Worksheet 6

Name: ______________________________

1. The set of *Gaussian integers* is
$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}.$$
Which of the following terms describe the Gaussian integers?

   rng, (commutative) ring, integral domain, field, $\mathbb{Z}$-module, $\mathbb{R}$-vector space

2. Let X be a set. Which of the following terms describe $(\mathcal{P}(X), \Delta, \cap)$?

   rng, (commutative) ring, integral domain, field

3. Which of the following terms describe
$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}?$$

   rng, (commutative) ring, integral domain, field

4. Let k be a field. A *polynomial* over k with indeterminate X is an expression of the form
$$c_0 + c_1 X + c_2 X^2 + \cdots + c_n X^n,$$
where $c_0, \ldots, c_n \in k$ and X is a formal symbol. For example,
$$2X^2 + 3, \qquad X^9, \qquad X^2 + 3X - 2$$
are polynomials over $\mathbb{R}$. Write $k[X]$ for the set of polynomials in X over k equipped with the familiar polynomial addition and multiplication operations. Which of the following terms describe $k[X]$?

   rng, (commutative) ring, integral domain, field

5. Let $(G, +)$ be an abelian group and define the operation
$$\cdot : \mathbb{Z} \times G \to G$$
by
$$k \cdot g = \underbrace{g + \cdots + g}_{k \text{ times}}$$
if $k > 0$, by
$$k \cdot g = \underbrace{(-g) + \cdots + (-g)}_{k \text{ times}}$$
if $k < 0$, and by $0 \cdot g = 0$.[1] Is G a $\mathbb{Z}$-module?

[1] That is, $0_{\mathbb{Z}} \cdot g = 0_G$, where $0_G \in G$ is the identity in G.

6. What are the additive inverses of $0, 1, 2, 3, 4 \in \mathbb{Z}_5$?
7. What are the multiplicative inverses of $1, 2, 3, 4 \in \mathbb{Z}_5$?
8. Let k be a field and fix $n \in \mathbb{N}$. Is

$$k^n = \{(a_1, \ldots, a_n) \mid a_1, \ldots a_n \in k\}$$

necessarily a k-vector space?[2]

9. Let $(R, +, \cdot)$ be a ring with additive identity $0 \in R$. That is, 0 is the (unique) element in R satisfying

$$\forall a \in R : a + 0 = a = 0 + a.$$

Prove that $0 \cdot a = 0 = a \cdot 0$.

Hint. Use the distributive property.

10. Consider the operation

$$\cdot : \mathbb{F}_2 \times \mathbb{R} \to \mathbb{R}$$

given by

$$\begin{aligned} 0 \cdot x &= 0 \\ 1 \cdot x &= x \end{aligned}$$

for $x \in \mathbb{R}$. Is $(\mathbb{R}, +, \cdot)$ an $\mathbb{F}_2$-vector space?

[2]By convention, we define $k^0 = \{0_k\}$