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6.5 Conditional Distributions: Continuous Case (p. 250

Def. For continuous random variables X and Y, the conditional probability density function of X given $Y=y$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \qquad \text{as long as } f_Y(y) > 0$$

where $f(x,y)$ is the joint pdf for X and Y and $f_Y(y)$ is the marginal pdf for Y.

Def. The conditional cumulative distribution function of X given $Y=y$ is

$$F_{X|Y}(a|y) = P\{X \le a \mid Y=y\} = \int_{-\infty}^{a} f_{X|Y}(x|y)\,dx$$

[see p. 251

Interpretation:

$$f_{X|Y}(x|y)\cdot dx = \frac{f(x,y)\,dx\,dy}{f_Y(y)\,dy}$$

$$\approx \frac{P\{x \le X \le x+dx,\ y \le Y \le y+dy\}}{P\{y \le Y \le y+dy\}}$$

see book pp. 250-251

So for small dx, dy

$f_{X|Y}(x|y)\,dx$ represents the conditional probability that X is on $[x, x+dx]$ given that Y is on $[y, y+dy]$

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EXAMPLE (Problem $\boxed{6.41}$)

The joint density function of X and Y is

$$f(x,y) = x e^{-x(y+1)} \qquad \begin{matrix} x>0 \\ y>0 \end{matrix} \qquad 0 \text{ otherwise}$$

a) Find the conditional density of X given $Y=y$

AND

the conditional density of Y given $X=x$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \qquad \text{AND} \qquad f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

- $$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} x e^{-x(y+1)}\,dx$$

$$= \cancel{-\frac{x e^{-x(y+1)}}{(y+1)}}\Bigg|_0^{\infty} + \int_0^{\infty} \frac{e^{-x(y+1)}}{(y+1)}\,dx \qquad \begin{matrix} u = x & dv = e^{-x(y+1)}dx \\ du = dx & v = -\frac{e^{-x(y+1)}}{(y+1)} \end{matrix}$$

$$= \frac{1}{(y+1)}\left[-\frac{e^{-x(y+1)}}{(y+1)}\right]_0^{\infty} = \frac{1}{(y+1)^2} \qquad y>0$$

So $$\boxed{f_{X|Y}(x|y) = \frac{x e^{-x(y+1)}}{\left(\frac{1}{(y+1)^2}\right)} = x(y+1)^2 e^{-x(y+1)} \qquad x>0,\ y>0}$$

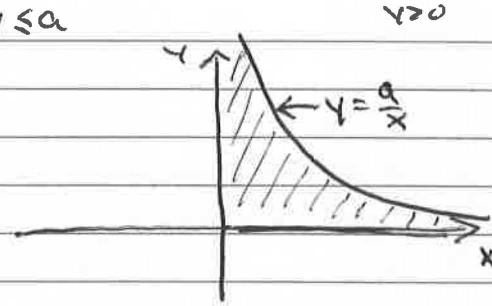
- $$f_X(x) = \int_0^{\infty} x e^{-x(y+1)}\,dy = -e^{-x(y+1)}\Big|_0^{\infty} = e^{-x}$$

So $$\boxed{f_{Y|X}(y|x) = \frac{x e^{-x(y+1)}}{e^{-x}} = x e^{-xy} \qquad x>0,\ y>0}$$

b) Find the density function of $Z_1 = XY$

$$F_{Z_1}(a) = P\{Z_1 \le a\}$$

$$= P\{XY \le a\} = \iint_{\substack{xy \le a}} f(x,y)\,dx\,dy \qquad \begin{matrix} x>0 \\ y>0 \end{matrix}$$



$$= \int_0^\infty \int_0^{\frac{a}{y}} f(x,y)\,dx\,dy$$

$$= \int_0^\infty \int_0^{\frac{a}{y}} x e^{-x(y+1)}\,dx\,dy$$

OR (for easier integration)

$$= \int_0^\infty \int_0^{\frac{a}{x}} x e^{-x(y+1)}\,dy\,dx$$

$$= \int_0^\infty \left[-e^{-x(y+1)} \right]_{y=0}^{y=\frac{a}{x}} dx$$

$$= \int_0^\infty -e^{-x\left(\frac{a}{x}+1\right)} + e^{-x}\,dx$$

$$= \int_0^\infty \left(-e^{-a-x} + e^{-x}\right) dx$$

$$e^{-(a+x)} - e^{-x} \Big|_0^\infty = -(e^{-a}-1) = 1-e^{-a}$$

$$\boxed{F_{Z_1}(a) = 1-e^{-a}} \quad \text{then} \quad \boxed{f_{Z_1}(a) = \frac{d}{da}F_{Z_1}(a) = e^{-a}}$$

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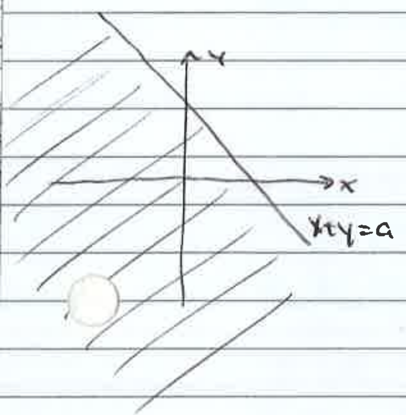
6.3 Sums of Independent Random Variables

Suppose X and Y are continuous, independent, random variables. What is the cumulative distribution function for the sum $X+Y$?

Note: $f(x,y) = f_X(x)\, f_Y(y)$ since X and Y are indep.

$$F_{X+Y}(a) = P\{X+Y \le a\} = \iint_{x+y\le a} f_X(x)\, f_Y(y)\,dx\,dy$$



$$= \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x)\, f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} f_Y(y)\underbrace{\left[\int_{-\infty}^{a-y} f_X(x)\,dx\right]}_{=F_X(a-y)} dy$$

OR

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{a-x} f_X f_Y\,dy\,dx$$

$$= \int_{-\infty}^{+\infty} f_X(x)\underbrace{\left[\int_{-\infty}^{a-x} f_Y(y)\,dy\right]}_{F_Y(a-x)} dx$$

$$F_{X+Y}(a) = \int_{-\infty}^{+\infty} F_X(a-y)\, f_Y(y)\,dy = \int_{-\infty}^{+\infty} F_Y(a-x)\, f_X(x)\,dx$$

Then

$$f_{X+Y}(a) = \frac{d}{da}F_{X+Y}(a) = \int_{-\infty}^{+\infty}\frac{d\,F_X(a-y)}{da} f_Y(y)\,dy = \int_{-\infty}^{+\infty}\frac{d}{da}F_Y(a-x)\, f_X(x)\,dx$$

$$\boxed{f_{X+Y}(a) = \int_{-\infty}^{+\infty} f_X(a-y)\, f_Y(y)\,dy} = \boxed{\int_{-\infty}^{+\infty} f_Y(a-x)\, f_X(x)\,dx}$$

So f_{X+Y} is the convolution of f_X and f_Y.

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EXAMPLE (Book example 3a)

Suppose X and Y are independent and each uniformly distributed on $[0,1]$. So

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\, dy$

$y \le a$, $y \ge a-1$

$$f_X(a-y) = \begin{cases} 1 & \text{when } 0 \le a-y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

using $f_Y(y)$ info

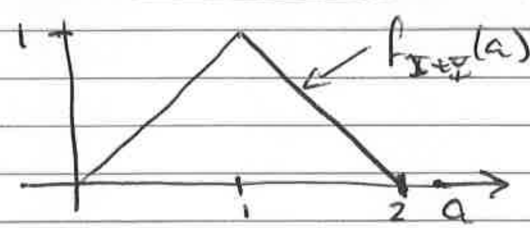
$= \int_0^1 f_X(a-y)\, dy$ let $t = a-y$, $dt = -dy$

easier...

$$= \int_a^{a-1} - f_X(t)\, dt = \begin{cases} 0 & a < 0 \\ -\int_a^0 1 \cdot dt = a & 0 \le a \le 1 \\ -\int_1^{a-1} 1 \cdot dt = 2-a & 1 < a \le 2 \\ 0 & a > 2 \end{cases}$$

$= \int_{a-1}^{a} f_X(t)\, dt \longrightarrow$

So $f_{X+Y}(a) = \begin{cases} a & 0 \le a \le 1 \\ 2-a & 1 < a \le 2 \\ 0 & \text{otherwise} \end{cases}$



$X+Y$ has a triangular distribution.

This idea can be generalized to the case of sums of multiple independent, uniform on $[0,1]$, random variables. (see Book p. 241 Ch. 6). Here we outline the details for the case of 3;

EXAMPLE

Let X, Y, Z be independent and uniform on $[0,1]$.

Let $W = X + Y$ and recall

$$f_W(a) = \begin{cases} a & 0 \le a \le 1 \\ 2-a & 1 < a \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad f_W(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 < t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Then
$$f_{W+Z}(a) = \int_{-\infty}^{+\infty} f_W(a-y) f_Z(y)\, dy$$
$$= \int_0^1 f_W(a-y) \cdot 1\, dy = \int_{a-1}^{a} f_W(t)\, dt$$

Cases: $0 \le a \le 1$

$$f_{W+Z}(a) = \int_0^a f_W(t)\, dt = \int_0^a t\, dt = \boxed{\tfrac{1}{2}a^2}$$

$1 \le a \le 2$

$$f_{W+Z}(a) = \int_{a-1}^{a} f_W\, dt$$
$$= \int_{a-1}^{1} t\, dt + \int_1^a (2-t)\, dt$$
$$= \tfrac{1}{2}\left[1-(a-1)^2\right] + \left(2t - \tfrac{1}{2}t^2\right)\Big|_1^a$$
$$= \tfrac{1}{2}\left[1-(a-1)^2\right] + 2a - \tfrac{1}{2}a^2 - \tfrac{3}{2}$$

*

$$= \frac{1}{2} - \frac{1}{2}(a-1)^2 + 2a - \frac{1}{2}a^2 - 3/2$$

$$= \cancel{1/2} - \frac{1}{2}a^2 + a - \cancel{1/2} + 2a - \frac{1}{2}a^2 - 3/2$$

$$= \boxed{-a^2 + 3a - 3/2}$$

<u>$2 \le a \le 3$</u>

$$f_{W+Z}(a) = \int_{a-1}^{2} (2-t)\,dt = 2t - \frac{1}{2}t^2 \Big|_{a-1}^{2} = (4-2) - \left(2(a-1) - \frac{1}{2}(a-1)^2\right)$$

$$= 2 - 2a + 2 + \frac{1}{2}(a^2 - 2a + 1)$$

$$= \boxed{\frac{1}{2}a^2 - 3a + \frac{9}{2}}$$

So

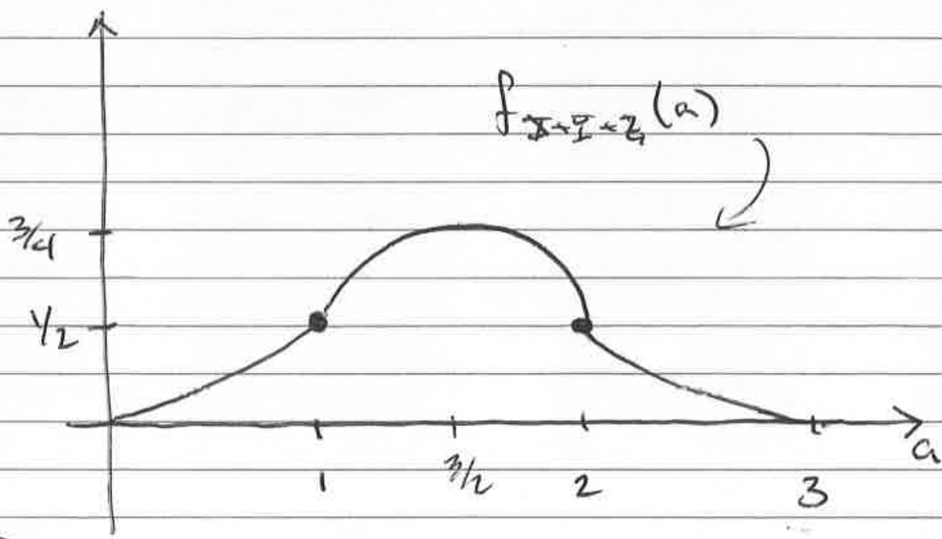
$$f_{\cancel{X}+Y+Z}(a) = \begin{cases} \frac{1}{2}a^2 & 0 \le a \le 1 \\ -a^2 + 3a - 3/2 & 1 \le a \le 2 \\ \frac{1}{2}a^2 - 3a + 9/2 & 2 \le a \le 3 \end{cases}$$

3 parabolas, piecewise

$-2a+3$

$-\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) - 3/2$

$\frac{3}{2}\left[-\frac{3}{2} + 3 - 1\right] = 3/4$



next one - cubics
X_1, X_2, X_3, X_4

See further details in book for independent variables ...

- sums of Gamma Random Variables (6.3.2)
- sums of Normal Random Variables (6.3.3)
- sums of Poisson Random Variables (6.3.4)
- sums of Binomial Random Variables (6.3.4)

Independent