Notes on $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$ Binomial Coefficient

- $\binom{n}{0} = \frac{n!}{n!\,0!} = \frac{1}{0!} = 1$ $\{0! = 1\}$

- $\binom{n}{r} = \binom{n}{n-r}$ ⇐ there is a symmetry here

e.g. # of ~~groups~~ combinations of r distinct objects drawn from an urn of n objects is the same as the # of combinations of n-r objects chosen from urn of n objects

(think of r chosen ⟺ n-r not chosen)

i.e. 9 players from team of 12

⇕

3 bench sitters from team of 12.

- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

$$= \frac{(n-1)!}{((n-1)-(r-1))!\,(r-1)!} + \frac{(n-1)!}{((n-1)-r)!\,r!}$$

$$= \frac{(n-1)!}{(n-r)!\,(r-1)!}\left(\frac{r}{r}\right) + \frac{(n-1)!}{(n-1-r)!\,r!}\left(\frac{n-r}{n-r}\right)$$

$$= \frac{(n-1)!\,(r+n-r)}{(n-r)!\,r!} = \frac{n!}{(n-r)!\,r!}$$

- The previous result connects to Pascal's triangle

- add entries in previous row (left/right) to get new entry.

								1									row 0
							1		1								row 1
						1		2		1							row 2
					1		3		3		1						
			$\binom{4}{0} \to$ 1			4		6		4		1 $\leftarrow \binom{4}{4}$					
			1		5		10		10		5		1				
		1		6		15		20		15		6		1			
	1		7		21		35		35		21		7		1		
1		8		28		56		70		56		28		8		1	

$\vdots$

- leftmost entry of row n is $\binom{n}{0} = 1$
- rightmost " " " " " $\binom{n}{n} = 1$
- e.g. entries in row $7 = n$

$$\binom{7}{0} = 1, \ \binom{7}{1} = 7, \ \binom{7}{2} = 21, \ \binom{7}{3} = 35$$

$$\binom{7}{4} = 35 \quad \binom{7}{5} = 21 \quad \binom{7}{6} = 7 \quad \binom{7}{7} = 1$$

- note: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

Binomial Theorem (p. 7 in Ross)

For non-negative integer n and real numbers x, y

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Proof: see text pp. 7-8

- The coefficient $\binom{n}{i}$ are called binomial coefficients

$$(x+y)^2 = 1 \cdot x^2 + 2xy + y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$\vdots$

from Pascal's Tri.

1 2 1

1 3 3 1

1 4 6 4 1

- $x = y = 1$ $\quad 2^n = \sum_{i=0}^{n} \binom{n}{i}$ $\leftarrow$ sum of entries in Pascal's triangle $= 2^n$ (row n)

- $x = 1$, $y = -1$ $\quad 0 = \sum_{n=0}^{n} \binom{n}{i} (-1)^{n-1}$ $\leftarrow$ Alt. signs (for $n > 0$ cases) of binomial coeff. add to zero.

1.5 Multinomial Coefficients 9-4-2018 (16)

Recall the MISSISSIPPI problem of finding the # of letter arrangements that can be made. This was effectively the problem of taking a set of 11 distinct items (spaces to put letters) and dividing them up into 4 groups (letters I, S, P, M) of sizes 4, 4, 2, and 1, respectively. We found that the number of arrangements was (note $4+4+2+1=11$)

$$\frac{11!}{4!\,4!\,2!\,1!}$$

The more general problem is:

Q: Given a set of n distinct items, divide these up into r groups of size $n_1, n_2, \ldots, n_r$ where $\sum_{i=1}^{r} n_i = n$.

A: First Group: $\binom{n}{n_1}$ choices for first group

Second Group: $\binom{n-n_1}{n_2}$ choices for second group

Third Group: $\binom{n-n_1-n_2}{n_3}$ choices for third group

$\vdots$

r^{th} Group: $\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$ choices for r^{th} group.

The choices made within each group are independent of the choices made in the other groups, so by basic principle of counting, the total number of divisions is

$$= \binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$

$$= \frac{n!}{\cancel{(n-n_1)!}\,n_1!} \cdot \frac{\cancel{(n-n_1)!}}{\cancel{(n-n_1-n_2)!}\,n_2!} \cdot \frac{\cancel{(n-n_1-n_2)!}}{\cancel{(n-n_1-n_2-n_3)!}\,n_3!} \cdots \frac{\cancel{(n-n_1-n_2-\cdots-n_{r-1})!}}{\underbrace{(n-n_1-n_2-\cdots-n_r)!}_{\substack{n = n_1+n_2+\cdots+n_r \\ \text{so this term is } 0! = 1}}\,n_r!}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_r!}$$

EX

15 members of a rowing club have one boat for eight rowers which needs one cox. The other 6 members will do an "erg" workout. How many arrangements are there (assuming ordering of those in the boat is not important)

Boat $\binom{15}{8} = \frac{15!}{7!\,8!}$

Cox $\binom{15-8}{1} = \frac{7!}{6!\,1!}$

Erg $\binom{15-8-1}{6} = \frac{6!}{0!\,6!}$

$$\left(\frac{15!}{\cancel{7!}\,8!}\right)\left(\frac{\cancel{7!}}{\cancel{6!}\,1!}\right)\left(\frac{\cancel{6!}}{0!\,6!}\right) = \boxed{\frac{15!}{8!\,1!\,6!}}$$

$= 45{,}045$

could have written these out in a different order cox, boat, erg

$$\binom{15}{1}\binom{14}{8}\binom{6}{6} = \cdots = \frac{15!}{8!\,1!\,6!}$$

If the ordering in the boat is important note that there are 8! permutations of each set of 8 people in the boat, so in this case the number of ~~possibilities~~ possibilities increases (by a factor of 8!) to

$$\frac{15!}{8!\,6!}\cdot 8! = \frac{15!}{6!} = 15\cdot 14\cdot \ldots \cdot 7$$

$$= \boxed{1{,}816{,}214{,}400}$$

Multinomial Thm (see p. 10, Ross)

$$(x_1+x_2+\ldots+x_r)^n = \sum_{(n_1,n_2,\ldots,n_r)} \frac{n!}{n_1!\,n_2!\cdots n_r!}\, x_1^{n_1} x_2^{n_2}\cdots x_r^{n_r}$$

where in the sum $(n_1, n_2, \ldots, n_r)$ are all vectors with $n_1+n_2+\ldots+n_r=n$

~~Ex Expand~~

notation

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\,n_2!\cdots n_r!}$$

if $n_1+n_2+\ldots+n_r=n$

Wrapping up ch 1

Recall example of

- Probability of <u>you</u> getting picked for a team of 3 from our class of, say, 25

Two approaches, (at least)

P that <u>you</u> get picked

ways this happens

you get picked on first pick $\frac{1}{25}$

OR

you get picked on 2^{nd} pick $\frac{24}{25}\frac{1}{24} = \frac{1}{25}$

OR

you get picked on 3^{rd} pick $\frac{24}{25}\frac{23}{24}\frac{1}{23} = \frac{1}{25}$

$$P_{picked} = \frac{1}{25} + \frac{1}{25} + \frac{1}{25} = \boxed{\frac{3}{25}}$$

$1 - P_{\text{you do not get picked}}$

$P_{\text{not pick}}$

How can this happen?

– three things must happen

not picked on 1st $\frac{24}{25}$

AND

not picked on 2^{nd} $\frac{23}{24}$

AND

not picked on third $\frac{22}{23}$

so

$$P_{\text{not pick}} = \frac{24}{25} \cdot \frac{23}{24} \cdot \frac{22}{23} = \frac{22}{25}$$

so

$$P_{\text{picked}} = 1 - \frac{22}{25} = \boxed{\frac{3}{25}}$$

EX

Let's now work out the probability of being dealt a pair from a deck with the missing Ace of Spades.

Recall (normal deck)

$$P_{52}^{pair} = \frac{\binom{13}{1}\binom{4}{2}}{\binom{52}{2}} = \frac{13 \cdot 6}{52 \cdot 51/2} = \frac{13 \cdot 6}{26 \cdot 51} = \boxed{\frac{1}{17}}$$

51-card deck

Total hands $= \binom{51}{2}$

Total pairs

ACES   NOT ACES

$$\binom{1}{1}\binom{3}{2} \qquad \binom{12}{1}\binom{4}{2}$$

$$P_{51}^{PAIR} = \frac{\binom{1}{1}\binom{3}{2} + \binom{12}{1}\binom{4}{2}}{\binom{51}{2}} = \frac{3 + 12 \cdot 6}{51 \cdot 50/2}$$

$$= \frac{3+72}{51 \cdot 25} = \frac{75}{51 \cdot 25} = \frac{3}{51} = \boxed{\frac{1}{17}}$$

Ch. 2 Axioms of Probability

2.1 Introduction

2.2 Sample Spaces and Events

Consider experiments such as ...

- flip a coin
- flip a coin twice
- draw a card from a deck
- roll a die

⋮

These are experiments where the outcome is not known in advance but the set of all possible outcomes is known in advance

Def: The set of all outcomes is called the

sample space.

EX
Sample space of flipping a coin is $S = \{H, T\}$

EX
Sample space of flipping a coin twice is

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

EX
Sample space of drawing a card from a deck is

$$S = \{ 2,3,4,5, \ldots, J, Q, K, A \text{ (Hearts)}, \quad 2,3,4, \ldots \text{ (Diamonds)}, \quad 2,3, \ldots K, A, \text{ (Clubs)}, \quad 2,3, \ldots, K, A \text{ (spades)} \}$$

EX

Sample space of rolling one die is

$$S = \{1,2,3,4,5,6\}$$

EX

Turn on a light bulb and measure the ~~[illegible]~~ # of hours until it burns out

$$S = \{x \in \mathbb{R} : x \geq 0\}$$

Def: An event is a subspace of the sample space.

Notes

- That is, an event consists of possible outcomes of the experiment. Not necessarily just one outcome and not necessarily all outcomes.

EX (Deck of Cards)

Let $E = \{A\heartsuit, A\diamondsuit, A\clubsuit, A\spadesuit\}$

= event that an ace is drawn from a deck of cards.

EX (One die)

Let $E = \{4,5,6\}$ = event of rolling a 4 or higher.

EX (Flip a coin twice)

Let $E = \{(H,H)\}$ = event of flipping heads two times in a row.

Further Notes

- If E and F are events in the same sample space S (so $E \subset S$, $F \subset S$)

union
- Then $E \cup F$ is an event in sample space S

intersection and $EF = E \cap F$ is an event " " " "

(if $E \cap F = \emptyset$ = Empty set — "null event" = event with no outcomes)

- $E^c = S \setminus E$ = complement of E in S is an event in S.

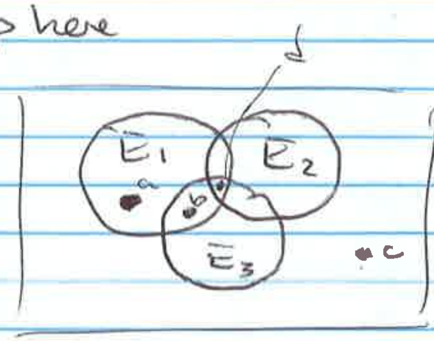
(i.e. $E \cup E^c = S$)

- If $E_1, E_2, E_3, \ldots$ are events in S then the union of these events $\bigcup_{n=1}^{\infty} E_n$ is the set of outcomes that are in E_n for at least one n.

Venn Diagram may help here

$a, b, d \in \bigcup_{n=1}^{3} E_n$

$c \notin \bigcup_{n=1}^{3} E_n$



- $\bigcap_{n=1}^{\infty} E_n$ is the set of outcomes in all of $E_1, E_2, E_3, \ldots$

$d \in \bigcap_{n=1}^{3} E_n$

- Review Basic Set Theory (see Ross, pp. 24-25)

Commutativity: $E \cup F = F \cup E$

$$EF = FE$$

Associativity: $(E \cup F) \cup G = E \cup (F \cup G)$

$$(EF)G = E(FG)$$

Distributivity: $(E \cup F)G = EG \cup FG$

$$EF \cup G = (E \cup G)(F \cup G)$$

DeMorgan's laws

$$(E \cup F)^c = E^c F^c$$

$$(EF)^c = E^c \cup F^c$$

Mutually Exclusive Events

If $EF = \emptyset$ ($\emptyset$ = event with no outcomes = empty set)

then events E and F are said to be <u>mutually exclusive</u>

<u>EX</u> (rolling die) $E = \{1,2,3\}$, $F = \{4,5,6\}$

$EF = \emptyset$ the event of rolling ≤ 3 and ≥ 4 has no outcomes