4.8 Other Discrete Probability Distributions

4.8.1 Geometric Random Variable

- independent Bernoulli trials, each with probability p of success, are performed until success occurs.
- Let X represent the number of trials required until success.
- So
$$P\{X=n\} = \underbrace{(1-p)^{n-1}}_{n-1 \text{ failures}} \cdot \underbrace{p}_{\text{success on } n\text{th trial}}$$

X is a geometric random variable

Comments

- $$\sum_{n=1}^{\infty} P\{X=n\} = \sum_{n=1}^{\infty} (1-p)^{n-1} \cdot p = p \cdot \frac{1}{1-(1-p)} = 1 \checkmark$$

(Example 8b) p. 148

- $$E[X] = \sum_{n=1}^{\infty} n(1-p)^{n-1} p$$
$$= \sum_{n=1}^{\infty} (n+1-1)(1-p)^{n-1} p$$
$$= \sum_{n=1}^{\infty} (1-p)^{n-1} p + \sum_{n=1}^{\infty} (n-1)(1-p)^{n-1} p$$
$$= 1 + \sum_{n=2}^{\infty} (n-1)(1-p)^{n-1} p$$

let $m=n-1$

Geometric Random Variable, X

$$E[X] = 1 + \sum_{m=1}^{\infty} m(1-p)^m p$$

$$= 1 + (1-p)\underbrace{\sum_{m=1}^{\infty} m(1-p)^{m-1} p}_{E[X]}$$

So $E[X] = 1 + (1-p)E[X]$

$E[X](1-1+p) = 1 \quad \Rightarrow \quad \boxed{E[X] = \frac{1}{p}}$

- $\boxed{Var(X) = \frac{1-p}{p^2}}$ (see Example 8c)

Example

Roll a fair die until a 6 appears. Let X = # of rolls required. ($p = \frac{1}{6}$ = probability of a success = roll a 6)

$$P\{X = i\} = \left(\frac{5}{6}\right)^{i-1} \cdot \left(\frac{1}{6}\right)$$ geometric random variable

$E[X] = \frac{1}{p} = \underline{\underline{6}}$ ← expected # of rolls to get a 6.

$$Var(X) = \frac{1-p}{p^2} = \frac{1 - 1/6}{\left(\frac{1}{6}\right)^2} = \frac{5}{6} \cdot 6^2 = \underline{\underline{30}}$$

$SD(X) = \sqrt{Var(X)} \approx \underline{\underline{5.477}}$

EXAMPLE - see Problem $\boxed{4.30}$

4.8.2 Negative Binomial Random Variable

- Independent trials each with probability p of being a success are performed until r successes are accumulated.

  - Let X = # of trials required

probability mass function $p(n) = P\{X=n\} = \underbrace{\binom{n-1}{r-1} p^{r-1} \cdot (1-p)^{n-r}}_{\text{r-1 successes in first n-1 trials}} \cdot \underbrace{p}_{\text{n}^{\text{th}}\text{ trial (last one) is a success.}}$

for $n \geq r$

X is a negative binomial random variable with parameters r and p.

Comments:

- A geometric random variable is a special case of the negative binomial random variable with $r = 1$.

- $\boxed{E[X] = \frac{r}{p}}$
- $\boxed{Var(X) = \frac{r(1-p)}{p^2}}$

see Example 8f (pp. 150-151)

see also notes p. 146

EXAMPLE (8g, p. 151)

Find $E[X]$ and $Var(X)$ of the number of times one must throw a die until 1 appears 4 times. Here $r=4$, $p=\frac{1}{6}$ so

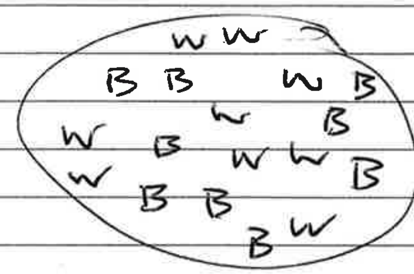
$E[X] = \frac{4}{(1/6)} = 24$ $\quad Var(X) = \frac{4(5/6)}{(1/6)^2} = 4 \cdot 5 \cdot 6 = 120$

4.8.3 The Hypergeometric Random Variable

- example - drawing balls from urn w/o replacement (see notes p. 107.1 →)

- n balls chosen randomly (without replacement) from an urn containing N balls of which

  m white balls
  $N-m$ black balls



- let X denote # of white balls selected

prob. mass function

white black

$$p(i) = P\{X = i\} = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{m}} \qquad i = 0, 1, \dots, n \qquad n \leq N$$

← total # of ways to choose n balls out of N

- we've seen already (notes p. 107.1) that

$$\boxed{E[X] = \frac{n \cdot m}{N}}$$

- also now (see text, p. 154)

$$\boxed{Var(X) = np(1-p)\left(1 - \frac{n-1}{N-1}\right)} \quad \text{where } p = \frac{m}{N}$$

- Recall that sampling with replacement corresponded to binomial random variable from p 107.1

$$P\{X=i\} = \frac{\binom{N_R}{i}\binom{N_G}{N-i}}{\binom{N_R+N_G}{N}} \quad \text{for } i=0,1,2,\ldots,N$$

urn w/ N_R red, N_G green balls

N balls chosen from urn w/o replacement.

here $N_R \to m$

$N_G \to N-m$

$N_R+N_G \to N$

$N \to n$