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Worksheet 4

Let R be a relation from A to B, let S be a relation from B to C, and let T be a relation from C to D.
Prove the following statements.

1. $I_A \circ R = R$

   Proof. Let $a \in A$ and $b \in B$ and observe:

$$\begin{aligned} a(I_a \circ R)b &\iff \exists a^{`} \in A : a = a^{`} \wedge a^{`}Rb \\ &\iff aRb \end{aligned}$$

$\square$

2. $R \circ I_A = R$

   Proof. Let $a \in A$ and $b \in B$ and observe:

$$\begin{aligned} a(R \circ I_a)b &\iff \exists b^{'} \in B : b = b^{'} \wedge aRb^{'} \\ &\iff aRb \end{aligned}$$

$\square$

3. $(R^{-1})^{-1} = R$

   Proof. Assume the relation R has an inverse and let $a \in A$ and $b \in B$:

$$\begin{aligned} a(R^{-1})^{-1}b &\iff bR^{-1}a \\ &\iff b(R^{-1})^{-1}a \end{aligned}$$

$\square$

4. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

   Proof. Let $a \in A$, $b \in B$, and $c \in C$

$$\begin{aligned} c(S \circ R)^{-1}a &\iff \exists c^{'} \in C : c = c^{'} \wedge c^{'}S^{-1}b \\ &\iff \exists b^{'} \in B : b = b^{'} \wedge b^{'}R^{-1}a \\ &\iff c(R^{-1} \circ S^{-1})a \end{aligned}$$

$\square$

5. $(T \circ S) \circ R = T \circ (S \circ R)$

Proof. Fix $a \in A$, $b \in B$, $c \in C$, and $d \in D$

$$\begin{aligned} a(T \circ S) \circ Rd &\iff \exists a^{'} \in A : a^{'} = a \wedge a^{'}Rb \\ &\iff \exists b^{'} \in B : b^{'} = b \wedge b^{'}Sc \\ &\iff \exists c^{'} \in C : c^{'} = c \wedge c^{'}Td \\ &\iff aT \circ (S \circ R)d \end{aligned}$$

$\square$

6. $Dom\, R = Rng\, R^{-1}$

   Proof. ($\subseteq$) Suppose R is invertible and fix $r \in Rng\, R^{-1}$. By definition of inverse it follows that $r \in Dom\, R$. $\square$

   Proof. ($\supseteq$) Fix $d \in Dom\, R$. By definition of domain it follows that $d \in Rng\, R^{-1}$. $\square$

7. $Rng\, R = Dom\, R^{-1}$

   Proof. ($\supseteq$) Suppose R is invertible and $r \in Rng\, R$. By the definition of inverse $r \in Dom\, R^{-1}$. $\square$

   Proof. ($\subseteq$) Fix $d \in Dom\, R^{-1}$. By the defintion of domain it follows that $d \in Rng\, R$. $\square$

For Question 8–10, suppose that $A = B = C$.

8. If R and S are equivalence relations, then $S \circ R$ is an equivalence relation.

   Proof. Suppose R is an equivalence relation from A to B and S is an equivalence relation from B to C and $A = B = C$.

$$\begin{aligned} S \circ R &\iff \forall a \in A : aSa \wedge aRa \\ &\iff \forall a, b, c \in A : (aSb \wedge bSc) => aSa \wedge (aRb \wedge bRc) => aRc \\ &\iff \forall a, b \in A : (aSb \wedge bSa) \wedge (aRb \wedge bRa) \end{aligned}$$

$\square$

9. If R is a partial order, then $R \circ R$ is a partial order.

   Proof. Suppose R is a partial order from A to B and $A = B$

$$\begin{aligned} R &\iff \forall a \in A : aRa \\ &\iff \forall a, b, c \in A : (aRb \wedge bRc) => aRc \\ &\iff \forall a, b \in A : (aRb \wedge bRa) => a = b \\ &\iff R \circ R \end{aligned}$$

$\square$

10. If R and S are partial orders, then it is not generally true that $S \circ R$ is a partial order.

    Proof. Let $R = \leq$ and $S = |$. Fix $a = 3$ and $b = 5$. Observe that $3 \leq 5$, however $3|5$ hence $S \circ R$ is not a partial order. $\square$

Bonus Questions Give an example of two relations R and S on a set A such that

11. $R \circ S \neq S \circ R$.

    Proof. Suppose $R = \leq$ and $S = |x|$. Fix $a = -9$ and $b = 5$. Observe that $-9(R \circ S)5 \neq -9(S \circ R)5$. $\square$

12. $S \circ R$ is an equivalence relation, but neither R nor S is an equivalence relation.

    Proof. $\square$