Math 351 - Spring 2025
Math 351 - Fall 2018

1/22/2025
8/28/2018



# Ch. 1 Combinatorial Analysis

We'll ~~starting~~ start thinking about probability for which we can count the number of possibilities or ways something can happen.

Flip one coin twice

| | | second coin H | second coin T |
|---|---|---|---|
| first coin | H | | |
| | T | | |

**EX** Flip a Coin

H is one of two outcomes
T is one of two outcomes

} two equally-likely outcomes

**EX** Roll a die. Six equally-likely outcomes

1, 2, 3, 4, 5, 6

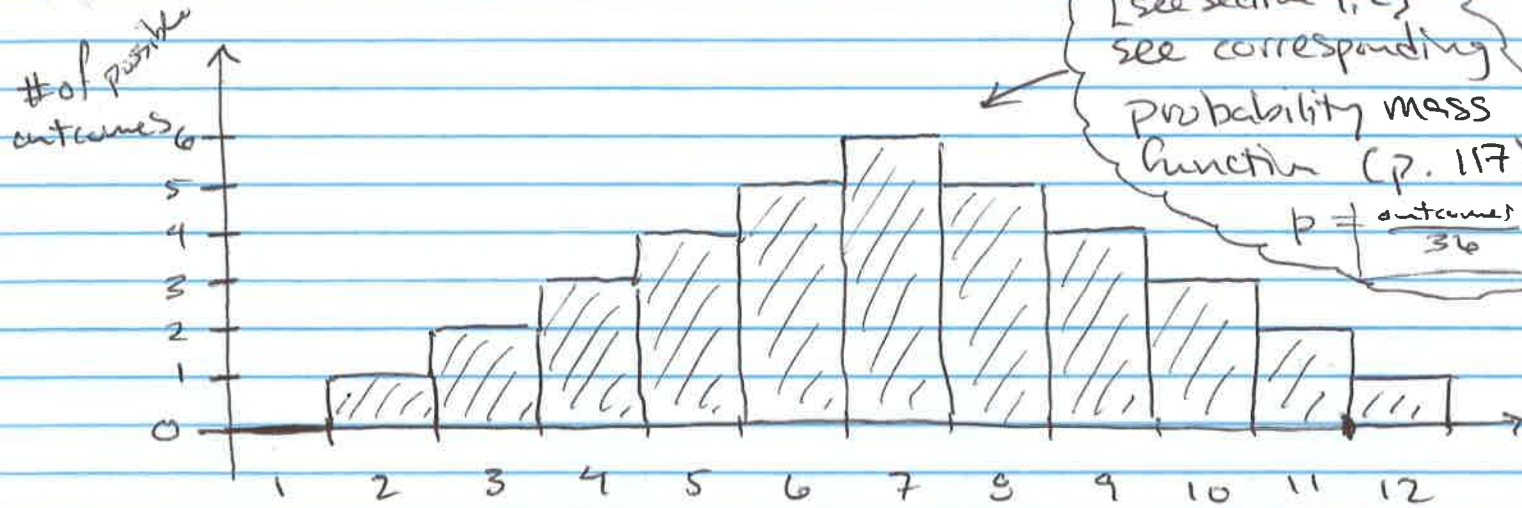
**EX** Roll Two Dice + Sum

How can we think about the outcome of the sum of both dice?

| | second die 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| first die 1 | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| 2 | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| 3 | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) |
| 4 | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) |
| 5 | (5,1) | (5,2) | (5,3) | (5,4) | (5,5) | (5,6) |
| 6 | (6,1) | (6,2) | (6,3) | (6,4) | (6,5) | (6,6) |

36 possible outcomes that are all equally-likely.

[see Section 4.2]
see corresponding probability mass function (p. 117)
$p = \frac{\text{outcomes}}{36}$




e.g. • 2 equally-likely ways to get sum = 3

{1,2}, {2,1}

• 6 equally-likely ways to get sum = 7

{1,6}, {2,5}, {3,4}, {4,3}, {5,2}, {6,1}

So... intuition says that we are more likely to roll a 7 than we are to roll a 3.

EX Urn

Suppose an urn contains six balls numbered 1-6, that are otherwise identical. Draw one ball, note its number, then replace it. Draw a second ball, note its number. Here we have the same set of 36 equally-likely outcomes as the previous example. This is sampling with replacement.

Spring 2025 - Math 357

INSTRUCTIONS:

1. Roll Dice (Both)
2. Sum the numbers
3. ✓ appropriate bin
4. pass Dice + sheet

} x4


| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Actual observations 2025 | 3 | 4 | 8 | 12 | 7 | 19 | 10 | 11 | 6 | 4 | 6 |
| From 2019 | 4 | 6 | 11 | 12 | 15 | 22 | 21 | 10 | 8 | 4 | 0 |
| total | 7 | 10 | 19 | 24 | 22 | 41 | 31 | 21 | 14 | 8 | 6 |

Sect 1.2 – 1.4 counting, permutations, combinations ) Presumably some overlap with Discrete Math (Math 125)

✓ 

## 1.2 Counting Principles: (see p.2)

**Generalized Basic Principle of Counting**

If $r$ "experiments" are to be performed and are such that the first experiment can result in $n_1$ possible outcomes, the second experiment can result in $n_2$ possible outcomes, etc. and the $r^{th}$ experiment can result in $n_r$ possible outcomes then there are a total of

$$n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_r$$

possible outcomes of the $r$ "experiments"

EX

"experiment" – roll die

$r=2$ : Roll 2 dice — 6 outcomes : die one; 6 outcomes : die two

total possible outcomes $= 6 \cdot 6 = 36$

"experiment": draw 1 ball from urn.

$r=2$ : : Draw two balls from urn with replacement. ← SAME

EX (Sampling without replacement) Balls #1-6 in urn.

$r=2$ Draw ~~[illegible]~~ a ball from the urn. Do not replace. ~~Draw~~ a second ball from the urn.

First ball: 6 outcomes possible

Second ball: 5 outcomes possible

$\rightarrow$ $30 = 6 \cdot 5$ possible outcomes

One way to visualize this is...

Second ball

| First ball | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| 2 | (2,1) | | (2,3) | (2,4) | (2,5) | (2,6) |
| 3 | (3,1) | (3,2) | | (3,4) | (3,5) | (3,6) |
| 4 | (4,1) | (4,2) | (4,3) | | (4,5) | (4,6) |
| 5 | (5,1) | (5,2) | (5,3) | (5,4) | | (5,6) |
| 6 | (6,1) | (6,2) | (6,3) | (6,4) | (6,5) | |

here we are distinguishing the first ball from the second

30 ~~[illegible]~~ equally-likely outcomes

(1,1), (2,2), etc. are not possible outcomes.

EX

Urn with $n$ numbered balls. Choose $r$... ~~without replacement~~

... with replacement: $\underbrace{n \cdot n \cdot n \cdot \ldots \cdot n}_{r \text{ terms}} = n^r$

\# of possible outcomes

... without replacement: $\underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{r \text{ terms}}$

$= \dfrac{n!}{(n-r)!}$

EX

How many different batting orders (of 9 batters) can be made with ~~a~~ a ~~~~ baseball team of 12 players?
softball

think about 3 and 3 & write out all possibilities

ABC
ACB
BAC
BCA
CAB
CBA

$\boxed{3\cdot2\cdot1}$

$$\underline{12}\quad\underline{11}\quad\underline{10}\quad\underline{9}\quad\underline{8}\quad\underline{7}\quad\underline{6}\quad\underline{5}\quad\underline{4}$$

$$12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4 = \frac{12!}{3!} = \frac{12!}{(12-9)!}$$

$$= \boxed{79,833,600}$$

**Permutations** - order is important

The number of arrangements of $n$ different objects, with no repetitions allowed, is

$$n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot3\cdot2\cdot1 = n!$$

Such arrangements are called permutations

EX Seating arrangements for a panel of 4 students, 3 faculty members, and 3 industry representatives. How many different seating arrangements are there if ~~the faculty~~ each group must sit together?

Assume seating in a row

(6)

Count in 4 steps:

all of these are permutations

(1) Arrange the groups : $3! = 3 \cdot 2 \cdot 1 = 6$ ways

(e.g.
F - S - I
S - F - I
F - I - S
S - I - F
I - F - S
I - S - F

(2) Arrange the ~~[illegible]~~ students : $= 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways

(3) " " faculty $= 3! = 6$

(4) " " industry reps $= 3! = 6$

So by the basic principle of counting

$$6 \cdot 24 \cdot 6 \cdot 6 = \underline{\underline{5184 \text{ ways}}}$$

**Combinations** - order is not important

A subset of $r$ elements out of a set of $n$ elements is a <u>combination</u> of $r$ out of the $n$ elements

<u>EX</u>
How many different teams of 9 players can be made from 12 total kids - (batting order, etc. is not important)?

Previously we said there were

$$\frac{12!}{3!}$$

different batting orders for 9 players chosen from 12.
But, if we do not care about the order we have
$9 \cdot 8 \cdot 7 \cdot \ldots \cdot 3 \cdot 2 \cdot 1$ different arrangements of
these players that we now are not distinguishing
as different. So our ~~result~~ answer must be
reduced by this factor.. So

$$\frac{12!}{3!} \frac{1}{9!}$$

different 9 player teams (~~ets~~ if we do not
care about batting order, position, etc.)

$$\# \text{teams} = \frac{12!}{3!\,9!} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = \boxed{220}$$

i.e. 220 different <u>combinations</u> of kids on the team.

e.g. a math competition team where the order of the students does not matter.

7.1

Smaller example (sometimes just try to write it all out)

How many different teams of 3 players can be made from 5 total kids (A, B, C, D, E)

$\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5 \cdot 4}{2} = 10$

| | | |
|---|---|---|
| ABC | BCD | CDE |
| ABD | BCE | |
| ABE | BDE | |
| ACD | | |
| ACE | | |
| ADE | | |

10 total

all possible ways with kid A

~~exclud A~~

exclude A

note $\binom{5}{3} = \frac{5!}{2!\,3!} = 10$ | i.e. $\left(\frac{5!}{2!}\right)\frac{1}{3!}$, $\frac{5!}{2!} = 5 \cdot 4 \cdot 3$

$\frac{5!}{2!}$ (order matters case)

$5 \cdot 4 \cdot 3 = 5 \cdot 4 \cdot 3 \cdot \frac{2 \cdot 1}{2 \cdot 1} = \frac{5!}{2!}$

$\boxed{\text{but for example this we consider equivalent to the 6 permutations "ABC", "ACB", "BAC", "BCA", "CAB", and "CBA"}}$