Worksheet 4 Answer Key

1. Let $a \in A$ and $b \in B$ and observe that

$$\begin{aligned} a(I_A \circ R)b &\iff \exists a' \in A : aI_A a' \wedge a'Rb \\ &\iff aRb. \end{aligned}$$

2. Fix $a \in A, b \in B$ and note that

$$\begin{aligned} a(R \circ I_B)b &\iff \exists a' \in A : aRb' \wedge b'I_B b \\ &\iff aRb. \end{aligned}$$

3. Choose $a \in A$ and $b \in B$. We have

$$\begin{aligned} a(R^{-1})^{-1}b &\iff bR^{-1}a \\ &\iff aRb. \end{aligned}$$

4. Fix $a \in A, c \in C$ and observe that

$$\begin{aligned} c(S \circ R)^{-1}a &\iff a(S \circ R)c \\ &\iff \exists b \in B : aRb \wedge bSc \\ &\iff \exists b \in B : bR^{-1}a \wedge cS^{-1}b \\ &\iff c(R^{-1} \circ S^{-1})a. \end{aligned}$$

5. Let $a \in A$ and $d \in D$. We have

$$\begin{aligned} a(T \circ S) \circ Rd &\iff \exists b \in B : aRb \wedge b(T \circ S)d \\ &\iff \exists b \in B : aRb \wedge (\exists c \in C : bSc \wedge cTd) \\ &\iff \exists b \in B : \exists c \in C : aRb \wedge bSc \wedge cTd \\ &\iff \exists c \in C : (\exists b \in B : aRb \wedge bSc) \wedge cTd \\ &\iff aT \circ (S \circ R)d. \end{aligned}$$

6. Fix $a \in A$ and observe that

$$\begin{aligned} a \in \operatorname{Dom} R &\iff \exists b \in B : aRb \\ &\iff \exists b \in B : bR^{-1}a \\ &\iff a \in \operatorname{Rng} R^{-1}. \end{aligned}$$

7. Fix $b \in B$ and observe that

$$\begin{aligned} b \in \operatorname{Rng} R &\iff \exists a \in A : aRb \\ &\iff \exists a \in A : bR^{-1}a \\ &\iff a \in \operatorname{Dom} R^{-1}. \end{aligned}$$

8. Let $A = \{1,2,3\}$ and define the relations

$$R = \{(1,1),(1,2),(2,1)(2,2),(3,3)\}$$
$$S = \{(1,1),(2,2),(2,3),(3,2),(3,3)\}.$$

Note that both R and S are equivalence relations on A, while

$$S \circ R = A^2 \backslash \{(3,1)\}$$

is neither transitive nor symmetric.

9. Fix $a, b, c \in A$.

If aRa then $a(R \circ R)a$, whence $(R \circ R)$ is reflexive.

Suppose that $a(R \circ R)b$ and $b(R \circ R)c$. Transitivity of R provides aRb and bRc, from which we obtain $a(R \circ R)c$. This establishes the transitivy of $R \circ R$.

Now suppose that $a(R \circ R)b$ and $b(R \circ R)a$. As above, the transitivity of R yields aRb and bRa, and antisymmetry provides $a = b$. We conclude that $R \circ R$ is antisymmetric.

10. Define the relations R and S on $A = \mathbb{Z}$ by

$$mRn \iff m \leq n$$
$$mSn \iff m \geq n.$$

Observe that both R and S are partial orders, while

$$S \circ R = \mathbb{Z} \times \mathbb{Z}$$

is not antisymmetric.

11. Let $A = \{1,2,3\}$ and let R and S be the relations of Question **??**. We have

$$\begin{aligned} S \circ R &= A^2 \backslash \{(3,1)\} \\ &\neq A^2 \backslash \{(1,3)\} \\ &= R \circ S. \end{aligned}$$

12. Let A, R, and S be given as in Question **??**.