Cardinality

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Math 300

Section 1

Equivalence of sets

Definition

Two sets A and B are *equivalent* (or in *one-to-one correspondence*) if there exists a bijection from A to B. In this case, we write $A \approx B$.

Informally, $A \approx B$ means that A and B have the same size.

Are they equivalent?

$$A = \{a, b, c\},\ B = \{1, 2, 3\}$$

Yes!

Are they equivalent?

$$A = \{a, b, c\},\ B = \{1, 2, 3, 4, 5\}$$

No!

Are they equivalent?

$$A = \{n \in \mathbb{N} \mid n \text{ even}\},\ B = \mathbb{N}$$

Yes!

Are they equivalent?

$$A = \mathbb{N},\ B = \mathbb{Z}$$

Yes!

Are they equivalent?

$$A = \mathbb{Z},\ B = \mathbb{Q}$$

Yes!

Are they equivalent?

$$A = \mathbb{Z},\ B = \mathbb{R}$$

No!

Is it true?

The countable union of countable sets is countable.

Yes!–wait No!–well, it depends. . .

Section 2

Cardinality

Definition

The *cardinality* of a finite set $A = \{a_1, \ldots, a_k\}$ is the number $k \in \mathbb{N}$ of elements in A.

We denote

$$\begin{aligned} \aleph_0 &= |\mathbb{N}| \\ \mathfrak{c} &= |\mathbb{R}|. \end{aligned}$$

and we write

$$\begin{aligned} |A| = |B| &\text{ when } \exists \text{ bijection } f : A \xrightarrow{\sim} B \\ |A| \leq |B| &\text{ when } \exists \text{ injection } f : A \hookrightarrow B \\ |A| < |B| &\text{ when } |A| \leq |B| \text{ and } |A| \neq |B| \end{aligned}$$

Let A and B be sets.

Theorem (Cantor–Schröder–Bernstein)

If $|A| \leq |B|$ *and* $|B| \leq |A|$, *then* $|A| = |B|$.

Proof.

Not today!

Proposition

We have $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

Proof sketch.

Define the function $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by $f(S) = 0.d_0 d_1 d_2 d_3 \ldots$ where $d_i = 1$ if $i \in S$ and $d_i = 0$ otherwise. As f is readily seen to be injective, we have $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Let $g : (0, 1) \to \mathcal{P}(\mathbb{N})$ be given by $g(0.b_0 b_1 b_2 \ldots) \subseteq \mathbb{N}$ include $i \in \mathbb{N}$ if and only if the binary digit $b_i = 1$. If $x \in (0, 1)$ has two distinct binary representations, let us take the one without the trailing digit 1. Since g is injective, we have $|(0, 1)| \leq |\mathcal{P}(\mathbb{N})|$. The result follows as $|\mathbb{R}| = |(0, 1)|$. $\square$

Let A be a set.

Theorem (Cantor's theorem)

We have $|A| < |\mathcal{P}(A)|$.

Proof.

Since

$$\begin{aligned} i : A &\to \mathcal{P}(A) \\ a &\mapsto \{a\} \end{aligned}$$

is an injection, it follows that $|A| \leq |\mathcal{P}(A)|$.

It remains to show that $|A| \neq |\mathcal{P}(A)|$.

Proof *(continued)*.

Suppose for a contradiction that $|A| = |\mathcal{P}(A)|$. In particular, there is a bijection $f : A \to \mathcal{P}(A)$. Put

$$B = \{a \in A \mid a \notin f(a)\}.$$

Since f is bijective, there is a $b \in A$ with $f(b) = B$. If $b \in f(b) = B$, then $b \notin B$. But if $b \notin f(B) = B$, then $b \in B$. This provides the desired contradiction. $\square$

Corollary

There is no set of all sets.

Proof.

Suppose to the contrary that S is the set of all sets. It follows that $\mathcal{P}(S) \subseteq S$, from which $|\mathcal{P}(S)| \leq |S|$ and thus $|\mathcal{P}(S)| = |S|$. This contradicts the fact that $|S| < |\mathcal{P}(S)|$. $\square$

Continuum hypothesis

Hypothesis (Continuum hypothesis)

There does not exist a set A with

$$\aleph_0 < |A| < \mathfrak{c}.$$

This statement can be neither proven nor disproven (in ZFC).

Section 3

Bonus topics

The axiom of choice

Axiom (Axiom of choice)

If $\mathcal{A}$ is a collection of nonempty sets, then there exists a function

$$F : \mathcal{A} \to \bigcup_{A \in \mathcal{A}} A$$

with $F(A) \in A$ for each $A \in \mathcal{A}$.

The function F is called a *choice function* (or *choice rule*, *selector*, *selection*).

Law of the excluded middle

Axiom (Law of the excluded middle)

$$\forall\phi : \phi \vee \neg\phi$$

This is implied by the axiom of choice.

Not to be confused with the

Principle (Principle of bivalence)

There are precisely two truth values. Every proposition is either true or false.

Well-ordering principle

Principle

Every nonempty subset $S \subseteq \mathbb{N}$ *has a least element.*

This must either be assumed or derived.