Writing Assignment 1

Baran Sevim

1 Groups

1.1 Basics

1.1.1 Definition

Definition 1. *A group is a non-empty set G together with a rule that assigns to each pair g, h of elements of G an element $g \times h$ such that*

- *$g \times h \in G$. We say that G is closed under $*$.*
- *$g * (h * k) = (g * h) * k$ for all $g, h, k \in G$. We say that $*$ is associative.*
- *There exists an identity element $e \in G$ such $e * g = g * e = g$ for all $g \in G$.*
- *Every element $g \in G$ has an inverse g^{-1} such that $g * g^{-1} = g^{-1} * g = e$.*

Add some context here on the usage of groups. An example of a group, an element, it's inverse, and the identity element would help.

1.2 Symmetries of Graphs

1.2.1 Definition: Graph

Definition 2. *A graph is a finite set of vertices joined by edges. We will assume that there is at most one edge joining two given vertices and no edge joins a vertex to itself. The valency of a vertex is the number of edges emerging from it.*

1.2.2 Definition: Symmetry

Definition 3. *A symmetry of a graph is a permutation of the vertices that preserves the edges. More precisely, let V denote the set of vertices of a graph. Thena symmetry is a bijection $f : V \mapsto V$ such that $f(v_1)$ and $f(v_2)$ are joined by an edge if and only if v_1 and v_2 are joined by an edge.*

This is great as a definition, but how does it relate to the definition of groups above?

Theorem 1. *The symmetries of a graph form a group.*

Proof. If $f : V \mapsto V$ and $g : V \mapsto V$ we define the group operation $f * g$ to be their composition (as maps), so $f * g = f \circ g$, i.e *do g first, then f*. The composition of symmetries is clearly a symmetry, so the operation is closed. Since the compositon of maps is associative

$$(f * g) * h := (f \circ g) \circ h = f \circ (g \circ h) := f * (g * h)$$

for all symmetries f, g, h. The identity map e which sends every vertex to itself is a symmetry, and obviously $e \circ f = f \circ e = f$ for all symmetries f. Lastly, if $f : V \mapsto V$ is a symmetry then it is bijective, so its inverse f^{-1} exists and is also a symmetry. It is characterized by $f \circ f^{-1} = f^{-1} \circ f = e$.

□

You use f * g but say it is their composition which is $f \circ g$. I might be wrong here though.

Theorem 2. *In a finite group, every element his finite order.*

Proof. Let $g \in G$. Consider the infinite sequence $g, g^2, g^3, \ldots$ If G is finite, then there must be repetitions in this infinite sequence. Hence there exists $m, n \in \mathbb{N}$ with $m > n$ such that $g^m = g^n$. By cancellation, $g^{m-n} = e$. □

1.3 Products

Definition 4. *The easiest way of making a new group from given ones.*

Theorem 3. *Let G, H be groups. The product $G \times H = \{(g, h) \mid g \in G, h \in H\}$*

- *The group operation is $(g, h) * (g', h') := (g * g'_G$*

Proof. □

Source: Wemyss, Michael (2011). Introduction to Group Theory, University of Glasgow
Latex code is below

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\documentclass{article}
\usepackage{amsmath, amsfonts, amssymb, amsthm}
\usepackage{fullpage}


\title{Writing Assignment 1}
\author{Baran Sevim}
\date{ }




\theoremstyle{plain}

\newtheorem{theorem}{Theorem}
\newtheorem{definition}{Definition}


\newcommand{\R}{\mathbb{R}}


\begin{document}
\maketitle

\section{Groups}
\subsection{Basics}

\subsubsection{Definition}
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\
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    \begin{itemize}
        \item \(g \times h \in G\). We say that \(G\) is \textit{closed} under \(*\).
    \end{itemize}
    \begin{itemize}
        \item \(g \ast (h \ast k) = (g \ast h) \ast k\) for all \(g, h, k \in G\). We say that \(*\) is
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    \end{itemize}
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    \end{itemize}

\

\end{definition}

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\begin{theorem}

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    \begin{itemize}
        \item The group operation is \((g, h) * (g', h') := (g * g'_G\)
    \end{itemize}
\end{theorem}

\begin{proof}

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