



Worksheet 6 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

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1 (★)

Prove that $\neg \text{isContr}(\emptyset)$.

Recall that, for any type A ,

$$\text{isContr}(A) \doteq \sum_{c:A} \prod_{a:A} c = a.$$

So we have the projection map

$$\sum_{c:\emptyset} (\prod_{a:A} c = a) \xrightarrow{\text{pr}_1} \emptyset$$

Which is a map $\text{isContr}(\emptyset) \rightarrow \emptyset$.

2 (★★)

Recall the *observational equality of natural numbers* $\text{Eq-}\mathbb{N} : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathcal{U}$

$$\begin{aligned} \text{Eq-}\mathbb{N} \ 0 \quad 0 &\doteq \mathbb{1} \\ \text{Eq-}\mathbb{N} \ (\text{suc } m) \ 0 &\doteq \emptyset \\ \text{Eq-}\mathbb{N} \ 0 \quad (\text{suc } n) &\doteq \emptyset \\ \text{Eq-}\mathbb{N} \ (\text{suc } m) \ (\text{suc } n) &\doteq \text{Eq-}\mathbb{N} \ m \ n \end{aligned}$$

Prove that, for every $n : \mathbb{N}$,

$$\text{Eq-}\mathbb{N} \ n \ (\text{suc } n) = \emptyset$$

Proceed by induction on n . For $n \doteq 0$, we have that $\text{Eq-}\mathbb{N} \ 0 \ 1$ is definitionally equal to \emptyset , so we use refl_\emptyset . Then, for some n , assuming

$$\text{ih} : (\text{Eq-}\mathbb{N} \ n \ (\text{suc } n)) = \emptyset$$

we have that $\text{Eq-}\mathbb{N} \ (\text{suc } n) \ (\text{suc}(\text{suc } n)) = \emptyset$ by the recursive definition of $\text{Eq-}\mathbb{N}$.

In Lecture 4, we did most of the proof that

$$\text{Eq-}\mathbb{N} \ m \ n \simeq m =_{\mathbb{N}} n.$$

Use this (and the fact proved above) to prove that $\neg(\text{isContr } \mathbb{N})$.

Suppose

$$(c, \varphi) : \text{isContr } \mathbb{N}.$$

So $c : \mathbb{N}$ and $\varphi : \prod_{n:\mathbb{N}} c = n$. So $\varphi(\text{succ } c) : c = (\text{succ } c)$. But we have

$$c = (\text{succ } c) \quad \simeq \quad \text{Eq-}\mathbb{N} \ c \ (\text{succ } c)$$

We showed that the right-hand side is \emptyset , and thus $\text{isContr}(\mathbb{N})$ implies \emptyset .

3 $(\star \star \star)$

Show that if A is contractible, then for any $x, y : A$, the identity type $x = y$ is also contractible.

Let $c : A$ be the center of contraction. It will suffice to show that $(x = y) \simeq \mathbb{1}$. For the map $f : (x = y) \rightarrow \mathbb{1}$, we can just use the constant \star map. For the other direction, $g : \mathbb{1} \rightarrow (x = y)$, we proceed by singleton induction and can therefore take $x \doteq c \doteq y$ and put

$$g(\star) \doteq \text{refl}_c.$$

To see that $g(f(p)) = p$ for any $p : x = y$, we can proceed by path induction and assume $x \doteq y$ and $p \doteq \text{refl}_x$. By singleton induction again, we can assume $x \doteq c$. So $p \doteq \text{refl}_c$, which is the same as $g(f(p))$. The other direction is simpler: for any $t : \mathbb{1}$, we prove that $f(g(t)) = t$ by singleton induction, i.e. assuming $t \doteq \star$. But $f(g(t)) \doteq \star$ by definition, so $\text{refl}_\star : f(g(t)) = t$.

4 $(\star \star \star)$

Recall the first projection function

$$\text{pr}_1 : \sum_{x:A} B(x) \rightarrow A$$

Show that pr_1 is an equivalence iff each $B(a)$ is contractible. *Hint: Use the results about identity types of Σ types we proved in a previous lecture.*

First the ‘only if’ direction: if \mathbf{pr}_1 is an equivalence, that means it’s a contractible map, i.e.

$$\mathbf{isContr}(\mathbf{fib}_{\mathbf{pr}_1}(a))$$

for each $a : A$. Then, given any $a : A$ and $b, b' : B(a)$, we have

$$((a, b), \mathbf{refl}_a) : \mathbf{fib}_{\mathbf{pr}_1}(a) \quad \text{and} \quad ((a, b'), \mathbf{refl}_a) : \mathbf{fib}_{\mathbf{pr}_1}(a).$$

Since $\mathbf{fib}_{\mathbf{pr}_1}(a)$ is contractible, we have

$$((a, b), \mathbf{refl}_a) = ((a, b'), \mathbf{refl}_a).$$

Using our result characterizing the identity type of Σ types (Lecture 5), this gives us a proof of

$$(a, b) = (a, b')$$

and, applying it again,

$$b = b'.$$

Thus we have that $B(a)$ is contractible.

In the other direction, suppose each $B(a)$ is contractible with center c_a . Then define an inverse $q : A \rightarrow \sum_{x:A} B(x)$ by

$$q(a) \doteq (a, c_a).$$

Then $\mathbf{pr}_1(q(a)) \doteq \mathbf{pr}_1(a, c_a) \doteq a$, so q is a section of \mathbf{pr}_1 . Furthermore, for any $(a, b) : \sum_{x:A} B(x)$, we have a proof that $b = c_a$, so, again using that identities of pairs are equivalent to pairs of identities, we get

$$(a, c_a) = (a, b).$$

The left-hand side is exactly $q(\mathbf{pr}_1(a, b))$, as desired.

Show that for any $a : A$, the map

$$\lambda((x, y), p). \mathbf{tr}_B(p, y) : \mathbf{fib}_{\mathbf{pr}_1}(a) \rightarrow B(a)$$

is an equivalence.

Call this map k . Now, pick some $b : B(a)$ and observe that $((a, b), \mathbf{refl}_a) : \mathbf{fib}_{\mathbf{pr}_1}(a)$ and that

$$k((a, b), \mathbf{refl}_a) \doteq \mathbf{tr}_B(\mathbf{refl}_a, b) \doteq b$$

So now pick some $((x, y), p) : \mathbf{fib}_{\mathbf{pr}_1}(b)$ and

$$q : \mathbf{tr}_B(p, y) = b.$$

Recall (again) from Lecture 5 that

$$(x, y) = (a, b) \simeq \sum_{p:x=a} \mathbf{tr}_B(p, y) = b$$

So, since we have $p : x = a$ and $q : \mathbf{tr}_B(p, y) = b$, we get a proof $r : (x, y) = (a, b)$. Finally, by the observations we made in lecture, to prove

$$((x, y), p) = ((a, b), \mathbf{refl}_a)$$

we just need to have our proof $r : (x, y) = (a, b)$ and then show

$$p = \mathbf{ap}_{\mathbf{pr}_1} r \cdot \mathbf{refl}_a.$$

But we can check that $\mathbf{ap}_{\mathbf{pr}_1} r$ gives us p , and $p \cdot \mathbf{refl}_a = p$.

5 (★★)

Construct for any map $f : A \rightarrow B$ an equivalence

$$e : A \simeq \sum_{y:B} \mathbf{fib}_f(y)$$

with a homotopy $H : f \sim \mathbf{pr}_1 \circ e$ witnessing that the triangle

$$\begin{array}{ccc} A & \xrightarrow{e} & \sum_{y:B} \mathbf{fib}_f(y) \\ & \searrow f & \swarrow \mathbf{pr}_1 \\ & B & \end{array}$$

commutes.

The natural definition for e is

$$e(a) \doteq (f(a), (a, \mathbf{refl}_{f(a)}))$$

The inverse for e is given by

$$e^{-1}(y, (x, p)) \doteq x.$$

Then $e^{-1}(e(a)) \doteq a$, so we have $e^{-1} \circ e \sim \mathbf{id}_A$. For the other composition, take any $x : A$. To prove that for any $y : B$ and $p : f(x) = y$

$$(y, (x, p)) = (f(x), (x, \mathbf{refl}_{f(x)}))$$

we use path induction. Therefore, we can take $y \doteq f(x)$ and $p \doteq \mathbf{refl}_{f(x)}$, and then we have that

$$e(e^{-1}(y, (x, p))) = (y, (x, p))$$

as desired.

The homotopy H is given by

$$H(x) \quad \doteq \quad \mathbf{refl}_{f(x)} \quad : \quad f(x) = \mathbf{pr}_1(e(x))$$