Some common types of discrete random variables are

- Bernoulli Random Variables } Section 4.6
- Binomial Random Variables } Section 4.6
  (urn with replacement)
- Poisson Random Variable Section 4.7
- Geometric Random Variable Section 4.8.1
- Negative Binomial Random Variable Section 4.8.2
- Hypergeometric Random Variable Section 4.8.3
  (urn w/o replacement)

4.6 Bernoulli Random Variables + Binomial Random Variables

Bernoulli Random Variable

Let X be a discrete random variable ~~with~~ that can take on the value of 0 or 1 (0 can be thought of as "failure" and 1 can be thought of as "success"). Further, the probability mass function is

$$p(0) = P\{X=0\} = 1-p$$

$$p(1) = P\{X=1\} = p$$

so p represents the probability of success.

Note: for Bernoulli Random Variable

$$E[X] = 0 \cdot p(0) + 1 \cdot p(1)$$
$$= 0 \cdot (1-p) + 1 \cdot p$$
$$E[X] = p$$

$$Var(X) = E[X^2] - (E[X])^2$$
$$= 0^2 \cdot (1-p) + 1^2 \cdot p - p^2$$
$$= p - p^2$$
$$Var(X) = p(1-p)$$

Binomial Random Variable

Suppose there are n independent trials, each with

... success with probability $= p$

... failure with probability $= 1-p$

Let X = # of successes in the n trials.

The probability mass function is

$$p(i) = P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i} \quad \text{for } i = 0, 1, \ldots, n$$

Such a random variable is called a Binomial random variable with parameters (n, p).

Comment: A Bernoulli random variable is a binomial random variable with parameters $(1, p)$ (i.e. one trial).

Note: For a Binomial Random Variable, X, with parameters (n,p)

$$E[X] = np$$
$$Var(X) = np(1-p)$$

} we'll derive these after some examples

EXAMPLE (Binomial Random Variable)

- draw N balls from an urn with N_R red balls and N_G green balls with replacement

X = # of red balls drawn

see notes p. 107.2

$$p(i) = P\{X = i\} = \left(\frac{N_R}{N_R+N_G}\right)^i \left(\frac{N_G}{N_R+N_G}\right)^{N-i} \binom{N}{i}$$

Let $p \equiv \dfrac{N_R}{N_R+N_G}$ = probability of drawing red ("success")

note: $\dfrac{N_G}{N_R+N_G} = \dfrac{N_G+N_R-N_R}{N_R+N_G} = 1-p$ ("failure")

so $p(i) = p^i(1-p)^i \binom{N}{i}$

EXAMPLE (Binomial)

Prof. Anderson is trimming trees in his ~~[illegible]~~ fenced-in back yard. His dog Bozeman is "helping". Occasionally Prof. A has to open the gate to the front yard to bring something out to the ~~curb~~ curb. Each time, he ~~tells~~ gives Bozeman the command "STAY" (i.e. stay in the back yard). Bozeman is moderately-well trained and 'stays' ~~with probability~~ successfully with probability p. Let X be a random variable representing the number of times Bozeman successfully stays in the back yard while the gate is open. Prof. A takes N trips out the gate.

$$p(i) = P\{X = i\} = p^i (1-p)^{N-i} \binom{N}{i}$$

assume indep. trials always with success probability p

Suppose $N = 12$

$p = \frac{1}{10}$ (BAD DOG)	$p = \frac{1}{2}$ (MEDIUM DOG)	$p = \frac{9}{10}$ (GOOD DOG)
$p(i) = \left(\frac{1}{10}\right)^i \left(\frac{9}{10}\right)^{12-i} \binom{12}{i}$	$p(i) = \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{12-i} \binom{12}{i}$	$p(i) = \left(\frac{9}{10}\right)^i \left(\frac{1}{10}\right)^{12-i} \binom{12}{i}$

What is $p(11)$ (11 successes...) (i.e. Bozeman only bolts to the front yard once)

$$p(11) = \left(\frac{1}{10}\right)^{11}\left(\frac{9}{10}\right)\binom{12}{11} = \frac{9 \cdot 12}{10^{12}} = 108 \times 10^{-12} = 1.08 \times 10^{-10}$$

$$p(11) = \left(\frac{1}{2}\right)^{12}\binom{12}{11} = \frac{12}{2^{12}} = \frac{12}{4096} = 0.00292$$

$$p(11) = \left(\frac{9}{10}\right)^{11}\frac{1}{10}\cdot 12 = \frac{12 \cdot 9^{11}}{10^{12}} = 0.3766$$

EX ~~[illegible]~~ (Binomial)

Toss a fair coin 10 times. X = # of heads appearing

$$P\{X=i\} = \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{10-i} \cdot \binom{10}{i} = \frac{\binom{10}{i}}{2^{10}}$$

$i =$	0	1	2	3	4	5	6	7	8	9	10
$P\{X=i\}$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

EX (Binomial)

~~Toss~~ Roll a fair die 10 times. X = # of 6's appearing

$$P\{X=i\} = \left(\frac{1}{6}\right)^i \left(\frac{5}{6}\right)^{10-i} \binom{10}{i}$$

$i =$	0	1	2	3	4	5	6	7	8	9	10
$P\{X=i\}$	~~0.0269~~ 0.1615	0.323	0.2907	0.1550	0.0543	0.0130	0.0022	2.48×10^{-4}	1.86×10^{-5}	8.27×10^{-7}	1.65×10^{-8}

Properties of Binomial Random Variables

Binomial Random Variable probability mass function

recall $$p(i) = P\{X=i\} = \binom{n}{i} p^i (1-p)^{n-i} \quad i=0,1,2,\ldots,n$$

First,

- $$E[X] = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} n \frac{(n-1)!}{(i-1)!(n-i)!} p\, p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$ let $j=i-1$

$$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-(j+1)}$$ let $m=n-1$

$$= np \underbrace{\boxed{\sum_{j=0}^{m} \binom{m}{j} p^j (1-p)^{m-j}}}_{=1}$$

$=1$ (sum of all probabilities for binomial random variable)

So $E[X] = np$ n times probability of success.

Next, moving towards $Var(X)$, ...

$$E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i^2 \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} (i^2 - i + i) \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \left(i(i-1) \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} + i \binom{n}{i} p^i (1-p)^{n-i} \right)$$

$$= \sum_{i=2}^{n} \frac{n(n-1)(n-2)!}{(i-2)!(n-i)!} p^i (1-p)^{n-i} + E[X]$$

$$= n(n-1)p^2 \sum_{i=2}^{n} \frac{(n-2)!}{(i-2)!(n-i)!} p^{i-2} (1-p)^{n-i} + E[X]$$

$$= n(n-1)p^2 \underbrace{\sum_{i=2}^{n} \binom{n-2}{i-2} p^{i-2} (1-p)^{n-i}}_{\text{let } j = i-2,\ m = n-2} + E[X]$$

$$= n(n-1)p^2 \underbrace{\boxed{\sum_{j=0}^{m} \binom{m}{j} p^j (1-p)^{m-j}}}_{=1} + E[X]$$

So $E[X^2] = n(n-1)p^2 + np$

Then

- $Var(X) = E[X^2] - (E[X])^2$

$$= n(n-1)p^2 + np - (np)^2$$

$$= np\Big((n-1)p + 1 - np\Big)$$

$$= np\Big(1-p\Big)$$

So for a Binomial Random Variable w/ parameters (n,p)

$$\boxed{\begin{aligned} E[X] &= np \\ Var(X) &= np(1-p) \end{aligned}}$$

See book, pp. 131-132 for

$$E[X^k] = np\,E[(Y+1)^{k-1}]$$

where Y = Binomial random variable w/ parameters $(n-1,p)$

Other Properties:

- $$\boxed{P\{X=k+1\} = \frac{p}{1-p}\,\frac{n-k}{k+1}\,P\{X=k\}}$$ ~ helpful in computing $P\{X=i\}$ values...

$$P\{X=i\} = \binom{n}{i} p^i (1-p)^{n-i}$$

This result follows directly using definition

$$P\{X = k+1\} = \binom{n}{k+1} p^{k+1} (1-p)^{n-(k+1)}$$

$$= \frac{n!}{(k+1)!\,(n-(k+1))!} p^{k+1} (1-p)^{n-(k+1)}$$

$$= \frac{1}{k+1} \frac{n-k}{n-k} \frac{n!}{k!\,(n-(k+1))!} p^{k} (1-p)^{n-k} \cdot \frac{p}{(1-p)}$$

$$= \frac{n-k}{k+1} \frac{p}{1-p} \boxed{\frac{n!}{k!\,(n-k)!} p^{k} (1-p)^{n-k}}$$

$$= P\{X = k\}$$

so $$\boxed{P\{X = k+1\} = \frac{n-k}{k+1} \frac{p}{1-p} P\{X = k\}}$$

Also, from this note that

- $P\{X = k+1\} > P\{X = k\}$ if $\frac{n-k}{k+1} \frac{p}{1-p} > 1$

$$(n-k)p > (k+1)(1-p)$$

$$np - \cancel{kp} > k + 1 - \cancel{kp} - p$$

$$k < np + p - 1$$