other notes

- $\binom{n}{0} = 1$ $\quad \frac{n!}{n!\,0!}$ $\quad$ note $0! = 1$
- $\binom{n}{r} = \binom{n}{n-r}$ symmetry
- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

see p(13)

1-27-2025

8-30-2018

Define "Binomial Coefficient" (also "$C(n,r)$")

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

(we say "n choose r")

which represents the number of combinations of n objects taken r at a time.

Note also symmetry

Note: The number of permutations of n objects taken r at a time is

$$\underbrace{\binom{n}{r}}_{\text{select the "players"}} \cdot \underbrace{r!}_{\text{now order the r players.}} = \frac{n!}{(n-r)!\,r!} \cdot r! = \frac{n!}{(n-r)!}$$

EX

- A club of 10 members wishes to choose a president, secretary, and tresurer. How many ways?

$$= \underset{(P)}{10} \cdot \underset{(S)}{9} \cdot \underset{(T)}{8} = 720$$

$\equiv \frac{10!}{(10-3)!} = \frac{10!}{7!}$ permut.

- A club of 10 members wishes to choose a 3 person "committee" (order on committee does not matter).

$$= \binom{10}{3} = \frac{10!}{7!\,3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = \frac{720}{6} = 120$$

EX

a) How many ways are there to choose a 3-student team from our class of 25 students

ans: $\binom{25}{3} = \frac{\underline{25} \cdot \underline{24} \cdot \underline{23}}{3!}$ ← number of permutations of a group of 3 students

(Binomial Coefficient)

$= \frac{25!}{22!} \cdot \frac{1}{3!}$

b) What is the probability that you will be selected if the students are selected randomly.

Thoughts ... • maybe $\frac{1}{25}$? ✗ no. This would be true for a team of 1

• note if it was a team of 25 students you would for sure be selected, so it somehow depends on team size.

• Maybe $\frac{3}{25}$? yes. But let's figure this out...

$P = \frac{\text{\# of teams you are on}}{\text{total \# of teams}}$

$= \frac{\binom{1}{1} \cdot \binom{24}{2}}{\binom{25}{3}}$

you must be chosen (from a group of 1)

any other 2 from 24

total

$$P = \frac{\binom{1}{1}\binom{24}{2}}{\binom{25}{3}} = \frac{\frac{24!}{22!\,2!}}{\frac{25!}{22!\,3!}} = \frac{3!}{2!}\,\frac{1}{25} = \frac{3}{25}$$

Does this make sense? What if the class size was N?

$$P = \frac{\binom{1}{1}\binom{N-1}{2}}{\binom{N}{3}} = \frac{\frac{(N-1)!}{(N-3)!\,2!}}{\frac{N!}{(N-3)!\,3!}} = \frac{3!}{2!}\,\frac{1}{N} = \frac{3}{N}$$

So, generally $\boxed{P = \frac{3}{N}}$

Maybe write out some cases

- $N=3$: $P=1$ ✓ this makes sense
- $N=4$ $P=3/4$ possible teams from A, B, C, D

ABC
ABD
ACD
BCD

person "A" (you) is in 3 of 4. ✓

- $N=5$ $P=3/5$

A, B, C, D, E

ABC	BCD
ABD	BCE
ABE	BDE
ACD	CDE
ACE	
ADE	

"A" is in $\frac{6}{10} = \frac{3}{5}$ ✓

One other thought...

$$P = 1 - P_{\text{not selected}}$$

$$= 1 - \underbrace{\left(\frac{\cancel{24}}{25} \cdot \frac{\cancel{23}}{\cancel{24}} \cdot \frac{22}{\cancel{23}}\right)}_{} = 1 - \frac{22}{25} = \frac{3}{25}$$

anybody but you

then anybody but you

then, anybody but you

Note: all 3 of these events must happen for you to not get picked (so multiplication)

Other thoughts...

we can also think cases

How can you get picked ~~to~~ end up on the team...

1st OR 2nd OR 3rd

$\frac{1}{25}$ then anything can happen

chance you get picked first

$\frac{24}{25}\;\frac{1}{24}$

Not picked first

Picked 2nd

$\frac{24}{25}\;\frac{23}{24}\;\frac{1}{23}$

not picked first

not picked second

picked 3rd

$$\text{so } P = \frac{1}{25} + \frac{24}{25}\frac{1}{24} + \frac{24}{25}\cdot\frac{23}{24}\cdot\frac{1}{23} = \frac{1}{25} + \frac{1}{25} + \frac{1}{25}$$

$$= \frac{3}{25}$$

Cards

"Standard" Deck of 52 cards

♣ 13 clubs (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)
♠ 13 spades (")
♥ 13 hearts (")
♦ 13 diamonds (")

EX 1

- How many different two-card hands are there? (order of the cards does not matter)

$$N_{TOT} = \binom{52}{2} = \frac{52 \cdot 51}{2} = 26 \cdot 51 = 1326$$

(Just like picking a calculus team of 3 from class of 25)

- How many two card hands are "pairs" (e.g. (A♣ and A♥), or (4♦ and 4♠))

$$N_{PAIR} = \underbrace{\binom{13}{1}}_{\text{"pick" the denomination}} \cdot \underbrace{\binom{4}{2}}_{\text{pick any 2 of the four suits}} = 13 \cdot \frac{4!}{2!2!} = 13 \cdot 3 \cdot 2 = 78$$

- Probability of getting a pair

$$P = \frac{N_{PAIR}}{N_{TOT}} = \frac{\binom{13}{1}\binom{4}{2}}{\binom{52}{2}} = \frac{1}{17} = \frac{13 \cdot 3 \cdot 2}{26 \cdot 51}$$

≈ 0.0588

Another way to do this calculation

$$N_{PAIR} = \underline{52} \cdot \underline{3} \Big/ 2$$

any card

only 3 other cards would provide a pair

divide by # of permutations of 2 cards since order of cards does not matter

$$= \frac{52 \cdot 3}{2} = 26 \cdot 3 = \underline{\underline{78}}$$

(then divide by N_{TOT} to get $P = 1/17$).

Is getting a pair more likely if I give you five cards?

EX 2

- How many different 5 card (Poker) hands are there?

pick 5 from 52

$$\boxed{N_{TOT} = \binom{52}{5}} = \frac{52!}{47!\,5!} = 2,598,960$$

- How many "one-pair" poker hands are there?

(e.g. K♦, K♠, 8♦, 7♥, 2♣)

$\underbrace{K♦, K♠}_{\text{"pair" (matches denomination)}}$, $\underbrace{8♦, 7♥, 2♣}_{\text{three other denominations all different and all different from the pair.}}$

count in steps...

$$N_{PAIR} = \underbrace{\binom{13}{1}}_{\text{pick the denomination of the pair}} \cdot \underbrace{\binom{4}{2}}_{\text{choose 2 of four suits}} \cdot \underbrace{\binom{12}{3}}_{\text{pick 3 other denominations from the remaining 12}} \cdot \underbrace{\binom{4}{1}\binom{4}{1}\binom{4}{1}}_{\text{independently I pick 1 of the suits for these 3 cards}}$$

$$\left\{ \text{or...} \quad \underbrace{13}_{\text{pick "K"}} \cdot \underbrace{\binom{4}{2}}_{\text{2 suits}} \frac{\underbrace{48}_{\text{any of 48 cards (not K)}} \cdot \underbrace{44}_{\text{(not K or \_)}} \cdot \underbrace{40}_{\text{not K or \_ or \_}}}{\underbrace{3!}_{\text{order doesn't matter}}} \right\}$$

8.7

$$N_{PAIR} = \binom{13}{}\left(\frac{4!}{2!2!}\right)\left(\frac{12!}{9!3!}\right) \cdot 4 \cdot 4 \cdot 4$$

$$= 13 \cdot \cancel{6} \cdot \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} \cdot 4 \cdot 4 \cdot 4$$

$$= 13 \cdot 12 \cdot 11 \cdot 10 \cdot 4^3$$

$$= \underline{1{,}098{,}240}$$

So

$$P_{PAIR} = \frac{N_{PAIR}}{N_{TOT}} = \frac{1{,}098{,}240}{2{,}598{,}960} \approx \boxed{0.423}$$

this is much higher than $P = \frac{1}{17} \approx 0.0588$

pair given only 2 cards

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EX

How many letter arrangements can be made from the letters of MISSISSIPPI?

(note: the I's are indistinguishable, the S's are as well, and P's too).

Sol 1

11 spaces _ _ _ _ _ _ _ _ _ _ _

- choose 4 spaces of the 11 for the I's: $\binom{11}{4}$
- choose 4 spaces of the remaining 7 for S's $\binom{7}{4}$
- choose 2 spaces of the remaining 3 for P's $\binom{3}{2}$
- choose 1 space of the remaing 1 for M $\binom{1}{1}$

↳ these are our 4 "experiments" so the total # of arrangements is (note the choices, or experiments, are independent).

$$\binom{11}{4}\cdot\binom{7}{4}\cdot\binom{3}{2}\cdot\binom{1}{1}$$

$$=\frac{11!}{\cancel{7!}\,4!}\cdot\frac{\cancel{7!}}{\cancel{3!}\,4!}\cdot\frac{\cancel{3!}}{\cancel{1!}\,2!}\cdot\frac{\cancel{1!}}{0!\,1!}=\frac{11!}{4!\,4!\,2!}$$

Sol 2 (put subscripts on the letters to distinguish them)

$M_1 I_1 S_1 S_2 I_2 S_3 S_4 I_3 P_1 P_2 I_4$

$$\frac{11\cdot\cancel{10}^{5}\cdot\cancel{9}^{3}\cdot\cancel{8}^{2}\cdots 5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1\cdot\cancel{2}\cdot 1}=\frac{10\cdot 5\cdot 3\cdot 2\cdot 7\cdot 6\cdot 5}{63{,}000}$$

Suppose there are x ways to arrange the unsubscripted letters (we want to find x)

For each of these x ways there are

- 4! ways to arrange the I's
- 4! " " " " S's
- 2! " " " " P's
- 1! " " " " M

The total # of ways to arrange the subscripted letters is 11! (since with the subscripts they are all distinct).

So $11! = x \cdot 4! \cdot 4! \cdot 2! \cdot 1!$

or $x = \dfrac{11!}{4!4!2!1!}$

see also general formula on page 4, section 1.3

EX

How many ~~ways~~ letter arrangements can be made from the letters of VIRGINIA ?

Sol 1

$$\binom{8}{3}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1}$$

I V R G N A

$= \dfrac{8!}{5!3!} 5! = \dfrac{8!}{3!}$

Sol 2

$$8! = x \cdot 3! \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$$

I V R G N A

$x = \dfrac{8!}{3!}$

P 16

11

Textbook Problem #20: A person has 8 friends, 5 of whom will be invited to a party.

a) How many choices if 2 friends are feuding and will not attend together?

Sol 1 - cases

- Bad friend 1 attends, 2 does not : $\binom{6}{4}$ i.e. choose 4 of remaining 6
- Bad friend 2 attends, 1 does not : $\binom{6}{4}$ .. .. ..
- Neither bad friend attends : $\binom{6}{5}$

These 3 scenarios represent disjoint or mutually exclusive ways to make 5 person party so

$$\text{Total} = \binom{6}{4} + \binom{6}{4} + \binom{6}{5} = 15 + 15 + 6 = 36$$

Sol 2

$\binom{8}{5}$ ways to make 5 person party but this includes cases where both bad friends attend, so we need to subtract something off. That is,

$\binom{8-2}{5-3} = \binom{6}{3}$ of these have both bad friends (i.e. choose 2 bad friends and 3 of the remaining 6). So $\binom{8}{5} - \binom{6}{3} = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} - \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = 56 - 20 = 36$

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b) How many ways if 2 of the friends will only attend together?

sol. 1

2 friends both go : $\binom{6}{3}$

2 friends do not go : $\binom{6}{5}$

$$\binom{6}{3} + \binom{6}{5} = \frac{6!}{3!\,3!} + \frac{6!}{1!\,5!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} + 6$$

$$= 20 + 6 = \boxed{26}$$

sol. 2

$$\binom{8}{5} - \binom{6}{4} - \binom{6}{4} = 56 - 15 - 15 = \boxed{26}$$

1 goes and other doesn't

2 goes and other doesnt