Math 351 - Spring 2025: Homework 6

Due: Wednesday, April 23, 2025

Instructions: Be sure to give explanations to your answers. I'm interested not only in whether you get the correct answer but also how you obtained it and your thought process along the way. Don't just write down a number even if the answer seems obvious.

1. The joint probability density function of X and Y is given by $f(x, y) = c$, where c is a positive constant, for $0 < y < x^3 < 1$ and $0 < x < 1$.
   (a) Find the value of c and sketch the region on which $f(x, y) > 0$.
   (b) Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$.
   (c) Are X and Y independent? Why or why not?

2. As in the Buffon's Needle Problem, consider a floor with floorboards that have width W such that the floor has parallel lines with the lines each spaced a distance W apart from the next line. Now suppose you have a collection of coins of varying sizes such that the coin radius is a uniformly distributed continuous random variable on the interval $[r_1, r_2]$ where r_1 is the smallest coin radius and r_2 is the largest coin radius. For $W = 1$, $r_1 = 1/4$ and $r_2 = 3/8$ find the probability that if you toss a coin from this collection it lands on the floor in such a way that it covers one of the lines. Use a joint probability argument to obtain your prediction.

3. The joint probability density function of X and Y is given by $f(x, y) = x^2y^2$, for $0 < x < 2$, $0 < y < 1$.
   (a) Find the marginal densities of X and Y.
   (b) Find $P\{X < Y\}$ and $P\{Y < X\}$.
   (c) Find the distribution function of $Z = X + Y$.

4. Two points are selected on a line of length L in the following way: The first point is selected randomly on one half of the line and the second point is selected randomly on the other half of the line. Thus, the first point is uniformly distributed on $(0, L/2)$ and the second point is uniformly distributed on $(L/2, L)$. Find the probability that the distance between the two points is greater than $3L/8$.

5. If X is uniformly distributed on $(0, 1)$ and Y is exponentially-distributed with parameter $\lambda = 1$, find the distribution function and the density function for $Z = X + Y$. Assume that X and Y are independent.

6. Suppose that three balls are chosen without replacement from an urn containing 5 white balls and 8 red balls. Let $X_i = 1$ if the i^{th} ball selected is white, and let it equal 0 otherwise. Find the joint probability mass function of (a) X_1 and X_2, and (b) X_1, X_2, and X_3.