Worksheet 3 Answer Key

We will prove 3, 4, 5, 6, 10, and 11, and disprove 1, 2, and 9.

1. Let $A = C = \{0\}$ and $B = \{1\}$. We have $\{0\} \cap \{1\} = \varnothing$ while $\{0\} \cap \{0\} \neq \varnothing$.

2. Let $A = C = \{0\}$ and $B = \{1\}$. We have $\{0\} \not\subseteq \{1\}$ and $\{1\} \not\subseteq \{0\}$, while $\{0\} \subseteq \{0\}$.

3. Suppose that $A \subseteq \varnothing$. From $\varnothing \subseteq A$, we conclude that $A = \varnothing$.

4. Fix $x \in A \cap B$. By the definition of intersection, we have $x \in A$ and $x \in B$. From the inclusion $A \subseteq C$, it follows that $x \in C$.

5. Let $a, a' \in A$ and suppose that $g(f(a)) = g(f(a'))$. By the injectivity of g we obtain $f(a) = f(a')$, and by the injectivity of f we conclude that $a = a'$.

6. Fix $c \in C$. The surjectivity of g implies the existence of a $b \in B$ with $g(b) = c$, while that of f yields an $a \in A$ with $f(a) = b$. Observe that, $g(f(a)) = g(b) = c$.

7. Consider $f : \mathbb{N} \to \mathbb{N}$ defined by
$$f(k) = k + 1.$$

8. Let $f : \mathbb{N} \to \mathbb{N}$ be given by
$$f(k) = \begin{cases} k - 1 & \text{if } k \geq 1 \\ 0 & \text{if } k = 0. \end{cases}$$

9. Let $f : \{0\} \hookrightarrow \mathbb{N}$ be the inclusion map, and let $g : \mathbb{N} \to \{0\}$ be the constant map with value 0. Then
$$\begin{aligned} g \circ f : \{0\} &\to \{0\} \\ 0 &\mapsto 0 \end{aligned}$$
is a bijection, while neither f nor g is a bijection.

10. Suppose for a contradiction that $f : A \to A$ is not injective. It follows that the size of the image of f is strictly less than the size of the domain A. Consequently, the image of f is not equal to A. This contradicts the surjectivity of f.

11. Let $x, x' \in A$ with $f(x) = f(x')$. By applying f to both sides of the preceding equality, we have
$$x = f \circ f(x) = f \circ f(x') = x'$$
whence f is injective. Now fix $y \in A$. From the identity $f(f(y)) = y$ we conclude that f is surjective.