3.2.7 Squeeze Theorem

Suppose $\{x_n\}, \{y_n\}, \{z_n\}$ are sequences such that

(*) $x_n \le y_n \le z_n \quad \forall n \in \mathbb{N}$.

Assume that $\lim x_n = \lim z_n$, then $\{y_n\}$ is convergent and

$$\lim x_n = \lim y_n = \lim z_n$$

Proof: Let $E := \lim x_n = \lim z_n$.

$\forall \epsilon > 0, \exists K(\epsilon) \in \mathbb{N}$:

$$|x_n - E| < \epsilon \quad \& \quad |z_n - E| < \epsilon,$$

$\forall n \ge K(\epsilon)$. Note that the above means

$$-\epsilon < x_n - E < \epsilon \quad \& \quad -\epsilon < z_n - E < \epsilon.$$

From (*)

$$x_n - E \le y_n - E \le z_n - E$$

So that

$$-\epsilon \le y_n - E < \epsilon,$$

or

$$|y_n - E| < \epsilon \quad \forall n \ge K(\epsilon),$$

i.e, $\lim y_n = E$. ∎

3.2.11 Theorem

Let $\{x_n\}$ be a sequence of positive real numbers such that

$$L := \lim \left(\frac{x_{n+1}}{x_n}\right) \text{ exists.}$$

If $L < 1$, then $\{x_n\}$ converges and $\lim x_n = 0$

Proof: Since $\frac{x_{n+1}}{x_n} > 0 \Rightarrow L \ge 0$. Choose $r \in (0,1)$ as $L < r < 1$ and define

$$\epsilon := r - L > 0$$

Since $\lim \frac{x_{n+1}}{x_n} = L$, then $\exists K(\epsilon)$:

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \epsilon \quad \forall n \ge K(\epsilon)$$

So that

$$\frac{x_{n+1}}{x_n} < L+\epsilon = r.$$

Hence for $n \geq K(\epsilon)$

$$0 < x_{n+1} < x_n \cdot r < x_{n-1} r^2 < \cdots < x_K r^{n-K+1}$$

Define $C = x_{K(\epsilon)} r^{-K}$ then

$$0 < x_{n+1} < C \cdot r^{n+1} \quad \forall n \geq K$$

Since $r \in (0,1) \Rightarrow \lim C r^{n+1} = 0$

$\Rightarrow$ by the squeeze theorem

$$\lim x_n = 0 \qquad \blacksquare$$

Example Let $x_n = \frac{b^n}{n!}$

$b > 1$ and note that $m! := m(m-1)(m-2)\cdots(1)$.

Then we prove $\lim x_n = 0$. Note.

$$\frac{x_{n+1}}{x_n} = \frac{\frac{b^{n+1}}{(n+1)!}}{\frac{b^n}{n!}} = \frac{b^{n+1}}{b^n} \cdot \frac{n!}{(n+1)!}$$

$$= b \cdot \frac{1}{n+1} \longrightarrow 0.$$

by the previous theorem $\lim x_n = 0$.

Section 3.3 (Monotone sequences)

- Convergence $\Rightarrow$ boundedness

- boundedness $\not\Rightarrow$ Convergence.

- $\left.\begin{array}{c}\text{boundedness} \\ + \\ \text{monotonicity}\end{array}\right\} \Rightarrow$ convergence.

Definition We say a sequence $\{x_n\}$ is increasing if

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots,$$

we say it is decreasing if

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$$

If $\{x_n\}$ is increasing or decreasing we say that $\{x_n\}$ is monotone.

3.3.2 Monotone Convergence Theorem Let $\{x_n\}$ be a monotone sequence. Then

$$\{x_n\} \text{ is convergent iff } \{x_n\} \text{ is bounded}.$$

Further

(a) If $\{x_n\}$ is a bounded increasing seq. $\lim x_n = \sup\{x_n : n \in \mathbb{N}\}$

(b) If $\{x_n\}$ is a bounded decreasing seq. $\lim x_n = \inf\{x_n : n \in \mathbb{N}\}$.

Proof: One direction is simple if $\{x_n\}$ is convergent we proved already that $\{x_n\}$ is bounded. We prove the other direction together with (a).

Since $\{x_n\}$ is bounded $\Rightarrow$ $|x_n| \leq M$ $\forall n \in \mathbb{N}$, and for some $M > 0$.

Then $\{x_n : n \in \mathbb{N}\}$ is bounded, and hence

$$x^* := \sup\{x_n : n \in \mathbb{N}\} \text{ exists}.$$

Let $\epsilon > 0$ be arbitrary $\Rightarrow$ $x^* - \epsilon$ is not an upper bound to $\{x_n : n \in \mathbb{N}\}$. Then $\exists K$: $x^* - \epsilon < x_K$.

Since $\{x_n\}$ is increasing

$$x^* - \epsilon < x_K \leq x_n \leq x^* < x^* + \epsilon \qquad \forall n \geq K.$$

So that $|x_n - x^*| < \epsilon$.

Since $\epsilon > 0$ was arbitrary $\Rightarrow$ $\lim x_n = x^*$ ∎

Examples

Let $x_1 > 1$ and $x_{n+1} := 2 - \frac{1}{x_n}$.

First note that $x_n > 1$ for all n.

For $n=1$, it is trivial, suppose it is true for $n-1$, then $x_{n-1} > 1$ so

$$x_n = 2 - \frac{1}{x_{n-1}} \quad \text{but} \quad 1 > \frac{1}{x_{n-1}}$$

$$-1 < -\frac{1}{x_{n-1}}$$

✓.

$$1 = 2-1 < 2 - \frac{1}{x_{n-1}}$$

- Is it increasing? We want to prove that $x_{n+1} \geq x_n$ so $2 - \frac{1}{x_n} \geq x_n$

or $2x_n - 1 - x_n^2 \geq 0 \Rightarrow -(x_n - 1)^2 \geq 0$ ✗

- Is it decreasing?

$$-(x_n-1)^2 \leq 0 \quad ✓$$

What is the limit?

$\{x_n\}$ is decreasing and bounded

$$x_n > 1 \quad \text{and} \quad x_1 \geq x_n$$

$$|x_n| \leq x_1.$$

The limit exists. Then $\{x_n\}$ and $\{x_{n+1}\}$ have the same limit

So from

$$x_{n+1} = 2 - \frac{1}{x_n}$$

Let $L = \lim x_n = \lim x_{n+1} \Rightarrow$

$$L = 2 - \frac{1}{L}$$

$$\Rightarrow \quad -(L-1)^2 = 0 \quad \Rightarrow \boxed{L=1}$$

3.4.1 Definition Let $\{x_n\}$ be a sequence and let $n_1 < n_2 < n_3 < \cdots < n_k < \cdots$ be a strictly increasing sequence. The sequence

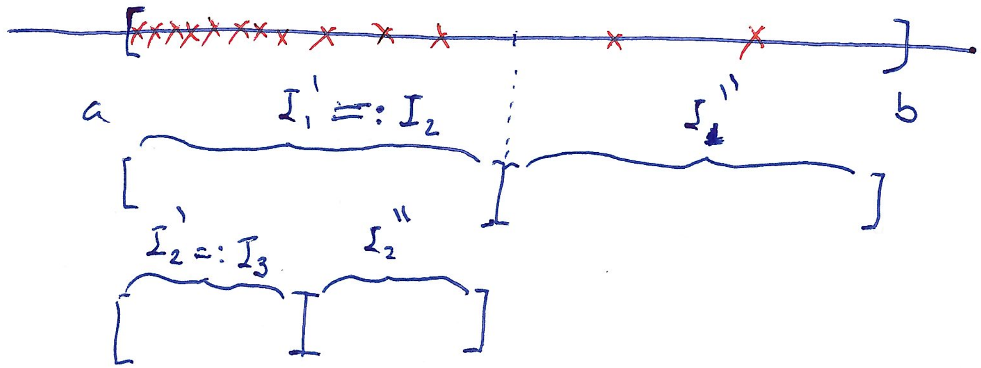
$$\{x_{n_k}\} = \{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\}$$

is called subsequence of $\{x_n\}$.

3.4.2 Theorem If $\{x_n\}$ converges to x then every subsequence of $\{x_n\}$ converges to the same limit.

3.4.8 The Bolzano-Weierstrass A bounded sequence of real numbers has a convergent subsequence.

IMPORTANT



define $n_1 := 1$

Proof Since $\{x_n : n\in \mathbb{N}\}$ is bounded is contained in $[a,b]$.

Step 1. Bisect $[a,b]$ into I_1' and I_1'', and divide $\{n\in\mathbb{N} : n>n_1\}$

$A_1 := \{n\in\mathbb{N} : n>n_1,\ x_n\in I_1'\}$ $B_1 = \{n\in\mathbb{N} : n>n_1,\ x_n\in I_1''\}$

If A_1 is infinite, define $I_2 := I_1'$, otherwise $I_2 := I_1''$.

Set n_2 as the smallest number in A_1 if ∞ or B_1 otherwise.

Step 2 Bisect I_2 into I_2' and I_2'', divide $\{n\in\mathbb{N} : n>n_2\}$

$A_2 := \{n\in\mathbb{N} : n>n_2,\ x_n\in I_2'\}$, $B_2 = \{n\in\mathbb{N} : n>n_2,\ x_n\in I_2''\}$

If A_2 is infinite, define $I_3 := I_2'$ otherwise $I_3 := I_2''$.

Set n_3 to the smallest number in A_2 if ∞ or B_2 otherwise.

Step 3. Continue the process ad-infinitum. Note $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ then by the nested interval property (2.5.2) $\exists\, \xi\in\mathbb{R} : \xi\in I_n, \forall n\in\mathbb{N}$

Further $|x_{n_k} - \xi| \le \frac{b-a}{2^{k-1}}$ then $\{x_{n_k}\}$ converges to ξ ∎

Section 3.5 The Cauchy Criterion

3.5.1 A sequence $\{x_n\}$ is said to be a Cauchy sequence if $\forall \epsilon > 0, \exists H(\epsilon) \in \mathbb{N} : \forall n, m \geq H(\epsilon)$

$$|x_n - x_m| < \epsilon .$$

Example $x_n = \dfrac{\sin(n)}{n^2}$, note

$$\left| x_n - x_m \right| \leq \frac{|\sin(n)|}{n^2} + \frac{|\sin(m)|}{m^2} \leq \frac{1}{n^2} + \frac{1}{m^2}$$

then, for $\epsilon > 0$ choose $H : \dfrac{1}{\sqrt{\frac{\epsilon}{2}}} < H$,

so that if $n, m \geq H$ then

$$|x_n - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon .$$

It is Cauchy.

Example $x_n = (-1)^n$. Suppose it is Cauchy then choose $\epsilon = 1$ and choose $n = 2k$ and $m = 2k+1 \Rightarrow$

$$|x_n - x_m| < 1 \Rightarrow |1 - (-1)| = 2 < 1$$

$\Rightarrow \Leftarrow$

$\underbrace{3.5.3}_{\text{Lemma}}$ If $\{x_n\}$ is convergent, it is Cauchy

Proof: Since $\lim x_n = x^* \Rightarrow$ for $\tilde{\epsilon} = \frac{\epsilon}{2}$, $\epsilon > 0$, $\exists K(\tilde{\epsilon}) : |x_n - x^*| < \tilde{\epsilon}$ for $n \geq K(\tilde{\epsilon})$. Then,

$$|x_n - x_m| = |x_n - x^* - (x_m - x^*)| \leq |x_n - x^*| + |x_m - x^*|$$
$$< 2\tilde{\epsilon} = \epsilon \quad \blacksquare$$

3.5.4 Lemma A Cauchy sequence is bounded.

Proof Let $\epsilon = 1$, $\exists H(1) = H : |x_n - x_H| < 1$ for $n \geq H$. Then $|x_n| \leq 1 + |x_H|$ for $n \geq H$.

Let

$$M := \sup \{ |x_1|, |x_2|, \ldots, |x_{H-1}|, |x_H| + 1 \}$$

then $|x_n| \leq M \quad \forall n \in \mathbb{N} \quad \blacksquare$

3.5.5 Cauchy Convergence Criteria

$\boxed{\text{A sequence } \{x_n\} \text{ is convergent IFF it is Cauchy.}}$

IMPORTANT.

Proof: We know convergent $\Rightarrow$ Cauchy. ✓

Suppose $\{x_n\}$ is Cauchy, $\forall \epsilon > 0$ $\exists$ $H(\frac{\epsilon}{2})$ such that $n, m \geq H(\frac{\epsilon}{2})$ then

$$(*) \qquad |x_n - x_m| < \frac{\epsilon}{2}.$$

Since $\{x_n\}$ is Cauchy $\Rightarrow$ bounded (by Lemma 3.5.4)

$\Rightarrow$ by Bolzano-Weierstrass $\exists$ a subsequence $\{x_{n_k}\}$ that converges to x^*. Then,

$\exists$ $K \geq H(\frac{\epsilon}{2})$ with $K \in \{n_1, n_2, \ldots\ldots\}$ such that

$$|x_K - x^*| < \frac{\epsilon}{2}.$$

So that (since $K \geq H(\frac{\epsilon}{2})$) by $(*)$

$$(**) \qquad |x_n - x_K| < \frac{\epsilon}{2} \quad \text{for } n \geq H(\frac{\epsilon}{2})$$

Hence for $n \geq H(\frac{\epsilon}{2})$

$$|x_n - x^*| = |(x_n - x_K) + (x_K - x^*)|$$

$$\leq |x_n - x_K| + |x_K - x^*|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\Rightarrow$ $\lim x_n = x^*$ since $\epsilon > 0$ was arbitrary ∎

Example) $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

~~31814~~ Let $m > n$ then,

$$x_m - x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m}$$

$$\geq \underbrace{\frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}}_{m-n \text{ times}} = \frac{m-n}{m}$$

$$= 1 - \frac{n}{m}$$

choose $m = 2n$, so

$$|x_{2n} - x_n| \geq \frac{1}{2}, \quad \Rightarrow \text{not Cauchy} ∎$$

3.5.7. Definition $\{x_n\}$ is contractive if $\exists: 0 < C < 1$:

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|.$$

For example consider $G: \mathbb{R} \to \mathbb{R}$ defined as

$$|G(x) - G(y)| \leq C|x-y|$$

with $C \in (0,1)$ $\Rightarrow$

$$x_{n+1} = G(x_n)$$

is contractive.

3.5.8 theorem Every contractive sequence is Cauchy and hence it is convergent.

Proof:

$$\boxed{|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|} \leq C^2|x_n - x_{n-1}|$$

$$\leq \cdots\cdots \leq \boxed{C^n|x_2 - x_1|}$$

Let $m > n$, then

$$x_m - x_n = (x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots\cdots + (x_{n+1} - x_n)$$

so

$$|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

$$\leq C^{m-2}|x_2 - x_1| + C^{m-3}|x_2 - x_1| + \cdots\cdots + C^{n-1}|x_2 - x_1|$$

$$\leq C^{n-1}|x_2 - x_1| \underbrace{\left(1 + C + \cdots\cdots + C^{m-n-1}\right)}_{\leq \frac{1}{1-C}}$$

$$\leq \frac{C^{n-1}}{1-C}|x_2 - x_1|.$$

$\Rightarrow$ $\{x_n\}$ is Cauchy