Worksheet 4

Let R be a relation from A to B, let S be a relation from B to C, and let T be a relation from C to D.
Prove the following statements.

1. $I_A \circ R = R$

   Proof. Let $a \in A$ and $b \in B$ and observe:

   $$\begin{aligned} a(I_a \circ R)b &\iff \exists a^{`} \in A : a = a^{`} \wedge a^{`}Rb \\ &\iff aRb \end{aligned}$$

   □

2. $R \circ I_A = R$

   Proof. Let $a \in A$ and $b \in B$ and observe:

   $$\begin{aligned} a(R \circ I_a)b &\iff \exists b^{'} \in B : b = b^{'} \wedge aRb^{'} \\ &\iff aRb \end{aligned}$$

   □

3. $(R^{-1})^{-1} = R$

   Proof. Fix $a \in A$ and $b \in B$:

   $$\begin{aligned} a(R^{-1})^{-1}b &\iff bR^{-1}a \\ &\iff b(R^{-1})^{-1}a \end{aligned}$$

   □

4. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

   Proof. Suppose $(c, a) \in (S \circ R)^{-1}$. Then by implication: $a(S \circ R)^{-1}$. Hence, there exists a $b \in B$ such that bSc and aRb and $cS^{-1}b$ and $bR^{-1}a$. Therefore $(c, a) \in (R^{-1} \circ S^{-1})$ and $(R^1 \circ S^{-1}) \subseteq (S \circ R)^{-1}$. The converse implication is obtained by retracing the given steps. □

5. $(T \circ S) \circ R = T \circ (S \circ R)$

Proof. Assume $(a,d) \in (T \circ S) \circ R$. It follows that $b \in B$ such that aRb and $b(T \circ S)d$. Hence there is a $c \in C$ such that bSc and cTd. This implies $a(S \circ R)c$, hence $aT \circ (S \circ R)d$. So we can conclude $T \circ (S \circ R) \subseteq (T \circ S) \circ R$. The converse implication is similar. □

6. $Dom\, R = Rng\, R^{-1}$

   Proof. ($\subseteq$) Fix $a \in A$ and observe that $a \in Dom\, R$. There there must exist $b \in B$ such that aRb and $bR^{-1}a$. Hence $a \in Rng\, R^{-1}$ and $Rng\, R^{-1} \subseteq Dom\, R$. □

   Proof. ($\supseteq$) Fix $a \in A$ and observe $a \in Rng\, R^{-1}$. There must be $b \in B$ such that bR^1a and aRb. Hence $a \in Dom\, R$ and $Dom\, R \subseteq Rng\, R^{-1}$. □

7. $Rng\, R = Dom\, R^{-1}$

   Proof. ($\supseteq$) Suppose $b \in Rng\, R$. This implies $a \in A$ such that aRb and bR^1a. Hence by the invertibility of R, $b \in Dom\, R^{-1}$ and $Dom\, R^{-1} \subseteq Rng\, R$. □

   Proof. ($\subseteq$) Fix $b \in Dom\, R^{-1}$. By implication we have $a \in A$ such that $bR^{-1}a$ and aRb. So it follows that $b \in Rng\, R$ and $Rng\, R \subseteq Dom\, R^{-1}$. □

For Question 8–10, suppose that $A = B = C$.

8. If R and S are equivalence relations, then $S \circ R$ is an equivalence relation.

   Proof. Suppose R is an equivalence relation from A to B and S is an equivalence relation from B to C and $A = B = C$.

$$\begin{aligned} S \circ R &\iff \forall a \in A : aSa \wedge aRa \\ &\iff \forall a,b,c \in A : (aSb \wedge bSc) => aSa \wedge (aRb \wedge bRc) => aRc \\ &\iff \forall a,b \in A : (aSb \wedge bSa) \wedge (aRb \wedge bRa) \end{aligned}$$

□

9. If R is a partial order, then $R \circ R$ is a partial order.

   Proof. Fix $a,b,c \in A$:

$$\begin{aligned} aRa \wedge aRc &\iff aRa \Rightarrow A(R \circ R)a \\ &\iff \forall a,b,c \in A : (aRb \wedge bRc) => aRc \\ &\iff \forall a,b \in A : (aRb \wedge bRa) => a = b \\ &\iff a(R \circ R)a \wedge a(R \circ R)c \end{aligned}$$

□

10. If R and S are partial orders, then it is not generally true that $S \circ R$ is a partial order.

    Proof. Let $R = \leq$ and $S = |$. Fix $a = 3$ and $b = 5$. Observe that $3 \leq 5$, however $3|5$ hence $S \circ R$ is not a partial order. $\square$

Bonus Questions Give an example of two relations R and S on a set A such that

11. $R \circ S \neq S \circ R$.

    Proof. Suppose $R = \leq$ and $S = |x|$. Fix $a = -9$ and $b = 5$. Observe that $-9(R \circ S)5 \neq -9(S \circ R)5$. $\square$

12. $S \circ R$ is an equivalence relation, but neither R nor S is an equivalence relation.

    Proof. $\square$