Solutions

Math 351 - Spring 2025: Homework 2

Due: Wednesday, February 4, 2025

Instructions: Be sure to give explanations to your answers. I'm interested not only in whether you get the correct answer but also how you obtained it and your thought process along the way. Don't just write down a number even if the answer seems obvious.

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1. A standard deck of 52 cards is dealt out. (a) What is the probability that the 10th card dealt is an King? (b) What is the probability that the first King is the 10th card?

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2. In the game of Poker Dice, five dice are rolled all at once. A straight is the event that the numbers $1, 2, 3, 4, 5$ appear or that the numbers $2, 3, 4, 5, 6$ appear. What is the probability of rolling a straight?

6

3. Two dice are rolled. Denote by E the event in which the sum of the two dice is even. Denote by F the event that at least one of the dice is a 5. Denote by G the event for which the numbers on the two dice are the same.

   Find $P(E)$, $P(F)$, $P(G)$, $P(E \cup F)$, $P(EF)$, $P(F \cup G)$, and $P(FG)$.

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4. An urn contains 8 red balls, 10 green balls, 10 yellow balls, and 9 blue balls. A set of three balls is randomly selected. (a) What is the probability that all three balls are red? (b) What is the probability that all three balls have different colors?

Math 351 - Spring 2025 HW 2 Solutions HW2.1

$\boxed{1}$ Standard Deck of 52 playing cards.

a) Probability that 10^{th} card is a King?

There are a few ways to think about this (including but not limited to the following)

- $\underline{51}\ \underline{50}\ \underline{49}\ \underline{48}\ \underline{47}\ \underline{46}\ \underline{45}\ \underline{44}\ \underline{43}\ |\ \underline{4}\ |\ \underbrace{42!}_{}$

can choose from 48 cards + 3 kings

(10^{th} card) Must be a King

rest of the cards (42 remaining)

Total ways to deal 52 cards $= 52!$

So $P = \dfrac{51! \cdot 4}{52!} = \dfrac{4}{52} = \boxed{\dfrac{1}{13}} \approx 0.0769$

Alternatively imagine a deck of 52 cards already shuffled. The probability that a King is in the 10^{th} position is

$$\frac{4}{52} = \frac{1}{13}$$

(i.e. 4 chances in 52 cards)

HW 2.2

(a) alternate

$$\binom{4}{1}\cdot\left[\underbrace{\binom{51}{9}\cdot 9!}\right]\left[\underbrace{\binom{42}{42}42!}\right]$$

# of arrangements with K in 10th spot

the 10th card must be a king

pick 9 other cards for the "first" 9 but note these can appear in 9! different orders

pick the 42 remaining cards for the "last" 42 again noting ordering options

$$= 4\cdot\left[\frac{51!}{42!\,9!}\,9!\right]\left[\frac{42!}{0!\,42!}\,42!\right]$$

$$= 4\cdot 51!$$

Total # of arrangements of 52 cards $= 52!$

so

$$P = \frac{4\cdot 51!}{52!} = \frac{4}{52} = \frac{1}{13}.$$

2.2.5
HW ~~2.3.5~~

1a) Another variation

4 cases

A: no kings in first 9

$$\frac{48}{52}\cdot\frac{47}{51}\cdot\frac{46}{50}\,\frac{45}{49}\,\frac{44}{48}\,\frac{43}{47}\,\frac{42}{46}\,\frac{41}{45}\,\frac{40}{44}\,\Bigg|\,\frac{4}{43}$$

B: 1 king in first 9

$$\left[\frac{48}{52}\,\frac{47}{51}\,\frac{46}{50}\,\frac{45}{49}\,\frac{44}{48}\,\frac{43}{47}\,\frac{42}{46}\,\frac{41}{45}\,\frac{4}{44}\times\binom{9}{1}\right]\frac{3}{43}$$

K

# of places where that K could appear in 9.

C: 2 kings in first 9

$$\left[\frac{48}{52}\,\frac{47}{51}\,\frac{46}{50}\,\frac{45}{49}\,\frac{44}{48}\,\frac{43}{47}\,\frac{42}{46}\,\frac{4}{45}\,\frac{3}{44}\times\binom{9}{2}\right]\frac{2}{43}$$

D: 3 kings in first 9

$$\left[\frac{48}{52}\,\frac{47}{51}\,\frac{46}{50}\,\frac{45}{49}\,\frac{44}{48}\,\frac{43}{47}\,\frac{4}{46}\,\frac{3}{45}\,\frac{2}{44}\times\binom{9}{3}\right]\frac{1}{43}$$

$$=\frac{(42\cdot41\cdot40\cdot4)+(42\cdot41\cdot4\cdot3)\times\binom{9}{1}+(42\cdot4\cdot3\cdot2)\times\binom{9}{2}+(4\cdot3\cdot2)\times\binom{9}{3}}{52\cdot51\cdot50\cdot49}$$

$$=\ \ldots\ =\frac{1}{13}$$

Also:

$$\frac{\binom{48}{9}}{\binom{52}{9}}\cdot\frac{\binom{4}{1}}{\binom{43}{1}}$$

$+$

$$\frac{\binom{4}{1}\binom{48}{8}}{\binom{52}{9}}\cdot\frac{\binom{3}{1}}{\binom{43}{1}}$$

$+$

$$\frac{\binom{4}{2}\binom{48}{7}}{\binom{52}{9}}\cdot\frac{\binom{2}{1}}{\binom{43}{1}}$$

$+$

$$\frac{\binom{4}{3}\binom{48}{6}}{\binom{52}{9}}\cdot\frac{\binom{1}{1}}{\binom{43}{1}}$$

HW 2.3

1b) What is the probability that the first King is the 10^{th} card?

- Note that no King can appear in the first 9 positions.

$$\underline{48}\ \underline{47}\ \underline{46}\ \underline{45}\ \underline{44}\ \underline{43}\ \underline{42}\ \underline{41}\ \underline{40} \Big| \underline{4} \Big| 42!$$

can't be a King but any other of the 48 cards would work

10^{th} card

remaining 42 cards

$$= 48! \cdot (42 \cdot 41 \cdot 40) \cdot 4$$

So

$$P = \frac{48! \cdot (42 \cdot 41 \cdot 40) \cdot 4}{52!}$$

same total # as part (a)

$$= \frac{\cancel{42}^{6} \cdot 41 \cdot \cancel{40}^{4} \cdot \cancel{4}}{\cancel{52}_{13} \cdot 51 \cdot \cancel{50}_{5} \cdot \cancel{49}_{7}} = \frac{6 \cdot 41 \cdot 4}{13 \cdot 5 \cdot 7 \cdot 51} = \frac{984}{23{,}205}$$

≈ 0.0424

Also

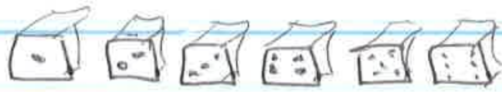
$$P = \underbrace{\frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} \cdot \frac{44}{48} \cdot \frac{43}{47} \cdot \frac{42}{46} \cdot \frac{41}{45} \cdot \frac{40}{44}}_{\text{first 9 not king}} \cdot \underbrace{\frac{4}{43}}_{\text{King}}$$

$$\frac{328}{7735}$$

$$= \frac{48!}{52!} \cdot 4 \cdot 42 \cdot 41 \cdot 40 \qquad = \frac{\binom{48}{9}}{\binom{52}{9}} \cdot \frac{\binom{4}{1}}{\binom{43}{1}}$$

HW 2.4

2 Poker Dice (5 dice - each 6 sided)



What is the probability of rolling a straight?

two ways: (A) 1, 2, 3, 4, 5

and

(B) 2, 3, 4, 5, 6

(A): $\frac{5\cdot4\cdot3\cdot2\cdot1}{6^5}$

\# of ways to get roll (A)

total # of outcomes

(B) same:

$\frac{5\cdot4\cdot3\cdot2\cdot1}{6^5}$

$$P_{straight} = 2\cdot\frac{5!}{6^5} = \frac{240}{7776} = \boxed{\frac{5}{162}} \approx 0.0309$$

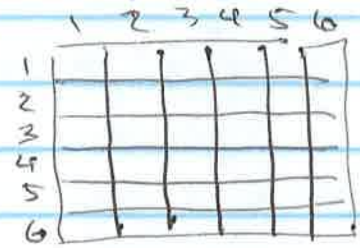
HW 2.5

3 Two dice rolled

E: sum is even

F: at least one die is "5"

G: both numbers match (e.g. (1,1), (2,2) ...)



- $P(E) = P(\text{sum} = 2, 4, 6, 8, 10, \text{or } 12)$

$$= \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36}$$

(2) (4) (6) (8) (10) (12)

$$= \frac{18}{36} = \frac{1}{2}$$

$\boxed{P(E) = 1/2}$

(first one a 5) (second one a 5) (both 5)

- $P(F) = \dfrac{1 \cdot 6 + 1 \cdot 6 - 1}{36} = \dfrac{6+6-1}{36} = \dfrac{11}{36}$

$\boxed{P(F) = \frac{11}{36}}$

- $\boxed{P(G) = \frac{6}{36} = \frac{1}{6}}$

$P(E \cup F) = P(E) + P(F) - P(EF)$

P(EF) =

- $P(EF) = P(\{5,1\}, \{5,3\}, \{5,5\}, \{1,5\}, \{3,5\}) = \boxed{\frac{5}{36}}$

- so $P(E \cup F) = \frac{1}{2} + \frac{11}{36} - \frac{5}{36} = \frac{18+11-5}{36} = \frac{24}{36} = \boxed{\frac{2}{3}}$

P(E∪F)

HW 2.6

- $P(FG) = P(\{5,5\}) = \frac{1}{36}$

at least one is 5 — both same

$$\boxed{P(FG) = \frac{1}{36}}$$

- $P(F \cup G) = P(F) + P(G) - P(FG)$

$$= \frac{11}{36} + \frac{6}{36} - \frac{1}{36} = \frac{16}{36} = \frac{4}{9}$$

$$\boxed{P(F \cup G) = \frac{4}{9}}$$

HW 2.7

$\boxed{4}$ URN: 8 red balls
10 green balls
10 yellow balls
9 blue balls

$\Big\}$ 37 total

3 balls randomly selected (no replacement)

a) $P(3\text{ red}) = \dfrac{\binom{8}{3}}{\binom{37}{3}} = \dfrac{\frac{8!}{5!\,3!}}{\frac{37!}{34!\,3!}} = \dfrac{8\cdot 7\cdot 6}{37\cdot 36\cdot 35}$

e.g. first ball has $\frac{8}{37}$ to be red then $\frac{7}{36}$ ... etc.

$= \dfrac{8}{37\cdot 6\cdot 5} \approx 0.007$

$= \dfrac{8}{1{,}110} = \dfrac{4}{555}$

b) $P(3\text{ different color})$ – note there are $\binom{4}{3} = 4$ color combinations

RGY	RGB	RYB	GYB
$\binom{8}{1}\binom{10}{1}\binom{10}{1}$	$\binom{8}{1}\binom{10}{1}\binom{9}{1}$	$\binom{8}{1}\binom{10}{1}\binom{9}{1}$	$\binom{10}{1}\binom{10}{1}\binom{9}{1}$
$= 8\cdot 10\cdot 10$	$8\cdot 10\cdot 9$	$8\cdot 10\cdot 9$	$10\cdot 10\cdot 9$

so $P(3\text{ different color}) = \dfrac{8\cdot 10\cdot 10 + 8\cdot 10\cdot 9 + 8\cdot 10\cdot 9 + 10\cdot 10\cdot 9}{\binom{37}{3}}$

HW 2.8

$$P(\text{3 different color}) = \frac{10(80+72+72+90)}{\left(\frac{37!}{34!\,3!}\right)}$$

$$= \frac{10(314)}{\left(\frac{37\cdot 36\cdot 35}{3!}\right)} = \frac{\overset{2}{\cancel{10}}\cdot \overset{157}{\cancel{314}}}{37\cdot \underset{7}{\cancel{35}}\cdot \underset{3}{\cancel{6}}} = \frac{2\cdot 157}{37\cdot 7\cdot 3}$$

$$= \boxed{\frac{314}{777}} \approx 0.404$$