

Worksheet 4

Let R be a relation from A to B , let S be a relation from B to C , and let T be a relation from C to D .

Prove the following statements.

1. $I_A \circ R = R$

Proof. Let $a \in A$ and $b \in B$ and observe:

$$\begin{aligned} a(I_A \circ R)b &\iff \exists a' \in A : a = a' \wedge a'Rb \\ &\iff aRb \end{aligned}$$

□

2. $R \circ I_A = R$

Proof. Let $a \in A$ and $b \in B$ and observe:

$$\begin{aligned} a(R \circ I_A)b &\iff \exists b' \in B : b = b' \wedge aRb' \\ &\iff aRb \end{aligned}$$

□

3. $(R^{-1})^{-1} = R$

Proof. Fix $a \in A$ and $b \in B$:

$$\begin{aligned} a(R^{-1})^{-1}b &\iff bR^{-1}a \\ &\iff b(R^{-1})^{-1}a \end{aligned}$$

□

4. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Proof. Suppose $(c, a) \in (S \circ R)^{-1}$. Then by implication: $a(S \circ R)^{-1}c$. Hence, there exists a $b \in B$ such that bSc and aRb and $cS^{-1}b$ and $bR^{-1}a$. Therefore $(c, a) \in (R^{-1} \circ S^{-1})$ and $(R^{-1} \circ S^{-1}) \subseteq (S \circ R)^{-1}$. The converse implication is obtained by retracing the given steps. □

5. $(T \circ S) \circ R = T \circ (S \circ R)$

Proof. Assume $(a, d) \in (T \circ S) \circ R$. It follows that $b \in B$ such that aRb and $b(T \circ S)d$. Hence there is a $c \in C$ such that bSc and cTd . This implies $a(S \circ R)c$, hence $aT \circ (S \circ R)d$. So we can conclude $T \circ (S \circ R) \subseteq (T \circ S) \circ R$. The converse implication is similar. \square

6. $Dom R = Rng R^{-1}$

Proof. (\subseteq) Fix $a \in A$ and observe that $a \in Dom R$. There there must exist $b \in B$ such that aRb and $bR^{-1}a$. Hence $a \in Rng R^{-1}$ and $Rng R^{-1} \subseteq Dom R$. \square

Proof. (\supseteq) Fix $a \in A$ and observe $a \in Rng R^{-1}$. There must be $b \in B$ such that $bR^{-1}a$ and aRb . Hence $a \in Dom R$ and $Dom R \subseteq Rng R^{-1}$. \square

7. $Rng R = Dom R^{-1}$

Proof. (\supseteq) Suppose $b \in Rng R$. This implies $a \in A$ such that aRb and $bR^{-1}a$. Hence by the invertibility of R , $b \in Dom R^{-1}$ and $Dom R^{-1} \subseteq Rng R$. \square

Proof. (\subseteq) Fix $b \in Dom R^{-1}$. By implication we have $a \in A$ such that $bR^{-1}a$ and aRb . So it follows that $b \in Rng R$ and $Rng R \subseteq Dom R^{-1}$. \square

For Question 8–10, suppose that $A = B = C$.

8. If R and S are equivalence relations, then $S \circ R$ is an equivalence relation.

Proof. Suppose R is an equivalence relation from A to B and S is an equivalence relation from B to C and $A = B = C$.

$$\begin{aligned} S \circ R &\iff \forall a \in A : aSa \wedge aRa \\ &\iff \forall a, b, c \in A : (aSb \wedge bSc) \Rightarrow aSa \wedge (aRb \wedge bRc) \Rightarrow aRc \\ &\iff \forall a, b \in A : (aSb \wedge bSa) \wedge (aRb \wedge bRa) \end{aligned}$$

\square

9. If R is a partial order, then $R \circ R$ is a partial order.

Proof. Fix $a, b, c \in A$:

$$\begin{aligned} aRc &\iff aRa \wedge bRb \wedge cRc \\ &\iff (aRb \wedge bRc) \Rightarrow aRc \\ &\iff (aRb \wedge bRa) \Rightarrow a = b \\ &\iff (a(R \circ R)b \wedge b(R \circ R)a) \Rightarrow a = b \\ &\iff a(R \circ R)a \wedge b(R \circ R)b \wedge c(R \circ R)c \\ &\iff (a(R \circ R)b \wedge b(R \circ R)c) \Rightarrow a(R \circ R)c \\ &\iff a(R \circ R)c \end{aligned}$$

□

10. If R and S are partial orders, then it is not generally true that $S \circ R$ is a partial order.

Proof. Let $R = \leq$ and $S = \geq$ therefore $a(S \circ R)c = a \geq b \leq c$ where b is $\text{MIN}(a, c)$. Fix $b = 3$. Observe that $3 \leq 3 \geq 5$ and $5 \leq 3 \geq 3$. Therefore $R \circ S$ is not antisymmetric. □

Bonus Questions Give an example of two relations R and S on a set A such that

11. $R \circ S \neq S \circ R$.

Proof. Suppose $R = \leq$ and $S = |x|$. Fix $a = -9$ and $b = 5$. Observe that $-9(R \circ S)5 \neq -9(S \circ R)5$. □

12. $S \circ R$ is an equivalence relation, but neither R nor S is an equivalence relation.

Proof. □