Natural Sciences Tripos Part IB
Mathematical Methods I
Michaelmas 2011

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November 16, 2011

6 Elementary analysis

6.1 Motivation

Analysis is the careful study of infinite processes such as limits, convergence, differential and integral calculus, and is one of the foundations of mathematics. This section covers some of the basic concepts including the important problem of the convergence of infinite series. It also introduces the remarkable properties of analytic functions of a complex variable.

6.2 Sequences

6.2.1 Limits of sequences

We first consider a *sequence* of real or complex numbers f_n, defined for all integers $n \geqslant n_0$. Possible behaviours as n increases are:

- f_n tends towards a particular value
- f_n does not tend to any value but remains limited in magnitude
- f_n is unlimited in magnitude

Definition 1. *The sequence f_n converges to the limit L as $n \to \infty$ if, for any positive number ϵ, $|f_n - L| < \epsilon$ for sufficiently large n.*

In other words *the members of the sequence are eventually contained within an arbitrarily small disk centred on L*. We write this as

$$\boxed{\lim_{n\to\infty} f_n = L} \qquad \text{or} \qquad \boxed{f_n \to L \quad \text{as } n \to \infty}$$

Note that L here is a finite number.

To say that a property holds *for sufficiently large* n means that there exists an integer N such that the property is true for all $n \geqslant N$.

Example:

$$\lim_{n\to\infty} n^{-\alpha} = 0 \qquad \text{for any } \alpha > 0$$

Proof:

$$|n^{-\alpha} - 0| < \epsilon \qquad \text{holds for all} \qquad n > \epsilon^{-1/\alpha}$$

If f_n does not tend to a limit it may nevertheless be *bounded*.

Definition 2. *The sequence f_n is bounded as $n \to \infty$ if there exists a positive number K such that $|f_n| < K$ for sufficiently large n.*

Example:

$$\left(\frac{n+1}{n}\right)\mathrm{e}^{\mathrm{i}n\alpha} \quad \text{is bounded as } n \to \infty \text{ for any real number } \alpha$$

Proof:

$$\left|\left(\frac{n+1}{n}\right)\mathrm{e}^{\mathrm{i}n\alpha}\right| = \frac{n+1}{n} < 2 \quad \text{for all } n \geqslant 2$$

6.2.2 Cauchy's principle of convergence

A necessary and sufficient condition for the sequence f_n to converge is that, for any positive number ϵ, $|f_{n+m} - f_n| < \epsilon$ for all positive integers m, for sufficiently large n. Note that this condition does not require a knowledge of the value of the limit L.

6.3 Convergence of series

6.3.1 Meaning of convergence

What is the meaning of an infinite series such as

$$\sum_{n=n_0}^{\infty} u_n$$

involving the addition of an infinite number of terms?

We define the *partial sum*

$$S_N = \sum_{n=n_0}^{N} u_n$$

The infinite series $\sum u_n$ is said to *converge* if the sequence of partial sums S_N tends to a limit S as $N \to \infty$. The value of the series is then S. Otherwise the series *diverges*.

Note that whether a series converges or diverges does not depend on the value of n_0 (i.e. on when the series begins) but only on the behaviour of the terms for large n.

According to Cauchy's principle of convergence, a necessary and sufficient condition for $\sum u_n$ to converge is that, for any positive number ϵ,

$$|S_{N+m} - S_N| = |u_{N+1} + u_{N+2} + \cdots + u_{N+m}| < \epsilon$$

for all positive integers m, for sufficiently large N.

6.3.2 Classic examples

The *geometric series* $\sum z^n$ has the partial sum

$$\sum_{n=0}^{N} z^n = \begin{cases} \frac{1-z^{N+1}}{1-z}, & z \neq 1 \\ N+1, & z = 1 \end{cases}$$

Therefore $\sum z^n$ converges if $|z| < 1$, and the sum is $1/(1-z)$. If $|z| \geqslant 1$ the series diverges.

The *harmonic series* $\sum n^{-1}$ diverges. Consider the partial sum

$$S_N = \sum_{n=1}^{N} \frac{1}{n} > \int_1^{N+1} \frac{\mathrm{d}x}{x} = \ln(N+1)$$

Therefore S_N increases without bound and does not tend to a limit as $N \to \infty$.

6.3.3 Absolute and conditional convergence

If $\sum |u_n|$ converges, then $\sum u_n$ also converges. $\sum u_n$ is then said to converge *absolutely*.

If $\sum |u_n|$ diverges, then $\sum u_n$ may or may not converge. If it does, it is said to converge *conditionally*.

[Proof of the first statement above: If $\sum |u_n|$ converges then, for any positive number ϵ,

$$|u_{N+1}| + |u_{N+2}| + \cdots + |u_{N+m}| < \epsilon$$

for all positive integers m, for sufficiently large N. But then

$$\begin{aligned} |u_{N+1} + u_{N+2} + \cdots + u_{N+m}| &\leqslant |u_{N+1}| + |u_{N+2}| + \cdots + |u_{N+m}| \\ &< \epsilon \end{aligned}$$

and so $\sum u_n$ also converges.]

6.3.4 Necessary condition for convergence

A necessary condition for $\sum u_n$ to converge is that $u_n \to 0$ as $n \to \infty$. Formally, this can be shown by noting that

$$u_n = S_n - S_{n-1}$$

If the series converges then

$$\lim S_n = \lim S_{n-1} = S$$

and therefore $\lim u_n = 0$.

This condition is *not* sufficient for convergence, as exemplified by the harmonic series.

6.3.5 The comparison test

This refers to series of non-negative real numbers. We write these as $|u_n|$ because the comparison test is most often applied in assessing the absolute convergence of a series of real or complex numbers.

If $\sum |v_n|$ converges and

$$|u_n| \leqslant |v_n| \qquad \text{for all } n$$

then $\sum |u_n|$ also converges. This follows from the fact that

$$\sum_{n=n_0}^{N} |u_n| \leqslant \sum_{n=n_0}^{N} |v_n|$$

and each partial sum is a non-decreasing sequence, which must either tend to a limit or increase without bound.

More generally, if $\sum |v_n|$ converges and

$$|u_n| \leqslant K|v_n|$$

for sufficiently large n, where K is a constant, then $\sum |u_n|$ also converges.

Conversely, if $\sum |v_n|$ diverges and

$$|u_n| \geqslant K|v_n|$$

for sufficiently large n, where K is a positive constant, then $\sum |u_n|$ also diverges.

In particular, if $\sum |v_n|$ converges (diverges) and

$$|u_n|/|v_n| \quad \text{tends to a non-zero limit as } n \to \infty$$

then $\sum |u_n|$ also converges (diverges).

6.3.6 D'Alembert's ratio test

This uses a comparison between a given series $\sum u_n$ of complex terms and a geometric series $\sum v_n = \sum r^n$, where $r > 0$.

The *absolute ratio of successive terms* is

$$r_n = \left| \frac{u_{n+1}}{u_n} \right|$$

Suppose that r_n tends to a limit r as $n \to \infty$. Then

- if $r < 1$, $\sum u_n$ converges absolutely
- if $r > 1$, $\sum u_n$ diverges (u_n does not tend to zero)
- if $r = 1$, a different test is required

Even if r_n does not tend to a limit, if $r_n \leqslant r$ for sufficiently large n, where $r < 1$ is a constant, then $\sum u_n$ converges absolutely.

Example: for the harmonic series $\sum n^{-1}$,

$$r_n = \frac{n}{n+1} \to 1 \quad \text{as } n \to \infty$$

A different test is required, such as the integral comparison test used above.

The ratio test is useless for series in which some of the terms are zero. However, it can easily be adapted by relabelling the series to remove the vanishing terms.

6.3.7 Cauchy's root test

The same conclusions as for the ratio test apply when instead

$$r_n = |u_n|^{1/n}$$

This result also follows from a comparison with a geometric series. It is more powerful than the ratio test but usually harder to apply.

6.4 Functions of a continuous variable

6.4.1 Limits and continuity

We now consider how a real or complex function $f(z)$ of a real or complex variable z behaves near a point z_0.

Definition 3. *The function $f(z)$ tends to the limit L as $z \to z_0$ if, for any positive number ϵ, there exists a positive number δ, depending on ϵ, such that $|f(z) - L| < \epsilon$ for all z such that $|z - z_0| < \delta$.*

We write this as

$$\boxed{\lim_{z \to z_0} f(z) = L} \qquad \text{or} \qquad \boxed{f(z) \to L \quad \text{as } z \to z_0}$$

The value of L would normally be $f(z_0)$. However, cases such as

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

must be expressed as limits because $\sin 0/0 = 0/0$ is indeterminate.

Definition 4. *The function $f(z)$ is continuous at the point $z = z_0$ if $f(z) \to f(z_0)$ as $z \to z_0$.*

Definition 5. *The function $f(z)$ is bounded as $z \to z_0$ if there exist positive numbers K and δ such that $|f(z)| < K$ for all z with $|z - z_0| < \delta$.*

Definition 6. *The function $f(z)$ tends to the limit L as $z \to \infty$ if, for any positive number ϵ, there exists a positive number R, depending on ϵ, such that $|f(z) - L| < \epsilon$ for all z with $|z| > R$.*

We write this as

$$\boxed{\lim_{z\to\infty} f(z) = L} \qquad \text{or} \qquad \boxed{f(z) \to L \quad \text{as } z \to \infty}$$

Definition 7. *The function $f(z)$ is bounded as $z \to \infty$ if there exist positive numbers K and R such that $|f(z)| < K$ for all z with $|z| > R$.*

There are different ways in which z can approach z_0 or ∞, especially in the complex plane. Sometimes the limit or bound applies only if approached in a particular way, e.g.

$$\lim_{x\to+\infty} \tanh x = 1, \qquad \lim_{x\to-\infty} \tanh x = -1$$

This notation implies that x is approaching positive or negative real infinity along the real axis. In the context of real variables $x \to \infty$ usually means specifically $x \to +\infty$.

A related notation for one-sided limits is exemplified by

$$\lim_{x\to 0^+} \frac{x(1+x)}{|x|} = 1, \qquad \lim_{x\to 0^-} \frac{x(1+x)}{|x|} = -1$$

6.4.2 The O notation

The useful symbols O, o and $\sim$ are used to compare the rates of growth or decay of different functions.

- $f(z) = O(g(z))$ as $z \to z_0$ means that

$$\frac{f(z)}{g(z)} \quad \text{is bounded as } z \to z_0$$

- $f(z) = o(g(z))$ as $z \to z_0$ means that

$$\frac{f(z)}{g(z)} \to 0 \quad \text{as } z \to z_0$$

- $f(z) \sim g(z)$ as $z \to z_0$ means that

$$\frac{f(z)}{g(z)} \to 1 \quad \text{as } z \to z_0$$

If $f(z) \sim g(z)$ we say that f is *asymptotically equal* to g. This should *not* be written as $f(z) \to g(z)$.

Notes:

- these definitions also apply when $z_0 = \infty$
- $f(z) = O(1)$ means that $f(z)$ is bounded
- either $f(z) = o(g(z))$ or $f(z) \sim g(z)$ implies $f(z) = O(g(z))$
- only $f(z) \sim g(z)$ is a symmetric relation

Examples:

$$A \cos z = O(1) \quad \text{as } z \to 0$$

$$A \sin z = O(z) = o(1) \quad \text{as } z \to 0$$

$$\ln x = o(x) \quad \text{as } x \to +\infty$$

$$\cosh x \sim \frac{1}{2}\mathrm{e}^x \quad \text{as } x \to +\infty$$

6.5 Taylor's theorem for functions of a real variable

Let $f(x)$ be a (real or complex) function of a real variable x, which is differentiable at least n times in the interval $x_0 \leqslant x \leqslant x_0 + h$. Then

$$\boxed{\begin{aligned} f(x_0 + h) =& f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots \\ & \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x_0) + R_n \end{aligned}}$$

where

$$R_n = \int_{x_0}^{x_0+h} \frac{(x_0 + h - x)^{n-1}}{(n-1)!} f^{(n)}(x)\,\mathrm{d}x$$

is the remainder after n terms of the Taylor series.

[Proof: integrate R_n by parts n times.]

The remainder term can be expressed in various ways. Lagrange's expression for the remainder is

$$R_n = \frac{h^n}{n!} f^{(n)}(\xi)$$

where ξ is an unknown number in the interval $x_0 < \xi < x_0 + h$. So

$$R_n = O(h^n)$$

If $f(x)$ is infinitely differentiable in $x_0 \leqslant x \leqslant x_0 + h$ (it is a *smooth* function) we can write an infinite Taylor series

$$\boxed{f(x_0 + h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x_0)}$$

which converges for sufficiently small h (as discussed below).

6.6 Analytic functions of a complex variable

6.6.1 Complex differentiability

Definition 8. *The derivative of the function $f(z)$ at the point $z = z_0$ is*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If this exists, the function $f(z)$ is differentiable at $z = z_0$.

Another way to write this is

$$\frac{df}{dz} \equiv f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

Requiring a function of a *complex* variable to be differentiable is a surprisingly strong constraint. The limit must be the same when $\delta z \to 0$ in any direction in the complex plane.

6.6.2 The Cauchy–Riemann equations

Separate $f = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$ into their real and imaginary parts:

$$f(z) = u(x, y) + \mathrm{i}v(x, y)$$

If $f'(z)$ exists we can calculate it by assuming that $\delta z = \delta x + \mathrm{i}\,\delta y$ approaches 0 along the real axis, so $\delta y = 0$:

$$\begin{aligned}
f'(z) &= \lim_{\delta x \to 0} \frac{f(z + \delta x) - f(z)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{u(x + \delta x, y) + \mathrm{i}v(x + \delta x, y) - u(x, y) - \mathrm{i}v(x, y)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{u(x + \delta x, y) - u(x, y)}{\delta x} + \mathrm{i} \lim_{\delta x \to 0} \frac{v(x + \delta x, y) - v(x, y)}{\delta x} \\
&= \frac{\partial u}{\partial x} + \mathrm{i}\,\frac{\partial v}{\partial x}
\end{aligned}$$

The derivative should have the same value if δz approaches 0 along the imaginary axis, so $\delta x = 0$:

$$
\begin{aligned}
f'(z) &= \lim_{\delta y \to 0} \frac{f(z + \mathrm{i}\,\delta y) - f(z)}{\mathrm{i}\,\delta y} \\
&= \lim_{\delta y \to 0} \frac{u(x, y + \delta y) + \mathrm{i}v(x, y + \delta y) - u(x, y) - \mathrm{i}v(x, y)}{\mathrm{i}\,\delta y} \\
&= -\mathrm{i} \lim_{\delta y \to 0} \frac{u(x, y + \delta y) - u(x, y)}{\delta y} + \lim_{\delta y \to 0} \frac{v(x, y + \delta y) - v(x, y)}{\delta y} \\
&= -\mathrm{i}\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}
\end{aligned}
$$

Comparing the real and imaginary parts of these expressions, we deduce the *Cauchy–Riemann equations*

$$
\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}
$$

These are necessary conditions for $f(z)$ to have a complex derivative. They are also sufficient conditions, provided that the partial derivatives are also continuous.

6.6.3 Analytic functions

If a function $f(z)$ has a complex derivative at every point z in a region R of the complex plane, it is said to be *analytic* in R. To be analytic at a point $z = z_0$, $f(z)$ must be differentiable throughout some neighbourhood $|z - z_0| < \epsilon$ of that point.

Examples of functions that are analytic in the whole complex plane (known as *entire functions*):

- $f(z) = c$, a complex constant
- $f(z) = z$, for which $u = x$ and $v = y$, and we easily verify the CR equations
- $f(z) = z^n$, where n is a positive integer

- $f(z) = P(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0$, a general polynomial function with complex coefficients

- $f(z) = \exp(z)$

In the case of the exponential function we have

$$f(z) = \mathrm{e}^z = \mathrm{e}^x \mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x \cos y + \mathrm{i}\,\mathrm{e}^x \sin y = u + \mathrm{i}v$$

The CR equations are satisfied for all x and y:

$$\frac{\partial u}{\partial x} = \mathrm{e}^x \cos y = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = \mathrm{e}^x \sin y = -\frac{\partial u}{\partial y}$$

The derivative of the exponential function is

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\,\frac{\partial v}{\partial x} = \mathrm{e}^x \cos y + \mathrm{i}\,\mathrm{e}^x \sin y = \mathrm{e}^z$$

as expected.

Sums, products and compositions of analytic functions are also analytic, e.g.

$$f(z) = z \exp(\mathrm{i}z^2) + z^3$$

The usual product, quotient and chain rules apply to complex derivatives of analytic functions. Familiar relations such as

$$\frac{\mathrm{d}}{\mathrm{d}z} z^n = n z^{n-1}, \qquad \frac{\mathrm{d}}{\mathrm{d}z} \sin z = \cos z, \qquad \frac{\mathrm{d}}{\mathrm{d}z} \cosh z = \sinh z$$

apply as usual.

Many complex functions are analytic everywhere in the complex plane except at isolated points, which are called the *singular points* or *singularities* of the function.

Examples:

- $f(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials. This is called a *rational function* and is analytic except at points where $Q(z) = 0$.

- $f(z) = z^c$, where c is a complex constant, is analytic except at $z = 0$ (unless c is a non-negative integer)

- $f(z) = \ln z$ is also analytic except at $z = 0$

The last two examples are in fact *multiple-valued functions*, which require special treatment (see next term).

Examples of non-analytic functions:

- $f(z) = \mathrm{Re}(z)$, for which $u = x$ and $v = 0$, so the CR equations are not satisfied anywhere

- $f(z) = z^*$, for which $u = x$ and $v = -y$

- $f(z) = |z|$, for which $u = (x^2 + y^2)^{1/2}$ and $v = 0$

- $f(z) = |z|^2$, for which $u = x^2 + y^2$ and $v = 0$

In the last case the CR equations are satisfied only at $x = y = 0$ and we can say that $f'(0) = 0$. However, $f(z)$ is not analytic even at $z = 0$ because it is not differentiable throughout any neighbourhood $|z| < \epsilon$ of 0.

6.6.4 Consequences of the Cauchy–Riemann equations

If we know the real part of an analytic function in some region, we can find its imaginary part (or vice versa) up to an additive constant by integrating the CR equations.

Example:

$$u(x, y) = x^2 - y^2$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \quad \Rightarrow \quad v = 2xy + g(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \Rightarrow \quad 2y + g'(x) = 2y \quad \Rightarrow \quad g'(x) = 0$$

Therefore $v(x, y) = 2xy + c$, where c is a real constant, and we recognize

$$f(z) = x^2 - y^2 + \mathrm{i}(2xy + c) = (x + \mathrm{i}y)^2 + \mathrm{i}c = z^2 + \mathrm{i}c$$

The real and imaginary parts of an analytic function satisfy Laplace's equation (they are *harmonic functions*):

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \\
&= 0
\end{aligned}$$

The proof that $\nabla^2 v = 0$ is similar. This provides a useful method for solving Laplace's equation in two dimensions. Furthermore,

$$\begin{aligned}
\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \\
&= \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \\
&= 0
\end{aligned}$$

and so the curves of constant u and those of constant v intersect at right angles. u and v are said to be *conjugate harmonic functions*.

6.7 Taylor series for analytic functions

If a function of a complex variable is analytic in a region R of the complex plane, not only is it differentiable everywhere in R, it is also differentiable any number of times. If $f(z)$ is analytic at $z = z_0$, it has an infinite Taylor series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n,\qquad a_n=\frac{f^{(n)}(z_0)}{n!}$$

which converges within some neighbourhood of z_0 (as discussed below). In fact this can be taken as a definition of analyticity.

6.8 Zeros, poles and essential singularities

6.8.1 Zeros of complex functions

The zeros of $f(z)$ are the points $z=z_0$ in the complex plane where $f(z_0)=0$. A zero is of order N if

$$f(z_0)=f'(z_0)=f''(z_0)=\cdots=f^{(N-1)}(z_0)=0\qquad\text{but}\quad f^{(N)}(z_0)\neq 0$$

The first non-zero term in the Taylor series of $f(z)$ about $z=z_0$ is then proportional to $(z-z_0)^N$. Indeed

$$f(z)\sim a_N(z-z_0)^N\qquad\text{as}\quad z\to z_0$$

A *simple zero* is a zero of order 1. A *double zero* is one of order 2, etc.

Examples:

- $f(z)=z$ has a simple zero at $z=0$
- $f(z)=(z-\mathrm{i})^2$ has a double zero at $z=\mathrm{i}$
- $f(z)=z^2-1=(z-1)(z+1)$ has simple zeros at $z=\pm 1$

Example . . . .

▷ *Find and classify the zeros of* $f(z) = \sinh z$.

$$0 = \sinh z = \frac{1}{2}(\mathrm{e}^z - \mathrm{e}^{-z})$$

$$\mathrm{e}^z = \mathrm{e}^{-z}$$

$$\mathrm{e}^{2z} = 1$$

$$2z = 2n\pi\mathrm{i}$$

$$z = n\pi\mathrm{i}, \qquad n \in \mathbf{Z}$$

$$f'(z) = \cosh z = \cos(n\pi) \neq 0 \quad \text{at these points}$$

$$\Rightarrow \quad \text{all simple zeros}$$

. . . .

6.8.2 Poles

If $g(z)$ is analytic and non-zero at $z = z_0$ then it has an expansion (locally, at least) of the form

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n \qquad \text{with} \quad b_0 \neq 0$$

Consider the function

$$f(z) = (z - z_0)^{-N} g(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n$$

with $a_n = b_{n+N}$ and so $a_{-N} \neq 0$. This is not a Taylor series because it includes negative powers of $z - z_0$, and $f(z)$ is not analytic at $z = z_0$. We say that $f(z)$ has a pole of order N. Note that

$$f(z) \sim a_{-N}(z - z_0)^{-N} \qquad \text{as} \quad z \to z_0$$

A *simple pole* is a pole of order 1. A *double pole* is one of order 2, etc.

Notes:

- if $f(z)$ has a zero of order N at $z = z_0$, then $1/f(z)$ has a pole of order N there, and vice versa

- if $f(z)$ is analytic and non-zero at $z = z_0$ and $g(z)$ has a zero of order N there, then $f(z)/g(z)$ has a pole of order N there

Example:

$$f(z) = \frac{2z}{(z+1)(z-\mathrm{i})^2}$$

has a simple pole at $z = -1$ and a double pole at $z = \mathrm{i}$ (as well as a simple zero at $z = 0$). The expansion about the double pole can be carried out by letting $z = \mathrm{i} + w$ and expanding in w:

$$\begin{aligned}
f(z) &= \frac{2(\mathrm{i}+w)}{(\mathrm{i}+w+1)w^2} \\
&= \frac{2\mathrm{i}(1-\mathrm{i}w)}{(\mathrm{i}+1)\left[1+\frac{1}{2}(1-\mathrm{i})w\right]w^2} \\
&= \frac{2\mathrm{i}}{(\mathrm{i}+1)w^2}(1-\mathrm{i}w)\left[1-\frac{1}{2}(1-\mathrm{i})w+O(w^2)\right] \\
&= (1+\mathrm{i})w^{-2}\left[1-\frac{1}{2}(1+\mathrm{i})w+O(w^2)\right] \\
&= (1+\mathrm{i})(z-\mathrm{i})^{-2} - \mathrm{i}(z-\mathrm{i})^{-1} + O(1) \qquad \text{as} \quad z \to \mathrm{i}
\end{aligned}$$

6.8.3 Essential singularities

It can be shown that any function that is analytic (and single-valued) throughout an annulus $a < |z - z_0| < b$ centred on a point $z = z_0$ has a unique *Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

which converges for all values of z within the annulus.

If $a = 0$, then $f(z)$ is analytic throughout the disk $|z - z_0| < b$ except possibly at $z = z_0$ itself, and the Laurent series determines the behaviour of $f(z)$ near $z = z_0$. There are three possibilities:

- if the first non-zero term in the Laurent series has $n \geqslant 0$, then $f(z)$ is analytic at $z = z_0$ and the series is just a Taylor series
- if the first non-zero term in the Laurent series has $n = -N < 0$, then $f(z)$ has a pole of order N at $z = z_0$
- otherwise, if the Laurent series involves an infinite number of terms with $n < 0$, then $f(z)$ has an *essential singularity* at $z = z_0$

A classic example of an essential singularity is $f(z) = \mathrm{e}^{1/z}$ at $z = 0$. Here we can generate the Laurent series from a Taylor series in $1/z$:

$$\mathrm{e}^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$$

The behaviour of a function near an essential singularity is remarkably complicated. *Picard's theorem* states that, in any neighbourhood of an essential singularity, the function takes all possible complex values (possibly with one exception) at infinitely many points. (In the case of $f(z) = \mathrm{e}^{1/z}$, the exceptional value 0 is never attained.)

6.8.4 Behaviour at infinity

We can examine the behaviour of a function $f(z)$ as $z \to \infty$ by defining a new variable $\zeta = 1/z$ and a new function $g(\zeta) = f(z)$. Then $z = \infty$ maps to a single point $\zeta = 0$, the *point at infinity*.

If $g(\zeta)$ has a zero, pole or essential singularity at $\zeta = 0$, then we can say that $f(z)$ has the corresponding property at $z = \infty$.

Examples:

$$f_1(z) = \mathrm{e}^z = \mathrm{e}^{1/\zeta} = g_1(\zeta)$$

has an essential singularity at $z = \infty$.

$$f_2(z) = z^2 = 1/\zeta^2 = g_2(\zeta)$$

has a double pole at $z = \infty$.

$$f_3(z) = e^{1/z} = e^{\zeta} = g_3(\zeta)$$

is analytic at $z = \infty$.

6.9 Convergence of power series

6.9.1 Circle of convergence

If the power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

converges for $z = z_1$, then it converges absolutely for all z such that $|z - z_0| < |z_1 - z_0|$.

[Proof: The necessary condition for convergence,

$$\lim a_n(z_1 - z_0)^n = 0$$

implies that

$$|a_n(z_1 - z_0)^n| < \epsilon$$

for sufficiently large n, for any $\epsilon > 0$. Therefore

$$|a_n(z - z_0)^n| < \epsilon\, r^n$$

for sufficiently large n, with

$$r = |(z - z_0)/(z_1 - z_0)| < 1$$

By comparison with the geometric series $\sum r^n$, $\sum |a_n(z - z_0)^n|$ converges.]

It follows that, if the power series diverges for $z = z_2$, then it diverges for all z such that $|z - z_0| > |z_2 - z_0|$.

Therefore there must exist a real, non-negative number R such that the series converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$. R is called the *radius of convergence* and may be zero (exceptionally), positive or infinite.

$|z-z_0| = R$ is the *circle of convergence*. The series converges inside it and diverges outside. On the circle, it may either converge or diverge.

6.9.2 Determination of the radius of convergence

The absolute ratio of successive terms in a power series is

$$r_n = \left|\frac{a_{n+1}}{a_n}\right| |z-z_0|$$

Suppose that $|a_{n+1}/a_n| \to L$ as $n \to \infty$. Then $r_n \to r = L|z-z_0|$. According to the ratio test, the series converges for $L|z-z_0| < 1$ and diverges for $L|z-z_0| > 1$. The radius of convergence is $R = 1/L$.

The same result, $R = 1/L$, follows from Cauchy's root test if instead $|a_n|^{1/n} \to L$ as $n \to \infty$.

The radius of convergence of the Taylor series of a function $f(z)$ about the point $z = z_0$ is equal to the distance of the nearest singular point of the function $f(z)$ from z_0. Since a convergent power series defines an analytic function, no singularity can lie inside the circle of convergence.

6.9.3 Examples

The following examples are generated from familiar Taylor series.

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{z^n}{n}$$

Here $|a_{n+1}/a_n| = n/(n+1) \to 1$ as $n \to \infty$, so $R = 1$. The series converges for $|z| < 1$ and diverges for $|z| > 1$. (In fact, on the circle

$|z| = 1$, the series converges except at the point $z = 1$.) The function has a singularity at $z = 1$ that limits the radius of convergence.

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots = z \sum_{n=0}^{\infty} \frac{1}{2n+1} (-z^2)^n$$

Thought of as a power series in $(-z^2)$, this has $|a_{n+1}/a_n| = (2n + 1)/(2n + 3) \to 1$ as $n \to \infty$. Therefore $R = 1$ in terms of $(-z^2)$. But since $|-z^2| = 1$ is equivalent to $|z| = 1$, the series converges for $|z| < 1$ and diverges for $|z| > 1$.

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Here $|a_{n+1}/a_n| = 1/(n + 1) \to 0$ as $n \to \infty$, so $R = \infty$. The series converges for all z; this is an entire function.