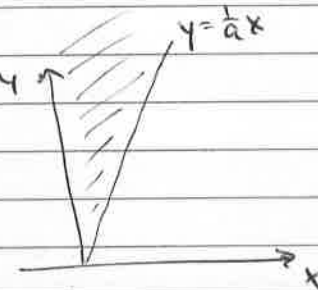
EXAMPLE (1e, p. 227)

The joint density function of X and Y is

$$f(x,y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the density function of the random variable $\frac{X}{Y}$

Want $F_{\frac{X}{Y}}(a) = P\left\{ \frac{X}{Y} \le a \right\} = P\{ X \le aY \}$



$$= \iint f(x,y)\,dx\,dy$$

$$= \int_0^\infty \int_0^{ay} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^\infty -e^{-(x+y)} \Big|_0^{ay}\,dy$$

$$= \int_0^\infty \left[-e^{-(ay+y)} + e^{-y} \right] dy$$

$$= \frac{e^{-(a+1)y}}{(a+1)} - e^{-y} \Big|_0^\infty = 0 - \left(\frac{e^0}{a+1} - 1 \right) = 1 - \frac{1}{a+1}$$

$$F_{\frac{X}{Y}}(a) = 1 - \frac{1}{a+1}$$

$$f_{\frac{X}{Y}}(a) = \frac{d}{da} F_{\frac{X}{Y}}(a) = +\frac{1}{(a+1)^2}$$

Joint probability distributions for n random variables

Multinomial Distribution (see p. 228, Example 1f)

- Suppose n independent and identical experiments are performed.
- Suppose that each experiment can result in one of r possible outcomes, with probabilities of these outcomes

$$p_1, p_2, p_3, \dots, p_r$$

with $\sum_{i=1}^{r} p_i = 1$.

- Let X_i = the number of the n experiments that result in outcome # i.

(e.g. roll a fair die 20 times — $n=20$, $r=6$, $p_1=p_2=\dots=p_6=\frac{1}{6}$, X_i = # of appearances of "i" in the 20 rolls)

- Then,

$$\boxed{P\{X_1=n_1, X_2=n_2, \dots, X_r=n_r\} = \frac{n!}{n_1!\,n_2!\cdots n_r!}\, p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}}$$

r discrete random variables $X_1, \dots, X_r$.

$n_1+n_2+\dots+n_r = n$

*

6.2 Independent Random Variables

Def: The R.V.s X and Y are said to be independent if for any two sets of real numbers A and B

$$P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{X \in B\}$$

- Recall our definition of independent events from Ch. 3 (Section 3.4, p. 75)
  - Events E and F are independent if

$$P(EF) = P(E) \cdot P(F)$$

i.e.

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(E) \cdot P(F)}{P(F)} = P(E)$$

$$P(F|E) = \frac{P(EF)}{P(E)} = \frac{P(E) \cdot P(F)}{P(E)} = P(F)$$

- An equivalent statement of independence of X and Y is

$$P\{X \leq a, Y \leq b\} = P\{X \leq a\} \cdot P\{Y \leq b\} \quad \text{for all } a, b$$

that is, X and Y are independent if

$$F(a,b) = F_X(a) \cdot F_Y(b) \quad = \text{product of marginal cdf's.}$$

for all a, b.

Discrete	Continuous
X and Y independent if	X and Y independent if
$p(x,y) = p_X(x) \cdot p_Y(y)$	$f(x,y) = f_X(x) \cdot f_Y(y)$
product of marginal pmf's	product of marginal pdf's.

Recall, in general the joint c.d.f. for X, Y (continuous)

(1) $$F(a,b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy \quad \text{for all } a,b$$

If X and Y are independent then

$$F(a,b) = F_X(a)\cdot F_Y(b) \quad \text{for all } a,b$$

so

$$F(a,b) = \int_{-\infty}^{a} f_X(x)\,dx \cdot \int_{-\infty}^{b} f_Y(y)\,dy$$

(2) $$= \int_{-\infty}^{b}\int_{-\infty}^{a} f_X(x)\cdot f_Y(y)\,dx\,dy \quad \text{for all } a,b$$

Comparing (1) with (2) says that for all a,b

$$\int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy = \int_{-\infty}^{b}\int_{-\infty}^{a} f_X(x)\,f_Y(y)\,dx\,dy$$

So this must apply "pointwise" (i.e. the integrands must match... So

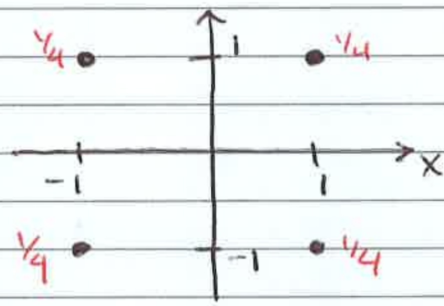
$$f(x,y) = \underbrace{f_X(x)\,f_Y(y)}_{\text{Product of marginal pdf's.}} \quad \text{for all } x,y$$

EXAMPLE (Discrete Case)

Consider discrete R.V.'s X and Y with joint pmf

$$p(x,y) = \frac{1}{4} \quad \text{if } (x,y) \in \{(-1,-1), (-1,1), (1,-1), (1,1)\}$$

i.e.



Note:

$$p_X(x) = \sum_y p(x,y) = \begin{cases} p(-1,-1) + p(-1,1) = 1/4 + 1/4 = \boxed{1/2} & x = -1 \\ p(1,-1) + p(1,1) = 1/4 + 1/4 = \boxed{\frac{1}{2}} & x = 1 \end{cases}$$

$$p_Y(y) = \sum_x p(x,y) = \begin{cases} p(-1,-1) + p(1,-1) = 1/4 + 1/4 = \boxed{1/2} & y = -1 \\ p(-1,1) + p(1,1) = 1/4 + 1/4 = \boxed{1/2} & y = 1 \end{cases}$$

Is it true that $p(x,y) = p_X(x) \cdot p_Y(y)$ for all x,y?

- $p(-1,-1) \stackrel{?}{=} p_X(-1) \cdot p_Y(-1)$

  $\frac{1}{4} = 1/2 \cdot 1/2$ ✓

- $p(-1,1) \stackrel{?}{=} p_X(-1) \cdot p_Y(1)$

  $1/4 = 1/2 \cdot 1/2$ ✓

- $p(1,-1) \stackrel{?}{=} p_X(1) \cdot p_Y(-1)$

  $1/4 = 1/2 \cdot 1/2$ ✓

- $p(1,1) \stackrel{?}{=} p_X(1) \cdot p_Y(1)$

  $1/4 = 1/2 \cdot 1/2$ ✓

So X and Y are independent R.V.'s

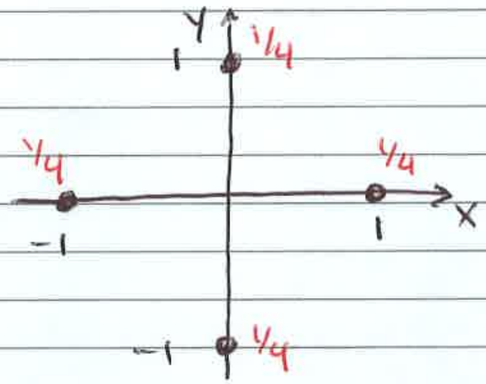
So.. knowledge of X does not inform you about the value of Y. And vice versa...

*

but now consider R.v.'s X and Y where

$$p(x,y) = 1/4 \quad \text{if} \quad (x,y) \in \{(1,0),(-1,0),(0,1),(0,-1)\}$$

i.e.



Here

$$p_X(x) = \sum_y p(x,y) = \begin{cases} \boxed{1/4} & x=-1 \\ 1/4 + 1/4 = \boxed{1/2} & x=0 \\ \boxed{1/4} & x=1 \end{cases}$$

and

$$p_Y(y) = \sum_x p(x,y) = \begin{cases} \boxed{1/4} & y=-1 \\ 1/4 + 1/4 = \boxed{1/2} & y=0 \\ \boxed{1/4} & y=1 \end{cases}$$

Is it true that $p(x,y) = p_X(x) \cdot p_Y(y)$ for all x, y?

NO, note that, for example,

$$p(1,0) = 1/4$$

not the same!

but

$$p_X(1) \cdot p_Y(0) = 1/4 \cdot 1/2 = 1/8$$

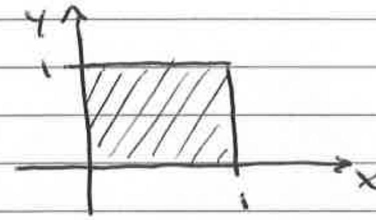
For this example, knowledge of X does give some information on the value of Y. And vice verse.

*

EXAMPLE (Continuous Case)

Consider continuous R.V.'s X and Y with joint pdf

$$f(x,y) = \begin{cases} 4xy & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



Note: $\int_0^1\int_0^1 4xy\,dx\,dy = \int_0^1 4\frac{x^2}{2}y\Big|_0^1 dy = \int_0^1 2y\,dy = 1$

Here

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_0^1 4xy\,dy = 4x\frac{y^2}{2}\Big|_0^1 = 2x$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \int_0^1 4xy\,dx = 4y\frac{x^2}{2}\Big|_0^1 = 2y$$

So, we can see that for all x,y

$$f_X(x)\,f_Y(y) = 2x \cdot 2y = 4xy = f(x,y)$$

So X and Y are independent.

EXAMPLE

Recall the ~~ex~~ continuous X and Y with

$$f(x,y) = \begin{cases} 2 & x \geq 0,\ y \geq 0,\ x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(see notes p. 202). There we found

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 2-2x & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$ (see notes p. 204-205)

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ 2-2y & 0 < y < 1 \\ 0 & y > 1 \end{cases}$$ (see notes p. 205)

clearly $f(x,y) \neq f_X(x) \cdot f_Y(y)$ for all x, y.

Therefore in this example, X and Y are not independent.

This should make sense. Given some information about X this then provides information about Y.