Worksheet 5

Name: ______________________________

1. Which of the following terms describe $(\mathbb{N}, +)$?

   magma, (commutative) semigroup, (commutative) monoid, (abelian) group

2. Fix a set A and consider the set

   $$\mathrm{Fun}(A, A) = \{f \mid f : A \to A\}$$

   of functions from A to itself, equipped with the binary product $\circ$ of function composition. Which of the following terms necessarily apply to $(\mathrm{Fun}(A, A), \circ)$?

   magma, (commutative) semigroup, (commutative) monoid, (abelian) group

3. Fix an integer $n \geq 1$ and equip the set $\mathbb{Z}_n = \{0, \ldots, n-1\}$ with multiplication modulo n,

   $$k * \ell = k\ell \mod n.$$

   Which of the following terms describe $(\mathbb{Z}_3, *)$? Which describe $(\mathbb{Z}_4, *)$?

   magma, (commutative) semigroup, (commutative) monoid, (abelian) group

4. Let A be a nonempty set and define the product $*$ by

   $$a * b = b.$$

   Show that every element $a \in A$ is a left identity in $(A, *)$. Explain why this does not contradict the uniqueness of identity elements that we proved in class.

5. Define a binary operation $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that every $x \in \mathbb{R}$ is a right identity for $*$.

6. Show that if a magma $(A, *)$ has a left identity $e_L \in A$ and a right identity $e_R \in A$ then $e_L = e_R$. Furthermore, prove that $e = e_L = e_R$ is an identity element for $(A, *)$.

7. A *zero element* for a binary operation $* : A \times A \to A$ is an element $z \in A$ satisfying

   $$\forall a \in A : a * z = z = z * a.$$

   Prove that if $z \in A$ is a zero for $*$, then z is unique with this property.

8. Let $(A, *)$ be a group. Prove that if $(A, *)$ has a zero element $z \in A$, then $A = \{z\}$.

9. A *band* is a semigroup $(A, *)$ with the property that

$$\forall a \in A : a * a = a.$$

   Define the relation $\leq$ on A by

$$a \leq b \iff a * b = a.$$

   Prove that $\leq$ is a partial order on A.

10. Suppose that $(A, *)$ is a group with three elements. Prove that $(A, *)$ is abelian.