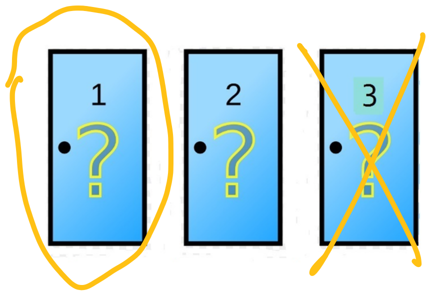
Lecture 03 EN.553.311

## 2.4 Conditional Probability

So far, we've looked at events occurring in a vacuum (totally on their own). What if we know more info? What if another event could influence our event?

Monty Hall problem / Let's Make a Deal / 21




3 doors: 2 goats
1 sportscar

Equally likely to start:

$P(D_1) = P(D_2) = P(D_3) = \frac{1}{3}$.

Say you pick $D_1$. The host then reveals a goat behind $D_3$. Do you stick with $D_1$ or do you switch to $D_2$?

You may think there's now a $\frac{1}{2}$ chance the car is behind either $D_1$ or $D_2$.
But this new information has changed the probability of switching leading to a car!
(You should switch, and we'll find out why...)

Def. Let $A, B$ be events on sample space $S$ with $P(B) > 0$. The conditional probability of $A$ given $B$ is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Note that the formula does not work if $P(B) = 0$.
We can't draw any conclusions about an event A based on an event B with no chance of occurring!

Back to Monty Hall...
You chose $D_1$.
If $D_1 =$ car, host opens either $D_2$ or $D_3$.

| Prize door, opened door | Probability |
|---|---|
| $D_1, D_2$ | $1/6$ |
| $D_1, D_3$ | $1/6$ |
| $D_2, D_3$ | $1/3$ |
| $D_3, D_2$ | $1/3$ |

$P(D_1) = \frac{1}{3}$

half the time host opens $D_2 \rightarrow \frac{1}{6}$
" " " " " $D_3 \rightarrow \frac{1}{6}$

If $D_2 =$ car, host must open $D_3$. $\rightarrow \frac{1}{3}$
" $D_3$ " " " " " $D_2 \rightarrow \frac{1}{3}$

let $A =$ event prize is behind $D_1$ (your door)
$B =$ event host opens $D_3$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$$

If you don't switch

Let $C$ = event prize is behind $D_2$ (not your door!)

$$P(C|B) = \frac{P(C \cap B)}{P(B)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$ if you switch!!

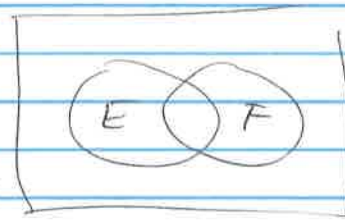
The additional info from the host opening the door means we should change our guess!

Sometimes we will use the conditional probability to compute the probability of an intersection. (sometimes $P(A|B)$ is given or easier to compute than $P(A \cap B)$.

## 3.3 Bayes's Formula

Let $E$ and $F$ be events. Then

$$E = EF \cup EF^c$$




since $EF$ and $EF^c$ are mutually exclusive events

$$P(E) = P(EF) + P(EF^c) \quad \text{by Axiom 3}$$

$$= P(E|F)\cdot P(F) + P(E|F^c)P(F^c)$$

but $P(F^c) + P(F) = 1$ so

$$\boxed{P(E) = P(E|F)P(F) + P(E|F^c)(1 - P(F))}$$

Ex

Recall Celine.

$P(F) = 1/2$ probability she takes French

$P(C) = 1/2$ probability she takes Chem.

coin flip

$P(A|C) = \frac{2}{3}$ ← given she takes Chem probability of an A

$P(A|F) = 1/2$ ← given she takes French probability of an A.

Suppose after the semester she tells you she got an A.

What is the probability that she took chemistry?

$= P(C|A)$ = probability she took Chem given she got an A (different from $P(A|C)$).

$$= \frac{P(AC)}{P(A)} = \frac{P(A|C)P(C)}{P(AC) + P(AC^c)}$$

From

$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$$

True statements that follow are...

$$P(F|E) = \frac{P(EF)}{P(E)}$$

$$\boxed{P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)}}$$

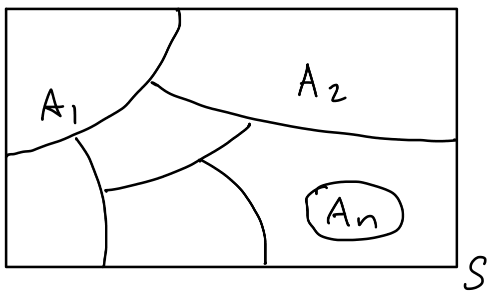
Also

$$P(F_c|E) = \frac{P(EF_c)}{P(E)}$$

$$\boxed{P(F_c|E) = \frac{P(E|F^c)P(F^c)}{P(E|F)P(F) + P(E|F^c)P(F^c)}}$$

these are the "$n=2$" version of Bayes's Theorem

Let's think about more than two events. We'll think about a partition to our sample space $S$:




The events $A_1, A_2, \ldots, A_n$ partition $S$ because every outcome in the sample space belongs to one and only one event $A_i$.

A partition is a mutually exclusive or disjoint collection of sets that together make up $S$.

Two theorems: 1 for unconditional probabilities,
1 for conditional probabilities.

Law of Total Probability (Thm 2.4.1 in text)

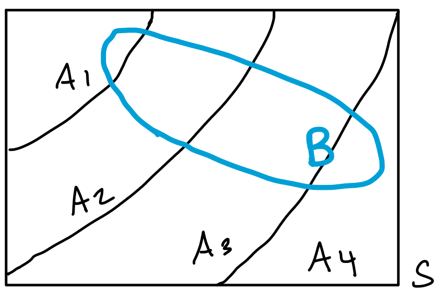
If $A_1, A_2, \ldots, A_n$ are disjoint events and

$$S = \bigcup_{i=1}^{n} A_i$$ (the $A_i$'s together make up $S$),

then for any event $B$,

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i) P(A_i).$$

Let's do a proof by picture when $n = 4$.




How can we write $B$ as the union of disjoint events?

$$B = (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3) \cup (B \cap A_4)$$

disjoint

Since $A_1, A_2, A_3, A_4$ form a partition of $S$.

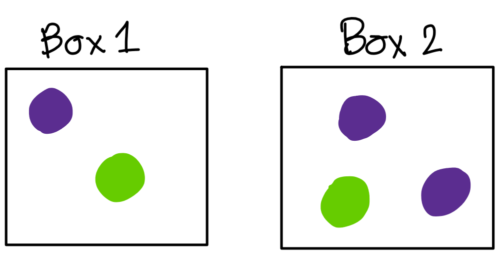
Then
$$P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) + P(B \cap A_4)$$
by countable additivity (axiom 3).

Using $P(A|C) = \dfrac{P(A \cap C)}{P(C)} \Rightarrow P(A \cap C) = P(C)\,P(A|C)$

$$P(B) = P(A_1)\,P(B|A_1) + P(A_2)\,P(B|A_2) + P(A_3)\,P(B|A_3) + P(A_4)\,P(B|A_4).$$

<u>Ex</u> Consider an experiment with 2 boxes with marbles. You toss a coin: heads $\Rightarrow$ select from Box 1

tails $\Rightarrow$ select from Box 2




Let $R$ be the event a purple marble is selected.

Let $B_i$ = event Box $i$ is selected

because these are determined by a coin flip,

$$P(B_1) = P(B_2) = \frac{1}{2}.$$

Conditional probabilities are straightforward:

$$P(R \mid B_1) = \frac{1}{2}, \quad P(R \mid B_2) = \frac{2}{3}.$$

Note that $B_1$ and $B_2$ partition the sample space and are disjoint. Thus,

$$P(R) = P(B_1)\,P(R|B_1) + P(B_2)\,P(R|B_2)$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}.$$

Now let's look at a theorem that allows us to switch the conditioning of events, ex. from $P(A|B)$, find $P(B|A)$.

<u>Bayes Rule</u> (Thm. 24.2 in text)

Let $A_1, A_2, \ldots, A_n$ be disjoint and partition the sample space $S$ (also referred to as <u>exhaustive</u>.)

let $B$ be any event. Then for any $A_i$,

$$P(A_i \mid B) = \frac{P(A_i)\,P(B \mid A_i)}{\underbrace{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots P(B|A_n)P(A_n)}_{P(B) \text{ from law of total probability}}}$$

so this reduces to

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}.$$

Ex: Dr. Bedekar loves to cook but sometimes she is too busy.

If there is traffic due to construction, there is a 50% chance she will order takeout.

If there is traffic not due to construction, there's a 30% chance she will order takeout.

If there is no traffic, there's a 10% chance she will order takeout.

On any given day, assume there is a 60% chance of no traffic, 20% chance of traffic, and a 20% chance of traffic from construction.

If Dr. Bedekar orders takeout, what's the probability

(a) there is traffic due to construction? $P(A_1 | B)$

(b) traffic not due to construction? $P(A_2 | B)$

(c) no traffic? $P(A_3 | B)$

Let $B$ = event she gets takeout.

$A_1$ = event there is traffic (C)

$A_2$ = " " " " (no C)

$A_3$ = " " " no traffic.

We have $P(A_1) = 0.2$, $P(A_2) = 0.2$, $P(A_3) = 0.6$

and

$P(B|A_1) = 0.5$, $P(B|A_2) = 0.3$, $P(B|A_3) = 0.1$

Using Bayes' rule:

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{\sum_{i=1}^{3} P(B|A_i)P(A_i)}$$

$$= \frac{(0.5)(0.2)}{(0.5)(0.2)+(0.3)(0.2)+(0.1)(0.6)} = \frac{0.1}{0.22}$$

$$\approx 45.5\%$$

Check:

$P(A_2|B) \approx 27.3\%$, $P(A_3|B) \approx 27.3\%$

Did not do any of the examples after this—all fair game!



$$P(C|A) = \frac{P(A|C)P(C)}{P(A|C)P(C) + P(A|C^c)P(C^c)}$$

note $C^c = F$

so

$$P(C|A) = \frac{P(A|C)P(C)}{P(A|C)P(C) + P(A|F)P(F)}$$

$$= \frac{(2/3)(1/2)}{(2/3)(1/2) + (1/2)(1/2)} = \frac{\frac{2}{3}}{\frac{2}{3} + \frac{1}{2}} = \frac{1}{1 + 3/4}$$

$$= \frac{4}{7}$$

probability she took chemistry

## Ch. 3 Problem 3.18

46% of voters in a city classify themselves as independents.

30% of " " " " " " liberals

24% " " " " " " conservatives.

In a recent election...

35% of independents voted
62% of liberals voted
58% of conservatives voted.

A voter is chosen at random. Given that this person voted...

What is the probability that he/she is

a) independent? b) liberal? c) conservative?

d) What percentage of voters participated in the election?

a) ans. $P(I|V)$ = probability of I = indep. given that they voted, V.

$$P(I|V) = \frac{P(IV)}{P(V)}$$

$$P(V) = P(IV) + P(LV) + P(CV)$$

(from $V = \underbrace{IV \cup LV \cup CV}_{\text{union of mutually exclusive events}}$)

so $$P(I|V) = \frac{P(IV)}{P(IV) + P(LV) + P(CV)}$$

$$P(IV) = P(V|I)\,P(I) = (.35)(.46) = 0.161$$
$$P(LV) = P(V|L) \cdot P(L) = (.62)(.3) = 0.186$$
$$P(CV) = P(V|C)\,P(C) = (.58)(.24) = 0.1392$$
$$0.4862$$

$$P(I|V) = \frac{0.161}{0.161 + 0.186 + 0.1392} = \frac{0.161}{0.4862} \approx .331$$

b) $$P(L|V) = \frac{P(LV)}{P(V)} = \frac{0.186}{0.4862} \approx .383$$

c) $$P(C|V) = \frac{P(CV)}{P(V)} = \frac{0.1392}{0.4862} \approx .286$$

d) $P(V) = 0.4862$

Generalized Version of Bayes's Formula.

Suppose events $F_1, F_2, \ldots, F_n$ are mutually exclusive (i.e. $F_iF_j = \phi$ if $i \neq j$) and together they make up the whole sample space. So

$$\bigcup_{i=1}^{n} F_i = S$$

Then we can write

$$E = EF_1 \cup EF_2 \cup \ldots \cup EF_n$$

and then, since $\underbrace{EF_i}_{\text{and } EF_j}$ are mutually exclusive for $i \neq j$

$$P(E) = P(EF_1) + P(EF_2) + \ldots + P(EF_n)$$

$$= \sum_{i=1}^{n} P(EF_i)$$

$$\boxed{P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i)}$$

Ross ch. 3.3 p. 69 eq. (3.4)

So

$$P(F_j|E) = \frac{P(EF_j)}{P(E)}$$

Prop. 3.1 Bayes in Ross p. 69

$$= \frac{P(EF_j)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)} = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}$$

## EXAMPLE 3a

Insurance company believes there are two classes of people, accident prone and not accident prone.

Statistics →

- accident prone person has probability 0.4 of having an accident in a 1-year period
- non-accident prone person has probability 0.2 of having an accident in a 1-year period.

Assume the 30% of the population is accident prone.

Q: What is the probability that a new policy holder will have an accident with a year of purchasing a policy

$$P(A_1) = \underbrace{P(A_1 | A)}_{0.4} \underbrace{P(A)}_{.3} + \underbrace{P(A_1 | A^c)}_{0.2} \underbrace{P(A^c)}_{0.7}$$

↑ prob. of new policy holder having accident

$$= 0.12 + 0.14 = \boxed{0.26}$$

Q: A new policy holder has an accident in first year. What is the probability that he/she is accident prone?

$$P(A | A_1) = \frac{P(AA_1)}{P(A_1)} = \frac{P(A_1 | A) P(A)}{P(A_1)} = \frac{(0.4)(0.3)}{0.26}$$

$$= \frac{0.12}{0.26} = \frac{12}{26} = \frac{6}{13}$$

Note:

$$P(A^c|A_1) = \frac{P(A^cA_1)}{P(A_1)} = \frac{P(A_1|A^c)P(A^c)}{P(A_1)}$$

probability of being not accident prone given that the person had an accident in the first year.

$$= \frac{(0.2)(0.7)}{0.26} = \frac{0.14}{0.26}$$

$$= \frac{14}{26} = \frac{7}{13} \quad \text{i.e. } 1 - \frac{6}{13}$$

Q: What if we make a different assumption about (*) the % of the population that is accident prone? (call it $p_A$)

note: the accident prone proportion of the population is relatively small in this assumption. (30%)

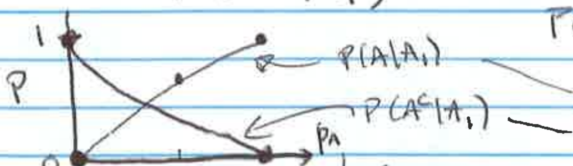
- $P(A_1) = P(A_1|A)P(A) + P(A_1|A^c)P(A^c)$

$$= (0.4)p_A + 0.2(1-p_A)$$

$$= 0.2 + 0.2p_A = 0.2(1+p_A)$$

- $P(A|A_1) = \frac{P(AA_1)}{P(A_1)} = \frac{P(A_1|A)P(A)}{0.2(1+p_A)} = \frac{0.4\,p_A}{0.2(1+p_A)} = \frac{2p_A}{1+p_A}$

- $P(A^c|A_1) = \frac{P(A^cA_1)}{P(A_1)} = \frac{P(A_1|A^c)P(A^c)}{0.2(1+p_A)} = \frac{0.2(1-p_A)}{0.2(1+p_A)} = \frac{1-p_A}{1+p_A}$




$P(A|A_1)$ = probability they are accident prone given an accident

$P(A^c|A_1)$ = probability they are not accident prone given an accident

$p_A = \frac{\%}{100}$ of population that is accident prone

EXAMPLE 3d p. 64

- A lab blood test is 95% effective at identifying a disease when it is actually present.
- "False positive" tests occur for 1% of healthy people tested.
- Suppose 0.5% of the population has the disease.

Q: What is the probability that a person has the disease given that the test result is positive?

let D = event that person tested has the disease

Let tpos = event that the test is positive

$$P(D|tpos) = \frac{P(D\,tpos)}{P(tpos)}$$

$D^c$ = event they do not have disease

$$= \frac{P(tpos|D)P(D)}{P(tpos|D)P(D) + P(tpos|D^c)P(D^c)}$$

$$= \frac{(0.95)(0.005)}{(0.95)(0.005) + (0.01)(.995)}$$

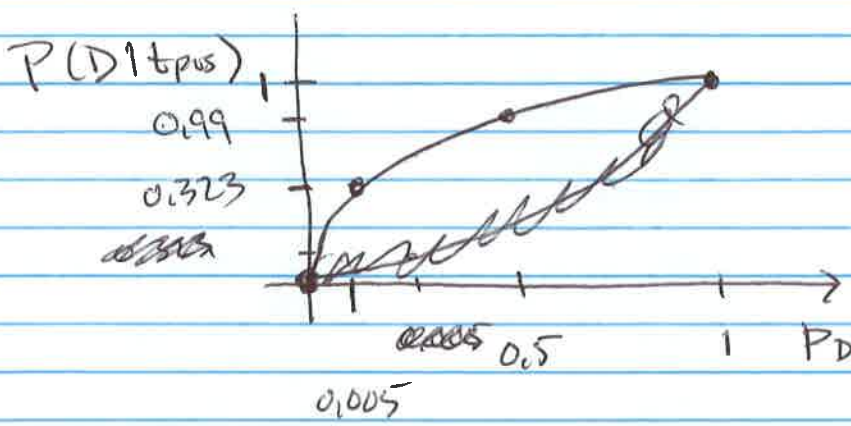
$$= \frac{0.00475}{0.00475 + 0.00995} = \frac{0.00475}{0.0147} \approx 0.323$$

relatively low it would seem..

What if $p_D$ ($\times 100\%$) is the ~~[illegible]~~ percentage of the population that has the disease?

$$P(D|t_{pos}) = \frac{(0.95)\, p_D}{0.95\, p_D + 0.01\,(1-p_D)}$$

$$= \frac{95\, p_D}{95\, p_D + 1 - p_D} = \frac{95\, p_D}{1 + 94\, p_D}$$




- if $p_D = 0.005$

  $P(D|t_{pos}) = 0.323$

- if $p_D = 0.5$

  $P(D|t_{pos}) = 0.99$

EX

Suppose you have two coins in your pocket (one fair coin and one two-headed coin). Suppose you pick a coin out at random and flip it.
If the coin flip shows H (heads) what is the probability that the coin was 2-headed?

Let 2 = event coin was two headed
F = " " " fair
H = " coin flip was heads
T = " " " " tails

- $$P(2|H) = \frac{P(2H)}{P(H)} = \frac{P(H|2)P(2)}{P(H|2)P(2) + P(H|F)P(F)}$$

$$= \frac{(1)(1/2)}{(1)(1/2) + (1/2)(1/2)} = \frac{1/2}{3/4} = \frac{2}{3}$$

- What if the coin taken out of the pocket was flipped twice and both times showed Heads. What is the probability it is the two-headed coin?

$$P(2|HH) = \frac{P(2HH)}{P(HH)} = \frac{P(HH|2)P(2)}{P(HH|2)P(2) + P(HH|F)P(F)}$$

$$= \frac{(1)(1/2)}{(1)(1/2) + (1/4)(1/2)} = \frac{4}{5}$$

- What if the coin was flipped k times and showed heads k times. What is the probability it is the 2-headed coin?

$$P(2 \mid kH) = \frac{P(kH \mid 2)P(2)}{\underbrace{P(kH \mid 2)P(2) + P(kH \mid F)P(F)}_{= P(kH)}}$$

$$= \frac{(1)(1/2)}{(1)(1/2) + (\frac{1}{2})^k \cdot (1/2)} = \frac{1}{1+(1/2)^k}$$

as $k \to \infty$ $P(2 \mid kH) \to 1$.

- What if the coin was flipped ~~k times~~ k+1 times and showed heads k times and tails 1 time. What is the probability it is the two headed coin?

$P(2 \mid kH1T) = 0$ ← two headed coin can never be tails.