Ch. 6 Jointly Distributed Random Variables

Here we are interested in problems with more than one random variable:

- Roll 2 dice: X = value on die #1
  Y = value on die #2

- Randomly chosen person: X = age of person
  Y = height of person
  Z =
  ⋮

- Measurement 1 of covid assay
- Measurement 2 of covid assay
  ⋮

Now we revisit some previous concepts for two (or more) random variables.

6.1 Joint Distribution Functions

Discrete Case

Def. If X and Y are random variables, the joint cumulative probability distribution function (joint cdf) of X and Y is

$$F(a,b) = P\{X \le a, Y \le b\} \qquad -\infty < a,b < \infty$$

EX Flip a fair coin twice (Discrete X, Discrete Y)

possible outcomes (H,H), (H,T), (T,H), (T,T)

Let $X = \begin{cases} 1 & \text{if first coin is H} \\ 0 & \text{otherwise} \end{cases}$

$Y = \begin{cases} 1 & \text{if second coin is H} \\ 0 & \text{otherwise} \end{cases}$

applicable for discrete R.V. X and Y *

Define the joint probability mass function (for discrete X, Y)

$$p(x,y) = P\{X = x, Y = y\}$$ for $x, y \in \{0,1\}$

where $p(0,0) = P\{X=0, Y=0\} = 1/4$ T, T

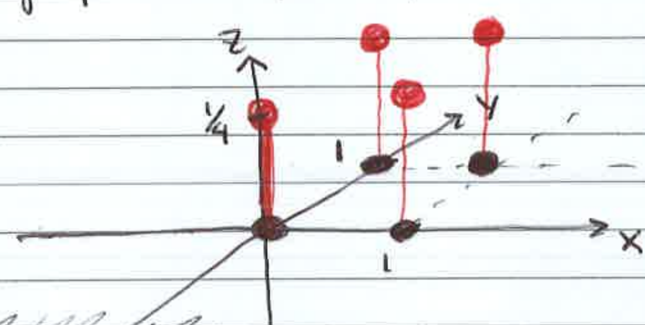
$p(1,0) = P\{X=1, Y=0\} = 1/4$ H, T

$p(0,1) = P\{X=0, Y=1\} = 1/4$ T, H

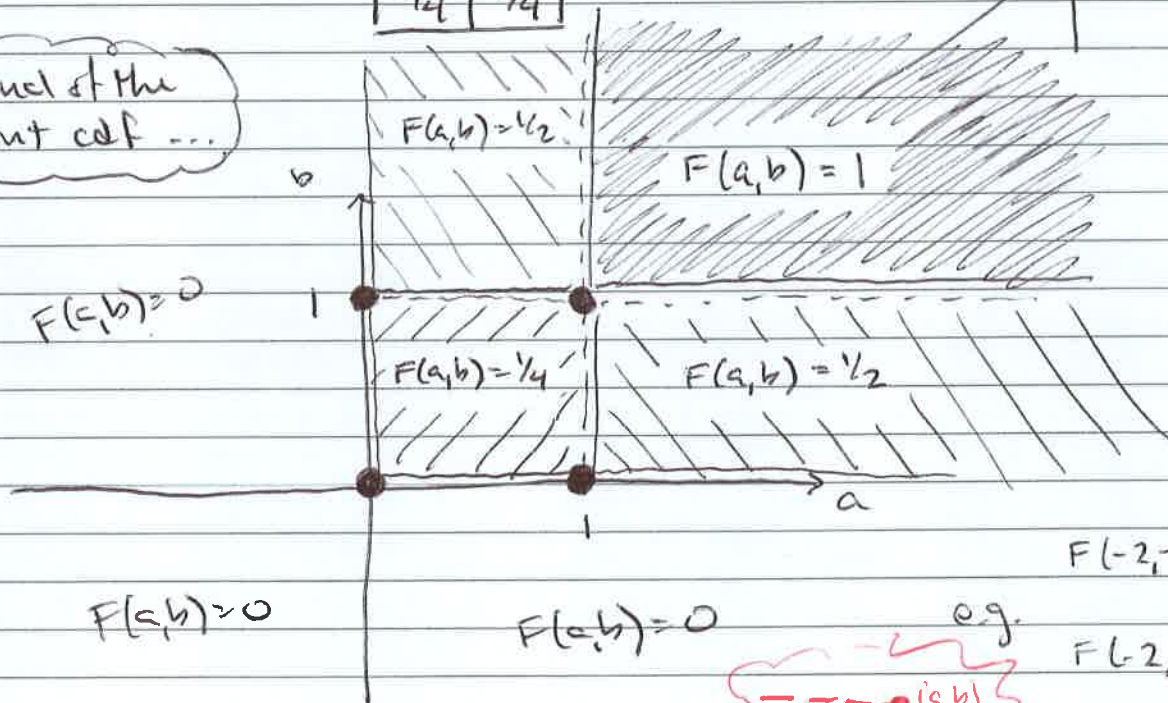
$p(1,1) = P\{X=1, Y=1\} = 1/4$ H, H

This can be expressed in a table, graph ($z = p(x,y)$)

$x=$ ↓ \ $y=$ →	0	1
0	1/4	1/4
1	1/4	1/4

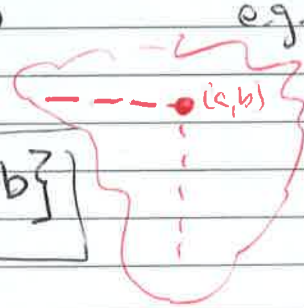


Visual of the joint cdf ...



$$\boxed{F(a,b) = P\{X \le a, Y \le b\}}$$

e.g.



$F(-2,-2) = P\{X \le -2, Y \le -2\}$
$= 0$

$F(-2,1) = P\{X \le -2, Y \le 1\}$
$= 0$

$F(1/2, 1/2) = P\{X \le 1/2, Y \le 1/2\}$
$= P\{X=0, Y=0\}$
$= 1/4$

$F(2, 1/2) = P\{X \le 2, Y \le 1/2\}$
$1/2 = \; = P\{X=0, Y=0\} + P\{X=1, Y=0\}$

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Modification, if two different unfair coins are used instead...

First coin ~~lands~~ lands on H with probability $\frac{2}{3}$

second " " " H " " $4/5$

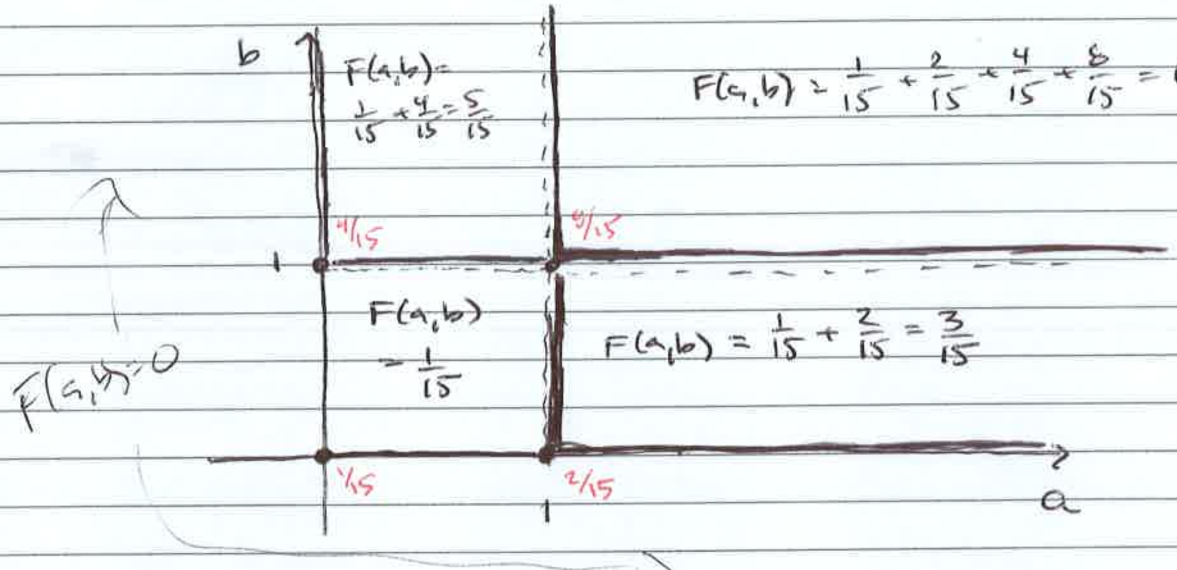
T,T $\quad p(0,0) = \frac{1}{3}\cdot\frac{1}{5} = \frac{1}{15}$

H,T $\quad p(1,0) = \frac{2}{3}\cdot\frac{1}{5} = \frac{2}{15}$

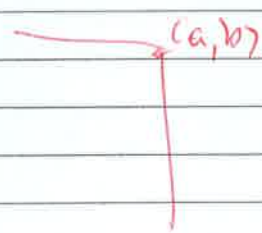
T,H $\quad p(0,1) = \frac{1}{3}\cdot\frac{4}{5} = \frac{4}{15}$

H,H $\quad p(1,1) = \frac{2}{3}\cdot\frac{4}{5} = \frac{8}{15}$

X \ Y	0	1
0	$\frac{1}{15}$	$\frac{4}{15}$
1	$\frac{2}{15}$	$\frac{8}{15}$



$F(a,b) = P\{X \le a, Y \le b\}$



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From the joint c.d.f. we can recover the cdf for X and the cdf for Y

$$F(a,b) = P\{X \le a,\ Y \le b\}$$ joint cdf for X and Y

cdf for X:

i.e. all y

This is called the marginal cumulative distribution function for X

$$\begin{aligned} F_X(a) &= P\{X \le a\} = P\{X \le a,\ Y < \infty\} \\ &= P\left\{\lim_{b\to\infty}\{X \le a,\ Y \le b\}\right\} \\ &= \lim_{b\to\infty} P\{X \le a,\ Y \le b\} \\ &= \lim_{b\to\infty} F(a,b) \end{aligned}$$

Similarly, $F_Y(b) = \lim_{a\to\infty} F(a,b)$ is the marginal cdf for Y.

For discrete R.V. X and Y we defined

$$p(x,y) = P\{X = x,\ Y = y\}$$ joint probability mass function

The marginal pmf for X ~~and~~ and for Y are

$$p_X(x) = \sum_{y:\,p(x,y)>0} p(x,y)$$ ← sum over y

$$p_Y(y) = \sum_{x:\,p(x,y)>0} p(x,y)$$ ← sum over x.

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For our examples...

FAIR COIN TWICE

$$F_X(a) = \lim_{b \to \infty} F(a,b) = \begin{cases} 0 & \text{if } a < 0 \\ 1/2 & \text{if } 0 \le a < 1 \\ 1 & \text{if } a \ge 1 \end{cases}$$

$$F_Y(b) = \lim_{a \to \infty} F(a,b) = \begin{cases} 0 & \text{if } b < 0 \\ 1/2 & \text{if } 0 \le b < 1 \\ 1 & \text{if } b \ge 1 \end{cases}$$

$$p_X(x) = \sum_{y : p(x,y) > 0} p(x,y) = \begin{cases} \frac{1}{4} + \frac{1}{4} = \frac{1}{2} & x = 0 \\ \frac{1}{4} + \frac{1}{4} = 1/2 & x = 1 \end{cases}$$

$$p_Y(y) = \sum_{x : p(x,y) > 0} p(x,y) = \begin{cases} \frac{1}{4} + \frac{1}{4} = 1/2 & y = 0 \\ \frac{1}{4} + \frac{1}{4} = 1/2 & y = 1 \end{cases}$$

UNFAIR COINS

$$F_X(a) = \lim_{b \to \infty} F(a,b) = \begin{cases} 0 & a < 0 \\ 5/15 = 1/3 & 0 \le a < 1 \\ 1 & a \ge 1 \end{cases}$$

$$F_Y(b) = \lim_{a \to \infty} F(a,b) = \begin{cases} 0 & b < 0 \\ \frac{3}{15} = 1/5 & 0 \le b < 1 \\ 1 & b \ge 1 \end{cases}$$

$$p_X(x) = \sum_{y} p(x,y) = \begin{cases} \frac{1}{15} + \frac{4}{15} = \frac{5}{15} = \frac{1}{3} & x = 0 \\ \frac{2}{15} + \frac{8}{15} = \frac{10}{15} = \frac{2}{3} & x = 1 \end{cases}$$

$$p_Y(y) = \sum_{x} p(x,y) = \begin{cases} \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \frac{1}{5} & y = 0 \\ \frac{4}{15} + \frac{8}{15} = \frac{12}{15} = \frac{4}{5} & y = 1 \end{cases}$$

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Continuous Case

Next, lets think about the case of continuous R.V.'s X, Y.

Def. Random variables X and Y are jointly continuous if there exists a function $f(x,y)$ defined for all real x and y having the property that for every set C in the x,y plane

$$P\{(X,Y)\in C\} = \iint_C f(x,y)\,dx\,dy$$

very similar to definition of single continuous r.v. $P\{X\in B\} = \int_B f(x)\,dx$

- $f(x,y)$ here is the joint probability density function of X and Y (joint pdf)

- The joint cumulative distribution function has

$$F(a,b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy = P\{X\le a, Y\le b\}$$

- Marginal cdf's (and marginal pdf's)

$$F_X(a) = \lim_{b\to\infty} F(a,b) = \int_{-\infty}^{+\infty}\int_{-\infty}^{a} f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{a}\underbrace{\left[\int_{-\infty}^{+\infty} f(x,y)\,dy\right]}_{\equiv f_X(x)}dx = \int_{-\infty}^{a} f_X(x)\,dx$$

$\equiv f_X(x)$ ← marginal probability density function (for X)

matches definition from ch 5 (p.177) for single R.V. X

Similarly,

$$F_Y(b) = \lim_{a\to\infty} F(a,b) = \int_{-\infty}^{b}\underbrace{\left[\int_{-\infty}^{+\infty} f(x,y)\,dx\right]}_{f_Y(y)}dy = \int_{-\infty}^{b} f_Y(y)\,dy$$

- Differentiating the joint cdf w.r.t. a and b

$$\frac{\partial F}{\partial b} = \int_{-\infty}^{a} f(x,b)\,dx$$

$$\boxed{\frac{\partial^2 F}{\partial b \partial a} = f(a,b)}$$

applies where F is differentiable...

- Also

$$P\{a \le X \le a+da,\ b \le Y \le b+db\} = \int_b^{b+db}\int_a^{a+da} f(x,y)\,dx\,dy \approx f(a,b)\,da\,db$$

for small da, db

(Recall in the single variable case with

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = P\{X \le x\}$$

we noted

$$P\{x \le X \le x+dx\} = \int_x^{x+dx} f(t)\,dt \approx f(x)\,dx$$

for small dx)

So $f(a,b)$ is a probability "per unit area" in (a,b) plane.

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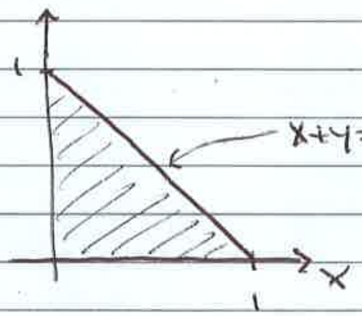
EXAMPLE

Suppose X and Y have the joint p.d.f.

$$f(x,y) = \begin{cases} c & , \; x \geq 0,\; y \geq 0,\; x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



a) Find c. (constant)

b) Find the joint cdf of X and Y

a) To find c we require

$$1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} c\,dx\,dy$$

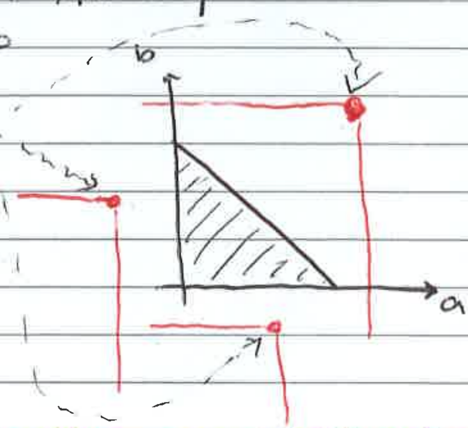
$$= \int_0^1 c\cdot(1-y)\,dy$$

$$= c\left(y - \tfrac{1}{2}y^2\right)\Big|_0^1 = c\left(1-\tfrac{1}{2}\right) = \frac{c}{2}$$

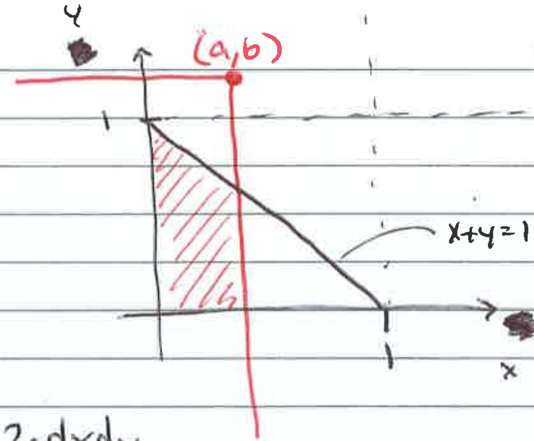
so choose $c=2$

b) $F(a,b) = P\{X \leq a, Y \leq b\} = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy$

cases:
- if $a<0$ $\quad F(a,b)=0$
- if $b<0$ $\quad F(a,b)=0$
- if $a,b \geq 1$ $\quad F(a,b)=1$
- what if a and/or b ~~occur~~ on $[0,1]$?



- $\underline{b \geq 1 \qquad 0 \leq a \leq 1}$



$$F(a,b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x,y)\,dx\,dy$$

$$= \int_{0}^{1-a} \int_{0}^{a} 2 \cdot dx\,dy + \int_{1-a}^{1} \int_{0}^{1-y} 2 \cdot dx\,dy$$

$$= 2(1-a)a + 2\int_{1-a}^{1} (1-y)\,dy$$

$$= 2(1-a)a + 2\left[y - \tfrac{1}{2}y^2\right]_{1-a}^{1}$$

$$= 2a - 2a^2 + 2\left[\left(1-\tfrac{1}{2}\right) - \left((1-a) - \tfrac{1}{2}(1-a)^2\right)\right]$$

$$= 2a - 2a^2 + 2\left[\tfrac{1}{2} - 1 + a + \tfrac{1}{2}(1-a)^2\right]$$

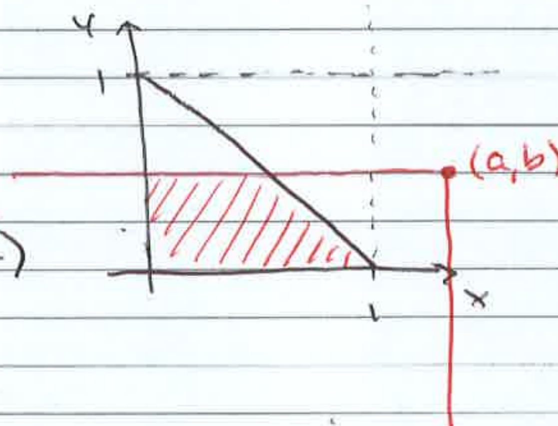
$$= 2a - 2a^2 + 1 - 2 + 2a + (1 - 2a + a^2)$$

$$\boxed{F(a,b) = 2a - a^2}$$

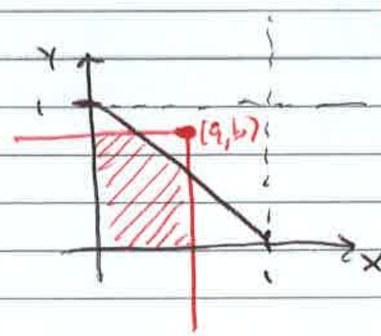
- $\underline{a \geq 1, \quad 0 \leq b \leq 1}$

$$\boxed{F(a,b) = 2b - b^2}$$

(swap a,b in previous calculation)



- $0 \leq a \leq 1, \quad 0 \leq b \leq 1, \quad a+b > 1$



$$F(a,b) = \int_{0}^{1-a} \int_{0}^{a} 2\,dx\,dy + \int_{1-a}^{b} \int_{0}^{1-y} 2\,dx\,dy$$

$$= 2a(1-a) + 2\int_{1-a}^{b} 1-y\,dy$$

$$= 2a(1-a) + 2\left(y - \tfrac{1}{2}y^2\right)\Big|_{1-a}^{b}$$

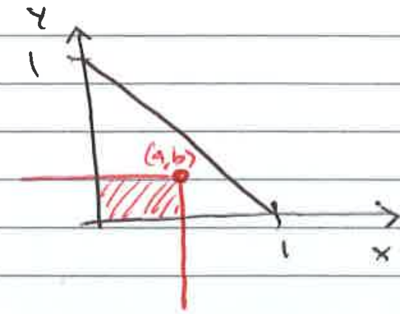
$$= 2a - 2a^2 + 2\left[\left(b - \tfrac{1}{2}b^2\right) - \left((1-a) - \tfrac{1}{2}(1-a)^2\right)\right] = 2a - 2a^2 + 2b - b^2 - 2(1-a) + (1-a)^2$$

$$= 2a - 2a^2 + 2b - b^2 - 2 + 2a + 1 - 2a + a^2$$

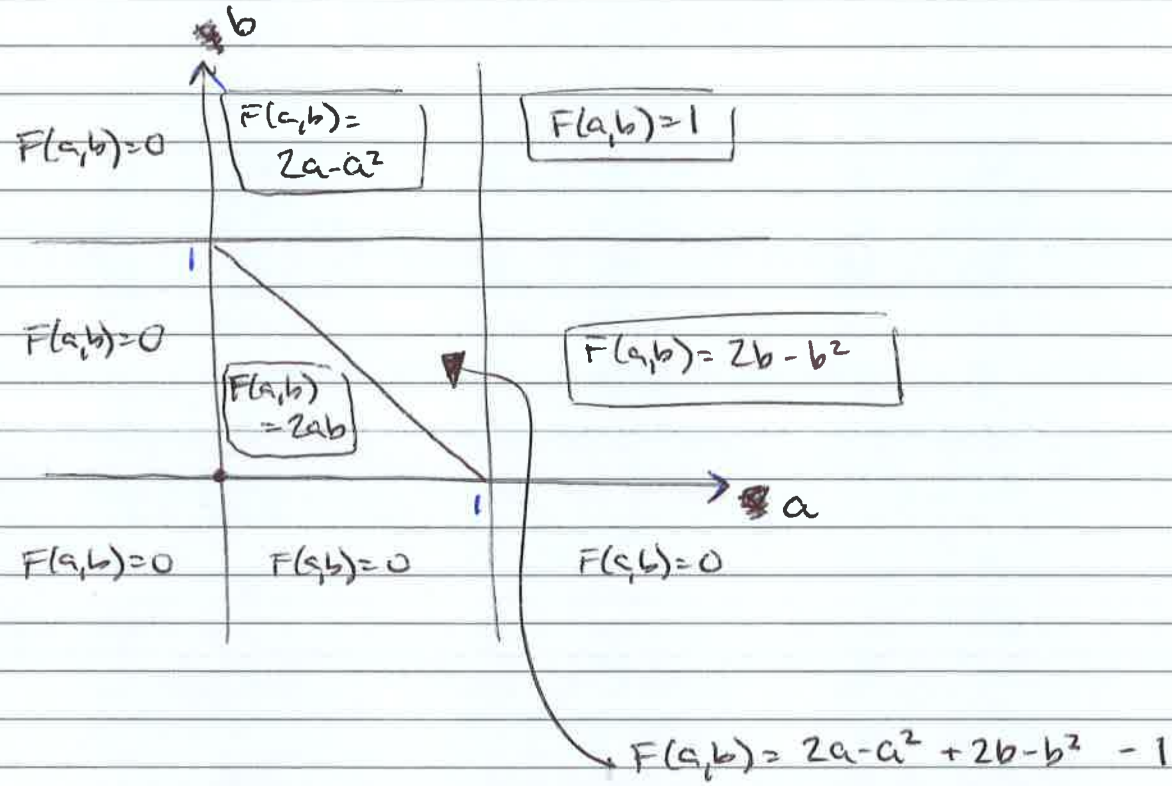
$$= \boxed{2a - a^2 + 2b - b^2 - 1}$$

* $0 \le a \le 1$, $0 \le b \le 1$, $a+b<1$

$$\boxed{F(a,b) = 2\cdot a \cdot b} \quad \text{(rectangle)}$$



So...



Then, for example, the <u>marginal c.d.f. of X</u> is

$$P\{X \le x\} = F_X(x) = P\{X \le x, Y < \infty\} = \lim_{b\to\infty} P\{X \le x, Y \le b\}$$

$$= \lim_{b\to\infty} F(x,b)$$

so

$$F_X(x) = \begin{cases} 0 & x<0 \\ 2x - x^2 & 0 \le x \le 1 \\ 1 & x>1 \end{cases}$$

The marginal pdf of X is

$$f_X(x) = \frac{d}{dx}F_X = \begin{cases} 0 & x<0 \\ 2-2x & 0<x<1 \\ 0 & x>1 \end{cases}$$

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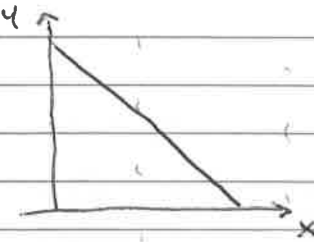
Alternatively,



$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$$

$$= \begin{cases} 0 & \text{if } x<0 \\ \int_0^{1-x} 2\,dy & \text{if } 0\le x\le 1 \\ 0 & \text{if } x>1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x<0 \\ 2(1-x) & \text{if } 0\le x\le 1 \\ 0 & \text{if } x>1 \end{cases}$$

Similarly, the marginal c.d.f. of Y is

$$P\{Y\le y\} = F_Y(y) = \lim_{a\to\infty} F(a,y) = \begin{cases} 0 & y<0 \\ 2y-y^2 & 0\le y\le 1 \\ 1 & y>1 \end{cases}$$

then the marginal pdf of Y is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 0 & y<0 \\ 2-2y & 0<y<1 \\ 0 & y>1 \end{cases}$$

Alternatively,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \begin{cases} 0 & \text{if } y<0 \\ \int_0^{1-y} 2\,dx & \text{if } 0\le y\le 1 \\ 0 & \text{if } y>1 \end{cases} = \begin{cases} 0 & \text{if } y<0 \\ 2(1-y) & \text{if } 0\le y\le 1 \\ 0 & \text{if } y>1 \end{cases}$$