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Worksheet 4

Let R be a relation from A to B, let S be a relation from B to C, and let T be a relation from C to D.

Prove the following statements.

1. $I_A \circ R = R$

    Proof. Let $a \in A$ and $b \in B$ and observe:

    $$a(I_a \circ R)b \iff \exists a^{'} \in A : a = a^{'} \wedge a^{'}Rb$$
    $$\iff aRb$$

    $\square$

2. $R \circ I_A = R$

    Proof. Let $a \in A$ and $b \in B$ and observe:

    $$a(R \circ I_a)b \iff \exists b^{'} \in B : b = b^{'} \wedge aRb^{'}$$
    $$\iff aRb$$

    $\square$

3. $(R^{-1})^{-1} = R$

    Proof. Assume the relation R has an inverse and let $a \in A$ and $b \in B$:

    $$a(R^{-1})^{-1}b \iff bR^{-1}a$$
    $$\iff b(R^{-1})^{-1}a$$

    $\square$

4. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

    Proof. Let $a \in A$, $b \in B$, and $c \in C$

    $$c(S \circ R)^{-1}a \iff \exists c^{'} \in C : c = c^{'} \wedge c^{'}S^{-1}b$$
    $$\iff \exists b^{'} \in B : b = b^{'} \wedge b^{'}R^{-1}a$$
    $$\iff c(R^{-1} \circ S^{-1})a$$

    $\square$

5. $(T \circ S) \circ R = T \circ (S \circ R)$

Proof. Fix $a \in A$, $b \in B$, $c \in C$, and $d \in D$

$$
\begin{aligned}
a(T \circ S) \circ Rd &\iff \exists a' \in A : a' = a \wedge a' Rb \\
&\iff \exists b' \in B : b' = b \wedge b' Sc \\
&\iff \exists c' \in C : c' = c \wedge c' Td \\
&\iff aT \circ (S \circ R)d
\end{aligned}
$$

$\square$

6. $DomR = RngR^{-1}$

Proof. Suppose R is invertible and let $d \in DomR$

$$
\begin{aligned}
DomR &\iff \exists r \in RngR^{-1} : r = d \\
&\iff RngR^{-1}
\end{aligned}
$$

$\square$

7. $RngR = DomR^{-1}$

Proof. Suppose R is invertible and $r \in RngR$

$$TBD$$

$\square$

For Question 8–10, suppose that $A = B = C$.

8. If R and S are equivalence relations, then $S \circ R$ is an equivalence relation.

Proof. $\square$

9. If R is a partial order, then $R \circ R$ is a partial order.

Proof. $\square$

10. If R and S are partial orders, then it is not generally true that $S \circ R$ is a partial order.

Proof. $\square$

Bonus Questions Give an example of two relations R and S on a set A such that

11. $R \circ S \neq S \circ R$.

Proof. □

12. $S \circ R$ is an equivalence relation, but neither R nor S is an equivalence relation.

Proof. □