldots, \ell_n\}$ can be colored so that no two adjacent regions have the same color. The line ℓ_{n+1} divides $\mathbb{R}^2$ into two half-planes. On one side, invert the colors of $\mathcal{C}'$. On the other side, do nothing. The resulting coloring of $\mathcal{C}$ satisfies the claim. $\square$

Proof!

Claim

Everyone has the same birthday.

We will show that every set P of people satisfies the property that, for all $p, q \in P$, p and q have the same birthday. We will proceed by induction on the size of P.

Proof?

If $P = \{p\}$, then it trivially follows that p and q have the same birthday for all $p, q \in P$.
Now suppose that $|P| \geq 2$ and that the claim holds for all sets of size $|P| - 1$. Choose nonempty subsets $X, Y \subseteq P$ with $|X| = |Y| = |P| - 1$, so that $X \cup Y = P$. If $p \in X \cap Y$ then every $q \in X \cup Y$ has the same birthday as p. $\square$

Spoof!

Claim

Every chocolate bar with a total of n squares takes precisely $n-1$ breaks to separate it into single squares.

Proof?

If $n=1$, then $n-1=0$ breaks are needed.
Suppose that $n \geq 2$ and that the claim holds for all chocolate bars of size $m < n$. It takes a single break to separate the bar into two bars of size $m, m' < n$. For each resulting bar, it takes precisely $m-1$ and $m'-1$ breaks to separate them into single squares, respectively. In all, it takes precisely

$$1 + (m-1) + (m'-1) = n-1$$

breaks to separate a size n bar into single squares. $\square$

Proof!

Claim

For all $n \in \mathbb{N}$, we have $2^n = 1$.

Proof?

The identity $2^0 = 1$ establishes the claim for $n = 0$.
Suppose that $n \geq 0$ and that $2^m = 1$ for all $m \leq n$. We have

$$2^{n+1} = \frac{2^n \cdot 2^n}{2^{n-1}} = \frac{1 \cdot 1}{1} = 1.$$

$\square$

Spoof!

The *Fibonacci sequence* is determined by $a_0 = 0$, $a_1 = 1$, and the recurrence relation $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$.

Claim

For all $n \geq 0$, $a_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$ *where* $\phi = \frac{1+\sqrt{5}}{2}$ *and* $\psi = \frac{1-\sqrt{5}}{2}$.

Proof?

The cases $n = 0$ and $n = 1$ are easily verified.
Fix $n \geq 1$ and suppose the claim is true for all $m \leq n$. We have

$$\begin{aligned} a_{n+1} &= a_n + a_{n-1} \\ &= \frac{\phi^n - \psi^n}{\sqrt{5}} + \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} \\ &= \frac{\phi^{n-1}(\phi+1) - \psi^{n-1}(\psi+1)}{\sqrt{5}} \\ &= \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}}. \end{aligned}$$

Proof! □

Claim

A chessboard with a single square removed can be completely covered with L-shaped tiles.

We will establish a stronger claim: namely, that a $2^n \times 2^n$ chessboard with $n \geq 1$ can be covered with L-shaped tiles.

Proof?

The claim is clearly true when $n = 1$.

Suppose that $n \geq 1$ and that the assertion is true for all $2^n \times 2^n$ chessboards. Given a $2^{n+1} \times 2^{n+1}$ chessboard with a single square removed, divide it into four $2^n \times 2^n$ boards. Cover the center of the $2^{n+1} \times 2^{n+1}$ board with an L-shaped tile so that each consitutent $2^n \times 2^n$ board has a single square either covered or removed. The result now follows by the inductive hypothesis. □

Proof!

Claim

We have $(x^n)' = 0$ *for all* $n \geq 0$.

Proof?

From $1' = 0$, the result obtains for $n = 0$.
Let $n \geq 0$ and assume that $(x^m)' = 0$ for all $m \leq n$. We have

$$\begin{aligned}(x^{n+1})' &= (x^n \cdot x^1)' \\ &= (x^n)' \cdot 1 + x^n \cdot (x^1)' \\ &= 0\end{aligned}$$

$\square$

Spoof!

Definition

A subset $S \subseteq \mathbb{R}$ is said to be *bounded above* when there is a $b \in \mathbb{R}$ with $b \geq s$ for all $s \in S$.

Claim

Every subset $S \subseteq \mathbb{R}$ is bounded above.

Proof?

If $S = \varnothing$, then every $b \in \mathbb{R}$ is an upper bound for S.
Let $n \geq 0$ and suppose that every subset $S' \subseteq \mathbb{R}$ is bounded. Let $S = \{s_1, \ldots, s_n, s_{n+1}\} \subseteq \mathbb{R}$. It follows by assumption that $S' = \{s_1, \ldots, s_n\}$ has an upper bound $b' \in \mathbb{R}$. It is readily seen that the greater of b' and s_{n+1} is an upper bound for S. $\square$

Spoof!