Math 300

Corse Notes

Contents

3 Sets operations and functions

3.1 Assorted abbreviations

abbr.	Latin	meaning
e.g.	*exempli gratia*	for example
i.e.	*id est*	that is
viz.	*videlicet*	namely
cf.	*confer*	compare (*erroneously:* see)
ff.	*foliis*	following
ibid.	*ibidem*	in the same place (followed by page number)
op. cit.	*opere citato*	in the work cited (in the same work)
loc. cit.	*loco citato*	in the place cited (on the same page)
QED	*quod erat demonstrandum*	that which was to be shown

3.2 Union, intersection, containment, and complement

Let A and B be sets.

Definition. The *union* of A and B is

$$A \cup B = \big\{x \,|\, x \in A \text{ or } x \in B\big\}.$$

Example. If A and B are the sets of even and odd integers, respectively, then $A \cup B = \mathbb{Z}$.

Definition. The *intersection* of A and B is

$$A \cap B = \big\{x \,|\, x \in A \text{ and } x \in B\big\}.$$

Example. We have

$$\mathbb{N} = \mathbb{Z} \cap \mathbb{R}_{\geq 0}$$

where $R_{\geq 0} = \{x \in \mathbb{R} \,|\, x \geq 0\}$ is the set of nonnegative real numbers.

Definition. We say that A and B are *disjoint* when $A \cap B = \varnothing$.

Example. Every set A is disjoint from the empty set $\varnothing$.

Definition. We say that A is a *subset* of B if

$$\forall x : \big(x \in A \to x \in B\big).$$

In this case, we write $A \subseteq B$.

Examples. We have

i. $\varnothing \subseteq A$ for every set A,

ii. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

Definition. The *difference* of A and B is

$$B \backslash A = \{x \in B \,|\, x \notin A\}.$$

Example. The set of irrational numbers is $\mathbb{R} \backslash \mathbb{Q}$.

Definition. If $A \subseteq B$, then the *complement* of A in B is $A^c = B \backslash A$.

Example. The complement of the set of even integers is the set of odd integers.

Claim. *Let A, B, and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

To prove this, we will assume that $A \subseteq B$ and $B \subseteq C$, and we must deduce that $A \subseteq C$.

Proof. Fix $x \in A$. From $A \subseteq B$ we obtain $x \in B$, and from $B \subseteq C$ we conclude that $x \in C$. $\square$

3.3 First definitions and examples

Let A and B be sets.

Informal Definition. A *function* $f : A \to B$ is a rule that assigns to each $x \in A$ a unique $f(x) \in B$.

$$\forall x \in A : \exists! y \in B : y = f(x)$$

Remark. We sometimes write $x \mapsto f(x)$ to

Examples. i. Consider

$$\begin{aligned} f : \mathbb{N} &\to \mathbb{N} \\ k &\mapsto 2k. \end{aligned}$$

ii. The *identity function* on A is

$$\begin{aligned} f : A &\to A \\ x &\mapsto x. \end{aligned}$$

iii. The *constant function* $f : A \to B$ with value $b \in B$ is

$$\begin{aligned} f : A &\to B \\ x &\mapsto b. \end{aligned}$$

iv. The *empty function* $f : \varnothing \to B$ is completely determined by the value it assigns each element in $\varnothing$.

v. If $A \subseteq B$ then the associated *inclusion function* is

$$\begin{aligned} f : A &\to B \\ x &\mapsto x. \end{aligned}$$

vi. We may consider a property $P(x)$ that elements $x \in A$ can satisfy as a function

$$\begin{aligned} P : A &\to \mathbb{B} \\ x &\mapsto P(x) \end{aligned}$$

where $\mathbb{B} = \{\top, \bot\}$ is the Boolean domain, comprising the *truth values* true $\top$ and false $\bot$.

Definition. The *composition* of $f : A \to B$ and $g : B \to C$ is

$$\begin{aligned} g \circ f : A &\to C \\ x &\mapsto g(f(x)). \end{aligned}$$

3.4 Injectivity and surjectivity

Definition. The function $f : A \to B$ is said to be *injective* if $f(x) = f(y)$ implies $x = y$.

$$\forall x, y \in A : \big(f(x) = f(y)\big) \implies (x = y)$$

Claim. *The function $f : \mathbb{N} \to \mathbb{N}$ given by $f(k) = 2k$ is injective.*

Proof. Let $k, \ell \in \mathbb{N}$ and suppose that $f(k) = f(\ell)$. Dividing both sides of $2k = 2\ell$ by 2 yields $k = \ell$. □

Claim. *The constant function $f : \mathbb{R} \to \mathbb{Z}$ with value 0 is not injective.*

We must show that

$$\exists x, y \in \mathbb{R} : \big(f(x) = f(y)\big) \wedge (x \neq y).$$

Proof. We have $f(1) = 0 = f(2)$ but $1 \neq 2$. $\square$

Definition. The function $f : A \to B$ is called *surjective* when for every $y \in B$ there is an $x \in A$ with $f(x) = y$.

$$\forall y \in B : \exists x \in A : f(x) = y$$

Claim. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not surjective.*

We must show that

$$\exists y \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) \neq y$$

Proof. From $x^2 \geq 0$ for all $x \in \mathbb{R}$, it follows that $f(x) \neq -1$ for all $x \in \mathbb{R}$. $\square$

Definition. We say that $f : A \to B$ is *bijective* when it is both injective and surjective.

Claim. *The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is bijective.*

Proof. If $x, y \in \mathbb{R}$ satisfy $2x = 2y$, then division by 2 yields $x = y$. This proves injectivity.

To establish surjectivity, let $y \in \mathbb{R}$ be arbitrary and observe that $2(\frac{y}{2}) = y$. $\square$