

# More algebraic structures

Casey Blacker  
Math 300

- 1 Rings
- 2 Fields
- 3 Vector spaces
- 4 Modules

## Section 1

# Rings

## Definition

A *ring*  $(R, +, \cdot)$  comprises a set  $R$  and two binary operations  $+, \cdot : R \times R \rightarrow R$ , such that

- i.  $(R, +)$  is an abelian group,
- ii.  $(R, \cdot)$  is a monoid,
- iii. the operation  $\cdot$  *distributes* over  $+$ , that is, for all  $a, b, c \in R$ ,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

## Remarks

- i. We typically call  $+$  *addition* and  $\cdot$  *multiplication*.
- ii. The additive identity of  $R$  is conventionally denoted  $0 \in R$  and called *zero*, and the multiplicative identity by  $1 \in R$  and called *one*.

## Remarks

- i. If the multiplication  $\cdot$  is commutative, then  $(R, +, \cdot)$  is called a *commutative ring*.
- ii. If  $(R, +, \cdot)$  satisfies all the conditions of a ring except for the existence of a multiplicative identity  $1 \in R$ , then it is called a *rng*.

Challenge! Is it a ring?

Is it a ring?

$$(\mathbb{Z}, +, \cdot)$$

Yes!

Is it a ring?

$$(\mathbb{R}, +, \cdot)$$

Yes!

Is it a ring?

$$(\mathbb{N}, +, \cdot)$$

No!



Is it a ring?

$$(2\mathbb{Z}, +, \cdot)$$

No!

Is it a ring?

$$(\{0\}, +, \cdot)$$

Yes! The *zero ring*

### Definition

A *zero divisor* in a commutative ring  $(A, +, \cdot)$  is an element  $a \in A$  for which there exists a nonzero  $b \in A$  with  $ab = 0$ .

### Definition

A commutative ring  $(R, +, \cdot)$  is called an *integral domain* when

- it does not contain any nonzero zero divisor,
- $0 \neq 1$ .

Is it an integral domain?

$$(\mathbb{Z}, +, \cdot)$$

Yes!

Is it an integral domain?

$$(\{0\}, +, \cdot)$$

No!

Is it an integral domain?

$$(\mathbb{Z}_4, +, \cdot)$$

No!

## Section 2

# Fields

## Definition

An integral domain  $(R, +, \cdot)$  is called a *field* when every nonzero element  $a \in R \setminus \{0\}$  has a multiplicative inverse  $a^{-1} \in R$ .



Is it a field?

$$(\mathbb{Z}, +, \cdot)$$

No!

Is it a field?

$$(\mathbb{R}, +, \cdot)$$

Yes!

Is it a field?

$$(\{0\}, +, \cdot)$$

No!

Is it a field?

$$(\mathbb{Z}_2, +, \cdot)$$

Yes!

## Section 3

# Vector spaces

## Definition

Let  $k$  be field. A  $k$ -vector space  $(V, +, \cdot)$  comprises a set  $V$  together with operations

$$+ : V \times V \rightarrow V$$

$$\cdot : k \times V \rightarrow V$$

such that

- i.  $(V, +)$  is an abelian group
- ii. *scalar multiplication*  $\cdot$  and *vector addition*  $+$  satisfy

$$1 \cdot u = u$$

$$(a + b) \cdot u = (a \cdot u) + (b \cdot u)$$

$$a \cdot (b \cdot u) = (a \cdot b) \cdot u$$

$$a \cdot (u + v) = (a \cdot u) + (a \cdot v)$$

# Is it a vector space?

$$k = \mathbb{R}, V = \mathbb{R}$$

Yes!

# Is it a vector space?

$$k = \mathbb{R}, V = \mathbb{C}$$

Yes!



# Is it a vector space?

$$k = \mathbb{Z}, V = \mathbb{R}$$

No!

# Is it a vector space?

$$k = \mathbb{R}, V = \mathbb{R}^n$$

Yes!

## Section 4

# Modules

Fix a ring  $R$ .

### Definition

An  $R$ -module  $(V, +, \cdot)$  comprises a set  $V$  together with operations

$$+ : V \times V \rightarrow V$$

and

$$\cdot : R \times V \rightarrow V$$

that together satisfy the familiar vector space conditions.

### Remark

That is, an  $R$ -module is a vector space with a ring of scalars  $R$  rather than a field of scalars  $k$ .

Is it a module?

$k$ -vector space  $V$

Yes!

# Is it a module?

$$R = \mathbb{Z}, V = \mathbb{Z}^n$$

Yes!

## Is it a module?

$$R = \{0\}, \quad V = \{0\}$$

Yes!