

Worksheet 4

Let R be a relation from A to B , let S be a relation from B to C , and let T be a relation from C to D .

Prove the following statements.

1. $I_A \circ R = R$

Proof. Let $a \in A$ and $b \in B$ and observe:

$$\begin{aligned} a(I_A \circ R)b &\iff \exists a' \in A : a = a' \wedge a'Rb \\ &\iff aRb \end{aligned}$$

□

2. $R \circ I_A = R$

Proof. Let $a \in A$ and $b \in B$ and observe:

$$\begin{aligned} a(R \circ I_A)b &\iff \exists b' \in B : b = b' \wedge aRb' \\ &\iff aRb \end{aligned}$$

□

3. $(R^{-1})^{-1} = R$

Proof. Assume the relation R has an inverse and let $a \in A$ and $b \in B$:

$$\begin{aligned} a(R^{-1})^{-1}b &\iff bR^{-1}a \\ &\iff b(R^{-1})^{-1}a \end{aligned}$$

□

4. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Proof. Let $a \in A$, $b \in B$, and $c \in C$

$$\begin{aligned} c(S \circ R)^{-1}a &\iff \exists c' \in C : c = c' \wedge c'S^{-1}b \\ &\iff \exists b' \in B : b = b' \wedge b'R^{-1}a \\ &\iff c(R^{-1} \circ S^{-1})a \end{aligned}$$

□

5. $(T \circ S) \circ R = T \circ (S \circ R)$

Proof. Fix $a \in A$, $b \in B$, $c \in C$, and $d \in D$

$$\begin{aligned}
a(T \circ S) \circ Rd &\iff \exists a' \in A : a' = a \wedge a' Rb \\
&\iff \exists b' \in B : b' = b \wedge b' Sc \\
&\iff \exists c' \in C : c' = c \wedge c' Td \\
&\iff aT \circ (S \circ R)d
\end{aligned}$$

□

6. $Dom R = Rng R^{-1}$

Proof. (\subseteq) Suppose R is invertible and fix $r \in Rng R^{-1}$. By definition of inverse it follows that $r \in Dom R$. □

Proof. (\supseteq) Fix $d \in Dom R$. By definition of domain it follows that $d \in Rng R^{-1}$. □

7. $Rng R = Dom R^{-1}$

Proof. (\supseteq) Suppose R is invertible and $r \in Rng R$. By the definition of inverse $r \in Dom R^{-1}$. □

Proof. (\subseteq) Fix $d \in Dom R^{-1}$. By the definition of domain it follows that $d \in Rng R$. □

For Question 8–10, suppose that $A = B = C$.

8. If R and S are equivalence relations, then $S \circ R$ is an equivalence relation.

Proof. Suppose R is an equivalence relation from A to B and S is an equivalence relation from B to C and $A = B = C$.

$$\begin{aligned}
S \circ R &\iff \forall a \in A : aSa \wedge aRa \\
&\iff \forall a, b, c \in A : (aSb \wedge bSc) \Rightarrow aSa \wedge (aRb \wedge bRc) \Rightarrow aRc \\
&\iff \forall a, b \in A : (aSb \wedge bSa) \wedge (aRb \wedge bRa)
\end{aligned}$$

□

9. If R is a partial order, then $R \circ R$ is a partial order.

Proof. Suppose R is a partial order from A to B and $A = B$

$$\begin{aligned}
R &\iff \forall a \in A : aRa \\
&\iff \forall a, b, c \in A : (aRb \wedge bRc) \Rightarrow aRc \\
&\iff \forall a, b \in A : (aRb \wedge bRa) \Rightarrow a = b \\
&\iff R \circ R
\end{aligned}$$

□

10. If R and S are partial orders, then it is not generally true that $S \circ R$ is a partial order.

Proof. Let $R = \leq$ and $S = |$. Fix $a = 3$ and $b = 5$. Observe that $3 \leq 5$, however $3 \nmid 5$ hence $S \circ R$ is not a partial order. \square

Bonus Questions Give an example of two relations R and S on a set A such that

11. $R \circ S \neq S \circ R$.

Proof. Suppose $R = \leq$ and $S = |x|$. Fix $a = -3$ and $b = 5$. Observe that $-3(R \circ S)5 \neq -3(S \circ R)5$. \square

12. $S \circ R$ is an equivalence relation, but neither R nor S is an equivalence relation.

Proof. \square