ξ2. THE THEOREM OF SARD AND BROWN

In general, it is too much to hope that the set of critical values of a smooth map be finite. But this set will be "small," in the sense indicated by the next theorem, which was proved by A. Sard in 1942 following earlier work by A.P. Morse, (References [30], [24].)

Theorem. *let $f : U \to R^n$ be a smooth map, defined on an open set $U \subset R^m$, and let*

$$C = \{x \in U | \; rank \; df_x < n\}$$

Then the image $f(C) \subset R^n$ has Lebesgue measure zero *

Since a set of measure zero cannot contain any nonvacuous open set, it follows that the complement $R^n - f(C)$ must be everywhere dense† in R^n.

The proof will be given in ξ3. It is essential for the proof that f should

*In other words, given any $\xi > 0$, it is possible to cover $f(C)$ by a sequence of cubes in R^n having total n-dimensional volume less than ξ

†Proved by Arthur B. Brown in 1935. This result was rediscovered by Dubovickil in 1953 and by Thorn in 1954. (References [5], [8], [36].)

have many derivatives. (Compare Whitney [38].)
We will be mainly interested in the case $m \geq n$. If $m < n$, then clearly $C = U$; hence the theorem says simply that $f(U)$ has measure zero.
More generally consider a smooth map $f : M => N$, from a manifold of dimension m to a manifold of dimension n. Let C be the set of all $x \in \mathbb{M}$ such that

$$df_x : TM_x \to TN_{f(x)}$$

has rank less than n (i.e. is not onto). Then C will be called the set of critical points, $f(C)$ the set of critical values, and the complement $N - f(C)$ the set of regular values of f. (This agrees with our previous definitions in the case $m = n$.) Since M can be covered by a countable collection of neighborhoods each diffeomorphic to an open subset of R^m, we have:

Corollary *(A. B. Brown). The set of regular values of a smooth map $f : M \to N$ is everywhere dense in N.*

In order to exploit this corollary we will need the following:

Lemma 1. *If $f : M \to N$ is a smooth map between manifolds of dimension $m \geq n$, and if $y \in N$ is a regular value, then the set $f^{-1}(y) \subset M$ is a smooth manifold of dimension $m - n$.*

Proof. Let $x \in f^{-1}(y)$. Since y is a regular value, the derivative df_x must map TM_x onto TN_y. The null space $\mathfrak{R} \subset TM_x$ of df_x will therefore be an $(m-n)$-dimensional vector space. □

If $M \subset R^k$ choose a linear map $L : R^K \to R^{m-n}$ that is nonsingular on this subspace $\mathfrak{R} \subset TM_x \subset R^k$. Now define

$$F : M \to NXR^{m-n}$$

by $F(\xi) = (f(\xi), L(\xi))$. The derivative dF_x is clearly given by the formula

$$dF_x(v) = (df_x(v), L(v)).$$

Thus dF_x is nonsigular. Hence F maps some neighborhood U of x diffeomorphically onto a neighborhood V of $(y, L(x))$. Note that $f^{-1}(y)$ corresponds, under F, to hyperplane $yXR^{m-n} \cap V$. This proves that $f^{-1}(y)$ is a smooth manifold of dimension $m-n$. As an example we can give an easy proof that the unit sphere S^{n-1} is a smooth manifold. Consider the function $f : R^m \to R$ defined by

$$f(x) = x_1^2 + x_2^2 + ... + x_m^2.$$

Any $y \neq 0$ is a regular value, and the smooth manifold $f^{-1}(1)$ is the unit sphere. If M' is a manifold which is contained in M, it has already been noted that TM'_x for $x \in M'$ is a subscape of TM_x for $x \in M'$. The orthogonal complement of TM'_x in TM_x is then a vector space of dimension $m - m'$ called the *the space of normal vectors to M' in M at x*. In particular let $M' = f^{-1}(y)$ for a regular value y of $f : M \to N$.

Lemma 2. *The null space of $df_x : TM_x \to TN_y$ is precisely equal to the tangent space $TM'_x \subset TM_x$ of the submanifold $M' = f^{-1}(y)$. Hence df_x maps the orthogonal complement of TM'_x isomorphically onto TN_y.*

Proof. From the diagram

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & M \arrow[d] \\ y \arrow[r] & N \end{tikzcd}$$

we see that df_x maps the subspace $TM'_x \subset TM_x$ to zero. Counting dimensions we see that df_x *maps the space of normal vectors to M' isomorphically onto* TN_y. □

0.1 Manifolds with Boundary

The lemmas above can be sharpened so as to apply to a map defined on a smooth "manifold with boundary." Consider first the closed half-space

$$H^m = (x_1, ..., x_m) \in R^m | x_m \geq 0.$$

The boundary ∂H^m is defined to be the hyperplance $R^{m-1} \times 0 \subset R^M$.

DEFINITION *A subset $X \subset R^k$ is called a* smooth m-manifold with boundary *if each $x \in X$ has a neighbohood $U \cup X$ diffeomorphic to an open subset $V \cup H^m$ of H^m. The* boundary *∂X is the set of all points in X which correspond to points of ∂H^m under such a diffeomorphism.*

It is not hard to show that ∂x is a well-defined smooth manifold of dimension $m-1$. The *interior* $X - \partial X$ is a smooth manifold of dimension m. The tangent space TX_z is defined just as in $\xi 1$, so that TX_x is a full m-dimensional vector space, even if x is a boundary point. Here is one method for generating examples. Let M be a manfiold without boundary and let $g : M \to R$ have 0 as regular value.

Lemma 3. *The set of x in M with $g(x) \geq 0$ is a smooth manifold, with boundary equal to $g^{-1}(0)$.*

The proof is just like the proof of Lemma 1.