Worksheet 4

Let R be a relation from A to B, let S be a relation from B to C, and let T be a relation from C to D.
Prove the following statements.

1. $I_A \circ R = R$

   Proof. Let $a \in A$ and $b \in B$ and observe:

   $$\begin{aligned} a(I_a \circ R)b &\iff \exists a^{`} \in A : a = a^{`} \wedge a^{`}Rb \\ &\iff aRb \end{aligned}$$

   $\square$

2. $R \circ I_A = R$

   Proof. Let $a \in A$ and $b \in B$ and observe:

   $$\begin{aligned} a(R \circ I_a)b &\iff \exists b^{'} \in B : b = b^{'} \wedge aRb^{'} \\ &\iff aRb \end{aligned}$$

   $\square$

3. $(R^{-1})^{-1} = R$

   Proof. Assume the relation R has an inverse and let $a \in A$ and $b \in B$:

   $$\begin{aligned} a(R^{-1})^{-1}b &\iff bR^{-1}a \\ &\iff b(R^{-1})^{-1}a \end{aligned}$$

   $\square$

4. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

   Proof. Let $a \in A$, $b \in B$, and $c \in C$

   $$\begin{aligned} c(S \circ R)^{-1}a &\iff \exists c^{'} \in C : c = c^{'} \wedge c^{'}S^{-1}b \\ &\iff \exists b^{'} \in B : b = b^{'} \wedge b^{'}R^{-1}a \\ &\iff c(R^{-1} \circ S^{-1})a \end{aligned}$$

   $\square$

5. $(T \circ S) \circ R = T \circ (S \circ R)$

Proof. $\square$

6. $DomR = RngR^{-1}$

   Proof. $\square$

7. $RngR = DomR^{-1}$

   Proof. $\square$

For Question 8–10, suppose that $A = B = C$.

8. If R and S are equivalence relations, then $S \circ R$ is an equivalence relation.

   Proof. $\square$

9. If R is a partial order, then $R \circ R$ is a partial order.

   Proof. $\square$

10. If R and S are partial orders, then it is not generally true that $S \circ R$ is a partial order.

    Proof. $\square$

Bonus Questions Give an example of two relations R and S on a set A such that

11. $R \circ S \neq S \circ R$.

    Proof. $\square$

12. $S \circ R$ is an equivalence relation, but neither R nor S is an equivalence relation.

    Proof. $\square$