

# Worksheet 4

Let  $R$  be a relation from  $A$  to  $B$ , let  $S$  be a relation from  $B$  to  $C$ , and let  $T$  be a relation from  $C$  to  $D$ .

Prove the following statements.

1.  $I_A \circ R = R$

*Proof.* Let  $a \in A$  and  $b \in B$  and observe:

$$\begin{aligned} a(I_A \circ R)b &\iff \exists a' \in A : a = a' \wedge a'Rb \\ &\iff aRb \end{aligned}$$

□

2.  $R \circ I_A = R$

*Proof.* Let  $a \in A$  and  $b \in B$  and observe:

$$\begin{aligned} a(R \circ I_A)b &\iff \exists b' \in B : b = b' \wedge aRb' \\ &\iff aRb \end{aligned}$$

□

3.  $(R^{-1})^{-1} = R$

*Proof.* Assume the relation  $R$  has an inverse and let  $a \in A$  and  $b \in B$ :

$$\begin{aligned} a(R^{-1})^{-1}b &\iff bR^{-1}a \\ &\iff b(R^{-1})^{-1}a \end{aligned}$$

□

4.  $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

*Proof.* Let  $a \in A$ ,  $b \in B$ , and  $c \in C$

$$\begin{aligned} c(S \circ R)^{-1}a &\iff \exists c' \in C : c = c' \wedge c'S^{-1}b \\ &\iff \exists b' \in B : b = b' \wedge b'R^{-1}a \\ &\iff c(R^{-1} \circ S^{-1})a \end{aligned}$$

□

5.  $(T \circ S) \circ R = T \circ (S \circ R)$

*Proof.* Fix  $a \in A$ ,  $b \in B$ ,  $c \in C$ , and  $d \in D$

$$\begin{aligned} a(T \circ S) \circ Rd &\iff \exists a' \in A : a' = a \wedge a' Rb \\ &\iff \exists b' \in B : b' = b \wedge b' Sc \\ &\iff \exists c' \in C : c' = c \wedge c' Td \\ &\iff aT \circ (S \circ R)d \end{aligned}$$

□

6.  $\text{Dom } R = \text{Rng } R^{-1}$

*Proof.* ( $\subseteq$ ) Suppose  $R$  is invertible and fix  $r \in \text{Rng } R^{-1}$ . By definition of inverse it follows that  $r \in \text{Dom } R$ . □

*Proof.* ( $\supseteq$ ) Fix  $d \in \text{Dom } R$ . By definition of domain it follows that  $d \in \text{Rng } R^{-1}$ . □

7.  $\text{Rng } R = \text{Dom } R^{-1}$

*Proof.* ( $\supseteq$ ) Suppose  $R$  is invertible and  $r \in \text{Rng } R$ . By the definition of inverse  $r \in \text{Dom } R^{-1}$ . □

*Proof.* ( $\subseteq$ ) Fix  $d \in \text{Dom } R^{-1}$ . By the definition of domain it follows that  $d \in \text{Rng } R$ . □

For Question 8–10, suppose that  $A = B = C$ .

8. If  $R$  and  $S$  are equivalence relations, then  $S \circ R$  is an equivalence relation.

*Proof.* □

9. If  $R$  is a partial order, then  $R \circ R$  is a partial order.

*Proof.* □

10. If  $R$  and  $S$  are partial orders, then it is not generally true that  $S \circ R$  is a partial order.

*Proof.* □

**Bonus Questions** Give an example of two relations  $R$  and  $S$  on a set  $A$  such that

11.  $R \circ S \neq S \circ R$ .

*Proof.* □

12.  $S \circ R$  is an equivalence relation, but neither  $R$  nor  $S$  is an equivalence relation.

*Proof.* □