Centre de Recerca Matemàtica

Savings Optimisation Strategies

Alejandro Jiménez Rico

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A man got to have a code.

Omar Little

Abstract

In classical savings theory it is stated that the optimal way to ensure a risk level is to set a constant proportion of risky assets at all time. Moreover, most savings strategies try to optimize their performance using simple and widely employed risk measures that are neither scale-free nor location-free.

Throughout this work we will replicate the results of an alternative optimal strategy that changes proportion invested in risky assets along time; we will incorporate the concept of Pooled Fund to those schemes and will study the risk profile of all those strategies using Tail Distribution Modelling and Extreme Value Analysis.

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1 Introduction

The reason for saving is to utilise the present wealth of a saver in order to build a retirement plan that secures a safe stream of capital for when it may be needed. This future condition need can be thought as deterministic, as we usually think in a typical pension plan, when there is a fixed time scheme when the saver starts collecting his money and when it ends; or it can be subject to some non-deterministic eventuality, as in most insurance plans. This kind of investment is, thus, characterised by an initial period of time in which the investor is saving money followed by a period of consumption, once the investor satisfies that future condition.

Hence, every savings strategy should aim to maximise that final capital whilst *securing* it. In general, we could say that there is a trade-off between that maximisation of capital (return) and the degree of its security (risk). The balance between these two magnitudes is what savings strategies try to optimise.

It is obvious that every saver would like to maximise the return of his money. But if we accept that this can only be accomplished in expense of more risk, the investor has to decide which degree of risk is she able to tolerate, setting a risk limit. This decision upon the exposure to risk is what defines the *risk aversion* profile of every investor.

Most savings strategies are measured setting a fixed risk limit provided by a risk aversion profile, thus maximising the returns that can be extracted once that risk limit is provided. Thus the great interest canalized on testing and analysing the results of different savings strategies using the return and the risk as measures of performance.

Throughout this project we will compare the results obtained from two different approaches to saving strategies, and we will compare their risk and return. The set up for both of them would be the same. A simulated investor will save up a yearly fixed amount of money from the age of 30 years and, after that, she will consume the same amount for the next 30 years. Part of the saved money will be invested in risky assets whilst the rest will be invested in risk-free assets. The question to be addressed is the optimal proportion of the investment exposed to risky assets.

In the case that we use as a benchmark, called *Constant Proportion Portfolio Insurance* (CPPI), introduced by [1], a constant proportion of the wealth to be invested is allocated in risky assets, whereas the rest is allocated in non-risky assets. This means that the same proportion of the investment will be invested in the risky market, year over year.

The other case, developed in [2], consists in a different approach regarding that proportion of risky investment. Instead of a time-constant proportion, this model suggests a variable proportion that follows a formula that takes into account the present wealth of the investor. We will explain this formula in the following chapters.

2 Background On Risk Measures

When simulating a savings scheme, the first assumption is regarding risk. It is usual to make a stark distinction between risky and risk-free investments.

But what do we mean by *risk*? We define risk as the uncertainty of the outcome of a given investment. So if we save some money at a 0.1% bank deposit, we can fairly assume that this investment is risk-free, because we know how much money we will receive at the end. A different scenario could be to invest this money in some market-dependent asset with 1% expected return but no guarantee of that return.

But how do we reckon that degree of uncertainty? In stock markets is usual to assume that the price of stocks behaviour is a brownian motion (see [3, 4]), and as such we can define their movement by *trend* and *dispersion* parameters. Since the return obtained by investing in stock assets comes from the relative difference in price between purchase and selling, it is easy to deduce that if we assume that the price follows a geometric brownian motion with trend μ and dispersion σ, the expected return of our investment would be μ with standard deviation σ. Thanks to this mathematical scheme, we can numerically define the expected return and some degree of uncertainty on it that we will henceforth use as our starting theoretical point. So, the returns at time t are defined as:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}. \tag{1}$$

Where P_t is the price of a stock at time t and P_{t-1} is the price of the stock at time $t-1$.

Even though many different models can be assumed to describe the behaviour of the financial markets, the point of this section is to understand and exemplify the set up necessary in order to decide how do we measure risk.

Another way to look at the risk definition is not just looking at dispersion. Using the standard defilogreturnsnition of risk, any unexpected uplift from the expected return would be considered under the umbrella of 'risky outcomes'. Since it is kind of counter-intuitive to assume the probability of a positive outcome as 'risk', it is fair to assume risk as just one half of that dispersion, the negative part of the random variable[1].

If we take a look at the actual observed returns in the stock market - as shown in figure 1 we can see the frequency of different returns. With a little bit of imagination, we can notice a *Bell Curve* shape, and so we could think of a normal distribution of the logreturns. Where *logreturns* are defined as:

$$r_t = \ln(1 + R_t). \tag{2}$$

This bell-shaped distribution is important because many financial models assume normality; see Modern Portfolio Theory [5], efficient markets and the Black-Scholes option

[1]Losses are negative and profits are positive.

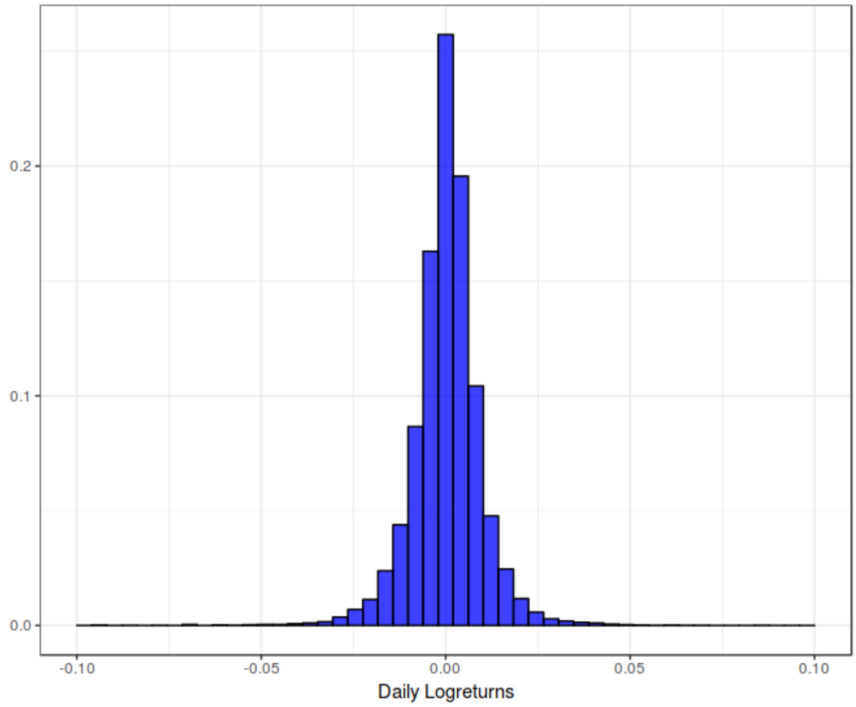

Figure 1: Daily logreturns of the Standard & Poors index [1950 - 2017]. Data from Yahoo Finance.

pricing model. And it is this very normality one of the main assumptions of the geometric brownian motion hypothesis of prices.

But given the irrational and unpredictable behaviour of the markets, we can see some flaws to this normality. Like the fat tails on the extremes of that histogram, fatter than they should, given normality. This little deviation implies that improbable events happen *a lot* more than expected, and this rises a philosophical doubt that undermines our understanding of risk. A normal distribution assumes that, given enough observations, all values in the sample will be distributed equally above and below the mean. Hence the convenience of using standard deviation as a measure of volatility, since it gives us some sense of how far away we can be from the mean. However, given the size of those extreme values, we can not entirely rely our assessment of risk upon the standard deviation; and thus we ought to study and gauge those fat tails further.

Expected Shortfall

In order to measure the importance and the impact of the tails of return distributions, it is common to compute what is called the *Expected Shortfall*. The Expected Shortfall (ES_α) at an α quantile of a given distribution X is defined as:

$$ES_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_\gamma(X)d\gamma, \tag{3}$$

Where $\alpha \in (0,1)$, $VaR_\gamma(X)$ is the $1-\gamma$ quantile of X and $\gamma \in (0,1)$. This means that the ES_α gives us the expected value of the returns distribution in the worst $\alpha\%$ cases.

Thus, the Expected Shortfall gives us a much more intuitive and reliable sense of the *risk* of any investment; in addition to its useful mathematical properties [6].

3 Optimisation of Investment Strategies

Most usually, classical savings theory states that the optimal approach to ensure the risk level of any given investment is to set a constant proportion of the wealth allocated in risky assets.

In this section we will explain how we simulate the result of such strategies and compare them to the methodology of the work developed in [2], in which they explain the idea of a *variable* proportion allocated in risky assets, instead of constant. We will replicate the results of that work and see how this change in the strategy can affect the performance of the investment plan.

3.1 CPPI Scheme

Firstly, we will start defining the *benchmark* model. The constant portfolio strategy follows the logic derived from constant relative risk exposure.

This methodology consists in investing a constant proportion π of the savings in risky assets (subject to volatility), whilst investing the rest $1 - \pi$ in risk-free assets. The point of this strategy is to present an intuitive straightforward way to control the risk exposure in savings strategies. The simplicity of this approach let us tweak π in order to make the investment best suited for the risk aversion profile of each investor individually.

Simulation

In order to simulate the performance of this kind of strategy, we start assuming that the risky assets follow a simplified geometric brownian motion, with *trend* α and *volatility* σ. Thus, if the saver invests x in this asset at day t, the wealth of the saver at the next day would be $x_{t+1} = (1 + N(\alpha, \sigma))x_t$.

This way, we construct the scenario of an investor, saving a fixed amount of money - which is denoted as a - throughout $T/2$ years, and that money being allocated $(1 - \pi)a$ in the risk-free asset, which we will set with return 0; and πa allocated in the risky asset, whose expected return is α and volatility is σ. Thus, if we set x_t as the wealth at any given time t, we can see that

$$x_{t+1} = (1 + N(\alpha, \sigma))x_t\pi + (1 - \pi)x_t + a. \tag{4}$$

At some point in time, our investor will stop saving money and will start consuming it (as in most pension plans), so we just convert that fixed amount of money a to *consumed* money instead of saved. Thus, the evolution of wealth turns out to be:

$$x_{t+1} = x_t(1 + N(\alpha, \sigma))\pi + (1 - \pi)x_t - a. \tag{5}$$

At the end of all T years, the final wealth X_T remaining to the saver is stored. By reproducing this same scheme, we manage to compute tens of thousands of different performances and make some statistics out of them.

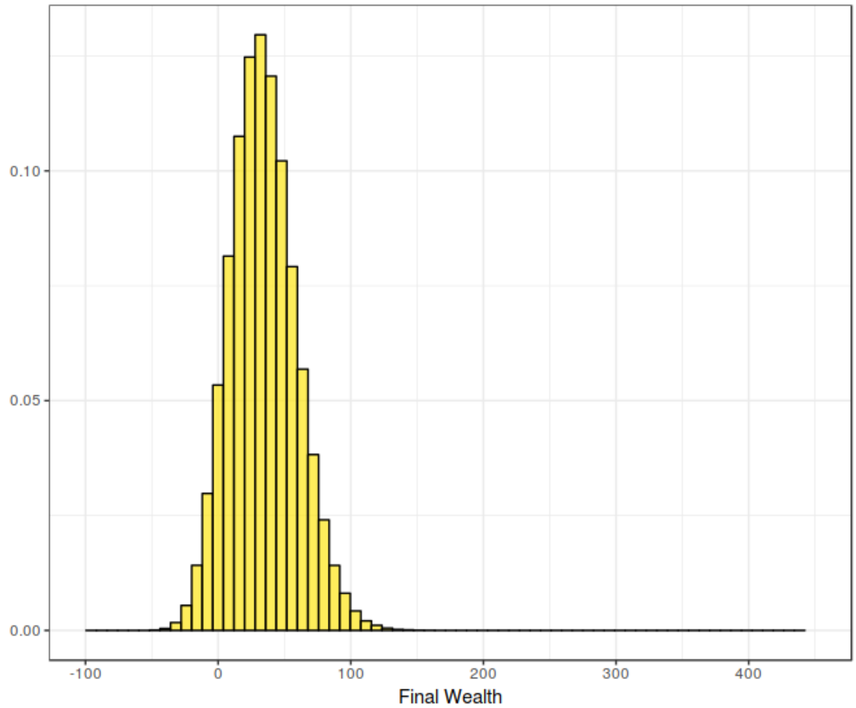

Figure 2: Results of the simulation for the CPPI model. Final wealth obtained for every simulation, where $T = 30$, $\mu = 0.0343$, $\sigma = 0.1544$, $a = 10$, $\pi = 0.1$

If we iterate this process over 10000 simulations, setting $\pi = 0.1$, $T = 60$, $\alpha = 0.0343$, $\sigma = 0.1544$ and $a = 10$ we get results as shown in figure 2. The first impression we get when looking at the histogram, is that it has the shape of a *Bell Curve*, thus following a *Normal Distribution*, but a simple *Kolmogorov-Smirnov test* would tells us that this is far from being true.

In the following snippet the code necessary to replicate these results in R is shown:

```
nsim <- 10000

pi <- 0.1
alpha <- 0.0343 # Expected return of the risky market
sigma <- 0.1544 # Expected volatility of the risky market
a <- 10 # Factor 'a'
years <- 60 # Total time

C <- append(rep(a, round(years/2)),rep(-a, round(years/2)))

X_T <- c()

for (j in 1:nsim){
```

```
  x <- c()
  x[1] <- a # Initial wealth

  for (i in 1:(years-1)){
    random <- rnorm(1, mean = alpha, sd = sigma)
    x[i+1] <- x[i]*(1+random)*pi + (1-pi)*x[i] + C[i+1]
  }
  X_T[j] <- x[years]

}
```

3.2 Alternative Scheme

Now that the CPPI model is presented and its logic understood, we can move upon to alternatives. One of the main characteristics of the CPPI model is that it is defined thanks to a constant, invariant π that settles the risk exposure of the investor. An interesting approach would not just change this parameter, but make it *variable*.

Normally, the simplest approach to ensure that an investment has a defined and controlled risk profile is to set a π constant proportion of the investment to be allocated in risky assets, as done in the CPPI. That proportion is defined by the risk aversion profile of the investor, but is invariable throughout the investment.

However, the work of [2] showed that, given some investment plan ending at time T with $X(T)$ final wealth, risk aversion profile defined by γ and a maximum possible allowed loss K, we can set the utility function

$$u_\gamma = \frac{1}{\gamma}(X(T) + K)^\gamma. \tag{6}$$

Whose expected value for any given present wealth $x = X(t)$ is defined as

$$\max_\pi \mathbb{E}(u_{X(T)} \mid x), \tag{7}$$

which can be maximized by a strategy that invests a relative amount in risky assets variable at any time $t \in [0, T)$, whose solution is:

$$\pi(t)X(t) = A(K + X(t) + g(t)). \tag{8}$$

Where $X(t)$ is the wealth at time t, A is a parameter that defines the risk aversion profile of the investor, K is the maximum loss the investor is capable to handle and $g(t)$ is the sum of all remaining inputs or outputs of money: $g(t) = \sum_{i=t}^{T} a_i$. [2].

Simulation

In order to analyse the alternative scheme, the process will be quite similar to the previous one. We set the normal behaviour of the price evolution of the risky asset, and fix all parameters. Thus, the wealth of the investor behaves as follows:

$$X_{t+1} = (1 + N(\alpha, \sigma))X_t\pi_t + (1 - \pi_t)X_t + C(t), \tag{9}$$

[2] In [7]'s work, the authors defined optimal savings strategies and analysed different updating periods for π.

where

$$\pi_t = \frac{A\Big(K + X_T + \sum_t^T C(t)\Big)}{X_T} \quad and \tag{10}$$

$$C(t) = \begin{cases} a & \text{if } t <= T/2 \\ -a & \text{if } t > T/2 \ . \end{cases}$$

Again, at the end of all T years, the final wealth X_T remaining to the saver it is stored, and then all the process is repeated. This way we manage to compute tens of thousands of different performances and make some statistics out of them.

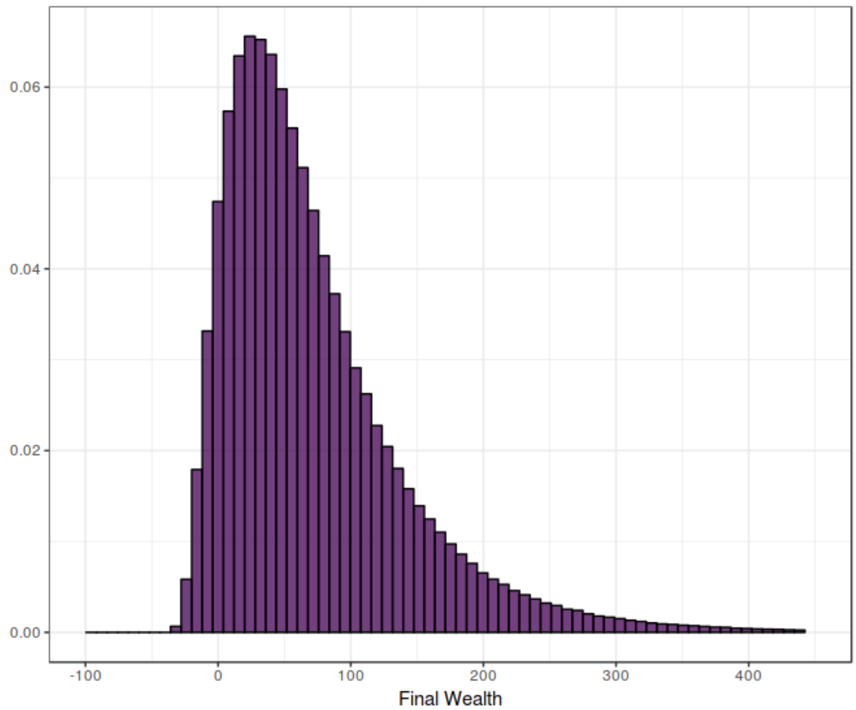

Figure 3: Results of the simulation for the *Alternative* model. Final wealth obtained for every simulation, where $T = 30$, $\mu = 0.0343$, $\sigma = 0.1544$, $a = 10$, $\pi = 0.1$.

In the case of figure 3, where the right tail corresponds to large values of wealth, whereas the left part deals with losses; we can see how outrageously obvious is that this it does *not* follow a Normal Distribution.

3.3 Comparison

At this moment, we have understood and tested both strategies, and it is moment to contrast each other and highlight their differences.

First of all we plot together the frequency histograms for the final wealth distribution for both models.

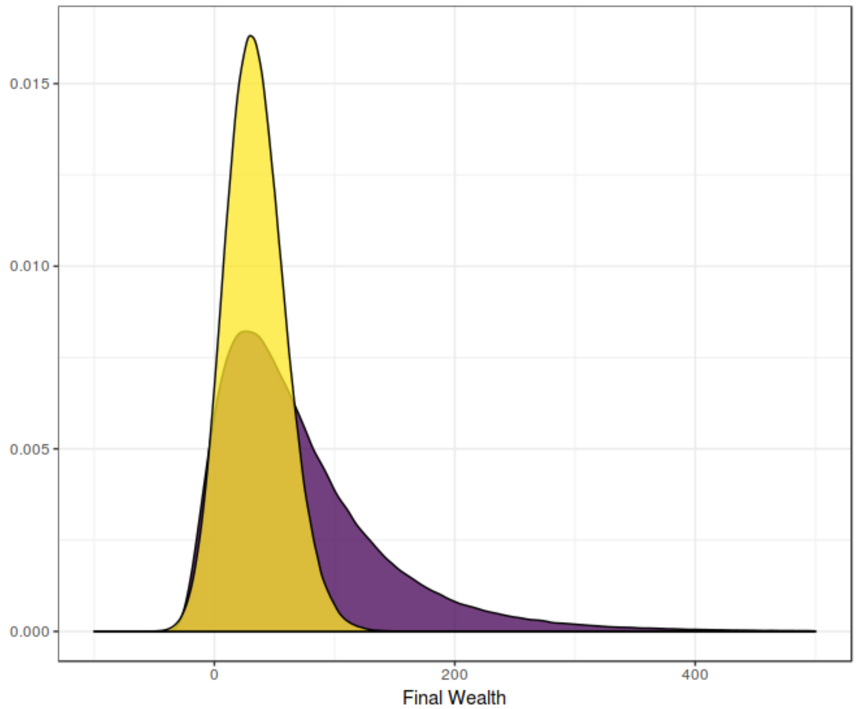

Figure 4: Results of the simulation for the both models. Final wealth obtained for every simulation. The yellow one is the CPPI, and the purple one is the Alternative.

Looking at Figure 4 we may notice that they follow a considerably different distribution. We could say that the alternative strategy presents more dispersion, even though it is always a *positive* deviation from the mean.

In section 2 we have discussed a little bit some implications of the definition of risk. Affirm that the alternative strategy presents more risk, just because its result is more disperse, would may seem a little simplistic.

On the other hand, we can take a look at those rare cases when the final wealth happens to be negative. These are the cases worth exploring, for it is the scenario every investor is afraid of: losing money. Zooming in into the negative zone, as shown in figure 5, we can focus in the difference between the two models. Even though we can see some spurious differences in some places, the most honest answer is that it is not clear whose result is less risky.

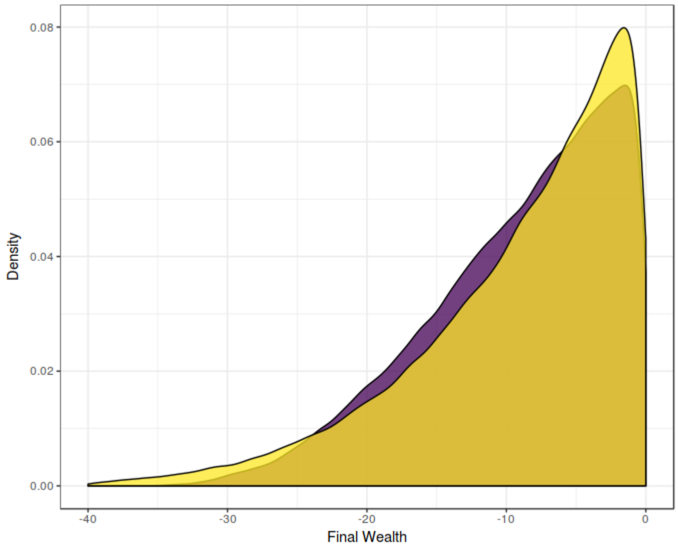

Figure 5: Results of the simulation for the both models. Final wealth, filtered by negative values obtained for every simulation. The yellow one is the CPPI, and the purple one is the Alternative.

In order to compare the degree of risk taken by every model, we make use of the *Expected Shortfall*, as explained in section 2. If we set this parameter as the level of risk and we set it constant in both models, we can compare their returns. In [2] it is shown that the Alternative model is able to set its Expected Shortfall (ES) by its K, using this formula:

$$\frac{K}{ES} = \frac{1}{-1 + (1-\theta)^{-1} e^{\alpha A T} \Phi\Big(\Phi^{-1}(1-\theta) - A\sigma\sqrt{T}\Big)} \tag{11}$$

Where α is the expected return of the risky asset, σ is the standard deviation of the risky asset, θ is the quantile on which the Expected Shortfall is measured, A is the parameter that settles the risk aversion profile and Φ represents the standard Normal distribution function.

Therefore, the approach is the following: We set a constant proportion π for the CPPI model, we simulate it many times and measure the ES for the results. Then we find the K in order to set the same ES on the Alternative model. This way we can simulate both models making sure they will assume the same risk, and thus we can freely compare their returns.

Setting $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$, $A = 0.5$ and number of simulations $N = 100,000$ we find the results shown in Table 2, in which we can see the outperformance of the alternative strategy, for many different levels of risk.

Table 1: Results of 100,000 simulations using different risk levels, with $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$, $A = 0.5$ and therefore $ES/K = -3.3$. These results match approximately those presented in [2], on Table 1 at page 8. Any differences are attributable to the intrinsic randomness of simulations.

π	**ES**	K	**CPPI ret**	**Alt ret**	**diff**	**equiv** π
0.1	-12.47	40.59	0.33	0.51	0.18	0.17
0.2	-27.03	87.98	0.65	0.98	0.38	0.30
0.3	-42.99	139.95	0.95	1.42	0.46	0.45
0.4	-60.13	195.71	1.23	1.83	0.60	0.61
0.5	-78.25	255.02	1.5	2.22	0.71	0.69
0.6	-99.96	325.36	1.76	2.63	0.87	0.92
0.7	-120.61	392.56	2.00	3.00	1.00	*
0.8	-144.71	471.02	2.23	3.40	1.17	*
0.9	-172.94	562.92	2.39	3.81	1.42	*
1	-205.09	667.57	2.58	4.27	1.70	*

The interesting conclusion that can be extracted from Table 2 is the value of the last column. It states that the necessary value of π that the CPPI should get in order to equal the return of the Alternative strategy, what we call the equivalent π. Thus, if the Expected Shortfall was settled by some π and the required equivalent π is greatar than that, we are saying that the CPPI should undertake in more risk in order to equal the return of the Alternative Scheme.

This equivalent π thus works as a succint metric to compare the performance of this two strategies, taking into account both return and risk. Hence, as long as the equivalent *pi* is greater than the initial π, we can say that the Alternative Strategy is outperforming the CPPI. Which is clear for many values of initial π and a fixed value for A.

However, in Figure 6 we can show the iterated simulation of Table 2, for many different values of π and many values of A. It is quite interesting to notice how this *outperformance* occurs whilst the continuous line is above the dashed line. And their downhill tendency suggests that the Alternative Strategy is more worth ir for smaller values of A.

Since the value of A is interpreted as the risk aversion of the investor, we could argue that the riskier the inversor the less worth it the Alternative Scheme becomes, until a point (usually around $A = 1.5$) where the CPPI is outperforming the Alternative Strategy.

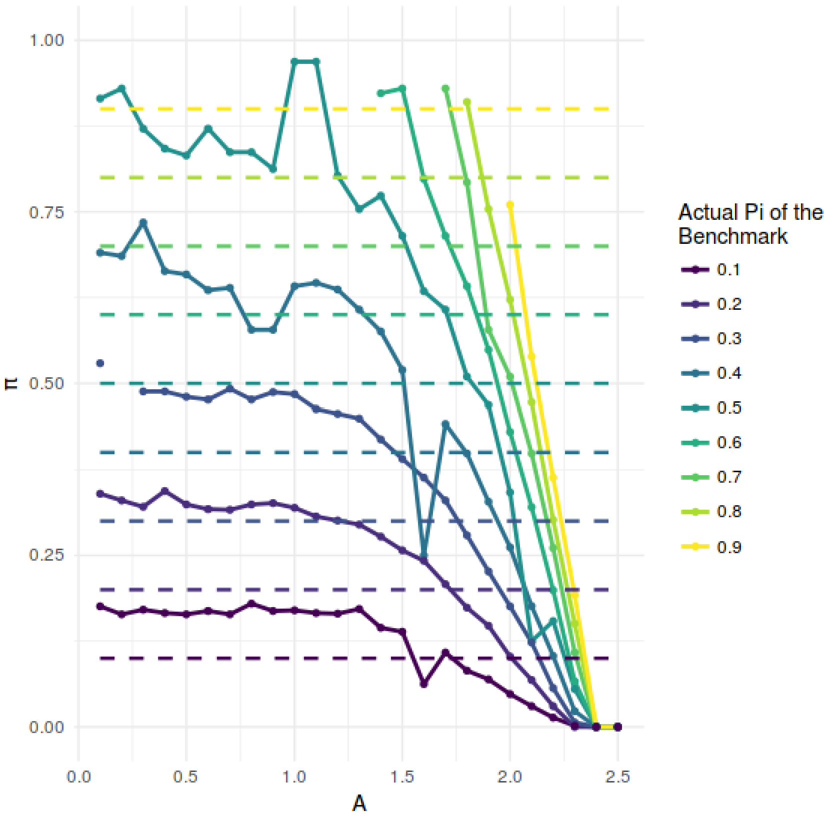

Figure 6: Plot that shows the relation between equivalent π and A, for many different initial π. From the results of $100,000$ simulations for each curve, with $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$.

4 Optimisation of Pooled Funds Investment Strategies

So far, we have managed to test two different strategies. We have analyzed their return and risk in a simulated scenario, within the context of a pension plan. In order to assess the risk of the strategies, so far we have been measuring the market risk of the inestment, by computing the *Expected Shortfall*. However, in real-scenario pension plans, there are plenty of other risks that should be taken into consideration. One important risk, not rarely underestimated, is the so-called **Longevity Risk**. Longevity risk is the risk of retirees that will live longer than expected and will thus exhaust all their savings. This risk might doom some individuals to utter poverty or to burden relatives.

Recently, two worldwide phenomena ought to be highlighted. The collapse in low-risk assets returns as government bonds or blue chip stocks. And the observed demographic transition Caldwell [8] and Bongaarts [9], in which both birth rates and death rates are plumbing down; increasing the life expectancy of elder individuals. The combination of these two factors is leading to an increase in longevity risk that the pension plans providers are facing, rising pension premiums and stagnating disposable incomes by savers and pushing them to work longer years before retirement.

As a response to this challenging, the work of [10] and [11] suggested a different approach to face longevity risk, the concept of **Pooled Funds**.

Pooled Funds are funds formed by many different individual savers that aggregate their savings together. Alongside other advantages, pooled funds benefit from economies of scale, cheaper diversification and a more efficient management of longevity risk.

In this section we will study the application of both CPPI and Alternative schemes that we have developed in previous sections under the framework of a pooled fund.

4.1 Simulating the Pooled Fund

In order to simulate the pooled fund, we will construct a simple scenario where many investors of the same age start investing at the same time. We will take real death probabilities at each age, and we will simulate the death of some of the savers.

When savers die, some proportion w of their saved money stays in the pool, benefiting the survivors. The rest is extracted from the pool, to their family or inheritors. In order to simulate the probability of death for each individual, we took the empirical measures of death probability for every age from the Spanish Government. This way, we can assume that from a starting number of persons n of the same age, and thus with the same death probability p, the number of persons that would die X before the next year should follow a *Binomial Distribution*. Thus the probability of k deaths is:

$$Pr(X = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}. \tag{12}$$

Now we can now make use of the algorithm described in [12] to generate random numbers k that follow a Binomial Distribution, and assume those k to be the simulated deaths for the following year. Once we have k, we can compute how much proportional wealth M has been given to every investor of the common pool by:

$$M = \frac{k}{n} w. \tag{13}$$

The point of constructing the Pooled Fund in such a way, is that we are not altering the core principles of the assumptions taken building the Alternative Model. The relative proportion π of wealth to be allocated in risky assets remains untouched, we are just altering the $x = X(t)$ present in Equation 7. Resulting in

$$\max_{\pi} \mathbb{E}(u_{X(T)} \mid x(1 + M)). \tag{14}$$

And thus the wealth of the investor will behave as

$$dX = \alpha\pi(t)X(t)dt + \sigma\pi(t)X(t)dW(t) + dC(t) + dM(t). \tag{15}$$

In the following snippet we show the necessary R code to perform this simulation for both CPPI and Alternative schemes:

```

### CPPI ###

C <- append(rep(a, round(years/2)),rep(-a, round(years/2)))
mort_table <- fread("mortality.csv")/1000
X_T <- c()

for (j in 1:nsim){
  x <- c()
  x[1] <- a # Initial wealth
  number_humans_alive <- starting_humans

  for (i in 1:(years-1)){
    prob_mort <- mort_table$total[i+starting_age-1]
    number_deads <- rbinom(1,number_humans_alive,prob_mort)
    number_humans_alive <- number_humans_alive - number_deads


    random <- rnorm(1, mean = alpha, sd = sigma)
    x[i+1] <- x[i]*(1+random)*pi + (1-pi)*x[i] + C[i+1] +
        (x[i]*number_deads/number_humans_alive)*w
  }
  X_T[j] <- x[years]
}
```

```
### Alternative ###

C <- append(rep(a, round(years/2)),rep(-a, round(years/2)))
mort_table <- fread("mortality.csv")/1000
X_T <- c()

for (j in 1:nsim){
  x <- c()
  x[1] <- a # Initial wealth
  number_humans_alive <- starting_humans

  for (i in 1:(years-1)){
    prob_mort <- mort_table$total[i+starting_age-1]
    number_deads <- rbinom(1,number_humans_alive,prob_mort)
    number_humans_alive <- number_humans_alive - number_deads

    pi <- fpi(A,K,X,C,i)
    random <- rnorm(1, mean = alpha, sd = sigma)
    x[i+1] <- x[i]*(1+random)*pi + (1-pi)*x[i] + C[i+1] +
        (x[i]*number_deads/number_humans_alive)*w
  }
  X_T[j] <- x[years]
}
```

4.2 Results

Now we are going to replicate the measures taken in the previous section but with the added conditioni of mortality and see whether or not the differences spotted between the CPPI and the Alternative strategies remain similar.

Table 2: Results of 100,000 simulations using different risk levels, with $\alpha = 0.343$, $\sigma = 0.1544$, $a = 10$, $T = 60$, $A = 0.5$ and therefore $ES/K = -3.3$. These results match approximately those presented in [2], on Table 1 at page 8. Any differences are attributable to the intrinsic randomness of simulations.

π	**ES**	K	**CPPI ret**	**Alt ret**	**diff**	**equiv** π
0.1	-	-	-	-	-	-
0.2	-	-	-	-	-	-
0.3	-	-	-	-	-	-
0.4	22.13	-72.04	3.11	-0.08	-3.19	0
0.5	-27.70	90.18	3.45	2.80	-0.64	0
0.6	-98.78	321.52	3.80	4.96	1.16	2.37
0.7	-121.94	396.91	4.13	5.66	1.52	1.28
0.8	-143.56	467.30	4.74	6.03	1.29	3.01
0.9	-168.69	549.08	4.78	6.48	1.70	*
1	-248.76	809.72	5.24	7.82	2.57	*

Table 3: My caption

Pi	ES	K	cppi_ret	montses_ret	diff	pi_b

5 Tails Analysis

At this point we should have notice that the real danger of investments does not resides solely within the standard concept of dispersion (standard deviation), nor within the ratio between return and dispersion. The true risk dwells in the worst case scenarios of the distributions, the tails. Where accumulated or great losses may strike down the whole investment plan. Hence, in order to address the risk analysis of any savings strategy, we need to take a closer look at the tails of their distributions.

Taking a close look at Figure 4, we are not able to clearly state which one is *less risky*. So far, we have been comparing the values of Expected Shortfall in order to get a sense of the risk. But, since the ES is actually just the average of the tail of the distribution, we will now discuss how we can pursue a more in-depth analysis for the distribution of those tails.

In this section, we will go one step further and try to fit some specific distribution to those tails. This way we will be able to see in more detail the behaviour of the losses of our strategies, and set more accurate risk measures.

5.1 Definition of ***Tail***

First things first. What is *the* Tail of a distribution? The tail of a distribution is not a precisely defined term; it can have many different forms and definitions. Generally, the *tail* is considered a broad term to name the extreme parts of a distribution. In other words, there is not some specific place where you stop being in the middle of the distribution and start being in the tail, and where to put that line is up to every case.

Let's say that we plot a distribution Y, the loss of a savings plan (note that negative losses imply profits). We decide to put the distinction between middle and tail at some point named u, so the tail would be everything that lands within $Y > u$.

If we understand the tail of a distribution as a distribution itself, we can move that tail to the origin and normalise its area. Thus, being positively defined as

$$Z = (Y - u|Y > u) \tag{16}$$

.

Once we have this definition, we can easily define the probability density as

$$\begin{aligned} F_Z(z) = P(Z < z) &= P(Y - u < z|Y > u) && (17)\\ &= P(Y < z + u|Y > u) && (18)\\ &= \frac{P(Y < z + u, Y > u)}{P(Y > u)} && (19) \end{aligned}$$

And thus, deriving this expression we find that, for every $z > 0$,

$$f_z(z) = \frac{f_y(z+u)}{1 - F_y(u)}. \tag{20}$$

Interesting thing about this result, is that we can analytically relate the distribution of the tail $f_z(z)$ with the whole original distribution $f_y(z)$.

Therefore, we can say that, if the distribution of the losses obtained is Y, our tail could be defined as

$$F_u(y) = (Y - u \leq y | Y > u). \tag{21}$$

The Pickands-Balkema-de Haan theorem states that for a large class of distributions X, exists u such as that F_u is well approximated by the Generalized Pareto Distribution (GPD) G. Where the standard cumulative distribution function of the GPD can be defined as:

$$F_\xi(y) = \begin{cases} 1 - (1 + \xi y)^{-1/\xi} & : \xi \neq 0 \\ 1 - e^{-y} & : \xi = 0 \end{cases}.$$

Thus, we can seek that point u from which our tail could fit a GPD. That would imply that, from that point u forward, the distribution follows a GPD. The kind of GPD would be determined by the value of ξ. This number, usually called *Extreme Value Index*, that determines the shape of the distribution. The larger the value of ξ, the thicker the tail grows, and the slowly it converges to zero. In general:

- $\xi > 0$. The tail does not converge to zero, ever. Power-law
- $\xi = 0$. The tail converges to zero at infinity. Exponential
- $\xi < 0$. The tail converges to zero in a finite point.

In what risk analysis concerns, the value of ξ is crucial. Since we are studying the risk of a strategy looking at the tail of distributions where distribution are the frequency of losses, we could think that the wider or thicker the tail, the riskier the strategy.

5.2 Extreme Value Analysis: Fitting a General Pareto Distribution

Let us repeat the simulation for the CPPI, using $\pi = 0.1$, and plot the histogram of its losses as an example.

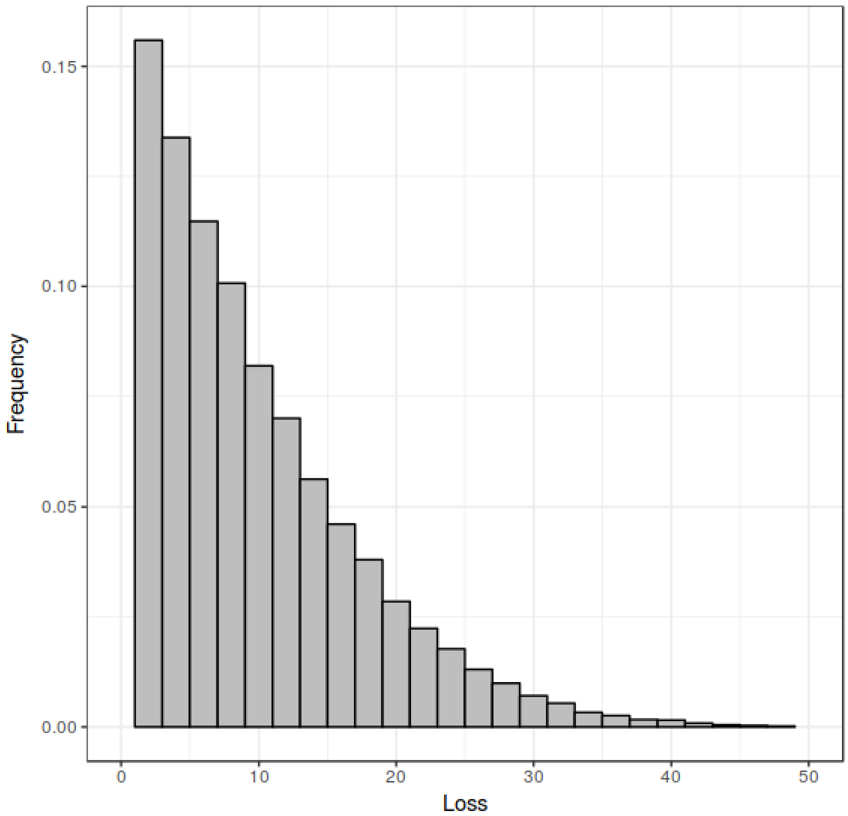

Figure 7: Frequency Histogram of the losses of the CPPI strategy. Using $\pi = 0.1$, $\alpha = 0.0343$, $\sigma = 0.1544$ and 1000000 simulations.

If we now plot that density as points in a logarithmic scale, we might see something like the following:

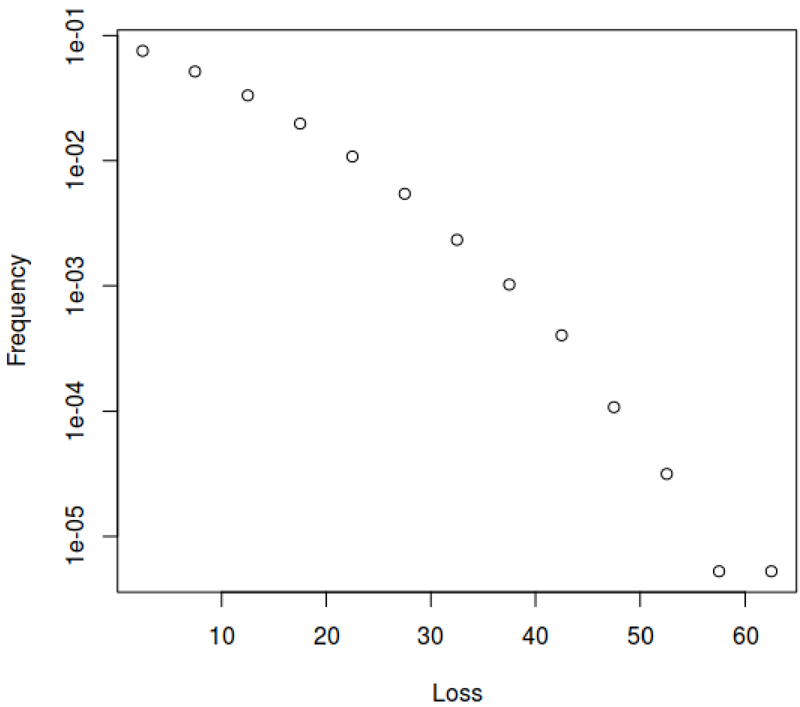

Figure 8: Density Distribution of the losses of the CPPI strategy in a logarithmic scale. Using $\pi = 0.1$, $\alpha = 0.0343$, $\sigma = 0.1544$ and 1000000 simulations.

Now, we could try to add here the Complementary Cumulative Distribution Function of a Generalized Pareto Distribution, and see if it fits to our distribution as it is now.

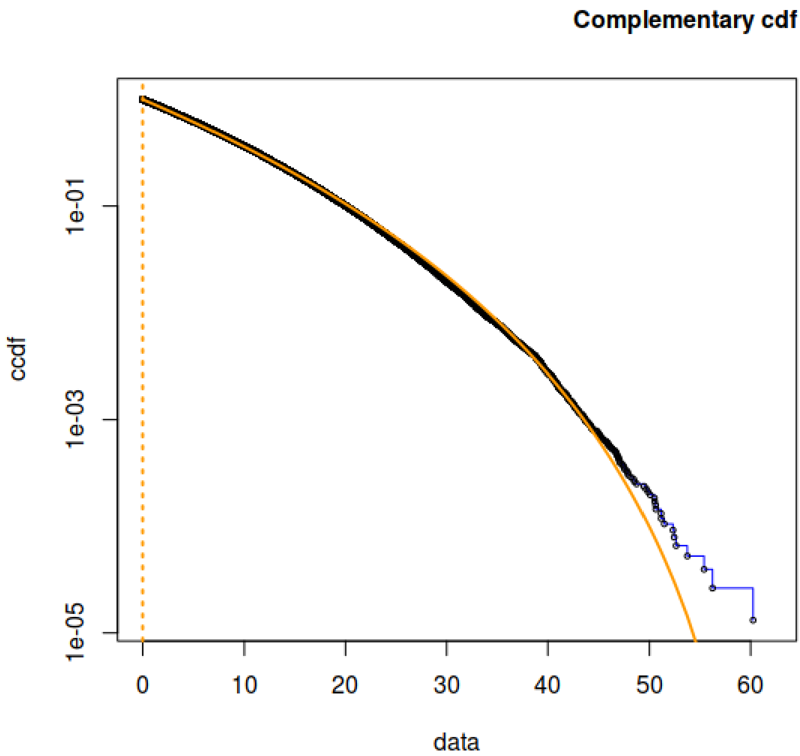

Figure 9: Density Distribution of the losses of the CPPI strategy in a logarithmic scale. Using $\pi = 0.1$, $\alpha = 0.0343$, $\sigma = 0.1544$ and 1000000 simulations.

Clearly, we can see how badly it fits for larger numbers. This indicates that our distribution is not following a GPD. Nevertheless, we can still find a proper u for which it would.

A usual method in order to seek that u threshold is to consecutively tryng increasing numbers until one fits the distribution properly. We know that for that value on, the distribution will follow a GPD.

In this example, the first value that we find satisfies this is $u = 42.59$. And the result can be visualized in the following plot:

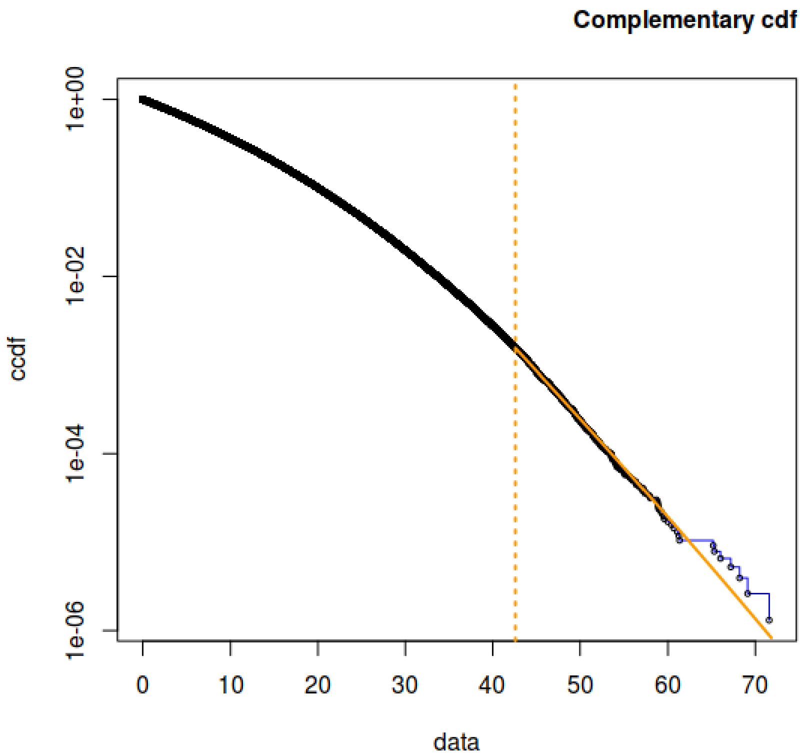

Figure 10: Exploratory empirical residual coefficient of variation of losses of the CPPI strategy in a logarithmic scale. Using $\pi = 0.1$, $\alpha = 0.0343$, $\sigma = 0.1544$ and 1000000 simulations.

The Extreme Value Index (EVI) obtained in this example is $\xi = 0.14$.

5.3 Convergence of the Expected Shortfall

So far, we have addressed the necessity of studying the tail of loss danger using the Expected Shortfall. It is quite straightforward and it easily assesses the risk of the tail. But since it is nothing more than a simple mean, it can arise some issues.

We talked about how the tail of a distribution can be considered a distribution by itself. Thus, the mean of the tail intends to be a summary statistic of this distribution. Despite that, we know that the arithmetic mean is not always a good summary statistic for every given distribution. There are some distributions from which the mean is not the best suited statistic to get a reliable feel of its *general tendency*, especially on very skewed distributions.

Moreover, without previous knowledge about the distribution we may encounter, it exists the possibility of facing a distribution whose arithmetic mean does not exist. This possibility arises a very disturbing problem: The computed value of the Expected

Shortfall should not exist. Since we are working with simulated data, and thus finite numbers, we will never face an infinite arithmetic mean. This implies that in order to ensure the reliability of the Expected Shortfall, we first need to check whether its computation makes sense or not.

In Figure 5 we see the density curve of the loss tails of both methods. If we compute the histogram and set the bars of the histogram as points, we can build a scatterplot. In Figure 11 we can see this scatter plot of their logarithmic density points. It is of great interest noticing that, whereas both tails follow a linear model quite accurately (after taking logarithms), the *Alternative* scheme has a much sharper descending trend. Which would mean that its *Expected Shortfall* is far less likely to be infinite in any analytical distribution. This suggests us that the computed value of the Expected Shortfall is much more reliable for the *Alternative* scheme than for the CPPI one.

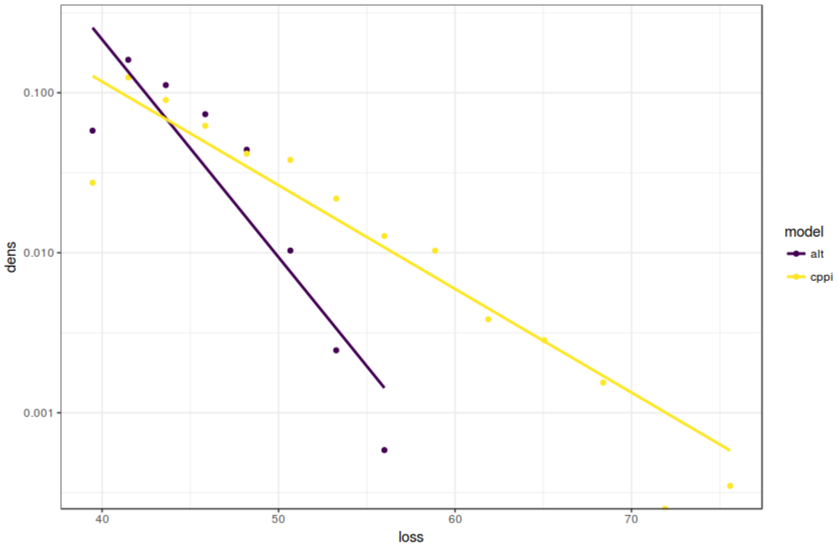

Figure 11: Scatter plot and linear regression of the logarithm of the density points of the *CPPI* and *Alternative* methods.

6 Results

A this point

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