Ch. 5 Continuous Random Variables

Def. We say that X is a continuous random variable if there exists a non-negative function f, defined on all real numbers $x \in (-\infty, +\infty)$ having the property that for any set B of real numbers

$$P\{X \in B\} = \int_B f(x)\,dx$$

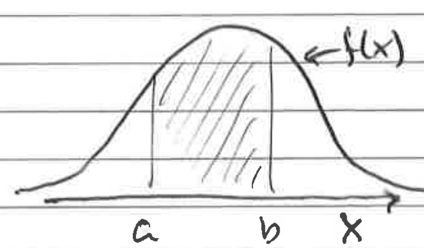
- The function f is called the probability density function of the random variable X
- B is a measurable set — eg. some interval of real numbers.

Note:

- $$P\{X \in (-\infty, +\infty)\} = \int_{-\infty}^{+\infty} f(x)\,dx = 1$$

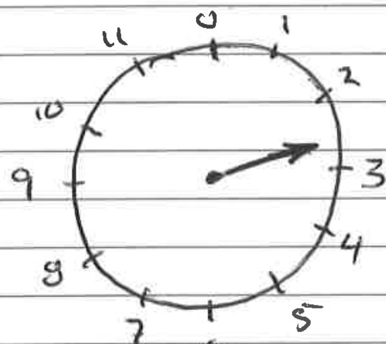
(i.e. $f(x)$ must have this property)

- $$P\{a \leq X \leq b\} = \int_a^b f(x)\,dx$$



- $P\{X = a\} = \int_a^a f(x)\,dx = 0$
- $P\{X < a\} = P\{X \leq a\} = \int_{-\infty}^{a} f(x)\,dx$

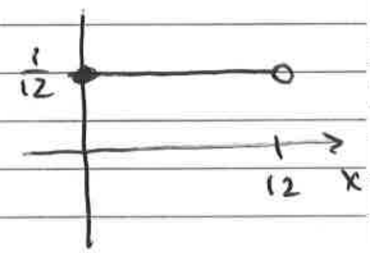
EX (Spinner)



Suppose a spinner can land on any real number $[0, 12)$. The probability that the spinner stops in an interval is proportional to the length of the interval.

Let X = random variable (continuous) representing the real number where the spinner stops.

Let
$$f(x) = \begin{cases} \frac{1}{12} & 0 \le x < 12 \\ 0 & \text{otherwise} \end{cases}$$



Note:

- $$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{12} \frac{1}{12}\,dx = 1$$

- The probability that the spinner stops in the interval $[a,b]$ $\quad a,b \in [0,12)$ is
$$P\{a \le X \le b\} = \int_a^b \frac{1}{12}\,dx = \frac{(b-a)}{12}$$

- $$P\{X = a\} = \int_a^a \frac{1}{12}\,dx = 0$$ (note: real #'s are uncountable)

The cumulative distribution function $F(x)$ for X is

$$F(x) = P\{X \le x\} = \int_{-\infty}^{x} f(t)\,dt$$

recall similar definition for discrete random variable X

continuous random variable with prob. density function f

By the Fundamental Theorem of Calculus (Part II ...)

$$\frac{dF}{dx} = f(x)$$

... the derivative of F, the cumulative dist. function, is the probability density function. That is, F is an antiderivative of f.

also

$$\lim_{x \to -\infty} F(x) = 0.$$

$$\lim_{x \to \infty} F(x) = 1$$

5.2 Expectation and Variance of Continuous Random Variables

For a continuous random variable X with probability density function $f(x)$, we define the expected value of X (expectation of X) as

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

Comments

- Compare to $E[X] = \sum_{x: p(x)>0} x\,p(x)$ for discrete R.V. X
- Observe $f(x)\,dx \approx P\{x \le X \le x + dx\}$ for small dx

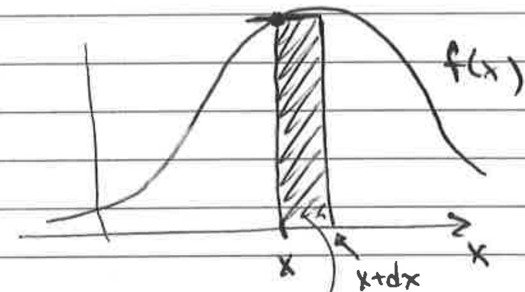
That is,

$$P\{x \le X \le x+dx\} = F(x+dx) - F(x)$$

$F =$ cumulative distribution function

$$f(x)\,dx \approx F(x+dx) - F(x)$$

$$f(x) \approx \underbrace{\frac{F(x+dx) - F(x)}{dx}}_{\approx F'(x)} \quad \text{for small } x$$



area $= f(x)\,dx$

$\approx P\{x \le X \le x+dx\}$

EX

Consider the continuous random variable X with probability density function:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

This is the pdf associated with an exponential random variable (see Sect 5.5)

- Comment: This pdf models the time ($t = x$) that one must wait for the next customer to enter the store, for example.

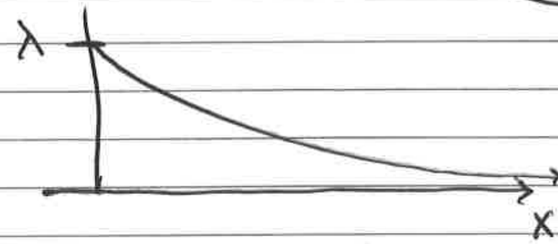
- $$P\{-\infty < X < \infty\} = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{\infty}$$
$$= 0 - (-1) = 1$$

- $$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \underbrace{x}_{u} \underbrace{\lambda e^{-\lambda x}\,dx}_{dv}$$

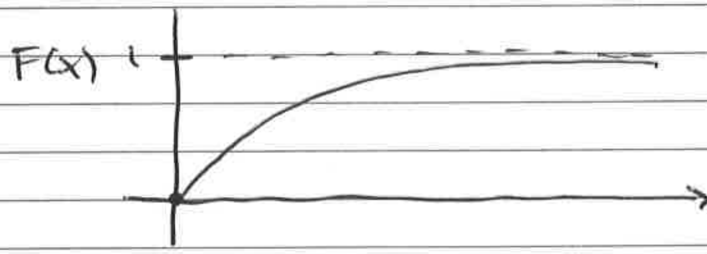
$E[X] = \frac{1}{\lambda}$

$$= x(-e^{-\lambda x})\Big|_0^{\infty} - \int_0^{\infty} -e^{-\lambda x}\,dx = \int_0^{\infty} e^{-\lambda x}\,dx = \frac{e^{-\lambda x}}{(-\lambda)}\Big|_0^{\infty} = \lambda^{-1}$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$F(x) = P\{X \leq x\} = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & \text{if } x \leq 0 \\ \int_0^x \lambda e^{-\lambda t}\,dt & \text{if } x > 0 \\ = -e^{-\lambda t}\Big|_0^x = \underline{\underline{1 - e^{-\lambda x}}} & \end{cases}$$



$F(0) = 0$

$\lim_{x \to \infty} F(x) = 1$

Functions of a continuous random variable, X. (See also Section 5.7 p. 208...)

Suppose X is a continuous random variable and let $Y = g(X)$, under certain conditions on g (see Theorem 7.1, p. 209 in Ross) we can get the probability density function of Y by first finding its cumulative distribution function and then differentiating it

(recall $\frac{dF}{dx} = f$ $\quad F = \text{c.d.f.}$, $f = \text{p.d.f.}$)

Here are some examples:

EX

Suppose X has a probability mass function

$$f_X = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$$

Let $Y = g(X) = X^2$. Find f_Y: the prob. mass function for Y.

Note: the cumulative distribution function for Y has

$$F_Y(y) = P\{Y \leq y\} = P\{X^2 \leq y\} = P\{-\sqrt{y} \leq X \leq \sqrt{y}\}$$

(for $y \geq 0$)

$$= F_X(+\sqrt{y}) - F_X(-\sqrt{y})$$

$$= \int_{-\infty}^{\sqrt{y}} f_X(t)\,dt - \int_{-\infty}^{-\sqrt{y}} f_X(t)\,dt$$

note $f_X = 0$ on $(-\infty, -\sqrt{y})$

$$= \int_0^{\sqrt{y}} e^{-t}\,dt - 0$$

$$F_Y(y) = -e^{-t}\Big|_0^{\sqrt{y}} = -e^{-\sqrt{y}} - (-1) = 1 - e^{-\sqrt{y}}$$

so $F_Y(y) = 1 - e^{-\sqrt{y}}$

then, since $\frac{dF_Y}{dy} = f_Y$

$$f_Y = \frac{d}{dy}\left(1 - e^{-\sqrt{y}}\right) = +e^{-\sqrt{y}} \cdot \tfrac{1}{2} y^{-1/2} = \frac{e^{-\sqrt{y}}}{2\sqrt{y}}$$

so

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{e^{-\sqrt{y}}}{2\sqrt{y}} & y > 0 \end{cases}$$

← repeat above argument for $y < 0$ to see $F_Y = 0$ and $f_Y = 0$.

EXAMPLE

Suppose X has a probability ~~mass~~ density function

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

← X is a uniform random variable on $(0,1)$ see Section 5.3 p. 184

Let $Y = g(X) = X^n$. For $0 < y < 1$

The cumulative dist. function $F_Y(y) = P\{Y \le y\} = P\{X^n \le y\}$

$$= P\{X \le y^{1/n}\}$$

$$= F_X(y^{1/n}) = \int_{-\infty}^{y^{1/n}} f_X(t)\,dt = \int_0^{y^{1/n}} 1 \cdot dt = y^{1/n}$$

then $f_Y = \frac{dF_Y}{dy} = \frac{1}{n} y^{\frac{1}{n}-1}$ for $0 < y < 1$

$$\Rightarrow f_Y = \begin{cases} \frac{1}{n} y^{\frac{1}{n}-1} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Note: if $y < 0$ or $y > 1$

$\downarrow$ $F_X(y^{1/n}) = 0$ $\quad$ $\downarrow$ $F_X(y^{1/n}) = 1$

and in either case $f_Y = \frac{dF_Y}{dy} = 0$

EX

same f_X as previous example, but $Y = g(X) = e^X$

for $y > 0$ $F_Y(y) = P\{Y \le y\} = P\{e^X \le y\} = P\{X \le \ln y\}$

$$= F_X(\ln y) = \int_{-\infty}^{\ln y} f_X(t)\,dt = \int_0^{\ln y} f_X(t)\,dt$$

$$= \begin{cases} \int_0^{\ln y} 0\,dt = 0 & \text{if } \ln y \le 0 \text{ (i.e. } y \le 1) \\ \int_0^{\ln y} 1 \cdot dt = \ln y & \text{if } 0 < \ln y \le 1 \text{ (i.e. } 1 < y \le e) \\ \int_0^{1} 1 \cdot dt = 1 & \text{if } \ln y > 1 \text{ (i.e. } y > e) \end{cases}$$

$$F_Y = \begin{cases} 0 & y \le 1 \\ \ln y & 1 < y \le e \\ 1 & y > e \end{cases}$$

Then $f_Y = \dfrac{dF_Y}{dy} = \begin{cases} 0 & y \le 1 \\ \frac{1}{y} & 1 < y \le e \\ 0 & y > e \end{cases}$

For further information and examples see Section 5.7 and Theorem 7.1 (pp. 208 – 210 in Ross).