Centre de Recerca Matemàtica

Savings Optimisation Strategies

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2018

A man got to have a code.

Omar Little

Abstract

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Contents

1 Introduction

The point of savings is to utilise the present wealth of a saver in order to build a retirement plan that secures a constant and safe stream of capital for when it may be needed. This future condition need can be though as deterministic, as we usually think in a typical pension plan, when there is a fixed time scheme when the saver starts collecting his money and when it ends; or it can be subject to some non-deterministic eventuallity, as in most insurance plans. This kind of investment is, thus, characterised by an initial period of time in which the investor is saving money followed by a period of consumption, once the investor satisfies that future condition.

Hence, every savings strategy should aim to maximise that final capital whilst *securing* it. In general, we could say that there is a trade-off between that maximisation of capital (return) and the degree of its security (risk). The balance between these two magnitudes is what savings strategies try to optimise.

It is obvious that every saver would like to maximise the return of his money. But if we accept that this can only be accomplished in expense of more risk, the investor has to decide which degree of risk is she able to tolerate, setting a risk limit. This decision upon the exposure to risk is what defines the *risk aversion* profile of every investor.

Most savings strategies are measured setting a fixed risk limit provided by a risk aversion profile, thus maximising the returns that can be extracted once that risk limit is provided. Thus the great interest canalized on testing and analysing the results of different savings strategies using the return and the risk as measures of performance.

Throughout this project we will compare the results obtained from two slightly different models, and we will compare their risk and return. The set up for both of them would be the same. A simulated investor will save up a yearly fixed amount of money from 30 years and, after that, she will consume the same amount for the next 30 years. Part of the saved money will be invested in risky assets whilst the rest will be invested in risk-free assets. The main difference between the models would be the proportion of the investment exposed to risky assets.

In the first model, called *Constant Proportion Portfolio Insurance* (CPPI), introduced by [1], a constant proportion of the invested money is allocated in risky assets, whereas the rest is allocated in non-risky assets. This means that the same proportion of the investment will be invested in the risky market, year over year.

The next model, developed in [2], consists in a different approach regarding that proportion of risky investment. Instead of a time-constant proportion, this model suggests a variable proportion that follows a formula that takes into account the present wealth of the investor. We will explain this formula in the following chapters.

2 On Risk measures

When simulating a savings scheme, the first assumption is regarding risk. It is usual make a stark distinction between risky and risk-free investments.

But what do we mean by *risk*? We define risk as the uncertainty of the outcome of a given investment. So if we save some money at a 0.1% bank deposit, we can farily assume that this investment is risk-free, because we know how much money we will recive at the end. Different scenario could be investing in some market-dependent asset with 1% expected return but no guarantee of that return.

But how do we measure that degree of uncertainty? In stock markets is usual to assume its behaviour as a brownian motion (see [3, 4]), and as such we can define its movement by *trend* and *dispersion* parameters. Since the return obtained by investing in stock assets comes from the relative difference in price between its purchase and its sell (see equation 1), it is easy to deduce that if we assume that price to follow a geometric brownian motion with trend μ and dispersion σ, the expected return of our investment would be μ with standard deviation σ. Thanks to this mathematical scheme, we can numerically define the expected return and some degree of uncertainty on it that we will henceforth use as our starting theoretical point.

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{1}$$

Even though many different models can be assumed to describe the behaviour of the financial markets, the point of this introduction is to understand and exemplify the set up necessary in order to decide how do we measure risk.

Another way to look at the risk definition is not just asking for uncertainty, but for negative uncertainty. Using the standard definition of risk, any unexpected uplift from the expected retur would be considered under the umbrella of 'risky outcomes'. Since it is kind of counterintuitive to assume the probability of a positive outcome as 'risk', it is fair to assume risk as just one half of that dispersion, the negative one.

If we take a look at the actual observed returns in the stock market - as shown in figure 1 we can see the frequency of different returns. With a little bit of imagination, we can notice a *Bell Curve* shape, and so we could think of a normal distribution of the logreturns. Where *logreturns* are defined as in equation 2.

$$r_t = \ln(1 + R_t) \tag{2}$$

This property is important because many financial models assume normality; see Modern Portfolio Theory [5], Efficient Markets and the Black-Scholes option pricing model. And it is this very normality one of the main assumptions of the geometric brownian motion of the price.

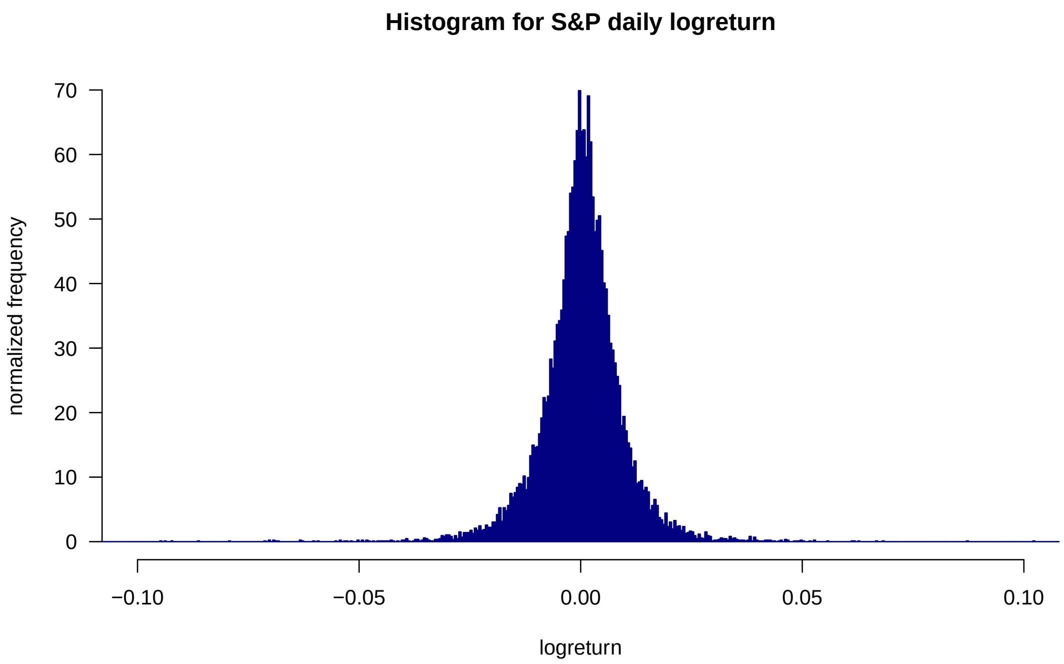

Figure 1: Daily logreturns of the Standard & Poors index. Data from Yahoo Finance

But given the irrational and unpredictable human behaviour, we can see some flaws to this normality. Like the fat tails on the extremes of that histogram, fatter than they should, given normality. This little flaw implies that improbable events happen *a lot* more than expected, and this arises a philosophical issue that undermines our understanding of risk. A normal distribution assumes that, given enough observations, all values in the sample will be distributed equally above and below the mean. Hence the convenience of using standard deviation as a measure of uncertainty, since it gives us some sense of how far away we can be from the mean. However, given the size of those extreme values, we can not rely all sense of uncertainty and dispersion upon the standard deviation, and thus we ought to study and gauge those fat tails in order to take good measure of the actual *risk* of the investment.

Expected Shortfall

In order to measure the importance and the impact of the tails of returns distributions, it is common to compute what is called the *Expected Shortfall*. The Expected Shortfall (ES_α) at an α quantile of a given distribution X is defined as:

$$ES_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_\gamma(X)d\gamma \tag{3}$$

Where $VaR_\gamma(X)$ is the $1-\gamma$ quantile of X. This means that the ES_α gives us the expected value of the returns distribution in the worst $\alpha\%$ cases.

Thus, the Expected Shortfall gives us a much more intuitive and reliable sense of the *risk* of any investment; in addition to its useful mathematical properties. [6]

3 Testing different Strategies

3.1 CPPI Model

Firstly, we will start explainning what will be the *benchmark* model. The constant portfolio strategy follows the logic derived from constant relative risk exposure.

This methodology consists in investing a constant proportion π of the savings in risky assets (subject to volatility), whilst invetsing the rest $1 - \pi$ in risk-free assets. The point of this strategy is to present an intuitive straightforward way to control the risk exposure in savings strategies. The simplicity of this approach let us tweak π in order to make the investment best suited for the risk aversion profile of each investor individually.

Simulation

In order to simulate the performance of this kind of strategy, we start assuming that the risky assets follow a simplified geometric brownian motion, with *trend* α and *volatility* σ. Thus, if the saver invests x in this asset at day t, the wealth of the saver at the next day would be $x_{t+1} = (1 + N(\alpha, \sigma))x_t$.

This way, we construct the scenario of an investor, saving a fixed amount of money a trhoughout $T/2$ years, and that money being allocated $(1 - \pi)a$ in the risk-free asset, which we will set with return 0; and πa allocated in the risky asset, wich expected return α and volatility σ. Thus, if we set x_t as the wealth at any given time t, we can see that

$$x_{t+1} = (1 + N(\alpha, \sigma))x_t\pi + (1 - \pi)x_t + a \tag{4}$$

At some point in time, our investor will saving money and will start consuming it (as in most pension plans), so we just convert that fixed amount of money a to *consumed* money instead of saved. Thus, the evolution of wealth turns to be

$$x_{t+1} = x_t(1 + N(\alpha, \sigma))\pi + (1 - \pi)x_t - a \tag{5}$$

At the end of all T years, the final wealth X_T reamining to the saver it is stored, and then all the process is repeated. This way we manage to compute tens of thousands of different performances and make some statistics out of them.

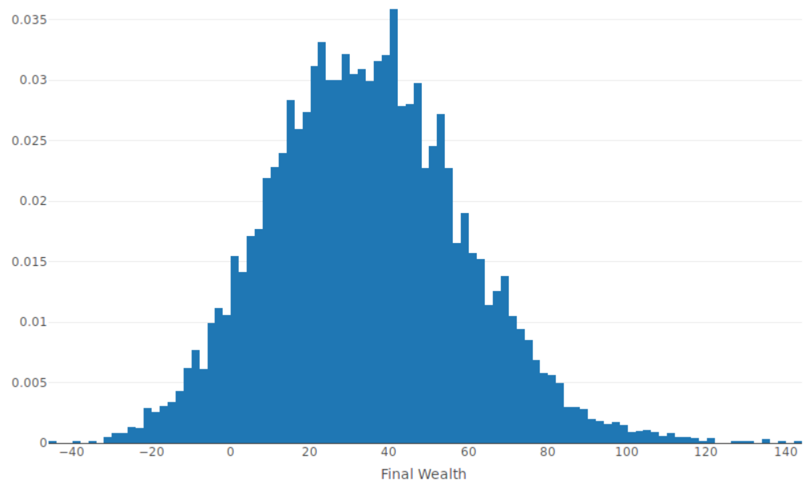

Figure 2: Results of the simulation for the CPPI model. Final wealth obtained for every simulation

If we iterate this process over 10000 simulations, setting $\pi = 0.1$, $T = 60$, $\alpha = 0.0343$, $\sigma = 0.1544$ and $a = 10$ we get results as shown in figure 2. The first impression we get when looking at the histogram, is that it has the shape of a *Bell Curve*, thus following a *Normal Distribution*, but a simple *Kolmogorov-Smirnov test* would tells us that this is far from being true.

In the following snippet is shown the code necessary to replicate these results in R:

```
nsim <- 10000

pi <- 0.1
alpha <- 0.0343 # Expected return of the risky market
sigma <- 0.1544 # Expected volatility of the risky market
a <- 10 # Factor 'a'
years <- 60 # Total time

C <- append(rep(a, round(years/2)),rep(-a, round(years/2)))

X_T <- c()

for (j in 1:nsim){
  x <- c()
  x[1] <- a # Initial wealth

  for (i in 1:(years-1)){
    random <- rnorm(1, mean = alpha, sd = sigma)
    x[i+1] <- x[i]*(1+random)*pi + (1-pi)*x[i] + C[i+1]
  }
  X_T[j] <- x[years]

}
```

3.2 Alternative Model

Now that the CPPI model is presented and its logic understood, we can move upon to alternatives. One of the main characteristics of the CPPI model is that is defined thanks to a constant, invariant π that settles the risk exposure of the investor. An interesting approach would not just to change this parameter, but to make it *variable*.

One interesting way to make the proportion of risky investments variable is to set a dependance on the accumulated wealth. In the work developed in [2] we can see the developement of a quite straightforward formula to decide the value of π. With this formula, the capital to be allocated in risky assets is defined as:

$$X(t)\pi = A(K + X(t) + g(t)) \tag{6}$$

Where $X(t)$ is the total wealth at time t, A is a parameter that defines the risk aversion profile of the investor, K is the maximum loss the investor is capable to handle and $g(t)$ is the sum of all remaining inputs or outputs of money: $g(t) = \sum_{i=t}^{T} a_i$

Simulation

In norder to perform the computation of this alternative model, the process will be quite similar to the previous one. We set the normal behaviour of the price evolution of the risky asset, and fix all parameters. Thus the wealth of the investor behaves as following:

$$X_{t+1} = (1 + N(\alpha, \sigma))X_t\pi_t + (1 - \pi_t)X_t + C(t) \tag{7}$$

Where

$$\pi_t = \frac{A\Big(K + X_T + \sum_t^T C(t)\Big)}{X_T} \tag{8}$$

$$C(t) = \begin{cases} a & \text{if } t <= T/2 \\ -a & \text{if } t > T/2 \end{cases}$$

Again, at the end of all T years, the final wealth X_T reamining to the saver it is stored, and then all the process is repeated. This way we manage to compute tens of thousands ofdifferent performances and make some statistics out of them.

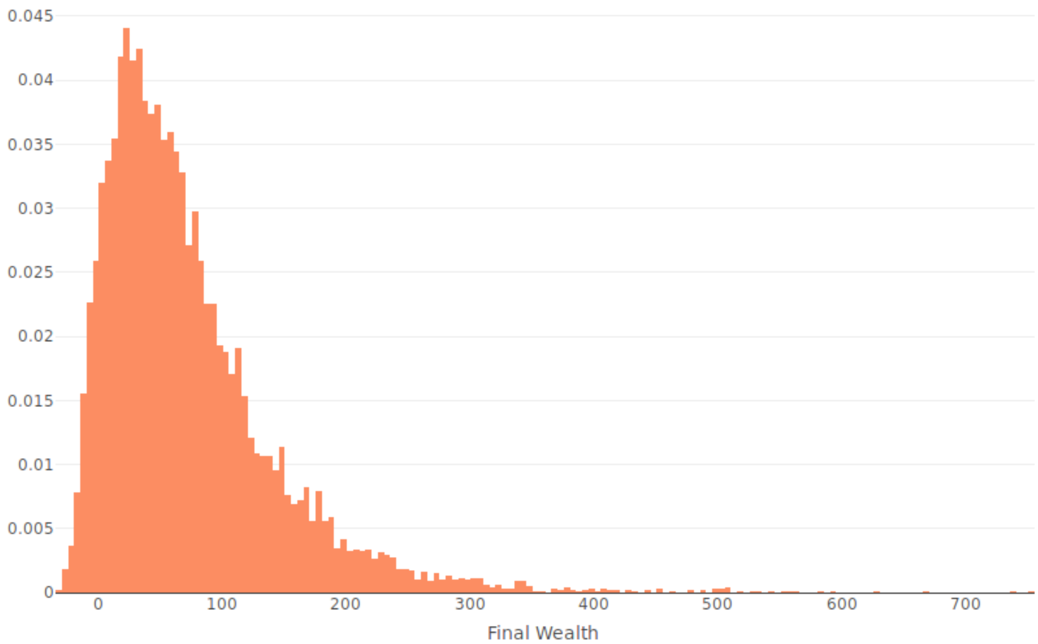

Figure 3: Results of the simulation for the *Alternative* model. Final wealth obtained for every simulation

In this case we can see how outrageously obvious is that this is *not* a Normal Distribution.

3.3 Comparison

At this moment, we have understood and tested both strategies, and it is moment to contrast each other and highlight their differences.

First of all we plot together the frequency histograms for the final wealth distribution for both models.

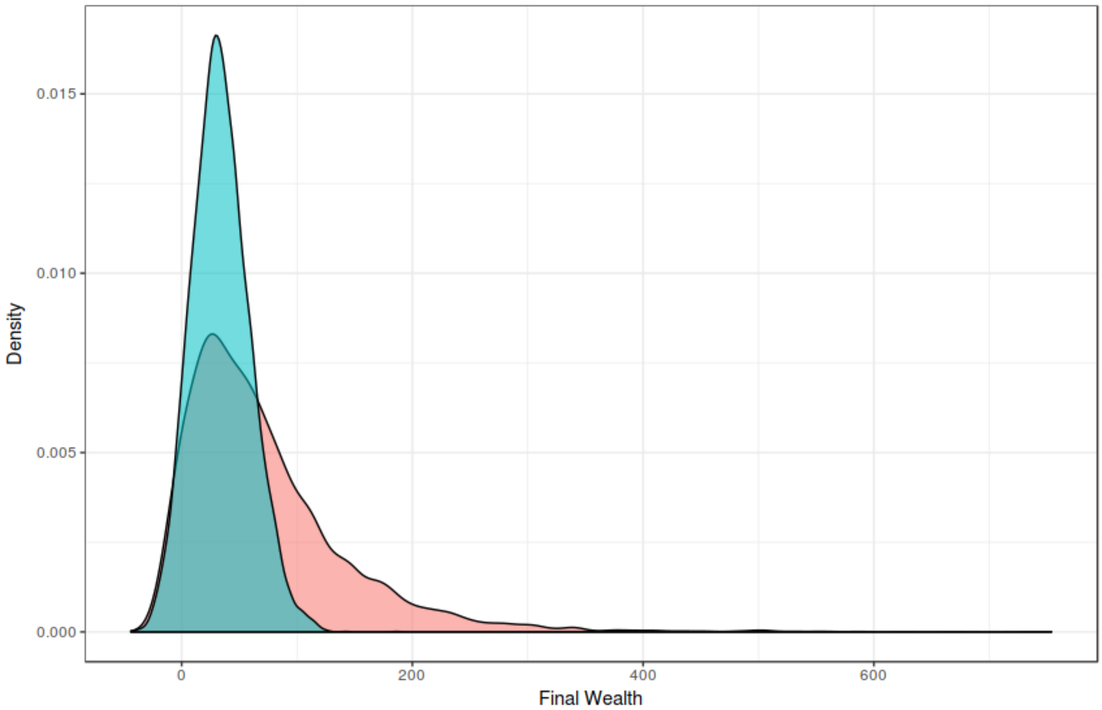

Figure 4: Results of the simulation for the both models. Final wealth obtained for every simulation. The blue one is the CPPI, and the red one is the Alternative.

Looking at Figure 4 we may notice that they follow a considerably different distribution. We could say that the alternative model presents more dispersion, even though it is always a *positive* deviation from the mean.

In chapter 2 we have discussed a little bit some implications of the definition of risk. Affirm that the alternative model presents more risk, just because its result is more disperse, would may seem a little simplistic.

On the other hand, we can take a look at those rare cases when the final wealth happens to be negative. These are the cases worth exploring, for it is the scenario every investor is afraid of: losing money. Zooming in into the negative zone, as shown in figure 5, we can focus in the difference between the two models. Even though we can see some spurious differences in some places, the most honest answer is that it is not clear whose result is less risky.

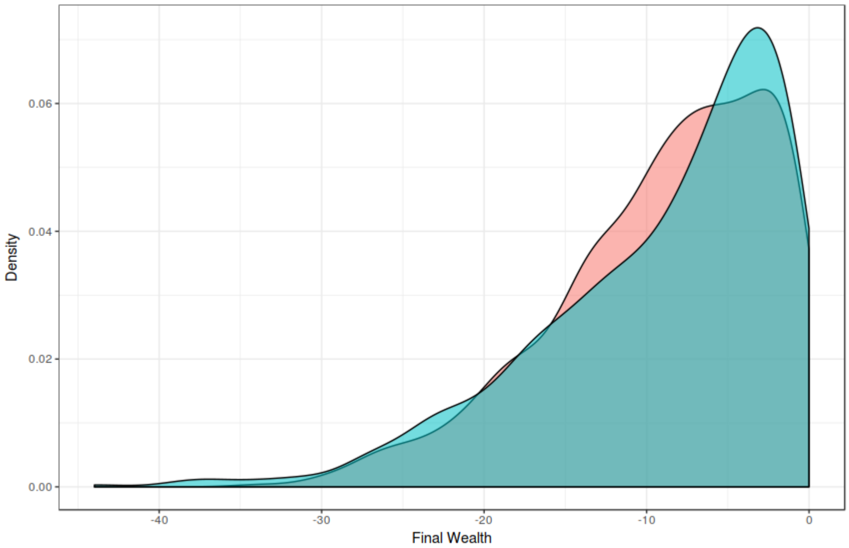

Figure 5: Results of the simulation for the both models. Final wealth, filtered by negative values obtained for every simulation. The blue one is the CPPI, and the red one is the Alternative.

In order to compare the degree of risk taken by every model, we make use of the *Expected Shortfall*, as explained in chapter 2. If we set this parameter as the level of risk and we set it constant in both models, we can compare their returns. In [2] it is shown that the Alternative model is able to set its Expected Shortfall (ES) by its K, using a closed formula.

Therefore, the approach is the following: We set a constant proportion π for the CPPI model, we simulate it many times and measure the ES for the results. Then we find the K in order to set the same ES on the Alternative model. This way we can simulate both models making sure they will assume the same risk, and thus we can freely compare their returns.

Setting $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$, $A = 0.5$ and number of simulations $N = 100000$ we find the results shown in table 1, in which we can see the outperformance of the alternative model, for many different levels of risk.

Table 1: Results of many simulation using different risk levels.

π	ES	K	CPPI ret	Alt ret	diff
0.1	-12.47	40.59	0.33	0.51	0.18
0.2	-27.03	87.98	0.65	0.98	0.38
0.3	-42.99	139.95	0.95	1.42	0.46
0.4	-60.13	195.71	1.23	1.83	0.60
0.5	-78.25	255.02	1.5	2.22	0.71
0.6	-99.96	325.36	1.76	2.63	0.87
0.7	-120.61	392.56	2.00	3.00	1.00
0.8	-144.71	471.02	2.23	3.40	1.17
0.9	-172.94	562.92	2.39	3.81	1.42
1	-205.09	667.57	2.58	4.27	1.70

4 Analysis of the Tails

At this point we should have notice that the real danger of investments does not resides solely within the standard concept of dispersion (standard deviation), nor within the ratio between return and dispersion. The true risk dwells in the worst case scenarios of the distributions, the tails. Where accumulated or great losses may strike down the whole investment plan. Hence, in order to address the risk analysis of any savings strategy, we need to take a closer look at the tails of their distributions.

Taking a close look at Figure 4, we are not able to clearly state which one is *less risky*. We can compare their Expected Shortfall, but we will now discuss why this approach could be improved. Acknowledging that those tails are, at the very least, comparable we can pursue a more in-depth analysis for the distribution of those tails.

4.1 Definition of *Tail*

The tail of a distribution is not a precisely defined term; it can have many different forms and definitions. Generally, the *tail* is considered a broad term to name the extreme parts of a distribution. In other words, there is not some specific place where you stop being in the middle of the distribution and start being in the tail, and where to put that line is up to every case.

Let's say that we plot a distribution Y, the loss of a savings plan (note that negative losses imply profits). We decide to put the distinction between middle and tail at some point named u, so the tail would be everything that lands within $Y > u$.

If we understand the tail of a distribution as a distribution itself, we can move that tail to the origin and normalise its area. Thus, being positively defined as

$$Z = (Y - u | Y > u) \tag{9}$$

.

Once we have this definition, we can easily define the probability density as

$$\begin{aligned} F_Z(z) &= P(Z < z) = P(Y - u < z | Y > u) && (10) \\ &= P(Y < z + u | Y > u) && (11) \\ &= \frac{P(Y < z + u, Y > u)}{P(Y > u)} && (12) \end{aligned}$$

And thus, deriving this expression we find that, for every $z > 0$,

$$f_z(z) = \frac{f_y(z+u)}{1 - F_y(u)}. \tag{13}$$

Interesting thing about this result, is that we can analytically relate the distribution of the tail $f_z(z)$ with the whole distribution $f_y(z)$

4.2 Convergence of the Expected Shortfall

So far, we have addressed the necessity of studying the tail of loss danger using the Expected Shortfall. It is quite straightforward and it easily assesses the risk of the tail. But since it is nothing more than a simple mean, it can arise some issues.

We talked about how the tail of a distribution can be considered a distribution by itself. Thus, the mean of the tail intends to be a summary statistic of this distribution. Despite that, we know that the arithmetic mean is not always a good summary statistic for every given distribution. There are some distributions from which the mean is not the best suited statistic to get a reliable feel of its *general tendency*, especially on very skewed distributions.

Moreover, without previous knowledge about the distribution we may encounter, it exists the possibility of facing a distribution whose arithmetic mean does not exist. This possibility arises a very disturbing problem: The computed value of the Expected Shortfall should not exist. Since we are working with simulated data, and thus finite numbers, we will never face an infinite arithmetic mean. This implies that in order to ensure the reliability of the Expected Shortfall, we first need to check whether its computation makes sense or not.

In Figure 5 we see the density curve of the loss tails of both models. If we compute the histogram and set the bars of the histogram as points, we can build a scatterplot. In Figure 6 we can see this scatter plot of their logarithmic density points. It is of great interest noticing that, whereas both tails follow a linear model quite accurately (after taking logarithms), the *Alternative* model has a much sharper descending trend. Which would mean that its *Expected Shortfall* is far less likely to be infinite in any analytical distribution. This suggests us that the computed value of the Expected Shortfall is much more reliable for the *Alternative* model than for the CPPI one.

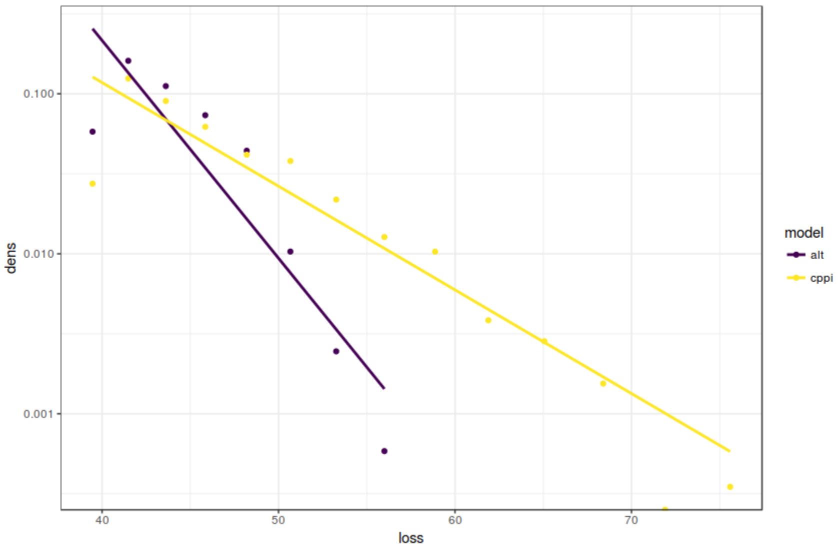

Figure 6: Scatter plot and linear regression of the logarithm of the density points of the *CPPI* and *Alternative* models.

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