Centre de Recerca Matemàtica

On Savings Optimisation Strategies

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2018

A man got to have a code.

Omar Little

Abstract

In classical savings theory it is stated that the optimal way to ensure a risk level is to set a constant proportion of risky assets at all time. This way, most savings strategies try to optimize their results based on a fixed risk aversion profile initially settled by the investor. Usually, funds arranging pension plans manage their fund as the result of the sum of many individually isolated investors, with no consideration upon the opportunities present in more collectively-managed schemes.

Throughout this work we will replicate the results of an alternative optimal strategy that changes proportion invested in risky assets along time, instead of setting an initial constan proportion; and we will confirm that this kind of strategy does not impede to set a fixed risk aversion profile. Moreover, we will incorporate the concept of Pooled Funds to those schemes and compare their performances using standard risk measures.

Finally, we will study the risk profile of all previously developed strategies on the article using Tail Distribution Modelling and Extreme Value Analysis in order to contrast the particularities of location and scale free risk measures.

Contents

1. Introduction

The reason for saving is to utilise the present wealth of a saver in order to build a retirement plan that secures a safe stream of capital for when it may be needed. This future condition need can be thought as deterministic, as we usually think in a typical pension plan, when there is a fixed time scheme when the saver starts collecting his money and when it ends; or it can be subject to some non-deterministic eventuality, as in most insurance plans. This kind of investment is, thus, characterised by an initial period of time in which the investor is saving money followed by a period of consumption, once the investor satisfies that future condition.

Hence, every savings strategy should aim at maximising that final capital whilst *securing* it. In general, we could say that there is a trade-off between that maximisation of capital (return) and the degree of its security (risk). The balance between these two magnitudes is what savings strategies try to optimise.

It is obvious that every saver would like to maximise the return of his money. But if we accept that this can only be accomplished at the expense of more risk, the investor has to decide which degree of risk is she able to tolerate, setting a risk limit. This decision upon the exposure to risk is what defines the *risk aversion* profile of every investor.

Most savings strategies are measured setting a fixed risk limit provided by a risk aversion profile, thus maximising the returns that can be extracted once that risk limit is provided. Thus the great interest canalized on testing and analysing the results of different savings strategies using the return and the risk as measures of performance.

Throughout this project we will compare the results obtained from two different approaches to saving strategies, and we will compare their risk and return. The set up for both of them would be the same. A simulated investor will save up a yearly fixed amount of money from the age of 30 years and, after that, she will consume the same amount for the next 30 years. Part of the saved money will be invested in risky assets whilst the rest will be invested in risk-free assets. The question to be addressed is the optimal proportion of the investment exposed to risky assets.

In the case that we use as a benchmark, called *Constant Proportion Portfolio Insurance* (CPPI), introduced by [1], a constant proportion of the wealth to be invested is allocated in risky assets, whereas the rest is allocated in non-risky assets. This means that the same proportion of the investment will be invested in the risky market, year over year.

The other case, developed in [2], consists of a different approach regarding that proportion of risky investment. Instead of a time-constant proportion, this model suggests a variable proportion that follows a formula that takes into account the present wealth of the investor. We will explain this formula in the following chapters.

Later on, we will include the concept of a *Pooled Fund*, in which the savers are not considered in isolation, but parts of a whole Fund. The main characteristic of considering the saver within a common Pooled Fund is the concept of mortality. There is always a risk that the saver does not live up to retire and collect back the fruit of her savings. Taking that risk into account, we will show how the wealth of those unlucky deceased investors can be distributed among the survivors. Moreover, we will test both previously described strategies in the context of a Pooled Fund and analyse their performance.

Throughout the majority of this work, we will make use of the risk adjusted ad hoc performance measure, recently developed in [3, 4], in order to evaluate our models. Yet in the final section we will develop and explain some other measures of Risk that are more sensible to worst case scenarios and larger losses, becoming another useful tool to assess risk and performance of Investment and Savings Strategies.

2. Background On Risk Measures

When simulating a savings scheme, the first assumption is regarding risk. It is usual to make a stark distinction between risky and risk-free investments.

But what do we mean by *risk*? We define risk as the uncertainty of the outcome of a given investment. So if we save some money at a 0.1% bank deposit, we can fairly assume that this investment is risk-free, because we know how much money we will receive at the end. A different scenario could be to invest this money in some market-dependent asset with 1% expected return but no guarantee of that return.

But how do we reckon that degree of uncertainty? In stock markets is usual to assume that the price of stocks behaviour is a brownian motion (see [5, 6]), and as such we can define their movement by *trend* and *dispersion* parameters. Since the return obtained by investing in stock assets comes from the relative difference in price between purchase and selling, it is easy to deduce that if we assume that the price follows a geometric brownian motion with trend μ and dispersion σ, the expected return of our investment would be μ with standard deviation σ. Thanks to this mathematical scheme, we can numerically define the expected return and some degree of uncertainty on it that we will henceforth use as our starting theoretical point. So, the returns at time t are defined as:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}. \tag{1}$$

Where P_t is the price of a stock at time t and P_{t-1} is the price of the stock at time $t-1$.

Even though many different models can be assumed to describe the behaviour of financial markets, the point of this section is to understand and exemplify the set up necessary in order to decide how we measure risk.

Another way to look at the risk definition is not just looking at dispersion. Using the standard defilogreturnsnition of risk, any unexpected uplift from the expected return would be considered under the umbrella of 'risky outcomes'. Since it is kind of counter-intuitive to assume the probability of a positive outcome as 'risk', it is fair to assume risk as just one half of that dispersion, the negative part of the random variable[1].

If we take a look at the actual observed returns in the stock market - as shown in figure 1 we can see the frequency of different returns. With a little bit of imagination, we can notice a *Bell Curve* shape, and so we could think of a normal distribution of the logreturns. Where *logreturns* are defined as:

$$r_t = \ln(1 + R_t). \tag{2}$$

[1]Losses are negative and profits are positive.

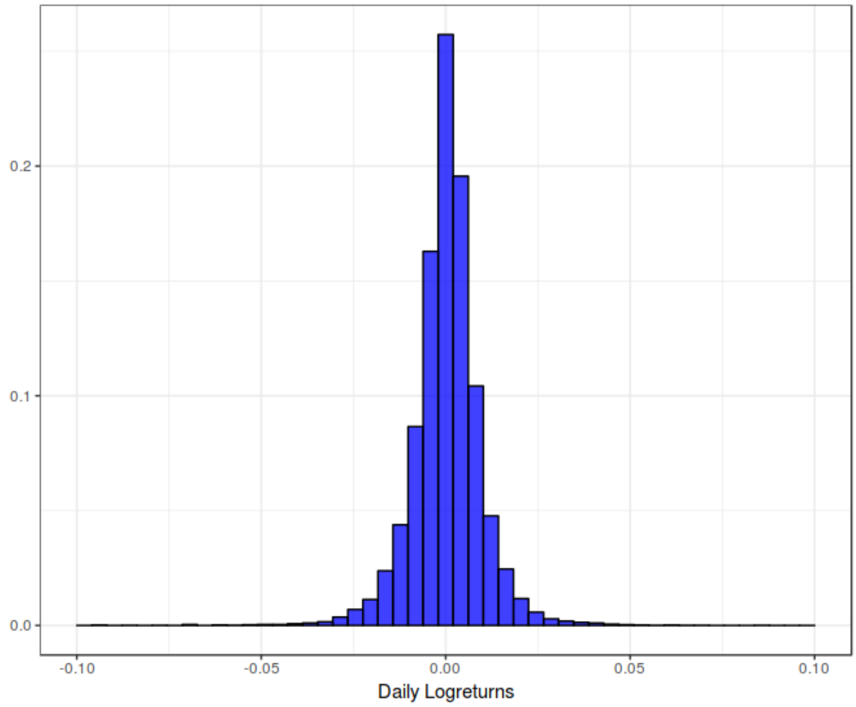


Figure 1: Daily logreturns of the Standard & Poors index [1950 - 2017]. Data from Yahoo Finance.

This bell-shaped distribution is important because many financial models assume normality; see Modern Portfolio Theory [7], efficient markets and the Black-Scholes option pricing model. And it is this very normality one of the main assumptions of the geometric brownian motion hypothesis of prices.

But given the irrational and unpredictable behaviour of the markets, we can see some flaws to this normality. Like the fat tails on the extremes of that histogram, fatter than they should, given normality. This little deviation implies that improbable events happen *a lot* more than expected, and this rises a philosophical doubt that undermines our understanding of risk. A normal distribution assumes that, given enough observations, all values in the sample will be distributed equally above and below the mean. Hence the convenience of using standard deviation as a measure of volatility, since it gives us some sense of how far away we can be from the mean. However, given the size of those extreme values, we can not entirely rely our assessment of risk upon the standard deviation; and thus we ought to study and gauge those fat tails further.

Expected Shortfall

In order to measure the importance and the impact of the tails of return distributions, it is common to compute what is called the *Expected Shortfall*. The Expected Shortfall

(ES_α) at an α quantile of a given distribution X is defined [8] as:

$$ES_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_\gamma(X)d\gamma, \tag{3}$$

Where $\alpha \in (0,1)$, $VaR_\gamma(X)$ is the $1-\gamma$ quantile of X and $\gamma \in (0,1)$. This means that the ES_α gives us the expected value of the returns distribution in the worst α% cases. Which, in case of a continuus X, can be expressed as

$$ES_\alpha = \mathbb{E}(X \mid X < VaR(X)). \tag{4}$$

Thus, the Expected Shortfall gives us a much more intuitive and reliable sense of the *risk* of any investment; in addition to its useful mathematical properties [9].

3. Optimisation of Investment Strategies

Most usually, classical savings theory states that the optimal approach to ensure the risk level of any given investment is to set a constant proportion of the wealth allocated in risky assets.

In this section we will explain how we simulate the result of such strategies and compare them to the methodology of the work developed in [2], in which they explain the idea of a *variable* proportion allocated in risky assets, instead of constant. We will replicate the results of that work and see how this change in the strategy can affect the performance of the investment plan.

3.1. CPPI Scheme

Firstly, we will start defining the *benchmark* model. The constant portfolio strategy follows the logic derived from constant relative risk exposure.

This methodology consists in investing a constant proportion π of the savings in risky assets (subject to volatility), whilst investing the rest $1 - \pi$ in risk-free assets. Thus, the time evolution of the wealth X of an investor would behave as following:

$$dX = \alpha\pi X(t)dt + \sigma\pi X(t)dW(t) + dC(t). \tag{5}$$

Where α is the expected return of the risky market, σ is the expected volatility of the risky market, $W(t)$ is a Wiener process and $C(t)$ is the allocated or consumed capital by the investor at every time step.

The point of this strategy is to present an intuitive straightforward way to control the risk exposure in savings strategies. The simplicity of this approach let us tweak π in order to make the investment best suited for the risk aversion profile of each investor individually.

Simulation

In order to simulate the performance of this kind of strategy, we start assuming that the risky assets follow a simplified geometric brownian motion, with *trend* α and *volatility* σ. Thus, if the saver invests x in this asset at day t, the wealth of the saver at the next day would be $x_{t+1} = (1 + N(\alpha, \sigma))x_t$.

This way, we construct the scenario of an investor, saving a fixed amount of money - which is denoted as a - throughout $T/2$ years. That money being allocated $(1 - \pi)a$ in the risk-free asset, which we will set with return 0; and πa allocated in the risky asset, whose expected return is α and volatility is σ. Thus, if we set x_t as the wealth at any given time t, we can see that

$$x_{t+1} = (1 + N(\alpha, \sigma))x_t\pi + (1 - \pi)x_t + a. \tag{6}$$

At some point in time, our investor will stop saving money and will start consuming it (as in most pension plans), so we just convert that fixed amount of money a to *consumed* money instead of saved. Thus, the evolution of wealth turns out to be:

$$x_{t+1} = x_t(1 + N(\alpha, \sigma))\pi + (1 - \pi)x_t - a. \tag{7}$$

At the end of all T years, the final wealth X_T remaining to the saver is stored. By reproducing this same scheme, we manage to compute tens of thousands of different performances and make some statistics out of them.

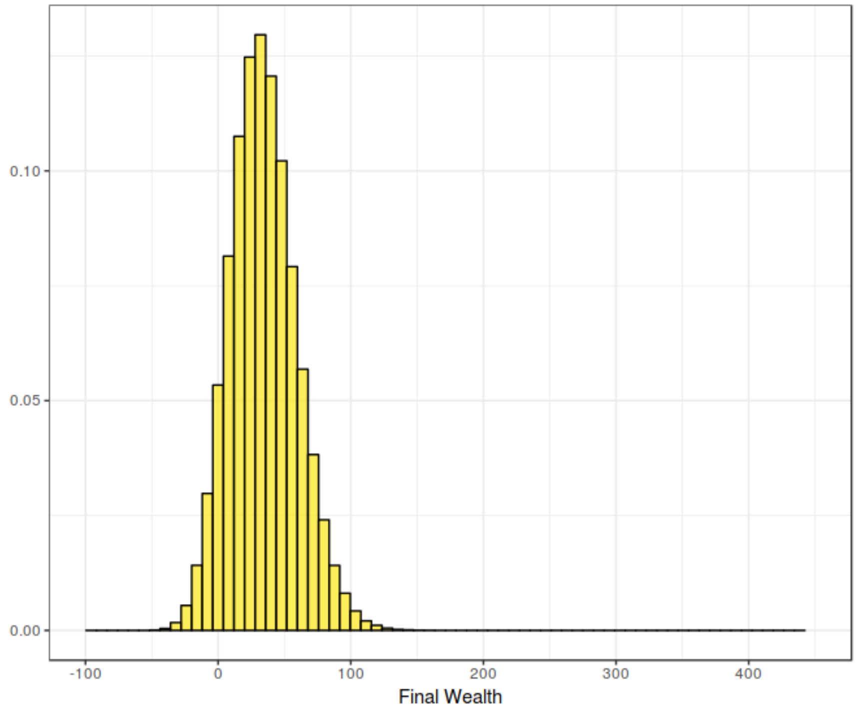


Figure 2: Results of the simulation for the CPPI model. Final wealth obtained for every simulation, over $10,000$ simulations, where $T = 60$, $\mu = 0.0343$, $\sigma = 0.1544$, $a = 10$, $\pi = 0.1$

The first impression we get when looking at the histogram from Figure2, is that it has the shape of a *Bell Curve*, thus following a *Normal Distribution* It seems quite symmetric and is slightly skewed towards positive results, what would imply modest positive results at the end of the plan.

The code necessary to replicate these results in R can be found in Appendix A

3.2. Alternative Scheme

Now that the CPPI model is presented and its logic understood, we can move upon to alternatives. One of the main characteristics of the CPPI model is that it is defined thanks to a constant, invariant π that settles the risk exposure of the investor. An interesting approach would not just change this parameter, but make it *variable*.

Normally, the simplest approach to ensure that an investment has a defined and controlled risk profile is to set a π constant proportion of the investment to be allocated in risky assets, as done in the CPPI. That proportion is defined by the risk aversion profile of the investor, but is invariable throughout the investment.

However, the work of [2] showed that, given some investment plan ending at time T with $X(T)$ final wealth, risk aversion profile defined by γ and a maximum possible allowed loss K, we can set the utility function

$$u_\gamma = \frac{1}{\gamma}(X(T) + K)^\gamma. \tag{8}$$

Whose expected value for any given present wealth $x = X(t)$ is defined as

$$\max_\pi \mathbb{E}(u_{X(T)} \mid x), \tag{9}$$

which can be maximized by a strategy that invests a relative amount in risky assets variable at any time $t \in [0, T)$, whose solution is:

$$\pi(t)X(t) = A\big(K + X(t) + g(t)\big). \tag{10}$$

Where $X(t)$ is the wealth at time t, A is a parameter that defines the risk aversion profile of the investor, K is the maximum loss the investor is capable to handle and $g(t)$ is the sum of all remaining inputs or outputs of money: $g(t) = \sum_{i=t}^{T} a_i$. [2]. Thus, the time evolution of the wealth X of an investor would behave as following:

$$dX = \alpha\pi(t)X(t)dt + \sigma\pi(t)X(t)dW(t) + dC(t). \tag{11}$$

Where α is the expected return of the risky market, σ is the expected volatility of the risky market, $W(t)$ is a Wiener process and $C(t)$ is the allocated or consumed capital by the investor at every time step.

[2] In [10]'s work, the authors defined optimal savings strategies and analysed different updating periods for π.

Simulation

In order to analyse the alternative scheme, the process will be quite similar to the previous one. We set the normal behaviour of the price evolution of the risky asset, and fix all parameters. Thus, the wealth of the investor behaves as follows:

$$X_{t+1} = (1 + N(\alpha, \sigma))X_t\pi_t + (1 - \pi_t)X_t + C(t), \tag{12}$$

where

$$\pi_t = \frac{A\Big(K + X_T + \sum_t^T C(t)\Big)}{X_T} \; and \tag{13}$$

$$C(t) = \begin{cases} a & \text{if } t <= T/2 \\ -a & \text{if } t > T/2\,. \end{cases}$$

Again, at the end of all T years, the final wealth X_T remaining to the saver it is stored, and then all the process is repeated. This way we manage to compute tens of thousands of different performances and make some statistics out of them.

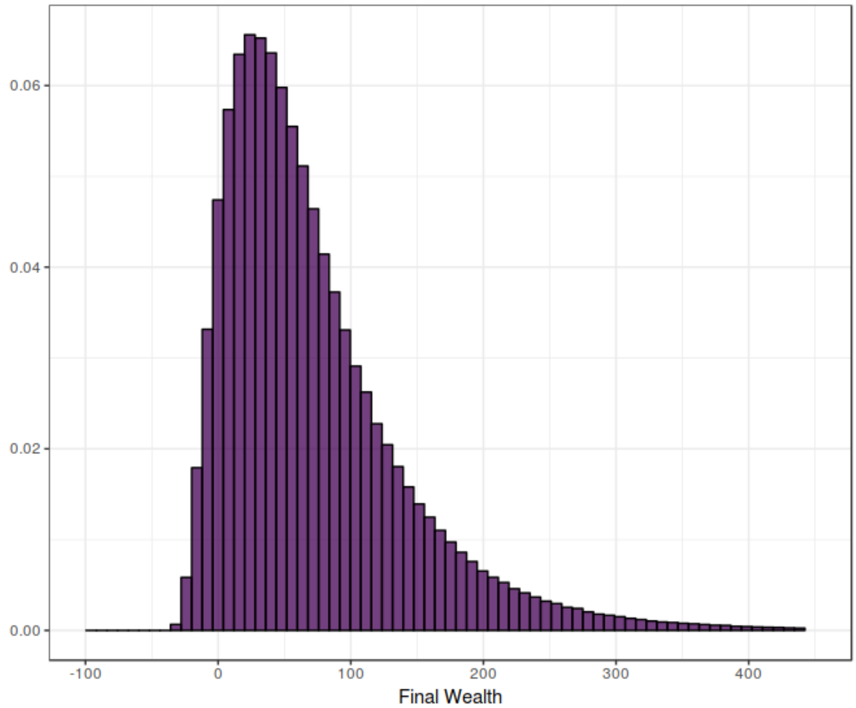


Figure 3: Results of the simulation for the *Alternative* model. Final wealth obtained for every simulation, where $T = 30$, $\mu = 0.0343$, $\sigma = 0.1544$, $a = 10$, $\pi = 0.1$.

In the case of figure 3, where the right tail corresponds to large values of wealth, whereas the left part deals with losses; we can see how outrageously obvious is that it does *not* follow a Normal Distribution.

The code necessary to replicate these results in R can be found in Appendix B.

3.3. Comparison

At this moment, we have understood and tested both strategies, and it is moment to contrast each other and highlight their differences.

First of all we plot together the frequency histograms for the final wealth distribution for both models.

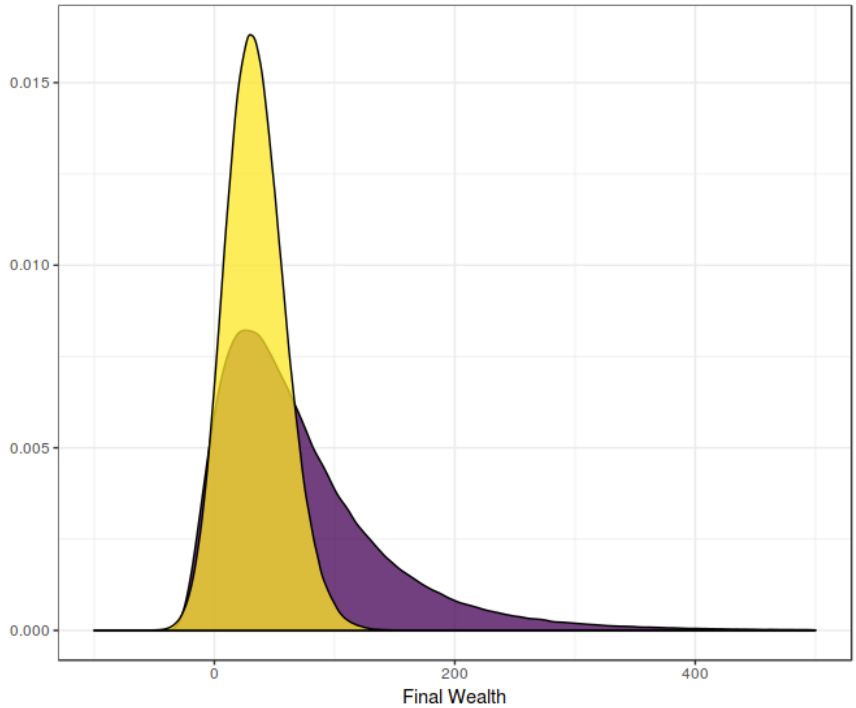


Figure 4: Results of the simulation for the both models. Final wealth obtained for every simulation. The yellow one is the CPPI, and the purple one is the Alternative.

Looking at Figure 4 we may notice that they follow a considerably different distribution. We could say that the alternative strategy presents more dispersion, even though it is always a *positive* deviation from the mean.

In section 2 we have discussed a little bit some implications of the definition of risk. Affirm that the alternative strategy presents more risk, just because its result is more disperse, may seem a little simplistic.

On the other hand, we can take a look at those rare cases when the final wealth happens to be negative. These are the cases worth exploring, for it is the scenario every investor is afraid of: losing money. Zooming in into the negative zone, as shown in figure 5, we can focus in the difference between the two models. Even though we can see some spurious differences in some places, the most honest answer is that it is not clear whose result is less risky.

In order to compare the degree of risk taken by every model, we make use of the *Expected Shortfall*, as explained in section 2. If we set this parameter as the level of risk

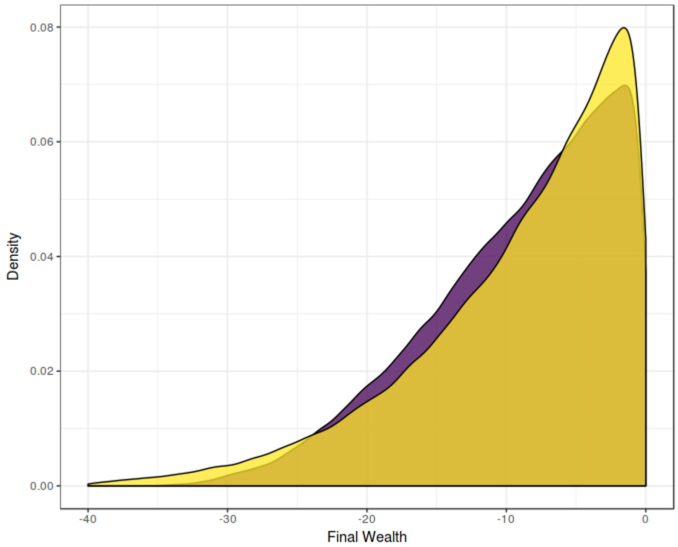


Figure 5: Results of the simulation for the both models. Final wealth, filtered by negative values obtained for every simulation. The yellow one is the CPPI, and the purple one is the Alternative.

and we set it constant in both models, we can compare their returns. In [2] it is shown that the Alternative model is able to set its Expected Shortfall (ES) by its K, using this formula:

$$\frac{K}{ES} = \frac{1}{-1 + (1-\theta)^{-1} e^{\alpha A T} \Phi\left(\Phi^{-1}(1-\theta) - A\sigma\sqrt{T}\right)}. \tag{14}$$

Where α is the expected return of the risky asset, σ is the standard deviation of the risky asset, θ is the quantile on which the Expected Shortfall is measured, A is the parameter that settles the risk aversion profile and Φ represents the standard Normal distribution function.

Therefore, the approach is the following: We set a constant proportion π for the CPPI model, we simulate it many times and measure the ES for the results. Then we find the K in order to set the same ES on the Alternative model. This way we can simulate both models making sure they will assume the same risk, and thus we can freely compare their returns.

Setting $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$, $A = 0.5$ and number of simulations $N = 100,000$ we find the results shown in Table 1, in which we can see the outperformance of the alternative strategy, for many different levels of risk.

The interesting conclusion that can be extracted from Table 1 is the value of the last column. It states the necessary value of π that the CPPI should get in order to equal the return of the Alternative strategy, what we call the equivalent π. Thus, if the Expected

Table 1: Results of 100,000 simulations using different risk levels, with $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$, $A = 0.5$ and therefore $ES/K = -3.3$. These results match approximately those presented in [2], on Table 1 at page 8. Any differences are attributable to the intrinsic randomness of simulations.

π	**ES**	K	**CPPI ret**	**Alt ret**	**diff**	**equiv** π
0.1	-12.47	40.59	0.33	0.51	0.18	0.17
0.2	-27.03	87.98	0.65	0.98	0.38	0.30
0.3	-42.99	139.95	0.95	1.42	0.46	0.45
0.4	-60.13	195.71	1.23	1.83	0.60	0.61
0.5	-78.25	255.02	1.5	2.22	0.71	0.69
0.6	-99.96	325.36	1.76	2.63	0.87	0.92
0.7	-120.61	392.56	2.00	3.00	1.00	*
0.8	-144.71	471.02	2.23	3.40	1.17	*
0.9	-172.94	562.92	2.39	3.81	1.42	*
1	-205.09	667.57	2.58	4.27	1.70	*

Shortfall was settled by some π and the required equivalent π is greater than that, we are saying that the CPPI should undertake in more risk in order to equal the return of the Alternative Scheme.

This equivalent π thus works as a succint metric to compare the performance of this two strategies, taking into account both return and risk. Hence, as long as the equivalent π is greater than the initial π, we can say that the Alternative Strategy is outperforming the CPPI. Which is clear for many values of initial π and a fixed value for A.

However, in Figure 6 we can show the iterated simulation of Table 1, for many different values of π and many values of A. It is quite interesting to notice how this *outperformance* occurs whilst the continuous line is above the dashed line. And their downhill tendency suggests that the Alternative Strategy is more worth ir for smaller values of A.

Since the value of A is interpreted as the inverse of the risk aversion [2] of the investor, we could argue that the riskier the inversor the less worth it the Alternative Scheme becomes, until a point (usually around $A = 1.5$) where the CPPI is outperforming the Alternative Strategy. Interestingly enough, this point is the same for all values of initial π. Indicating that, regardless of the chosen value for π the decision upon the election of one of the strategies relies on the risk aversion profile of each investor.

It is important to highlight that all the deductions derived from the study of these results is a consequence of the assumption that the Expected Shortfall of both schemes is equalise for each value of π. As concluded by Equation 14. Just to double-check, we have measured that this holds true for all simulations partaken in the results shown in Figure 8b. We might notice a slight deviation from that when the Expected Shortfall approaches 0, but nothing that could undermine the conclusions deduced from these results.

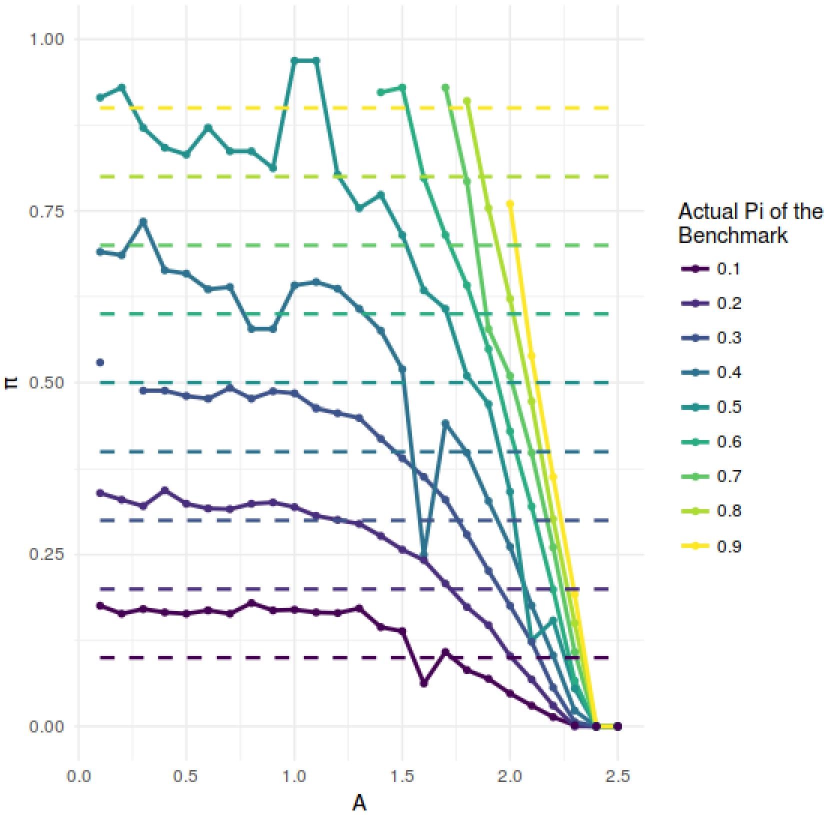


Figure 6: Plot that shows the relation between equivalent π and A, for many different initial π. From the results of 100,000 simulations for each curve, with $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$.

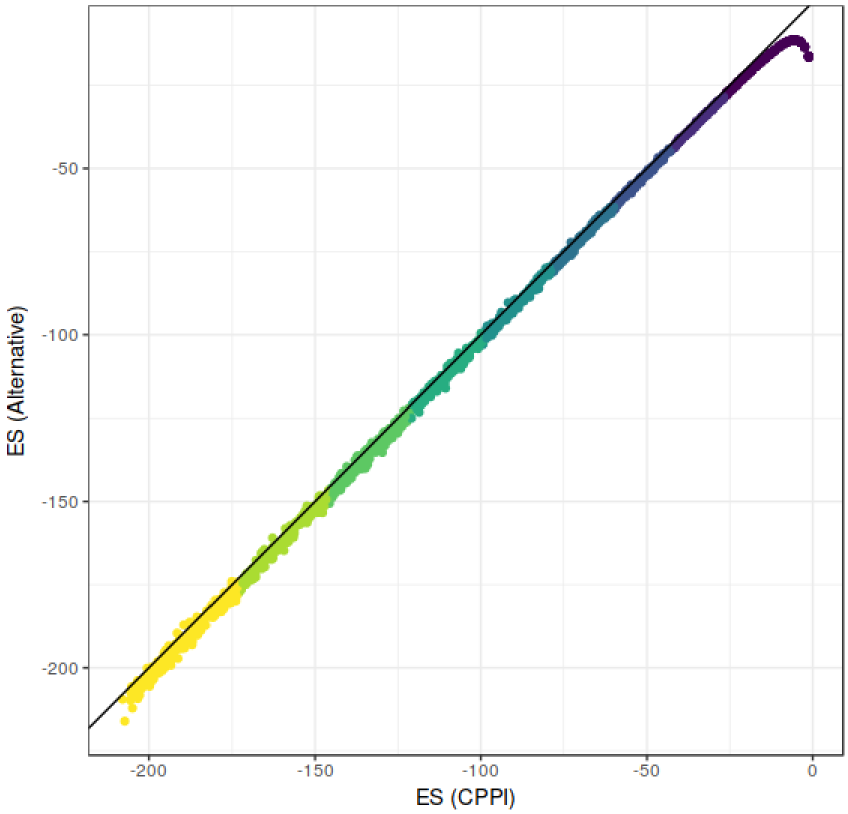
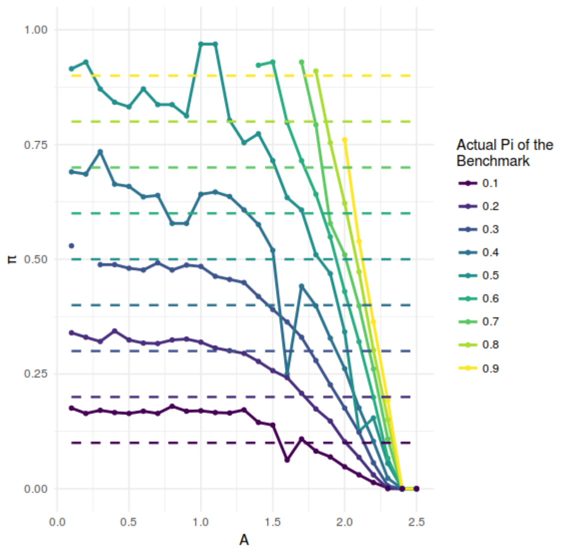
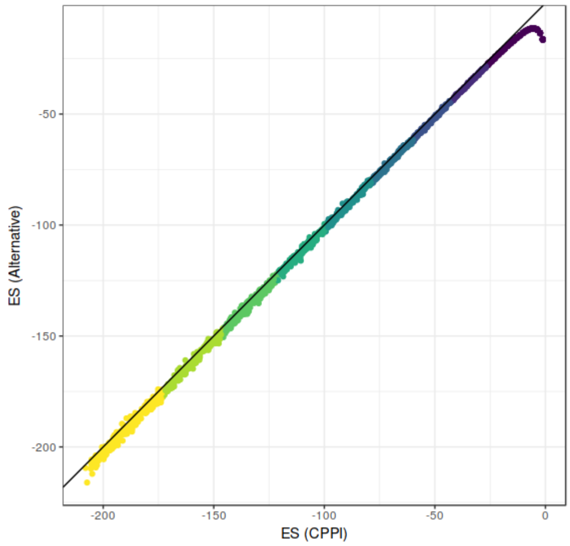


Figure 7: Scatterplot that shows the equivalence between the Expected Shortfall derived from the simulations of Figure 6. With $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$.



(a) Plot that shows the relation between equivalent π and A.



(b) Scatterplot that shows the equivalence between the Expected Shortfall derived from the simulations of Figure 8a.

Figure 8: Results from 20, 000, 000 simulations performed for many different values of π and A, with $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$.

4. Optimisation of Pooled Funds Investment Strategies

So far, we have managed to test two different strategies. We have analysed their return and risk in a simulated scenario, within the context of a pension plan. In order to assess the risk of the strategies, so far we have been measuring the market risk of the investment, by computing the *Expected Shortfall*. However, in real-scenario pension plans, there are plenty of other risks that should be taken into consideration. One important risk, not rarely underestimated, is the so-called **Longevity Risk**. Longevity risk is referred to the probability of retirees that will live longer than expected and will thus exhaust all their savings, and all the costs incurred. This risk might doom some individuals to utter poverty or to burden relatives.

Recently, two worldwide phenomena ought to be highlighted. The collapse in low-risk assets returns as government bonds or blue chip stocks. And the observed demographic transition Caldwell [11] and Bongaarts [12], in which both birth rates and death rates are plumbing down; increasing the life expectancy of elder individuals. The combination of these two factors is leading to an increase in longevity risk that pension plans providers are facing, rising pension premiums and stagnating disposable incomes by savers and pushing them to work longer years before retirement.

As a response to this challenge, the work of [13] and [14] suggested a different approach to face longevity risk, the concept of **Pooled Funds**.

Pooled Funds are funds formed by many different individual savers that aggregate their savings together. The main characteristic of Pooled Funds is the fact that it takes into account the survival rate of the savers. Sadly, not all investors live long enough to cash back the profits of their investments. Pooled Funds make a strength out of this and proportionally redistributes the invested money of the deceased among the rest of the investors. Alongside other advantages, pooled funds benefit from economies of scale, cheaper diversification and a more efficient management of longevity risk.

In this section we will study the application of both CPPI and Alternative schemes that we have developed in previous sections under the framework of a simplified pooled fund.

4.1. Simulating the Pooled Fund

In order to simulate the pooled fund, we will construct a simple scenario where many investors of the same age start investing at the same time. We will take real death probabilities at each age, and we will simulate the death of some of the savers.

When savers die, some proportion w of their saved money stays in the pool, benefiting the survivors. The rest is extracted from the pool, to their family or inheritors. In order to simulate the probability of death for each individual, we took the empirical measures of death probability for every age from the Spanish Government [15]. This way, we can assume that from a starting number of persons n of the same age, and thus with the same death probability p, the number of persons that would die X before the next year should follow a *Binomial Distribution*. Thus the probability of k deaths is:

$$Pr(X = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}. \tag{15}$$

Now we can now make use of the algorithm described in [16] to generate random numbers k that follow a Binomial Distribution, and assume those k to be the simulated deaths for the following year. Once we have k, we can compute how much proportional wealth M has been given to every investor of the common pool by:

$$M = \frac{k}{n} w. \tag{16}$$

The point of constructing the Pooled Fund in such a way, is that we are not altering the core principles of the assumptions taken building the Alternative Model. The relative proportion π of wealth to be allocated in risky assets remains untouched, we are just altering the $x = X(t)$ present in Equation 9. Resulting in

$$\max_{\pi} \mathbb{E}(u_{X(T)} \mid x(1 + M)). \tag{17}$$

And thus the wealth of the investor will behave as

$$dX = \alpha\pi(t)X(t)dt + \sigma\pi(t)X(t)dW(t) + dC(t) + dM(t)X(t)w. \tag{18}$$

In the snippet in Appendix C we show the necessary R code to perform this simulation for both CPPI and Alternative schemes.

4.2. Results

Now we are going to replicate the measures taken in the previous section but with the added condition of mortality and see whether or not the differences spotted between the CPPI and the Alternative strategies remain similar.

Firstly, we plot together the frequency histograms for the final wealth distribution for both models.

Looking at Figure 9 we might note that the results are quite similar. Nevertheless, we can spot some differences in the shape of the non-central zones of the figure, where the Alternative Strategy seems to show less dispersion.

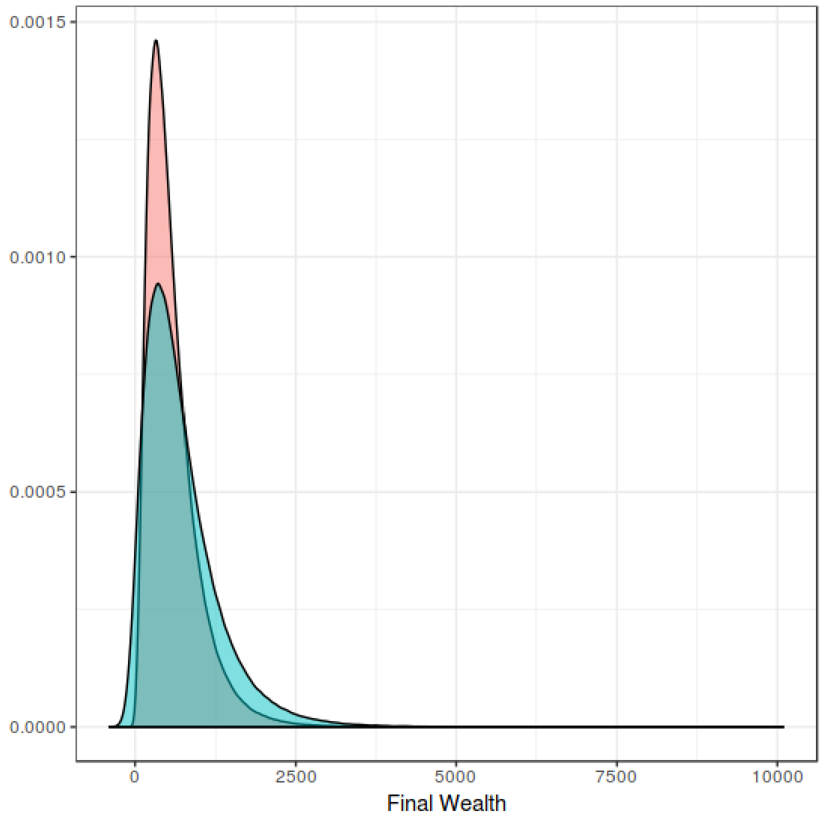


Figure 9: Results of $1,000,000$ simulations for the both models with mortality. Final wealth obtained for every simulation. The blue one is the CPPI, and the red one is the Alternative. With $\pi = 0.5$, $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$, $A = 0.5$.

With a quick glimpse at Figure 10 we can already notice the great difference in the loss distribution of both schemes. It is clear that the CPPI is far more risky than the Alternative.

Since the conclusions extracted from just one value of π may be misleading, let us iterate these simulations for many values of π and K. In order to equalise the level of risk of the result of both strategies, we ought to make use of Formula 14, as we have done before. The problem is that we can not be sure that the assumptions and deduction from [2] will be standing for this scenario that we are constructing. In fact, looking at the results of Figure 12b we can be sure that the measured Expected Shortfall are not the same for both strategies when in a Pooled Fund.

In order to pursue further results, the necessary value of K in order to ensure equal Expected Shortfall for both strategies has been found numerically, whose results are shown in Figure 12a, Figure 12b and Table 2.

As in the previous section, equivalent values of π have been working as a succint metric to compare the performance of this two strategies, taking into account both return and

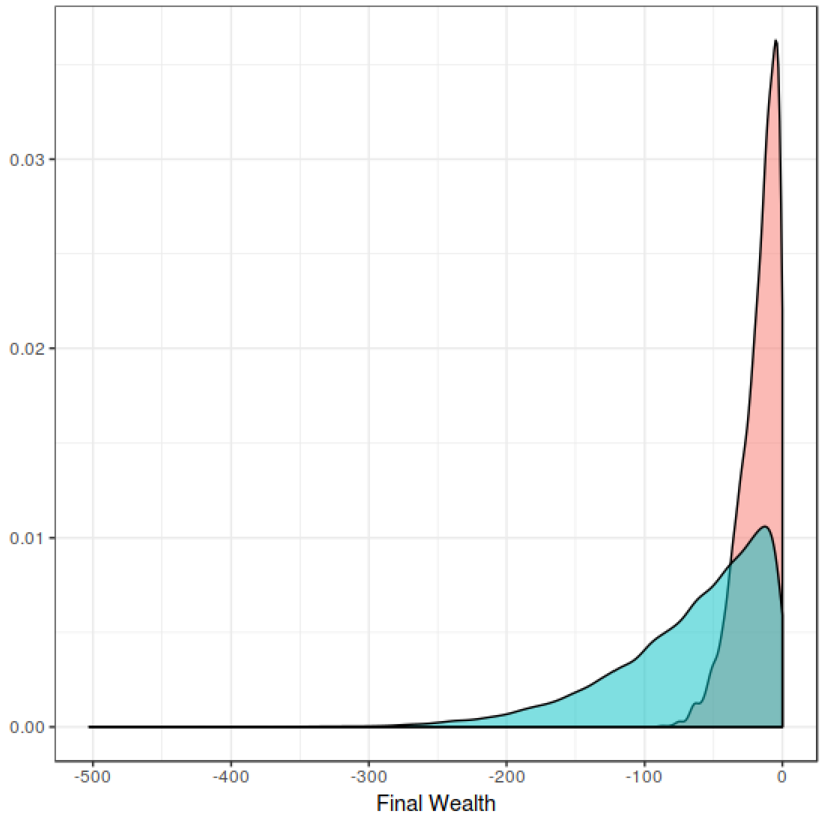


Figure 10: Results of $1,000,000$ simulations for the both models with mortality. Losses obtained for every simulation. The blue one is the CPPI, and the red one is the Alternative. With $\pi = 0.5$, $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$, $A = 0.5$.

risk. Hence, as long as the equivalent π remains above than the initial π, we can say that the Alternative Strategy is outperforming the CPPI. Which is clear for most of the values of initial π and a fixed value for A. Additionally, we can expand this results and find the iterated simulation of Table 2 for many values of A, as shown in Figure 12a.

The main difference we can note between Figure 7 and Figure **??** is that the equivalence between Expected Shortfall tends to be true, but is far more volatile in the case of a Poolued Fund. This leads to far more disperse result in Figure 12a than in Figure 8a. Despite that, we can glimpse some patterns within the results that let us observe some distinctions between the results without and with mortality. Note that the values that are not present in the plot of Figure 12a is because there were not value of equivalent π for the CPPI capable of equaling the results of the Alternative. Note also that this happens more often for higher values of π.

Attending at the obtained results, we can see that outside of a Pooled Fund, the decision of whether the CPPI strategy or the Alternative are more worth it, has to do with the value of A alone, not with π. On the contrary, within the context of a Pooled Fund, we can see that the Alternative model shines over the CPPI for higher values of π. In fact, for higher values of π, the Alternative strategy outperforms the CPPI so much

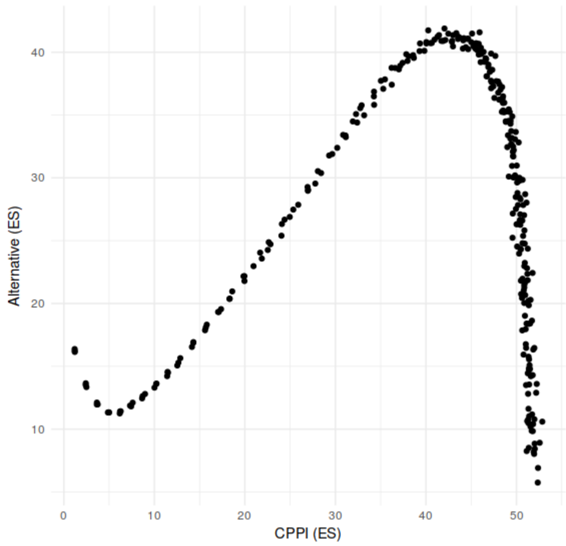
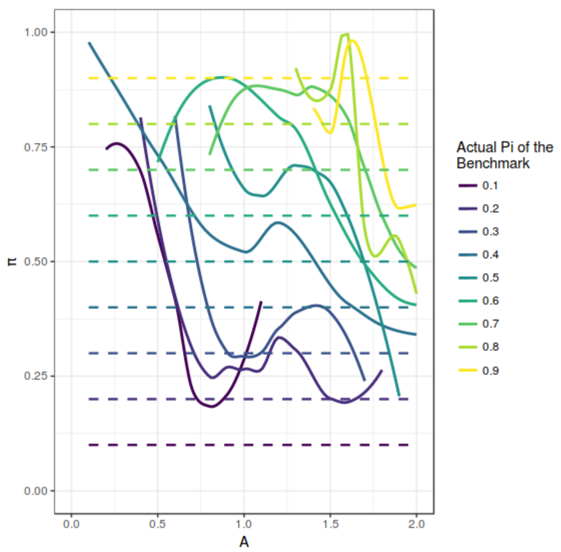
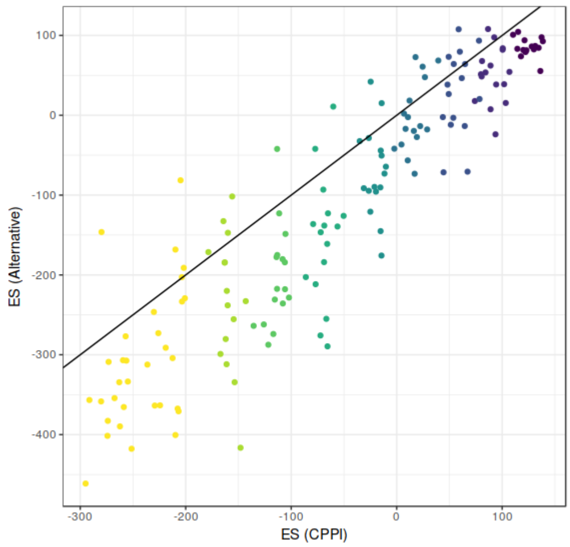


Figure 11: Expected Shortfall measured from the results of 100, 000, 000 simulations for the both models with mortality. With many values for π, A and K; $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$ and $A = 0.5$.



(a) Plot that shows the relation between equivalent π and A.



(b) Scatterplot that shows the equivalence between the Expected Shortfall derived from the simulations of Figure 8a.

Figure 12: Results from 20, 000, 000 simulations performed for many different values of π and A, with $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$.

Table 2: Results of 100, 000 simulations of both CPPI and Alternative models in a Pooled Fund, using different risk levels, with $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$, $A = 0.5$.

π	**ES**	K	**CPPI ret**	**Alt ret**	**diff**	**equiv** π
10	140.32	18.11	1.84	1.97	0.13	0.15
20	88.75	88.41	2.13	2.79	0.65	0.34
30	58.50	96.45	2.66	2.80	0.13	0.32
40	20.85	165.75	3.13	3.59	0.46	0.49
50	-28.11	232.00	3.46	4.17	0.70	0.64
60	-73.18	294.04	3.83	4.67	0.83	1.06
70	-118.58	381.23	4.24	5.40	1.15	1.63
80	-153.69	424.03	4.77	5.85	1.07	*
90	-223.58	524.09	4.88	6.77	1.88	*
100	-261.12	618.38	5.16	6.79	1.62	*

that often an equivalent value π_b for the CPPI to equal the return of the Alternative simply does not exist. This result is of great importance, specially considiering that both strategies incurr in the same risk level, as it is shown in Figure 12b.

As before, we can roughly observe a downhill behaviour of the plot. This trend indicates that, for higher levels of A, where A is interpretd as the inverse of the risk aversion, the Alternative strategy tends to be less relatively worth it, compared to the CPPI. Nonetheless, we should highlight a sweetspot around $A = 1.5$ where the plot gets to relative maximum for many levels of π.

5. Tails Analysis

At this point we should have notice that the real danger of investments does not resides solely within the standard concept of dispersion (standard deviation), nor within the ratio between return and dispersion. The true risk dwells in the worst case scenarios of the distributions, the tails. Where accumulated or great losses may strike down the whole investment plan. Hence, in order to address the risk analysis of any savings strategy, we need to take a closer look at the tails of their distributions.

Taking a close look at Figure 4, we are not able to clearly state which one is *less risky*. So far, we have been comparing the values of Expected Shortfall in order to get a sense of the risk. But, since the ES is actually just the average of the tail of the distribution, we will now discuss how we can pursue a more in-depth analysis for the distribution of those tails.

In this section, we will go one step further and try to fit some specific distribution to those tails. This way we will be able to see in more detail the behaviour of the losses of our strategies, and set more accurate risk measures.

5.1. Definition of *Tail*

First things first. What is *the* Tail of a distribution? The tail of a distribution is not a precisely defined term; it can have many different forms and definitions. Generally, the *tail* is considered a broad term to name the extreme parts of a distribution. In other words, there is not some specific place where you stop being in the middle of the distribution and start being in the tail, and where to put that line is up to every case.

Let's say that we plot a distribution Y, the loss of a savings plan (note that negative losses imply profits). We decide to put the distinction between middle and tail at some point named u, so the tail would be everything that lands within $Y > u$.

If we understand the tail of a distribution as a distribution itself, we can move that tail to the origin and normalise its area. Thus, being positively defined as

$$Z = (Y - u | Y > u) \tag{19}$$

.

Once we have this definition, we can easily define the probability density as

$$\begin{aligned}
F_Z(z) = P(Z < z) &= P(Y - u < z | Y > u) && (20)\\
&= P(Y < z + u | Y > u) && (21)\\
&= \frac{P(Y < z + u, Y > u)}{P(Y > u)} && (22)
\end{aligned}$$

And thus, deriving this expression we find that, for every $z > 0$,

$$f_z(z) = \frac{f_y(z+u)}{1 - F_y(u)}. \tag{23}$$

Interesting thing about this result, is that we can analytically relate the distribution of the tail $f_z(z)$ with the whole original distribution $f_y(z)$.

Therefore, we can say that, if the distribution of the losses obtained is Y, our tail could be defined as

$$F_u(y) = \big(Y - u \leq y | Y > u\big). \tag{24}$$

The Pickands-Balkema-de Haan theorem states that for a large class of distributions X, exists u such as that F_u is well approximated by the Generalized Pareto Distribution (GPD). Where the standard cumulative distribution function of the GPD can be defined as:

$$F_\xi(y) = \begin{cases} 1 - \left(1 + \xi \frac{y}{\psi}\right)^{-1/\xi} & : \xi \neq 0 \\ 1 - e^{-y/\psi} & : \xi = 0 \end{cases}.$$

Thus, we can seek that point u from which our tail could fit a GPD. That would imply that, from that point u forward, the distribution follows a GPD. The kind of GPD would be determined by the value of ξ. This number, usually called *Extreme Value Index*, that determines the shape of the distribution. The larger the value of ξ, the thicker the tail grows, and the slowly it converges to zero. In general:

- $\xi > 0$. The tail does not converge to zero, ever. Power-law
- $\xi = 0$. The tail converges to zero at infinity. Exponential
- $\xi < 0$. The tail converges to zero in a finite point.

In what risk analysis concerns, the value of ξ is crucial. Since we are studying the risk of a strategy looking at the tail of distributions where distribution are the frequency of losses, we could think that the wider or thicker the tail, the riskier the strategy.

5.2. Extreme Value Analysis: Fitting a General Pareto Distribution

Let us repeat the simulation for the CPPI, using $\pi = 0.1$, and plot the histogram of its losses as an example.

If we now plot that density as points in a logarithmic scale, we might see something like the following:

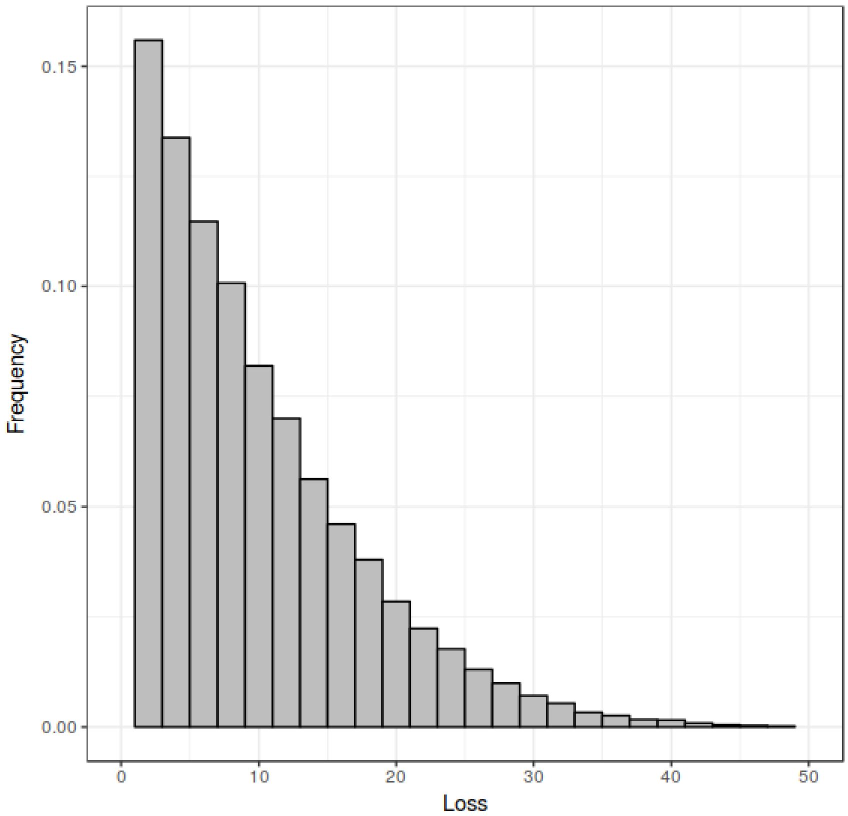


Figure 13: Frequency Histogram of the losses of the CPPI strategy. Using $\pi = 0.1$, $\alpha = 0.0343$, $\sigma = 0.1544$ and 1000000 simulations.

Now, we could try to add here the Complementary Cumulative Distribution Function of a Generalized Pareto Distribution, and see if it fits to our distribution as it is now.

Clearly, we can see how badly it fits for larger numbers. This indicates that our distribution is not following a GPD. Nevertheless, we can still find a proper u for which it would.

A usual method in order to seek that u threshold is to consecutively trying increasing numbers until one fits the distribution properly. We know that for that value on, the distribution will follow a GPD.

In this example, the first value that we find satisfies this is $u = 42.59$. And the result can be visualized in the following plot:

The Extreme Value Index (EVI) obtained in this example is $\xi = 0.14$. This value of ξ would suggest that the result of our model's distribution is of some kind with very fat tails, as a Power Law.

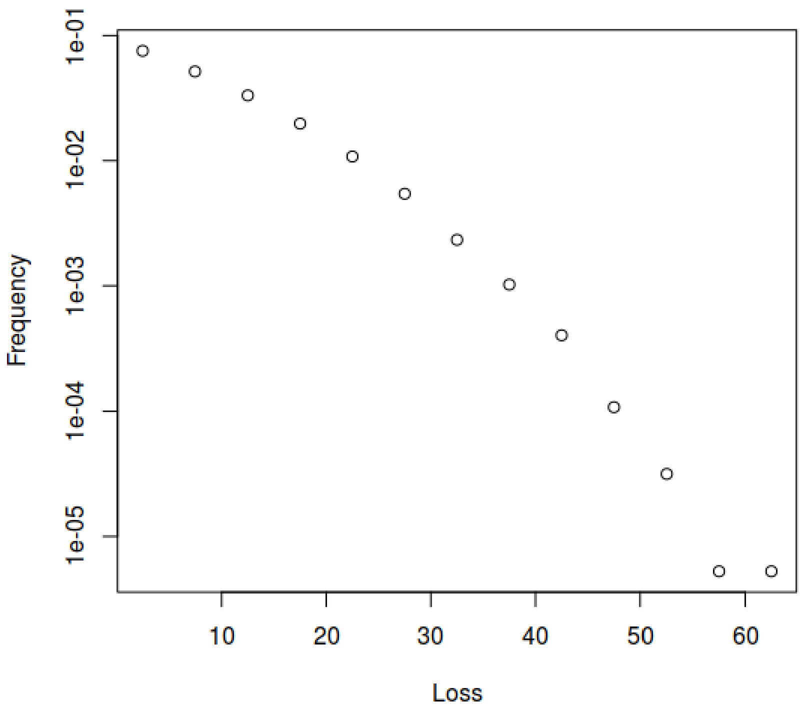


Figure 14: Density Distribution of the losses of the CPPI strategy in a logarithmic scale. Using $\pi = 0.1$, $\alpha = 0.0343$, $\sigma = 0.1544$ and 1000000 simulations.

5.3. Convergence of the Expected Shortfall

So far, we have addressed the necessity of studying the tail of loss danger using the Expected Shortfall. It is quite straightforward and it easily assesses the risk of the tail. But since in order to get the Expetced Shortfall, what we are computing is the average of the tail of the distribution (in this case, marking the origin of the Tail as its VaR). In our example, we have found that the tail might be following a Power Law. The theoretical mean of a power-law distributed quantity Y is, by defintion, given by

$$\overline{Y} = \int_{Y_{min}}^{\infty} Y p(Y)\,\mathrm{d}Y = \cdots = \frac{C}{2-\alpha} Y^{-\alpha+2}\Big|_{Y_{min}}^{\infty} \tag{25}$$

It needs to be noted that this expression diverges for $\alpha \leq 2$. That means that for that values a Power Law has no finite mean. But what does it mean for a distribution not to having a finite mean? We can surely get the actual data and calculate their average. And as we will always get a finite number. Only if we had an infinite number of simulations/experiments we would get a mean that actually diverges.

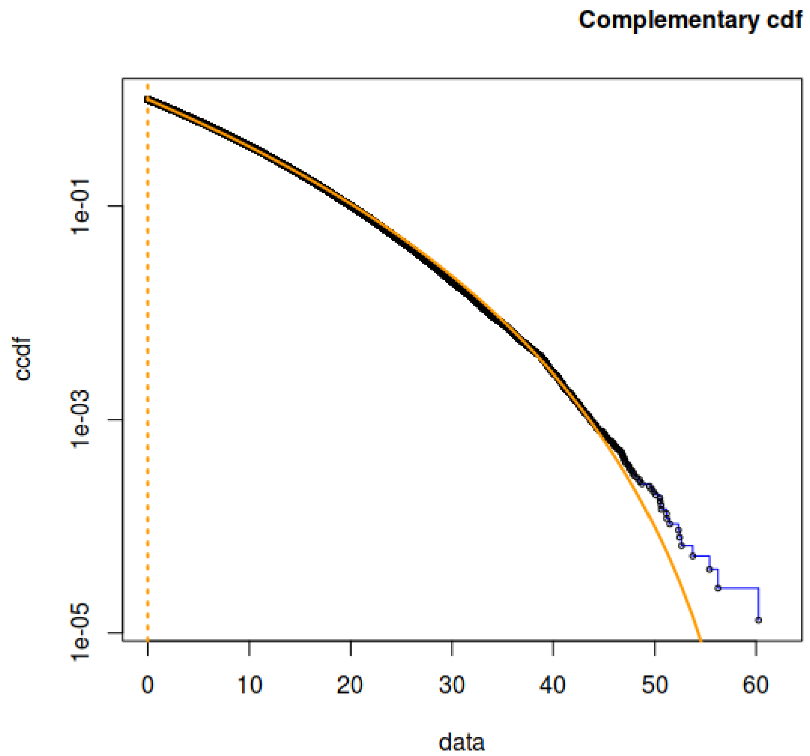


Figure 15: Density Distribution of the losses of the CPPI strategy in a logarithmic scale. Using $\pi = 0.1$, $\alpha = 0.0343$, $\sigma = 0.1544$ and 1000000 simulations.

Hence, if we were to repeat our finite experiment many times and calculate their mean for every repetition, the computed average of those many means should be - formally - divergent as well. Since we are actually capable of computing the mean, the consequence of the divergence found in Equation 25 is that these means may fluctuate a lot. We can say that the lack of convergence is indeed stating that the mean is not a well defined quantity, because it might vary enormously from one measurement to another. The formal divergence of $\overline{Y}$ is a red flag for us that should prevent us from computing empirical means, since they might not be reliable.

Since a Power Law can be expressed as a particular case of a Generalized Pareto Distribution, a general way of evaluating the convergence of the Expected Shortfall would be to fit a GPD to our distribution and compute their *Extreme Value Index* ξ. And since we know that $\alpha = \frac{1}{\xi}$, we can say that for $\xi \geq 1/2$, the Expected Shortfall does not converge.

In Figure 17 we can see a particular example in which most of the simulations did not spotted a situation where clearly $\xi \geq 1/2$. Despite that, whereas for negatives values of ξ we can fairly assume that the ES is going to converge, could not be the case for ξ close to 0 or even positive. In our results we have spotted most of them for smaller values of π. Therefore, we might not able to confirm that the ES might converge in those

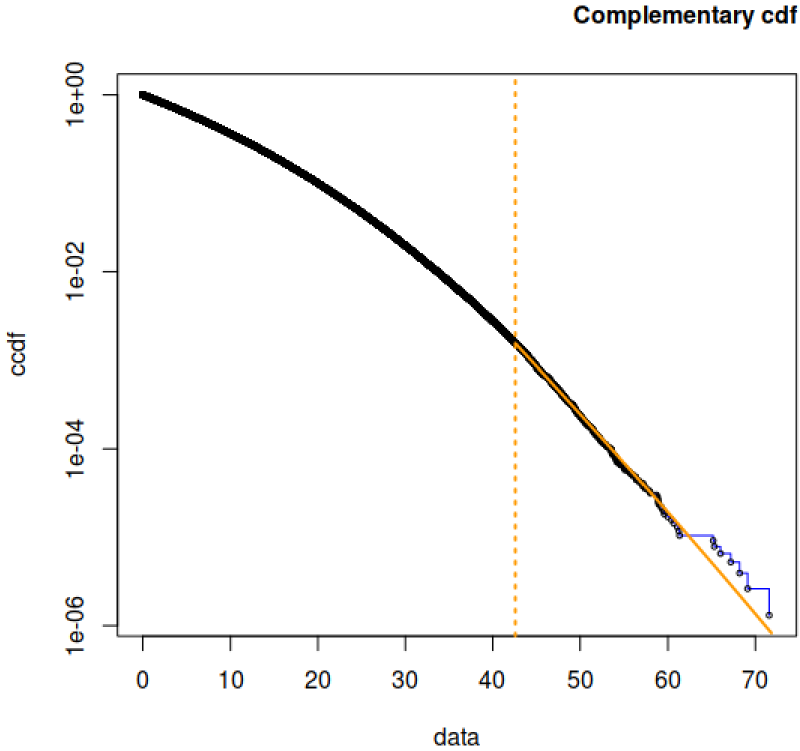


Figure 16: Exploratory empirical residual coefficient of variation of losses of the CPPI strategy in a logarithmic scale. Using $\pi = 0.1$, $\alpha = 0.0343$, $\sigma = 0.1544$ and 1000000 simulations.

situations. This kind of result arises doubts on the reliability of some of the previously computed values for the Expected Shortfall. We have indeed being using the Expected Shortfall as the standard reliable risk measure upon which we rely the whole analysis.

5.4. Extreme Value Index as Risk Measure

In financial investment strategies and, in savings strategies, one of the main concerns is the assessment of risk. In order to address that, we make use of what we call *risk measures*. We have discussed pros and cons of some of them and spotted situations where the standard ones might be misleading.

It is common practice to praise risk measures that falls within the category of *coherent risk measures*, and for good reasons, because gien a risk measure ρ, we define it as coherent if it satisfies the following properties:

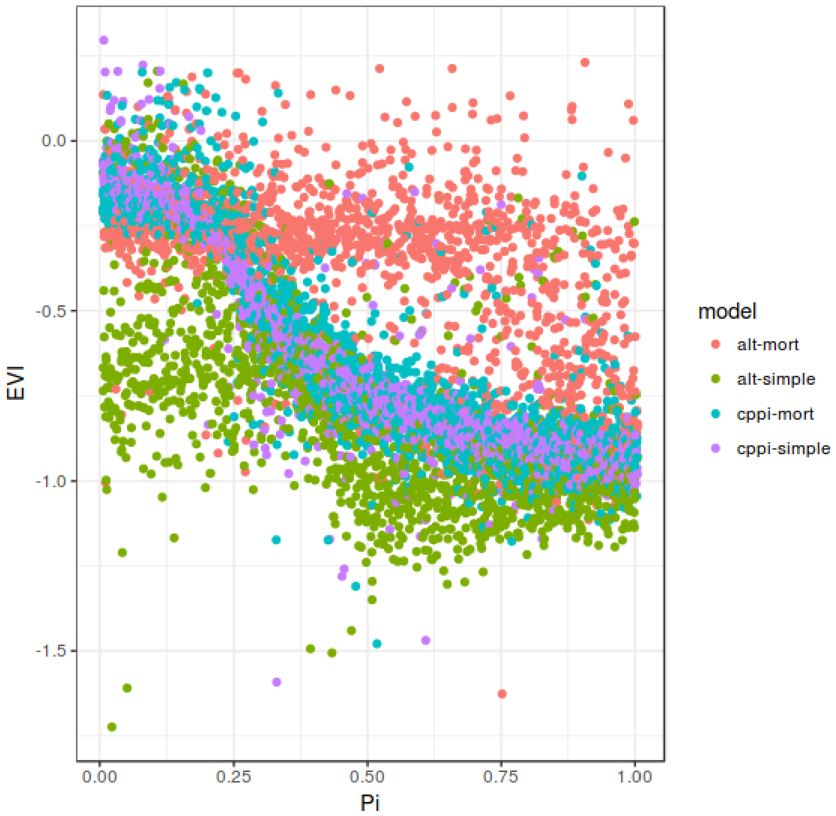


Figure 17: Extreme Value Index measured by varying values of π and A. Using $\alpha = 0.0343$, $\sigma = 0.1544$, and 1000000 simulations.

$$\begin{aligned} \rho(\lambda X) &= \lambda \rho(X) \\ \rho(\lambda + X) &= \lambda \rho(X) \\ P(X_1 < X_2) = 1 &\implies \rho(X_1) < \rho(X_2) \\ \rho(X_1 + X_2) &\leq \rho(X_1) + \rho(X_2). \end{aligned} \tag{26}$$

This properties ensure that the risk measure follows somehow a "reasonable" reasonable behaviour, that tends to coincide with common sense. The main problem with this kind of metrics is that they highly rely on Location and Scale. Thus, are unfitted to give any sense of the qualitatively sense or risk, apart from its magnitude. In order to address that, the work of **INSERT REF** proposed the definition of Scale-free and Location-Free risk measures, defined by the following properties instead, where X and ρ are limited to be positive:

$$
\begin{aligned}
\rho(\lambda X) &= \rho(X) \\
\rho(\lambda + X) &= \rho(X) \\
P(X_1 < X_2) = 1 &\implies \rho(X_1) < \rho(X_2) \\
\rho(X_1 + X_2) &\leq \rho(X_1) + \rho(X_2).
\end{aligned}
\tag{27}
$$

We can see than the 1st and 2nd properties are different. Moreover, both the distribution and the risk measure are restricted to be positive. Whilst the latter seems reasonable and seamlessly interpretable, for the simplicity that comes avoiding "negative risk" results; the former might seem a bit odd at first. However, as we have discussed, is somehow misleading to consider the whole distribution in order to assess risk, because we might end pondering the probability of unusually huge benefits as a "risk", whereas limiting the distribution to the losses might be more reasonable. Thus, a distribution of the losses would fit the restrictions for this new kind of risk measure.

Moreover, in **INSERT REF** the authors propose the *Extreme Value Index* as a Risk Measure. It surely satisfies the properties for a scale and location free risk measure, characterizes the worst case scenarios better than more standard measures as the *VaR* or the *Expected Shortfall*.

If the *VaR* and the *Expected Shortfall* gave us a sense of the location and scale of the worst scenario losses, the Extreme Value Index (EVI) can be interpreted as the thickness of the tail of the distribution. And this kind of assessment can be of great help, specially in those scenarios when the Expected Shortfall might not be reliable.

In Figure 18 it is shown that the relationship between the Expected Shortfall and the EVI is rather variable. Note that the Figure stands for positive and negative values of ES. This is kind of confusing, because the strategies in Pooled Funds showed such outstanding performance that their Expected Shortfall was of profits, instead of losses. Thus, we have settled negative Expected Shortfall as for losses and positive for profits. Qualitatively is interpreted as that the more negative the value of ES, the greater the risk.

It is interesting to contrast the clusters formed by different strategies in their relationship between EVI and ES. In particular, we can highlight the lower values of EVI with respect to ES for the Alternative strategy without mortality. This becomes less clear for the strategies in a Pooled Fund, even though most of the results where the Alternative strategy was applied in a Pooled Fund showed EVI dangerously close to 0.

In general, we can note a tendency for greater values of EVI for more positive values of ES. This is telling us that the less risk the ES is pointing, the less reliable this measure becomes, because it is getting closer to the non-convergence area $\xi >= 0.5$.

The result from the analysis of Location and Scale free risk measures indicates that extreme caution ought to be taken when making conclusions out of apparently less risky scenarios.

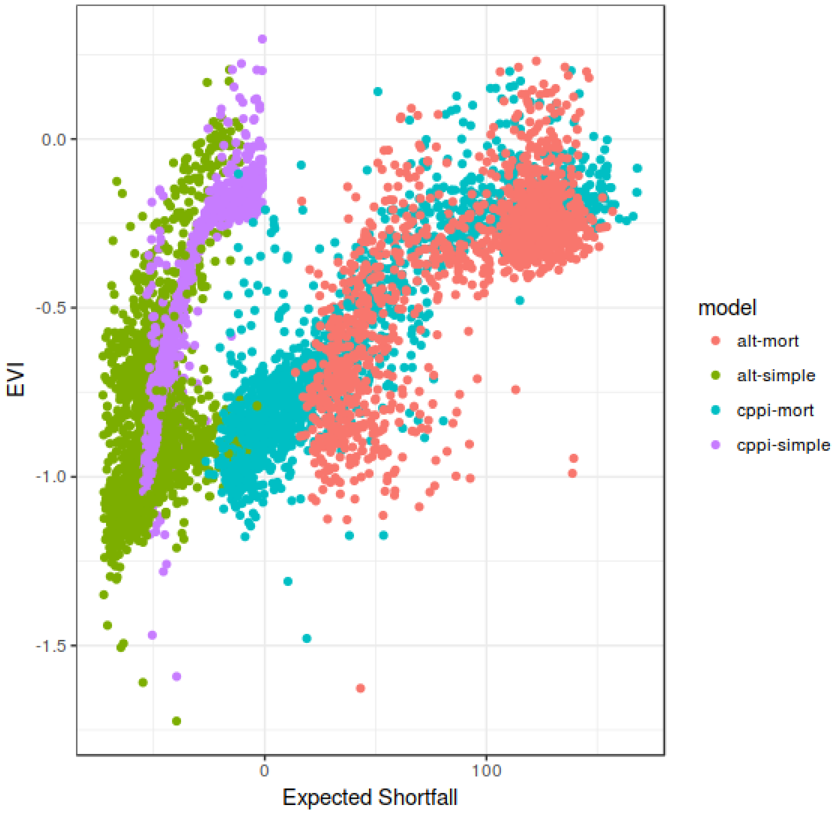


Figure 18: Extreme Value Index vs Expected Shortfall. Scatter plot of $10,000,000$ simulations of all four strategies, iterating over all values of $0 <= \pi <= 1$ and $0.1 <= A <= 2$.

6. Conclusions

As a result of this work we have given a bit of context and explanation to standard risk measures and utilised them to revise and expand the results obtained from [2]. It has been shown that for more values for π and A the same conclusions can be extracted: If a saver is capable of assuming up to K losses, then the optimal strategy is to invest A times the accumulated wealth in risky assets, where A is defined as the inverse of the risk aversion.

Additionally, following the logic provided by the comparison of the benchmark and the proposed strategy, the same analysis has been applied within the context of a Pooled Fund, that takes into account mortality. First, the purpose has been to present and simulate the results obtained of utilising such scheme, and we have seen that the computed returns presented in Table 2 corresponding to the Pooled Fund are consideraebly higher than the presented in Table 1 corresponding to the simpler approach. These results highlight the potential profits that the industry could gain from using such schemes in Pension Plans, and the necessary academic interest in studying them.

Then, we aimed to test whether or not the same conclusions and insights gotten from [2] can be extrapolated when adding Mortality and a Pooled Fund. The results presented in Figure 12a have shown us that there is a sweetspot on $1 \leq A \leq 1.5$ where the proposed alternative strategy outperforms the benchmark for $\pi = 0.6$, whereas for $\pi > 0.7$ the performance of the alternative strategy dwarves the returns obtained by the CPPI, as much as concluding that there is no achievable level of risk that lets the CPPI equal the higher results obtained by the alternative strategy.

Moreover, we have added to these previous analysis the usage of a new kind of risk measure based on the Extreme Value Analysis of the tails of the obtained distributions, providing a location and scale free risk measure, especially in situations where location-dependent measures are less reliable. The purpose has been to highlight the main differences and limitations of different risk measures and stress the necessity of adding location and scale free risk measures, as the Extreme Value Index, to the standard toolkit of risk analysis when optimising financial investments and savings strategies. When applied to the strategies of interest, we have been able to spot some interesting insights, as the decrease of the thickness of the tails obtained by the strategies where the Alternative Scheme is used. Indicating that even for the same values of Expected Shortfall, we can find that one strategy shows less risk than the other, helping the investor or institution contrast the risk level of different strategies.

An immediate consequence of adding mortality to the scheme of any strategy is that the investor the survives the long run manages to laregly increase their wealth. This shrinkes the risk of losses dramatically. This kind of risk has to be necessarily different in nature from the risk just derived from the dispersion of the expected return. From the results extracted from Figure 18 we might deduce that this condition is fairly better reflected by the Extreme Value Index rather than the Expected Shortfall, proving that the EVI can function as a location and scale free risk measure.

We conclude that pension plans providers could benefit greatly from using of both the Alternative Model and the context of Pooled Funds, separated or together, for most risk profiles.

Further research could be of extreme interest on the Pooled Fund scheme. The structure presented in this work is utterly simplistic and could be extended. Major improvements could reside in studying more levels for the percentage returned to the pool, when an investor dies (in this work we have always considered a 100%, even acknowledging that that could be unrealistic) or adding age variability to the simulated investors.

A. Simple CPPI Code

```
nsim <- 10000

pi <- 0.1
alpha <- 0.0343 # Expected return of the risky market
sigma <- 0.1544 # Expected volatility of the risky market
a <- 10 # Factor 'a'
years <- 60 # Total time

C <- append(rep(a, round(years/2)),rep(-a, round(years/2)))

X_T <- c()

for (j in 1:nsim){
  x <- c()
  x[1] <- a # Initial wealth

  for (i in 1:(years-1)){
    random <- rnorm(1, mean = alpha, sd = sigma)
    x[i+1] <- x[i]*(1+random)*pi + (1-pi)*x[i] + C[i+1]
  }
  X_T[j] <- x[years]

}
```

B. Simple Alternative Code

```
# Computation of 'pi' value function
fpi <- function(A, K, X, C, time){
  g <- sum(C[-c(1:time)])
  xpi <- A*(K + X + g)
  return(xpi)
}

nsim <- 10000

pi <- 0.1
alpha <- 0.0343 # Expected return of the risky market
sigma <- 0.1544 # Expected volatility of the risky market
a <- 10 # Factor 'a'
years <- 60 # Total time

C <- append(rep(a, round(years/2)),rep(-a, round(years/2)))

X_T <- c()

for (j in 1:nsim){
  x <- c()
  x[1] <- a # Initial wealth

  for (i in 1:(years-1)){
    random <- rnorm(1, mean = alpha, sd = sigma)
    pi <- fpi(A,K,X,C,i)
    x[i+1] <- x[i]*(1+random)*pi + (1-pi)*x[i] + C[i+1]
  }
  X_T[j] <- x[years]

}
```

C. Mortality Code

```

# Computation of 'pi' value function
fpi <- function(A, K, X, C, time){
  g <- sum(C[-c(1:time)])
  xpi <- A*(K + X + g)
  return(xpi)
}

  ### CPPI ###

  C <- append(rep(a, round(years/2)),rep(-a, round(years/2)))
  mort_table <- fread("mortality.csv")/1000
  X_T <- c()

  for (j in 1:nsim){
    x <- c()
    x[1] <- a # Initial wealth
    number_humans_alive <- starting_humans

    for (i in 1:(years-1)){
      prob_mort <- mort_table$total[i+starting_age-1]
      number_deads <- rbinom(1,number_humans_alive,prob_mort)
      number_humans_alive <- number_humans_alive - number_deads


      random <- rnorm(1, mean = alpha, sd = sigma)
      x[i+1] <- x[i]*(1+random)*pi + (1-pi)*x[i] + C[i+1] +
          (x[i]*number_deads/number_humans_alive)*w
    }
    X_T[j] <- x[years]
  }

  ### Alternative ###

  C <- append(rep(a, round(years/2)),rep(-a, round(years/2)))
  mort_table <- fread("mortality.csv")/1000
  X_T <- c()

  for (j in 1:nsim){
    x <- c()
    x[1] <- a # Initial wealth
    number_humans_alive <- starting_humans

    for (i in 1:(years-1)){
      prob_mort <- mort_table$total[i+starting_age-1]
      number_deads <- rbinom(1,number_humans_alive,prob_mort)
      number_humans_alive <- number_humans_alive - number_deads

      pi <- fpi(A,K,X,C,i)
      random <- rnorm(1, mean = alpha, sd = sigma)
```

```
      x[i+1] <- x[i]*(1+random)*pi + (1-pi)*x[i] + C[i+1] +
          (x[i]*number_deads/number_humans_alive)*w
    }
    X_T[j] <- x[years]
  }
```

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