Continuous functions

Casey Blacker
Math 300

Section 1

Formal writing: Whom

Possessive pronouns

subjective	I	you	he	she	they	who
objective	me	you	him	her	them	whom
possessive	mine	yours	his	hers	theirs	whose

Who or whom?

Who/Whom wrote the book?

Who!

Who or whom?

Who/Whom did you call?

Whom!

Who or whom?

I will give the chalk to the student who/whom gives the presentation.

who!

Who or whom?

The writer, who/whom I greatly admire, won the award.

whom!

Who or whom?

The writer, who/whom won the award, is greatly admired.

who!

Who or whom?

I will donate the book to whoever/whomever asks first.

whoever!

Who or whom?

Who/Whom did you elect?

Whom!

Who or whom?

Who/Whom won the election?

Who!

Section 2

Proofs with convergence

Definition

Let $(x_i)_i$ be a sequence in X and fix $x \in X$. We say that $(x_i)_i$ *converges* to x if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, x) \leq \varepsilon.$$

In this case, we write $x_i \to x$ or $\lim_{i \to \infty} x_i = x$ and we say that x is the *limit* of $(x_i)_i$.

Definition

If the sequence $(x_i)_i$ does not converge to any point $x \in X$, then $(x_i)_i$ is said to *diverge*.

Proposition

If $(x_i)_i$ is a constant sequence with value $x \in X$, then $x_i \to x$.

Proof.

Let $\varepsilon > 0$. For all $n \geq 1$, we have $d(x_n, x) = 0 \leq \varepsilon$. □

Proposition

If $x_i \to x$ *and* $x_i \to y$*, then* $x = y$*.*

Proof.

Suppose not. Then there is an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(x_n, x) \leq \frac{1}{3} d(x, y) \quad \text{and} \quad d(x_n, y) \leq \frac{1}{3} d(x, y).$$

Consequently,

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \leq \frac{2}{3} d(x, y).$$

This yields the desired contradiction. $\square$

Example

Consider the sequence of functions $(f_i)_i$ given by

$$f_i(x) = \begin{cases} i - i^3|x| & \text{if } |x| < \frac{1}{i^2} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $f_i \to 0$ with respect to the L^1-metric, and that f_i diverges with respect to the L^∞-metric.

Section 3

Definitions and examples

Let (X, d_X) and (Y, d_Y) be metric spaces.

Definition

A function $f : X \to Y$ is *continuous at* $x \in X$ when

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon.$$

Definition

We say that $f : X \to Y$ is *continuous* if it is continuous at every $x \in X$.

Proposition

If $f : X \to Y$ is the constant function with value $c_0 \in Y$, then f is continuous.

Verbose proof.

Fix $x \in X$ and let $\varepsilon > 0$. Put $\delta = 1$. Choose $y \in X$. Suppose that $d_X(x, y) < \delta$. It follows that $d_Y(f(x), f(y)) = 0 < \varepsilon$. □

Concise proof.

Observe that $d_Y(f(x), f(y)) = 0 < \varepsilon$ for all $\varepsilon > 0$. □

Proposition

The identity function $f : X \to X$ is continuous.

Proof.

Fix $x \in X$ and let $\varepsilon > 0$. Choose $y \in X$ such that $d_X(x, y) < \varepsilon$. It follows that $d_X(f(x), f(y)) < \varepsilon$. $\square$

Proposition

The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \textit{if } x \leq 0 \\ 1 & \textit{if } x > 0 \end{cases}$$

is discontinuous at 0*.*

Proof.

Let $\varepsilon = \frac{1}{2}$. Fix $\delta > 0$ and put $y = \frac{\delta}{2}$. We have $|0 - y| = \frac{\delta}{2} < \delta$ and $|f(0) - f(y)| = 1 > \varepsilon$. $\square$

Example

For each $x \in \mathbb{R}$, the *evaluation map*

$$\begin{aligned} \varepsilon_x : C_0(\mathbb{R}) &\to \mathbb{R} \\ f &\mapsto f(x) \end{aligned}$$

is continuous with respect to the L^∞-metric and discontinuous with respect to the L^1-metric.

Section 4

Proofs with continuous functions

Proposition

If d_X is the discrete metric on X, then every function $f : X \to Y$ is continuous.

Proof.

Let $x \in X$ and fix $\varepsilon > 0$. If $y \in X$ with $d_X(x, y) < \frac{1}{2}$, then $x = y$ and it follows that $d_Y(f(x), f(y)) = 0 < \varepsilon$. □

Proposition

If $f : X \to Y$ is continous at $x \in X$, and if $g : Y \to Z$ is continuous at $f(x)$, then $g \circ f : X \to Z$ is continuous at x.

Proof.

Fix $\varepsilon > 0$. Choose $\delta' > 0$ so that $d_Z\big(g(f(x)), g(z)\big) < \varepsilon$ whenever $d_Y(f(x), z) < \delta'$, and $\delta > 0$ subject to the condition that $d_Y(f(x), f(y)) < \delta'$ whenever $d_X(x, y) < \delta$. Observe that

$$\begin{aligned} d_X(x,y) < \delta &\implies d_Y(f(x), f(y)) < \delta' \\ &\implies d_Z\big((g \circ f)(x), (g \circ f)(y)\big) < \varepsilon. \end{aligned}$$

□

Section 5

Convergence and continuity

Theorem

The function $f : X \to Y$ is continuous at $x \in X$ if and only if $x_i \to x$ implies $f(x_i) \to f(x)$ for all sequences $(x_i)_i \subseteq X$.

Proof.

$(\implies)$. Suppose that $x_i \to x$. Fix $\varepsilon > 0$, choose $\delta > 0$ so that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$, and choose $N \in \mathbb{N}$ so that $d_X(x_i, x) < \delta$ whenever $n > N$. It follows that $d_Y(f(x_i), f(x)) < \varepsilon$ for all $n > N$.
$(\impliedby)$. Suppose not. Then there is an $\varepsilon > 0$ such that for all $\delta > 0$ there is a $y \in X$ with $d_X(x, y) < \delta$ and $d_Y(f(x), f(y)) \geq \varepsilon$. In particular, for every $N \in \mathbb{N}$ there is an $x_N \in X$ with $d_X(x, x_N) < \frac{1}{N}$ and $d_Y(f(x), f(x_N)) \geq \varepsilon$. It follows that $x_i \to x$ and $f(x_i) \not\to f(x)$. This yields the desired contradiction. $\square$