Worksheet 3

Let A, B, *and* C be sets. Prove or disprove the following statements.

1. If $A \cap B = \emptyset$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$

   Proof. Let $A = \{a\}$ and $C = \{a, c\}$ and $B = \{b\}$. Observe $\{a\} \cap \{b\} = \emptyset$ and $\{b\} \cap \{a, c\} = \emptyset$ while $\{a\} \cap \{a, c\} = \{a\}$ $\square$

2. If $A \not\subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$

   Proof. Let $A = \{a\}$ and $C = \{a, c\}$ and $B = \{b\}$ Observe $\{a\} \not\subseteq \{b\}$ and $\{b\} \not\subseteq \{a, c\}$, while $\{a\} \subset \{a, c\}$ $\square$

3. If $A \subseteq \emptyset$, then $a = \emptyset$

   Proof. Assume the negation $A \subseteq \emptyset$ and $A \neq \emptyset$. If $A \neq \emptyset$ then $A \not\subseteq \emptyset$ by definition of $\emptyset$ $\square$

4. If $A \subseteq C$ and $B \subseteq C$, then $A \cap B \subseteq C$

   Proof. Assume that $A \subseteq C$ and $B \subseteq C$ there for 2 cases can occur for $A \cap B \subseteq C$
   Case 1: $A \cap B = \emptyset$ there for $A \cap C \subseteq C$ as $\emptyset \subset C$
   Case 2: $A \cap B \neq \emptyset$ then $\forall e \in A \cap B : e \in C$ there for $A \cap B \subseteq C$ $\square$

5. If $f : A \to B$ is injective and $g : B \to C$ is injective, then $g \circ f : A \to C$ is injective.

   Proof. Assume that $\forall x, y \in A$ if $f(x) = f(y)$ then $x = y$ and the same for g. $f(A) \subseteq B$ and $g(B) \subseteq C$ there for as both f and g are injective, the subset of B passed from f to g will also be injective. Hence $g \circ f$ is injective. $\square$

6. If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is surjective

   Proof. By the definition of surjective f maps to all values in B, similarly g maps to all values in C. Hence $g \circ f$ maps to all values in C and is surjective. $\square$

7. Give an example of a function $f : A \to A$ that is injective but not surjective.

Proof. $g : b \mapsto 2b$ maps to only the even co-domain $\square$

8. Give an example of a function $g : A \to A$ that is surjective but not injective.

   Proof. $f : a \mapsto \sin(a)$ every number in the co-domain is covered, but multiple numbers in the domain map to the same value. $\square$

9. Let $f : A \to B$ and $g : B \to A$. If $g \circ f = id_a$, then both f and g are bijections.

   Proof. As previously proved the composition of two surjective functions are surjective and the same for injective, hence for $g \circ f$ to be bijective both f and g must also be bijective. $\square$

10. If $f : A \to A$ is surjective, and if A is a finite set, then f is injective.

    Proof. By definition of surjective $\forall a \in A : \exists b \in A : f(b) = a$. By definition of a function no parameter may map to more than one value. Hence, if the domain and co-domain are both a finite set and the function is surjective then the function must be injective. $\square$

11. If $f : A \to A$ satisfies the property that $f \circ f = id_a$ then f is a bijection.

    Proof. As previously proved the composition of two surjective functions are surjective and the same for injective, hence for $f \circ f$ to be bijective f must also be bijective. $\square$