Natural Sciences Tripos Part IB
Mathematical Methods I
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Dr M Wingate

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4 The Fourier transform

4.1 Motivation

A periodic signal can be analysed into its harmonic components by calculating its Fourier series. If the period is P, then the harmonics have frequencies n/P where n is an integer.

The Fourier transform generalizes this idea to functions that are not periodic. The 'harmonics' can then have any frequency.

The Fourier transform provides a complementary way of looking at a function. Certain operations on a function are more easily computed 'in the Fourier domain'. This idea is particularly useful in solving certain kinds of differential equation.

Furthermore, the Fourier transform has innumerable applications in diverse fields such as astronomy, optics, signal processing, data analysis, statistics and number theory.

4.2 Relation to Fourier series

A function $f(x)$ has period P if $f(x+P) = f(x)$ for all x. It can then be written as a *Fourier series*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(k_n x) + \sum_{n=1}^{\infty} b_n \sin(k_n x)$$

where

$$k_n = \frac{2\pi n}{P}$$

is the wavenumber of the nth harmonic.

Such a series is also used to write any function that is defined only on an interval of length P, e.g. $-P/2 < x < P/2$. The Fourier series gives the extension of the function by periodic repetition.

The Fourier coefficients are found from

$$a_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos(k_n x)\,\mathrm{d}x$$

$$b_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\sin(k_n x)\,\mathrm{d}x$$

Define

$$c_n = \begin{cases} (a_{-n} + \mathrm{i}b_{-n})/2, & n < 0 \\ a_0/2, & n = 0 \\ (a_n - \mathrm{i}b_n)/2, & n > 0 \end{cases}$$

Then the same result can be expressed more simply and compactly in the notation of the *complex Fourier series*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{\mathrm{i}k_n x}$$

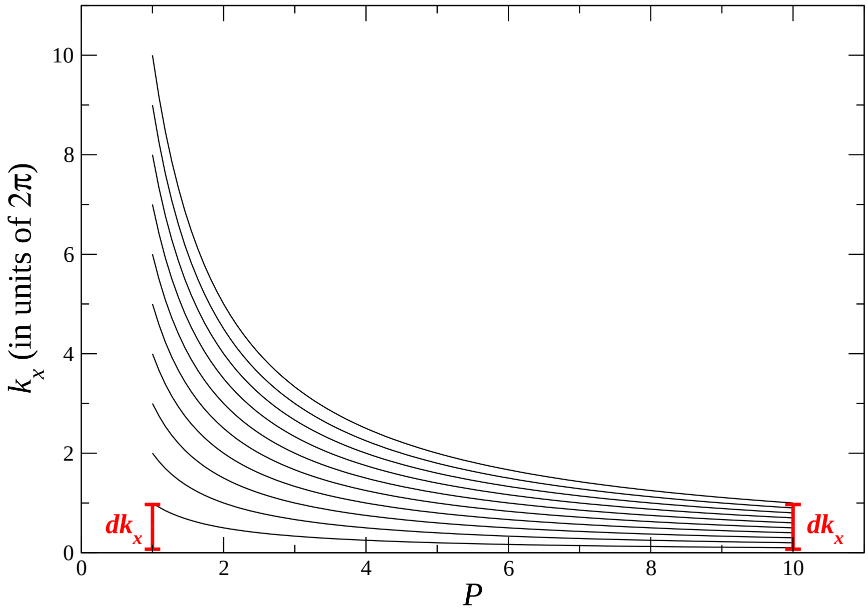
where, after multiplying the preceding equation by $\exp(-\mathrm{i}k_n x)$ and integrating we find (recall $k_n = 2\pi n/P$)

$$c_n = \frac{1}{P}\int_{-P/2}^{P/2} f(x)\,\mathrm{e}^{-\mathrm{i}k_n x}\,\mathrm{d}x$$

This expression for c_n can be verified using the *orthogonality relation*

$$\frac{1}{P}\int_{-P/2}^{P/2} \mathrm{e}^{\mathrm{i}(k_n-k_m)x}\,\mathrm{d}x = \delta_{mn}$$

which follows from an elementary integration.



To approach the Fourier transform, let $P \to \infty$ so that the function is defined on the entire real line without any periodicity. The wavenumbers $k_n = 2\pi n/P$ are replaced by a continuous variable k. We have to be careful as we switch from a sum over allowed values of k_n to an integral over k. Let $\mathrm{d}n$ be the number of allowed values in $\mathrm{d}k$

$$\mathrm{d}n = \frac{P}{2\pi}\,\mathrm{d}k$$

Then

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} c(k)\,e^{ikx}\mathrm{d}n \\ &= \int_{-\infty}^{\infty} c(k)\,e^{ikx}\,\frac{P}{2\pi}\,\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\,e^{ikx}\,\mathrm{d}k \end{aligned}$$

where we have defined We define

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$$

corresponding to Pc_n.

We therefore have the *forward Fourier transform* (Fourier analysis)

$$\boxed{\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x}$$

and the *inverse Fourier transform* (Fourier synthesis)

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k}$$

Notes:

- the Fourier transform operation is sometimes denoted by

$$\tilde{f}(k) = \mathcal{F}[f(x)], \qquad f(x) = \mathcal{F}^{-1}[\tilde{f}(k)]$$

- the variables are often called t and ω rather than x and k (time $\leftrightarrow$ angular frequency vs. position $\leftrightarrow$ wavenumber)

- it is sometimes useful to consider complex values of k

- for a rigorous proof, certain technical conditions on $f(x)$ are required:

A necessary condition for $\tilde{f}(k)$ to exist for all real values of k (in the sense of an ordinary function) is that $f(x) \to 0$ as $x \to \pm\infty$. Otherwise the Fourier integral does not converge (e.g. for $k = 0$).

A set of sufficient conditions for $\tilde{f}(k)$ to exist is that $f(x)$ have 'bounded variation', have a finite number of discontinuities and be 'absolutely integrable', i.e.

$$\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x < \infty.$$

However, we will see that Fourier transforms can be assigned in a wider sense to some functions that do not satisfy all of these conditions, e.g. $f(x) = 1$.

Warning 2. *Several different definitions of the Fourier transform are in use. They differ in the placement of the 2π factor and in the signs of the exponents. The definition used here is probably the most conventional.*

How to remember this convention:

- the sign of the exponent is different in the forward and inverse transforms

- the inverse transform means that the function $f(x)$ is synthesized from a linear combination of basis functions $\mathrm{e}^{\mathrm{i}kx}$
- the division by 2π always accompanies integration with respect to k

4.3 Examples

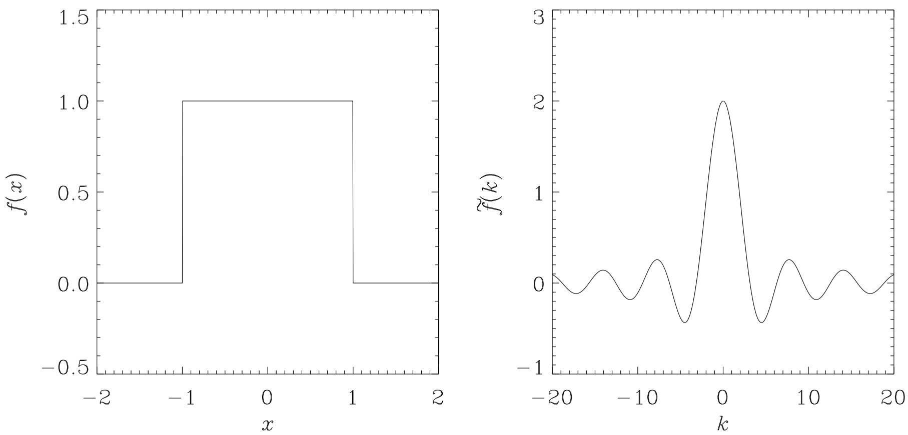
Example (1): top-hat function:

$$f(x) = \begin{cases} c, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{f}(k) = \int_a^b c\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{\mathrm{i}c}{k}\left(\mathrm{e}^{-\mathrm{i}kb} - \mathrm{e}^{-\mathrm{i}ka}\right)$$

e.g. if $a = -1$, $b = 1$ and $c = 1$:

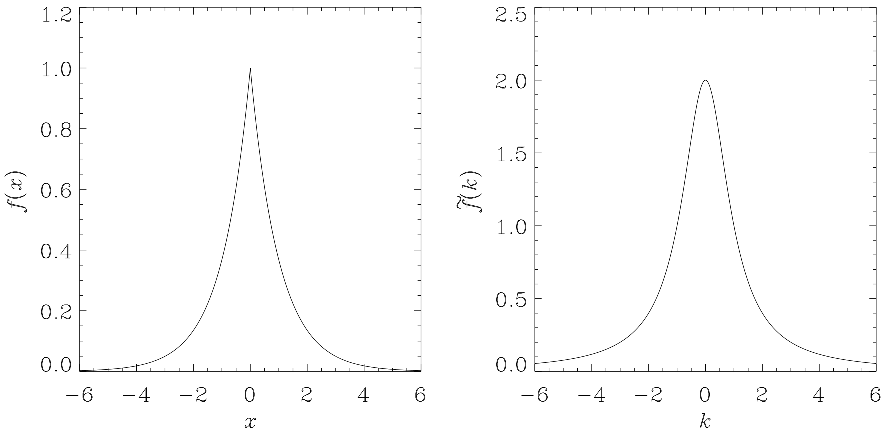
$$\tilde{f}(k) = \frac{\mathrm{i}}{k}\left(\mathrm{e}^{-\mathrm{i}k} - \mathrm{e}^{\mathrm{i}k}\right) = \frac{2\sin k}{k}$$



Example (2):

$$f(x) = \mathrm{e}^{-|x|}$$

$$\begin{aligned}
\tilde{f}(k) &= \int_{-\infty}^{0} \mathrm{e}^{x}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-x}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \frac{1}{1-\mathrm{i}k}\left[\mathrm{e}^{(1-\mathrm{i}k)x}\right]_{-\infty}^{0} - \frac{1}{1+\mathrm{i}k}\left[\mathrm{e}^{-(1+\mathrm{i}k)x}\right]_{0}^{\infty} \\
&= \frac{1}{1-\mathrm{i}k} + \frac{1}{1+\mathrm{i}k} \\
&= \frac{2}{1+k^2}
\end{aligned}$$



Example (3): Gaussian function (normal distribution):

$$f(x) = (2\pi\sigma_x^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma_x^2}\right)$$

$$\tilde{f}(k) = (2\pi\sigma_x^2)^{-1/2} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma_x^2} - \mathrm{i}kx\right) \, \mathrm{d}x$$

Change variable to

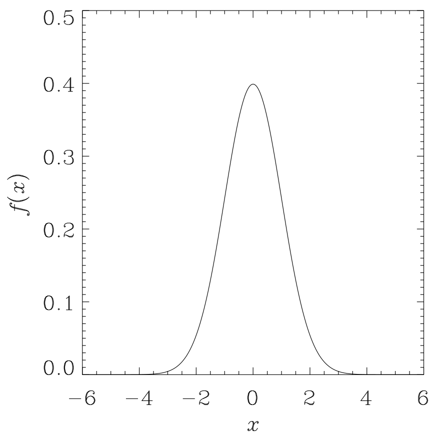
$$z = \frac{x}{\sigma_x} + \mathrm{i}\sigma_x k$$

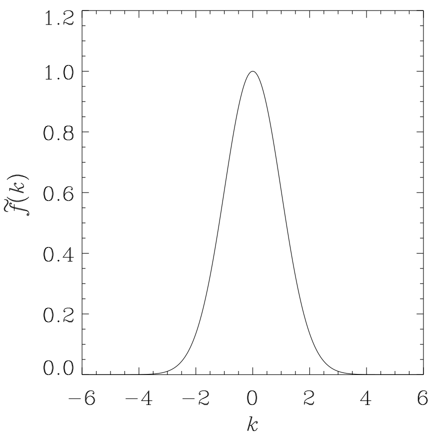
so that

$$-\frac{z^2}{2} = -\frac{x^2}{2\sigma_x^2} - \mathrm{i}kx + \frac{\sigma_x^2 k^2}{2}$$

Then

$$\begin{aligned}\tilde{f}(k) &= (2\pi\sigma_x^2)^{-1/2} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right) \, \mathrm{d}z \, \sigma_x \exp\left(-\frac{\sigma_x^2 k^2}{2}\right) \\ &= \exp\left(-\frac{\sigma_x^2 k^2}{2}\right)\end{aligned}$$





where we use the standard Gaussian integral

$$\int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z = (2\pi)^{1/2}$$

Actually there is a slight cheat here because z has an imaginary part. This will be explained next term.

The result is proportional to a standard Gaussian function of k:

$$\tilde{f}(k) \propto (2\pi\sigma_k^2)^{-1/2} \exp\left(-\frac{k^2}{2\sigma_k^2}\right)$$

of width (standard deviation) σ_k related to σ_x by

$$\sigma_k = \frac{1}{\sigma_x}$$

This illustrates a property of the Fourier transform: the narrower the function of x, the wider the function of k.

4.4 Basic properties of the Fourier transform

Linearity:

$$g(x) = \alpha f(x) \quad \Leftrightarrow \quad \tilde{g}(k) = \alpha \tilde{f}(k) \tag{1}$$

$$h(x) = f(x) + g(x) \quad \Leftrightarrow \quad \tilde{h}(k) = \tilde{f}(k) + \tilde{g}(k) \tag{2}$$

Rescaling (for real α):

$$g(x) = f(\alpha x) \quad \Leftrightarrow \quad \tilde{g}(k) = \frac{1}{|\alpha|}\tilde{f}\left(\frac{k}{\alpha}\right) \tag{3}$$

Shift/exponential (for real α):

$$g(x) = f(x - \alpha) \quad \Leftrightarrow \quad \tilde{g}(k) = \mathrm{e}^{-\mathrm{i}k\alpha}\tilde{f}(k) \tag{4}$$

$$g(x) = \mathrm{e}^{\mathrm{i}\alpha x} f(x) \quad \Leftrightarrow \quad \tilde{g}(k) = \tilde{f}(k - \alpha) \tag{5}$$

Differentiation/multiplication:

$$g(x) = f'(x) \quad \Leftrightarrow \quad \tilde{g}(k) = \mathrm{i}k\tilde{f}(k) \tag{6}$$

$$g(x) = xf(x) \quad \Leftrightarrow \quad \tilde{g}(k) = \mathrm{i}\tilde{f}'(k) \tag{7}$$

Duality:

$$g(x) = \tilde{f}(x) \quad \Leftrightarrow \quad \tilde{g}(k) = 2\pi f(-k) \tag{8}$$

i.e. transforming twice returns (almost) the same function

Complex conjugation and parity inversion (for real x and k):

$$g(x) = [f(x)]^* \quad \Leftrightarrow \quad \tilde{g}(k) = [\tilde{f}(-k)]^* \tag{9}$$

Symmetry:

$$f(-x) = \pm f(x) \quad \Leftrightarrow \quad \tilde{f}(-k) = \pm \tilde{f}(k) \tag{10}$$

Sample derivations: property (3):

$$
\begin{aligned}
g(x) &= f(\alpha x) \\
\tilde{g}(k) &= \int_{-\infty}^{\infty} f(\alpha x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \mathsf{sgn}\ (\alpha) \int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-\mathrm{i}ky/\alpha}\,\frac{\mathrm{d}y}{\alpha} \\
&= \frac{1}{|\alpha|} \int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-\mathrm{i}(k/\alpha)y}\,\mathrm{d}y \\
&= \frac{1}{|\alpha|}\tilde{f}\left(\frac{k}{\alpha}\right)
\end{aligned}
$$

Property (4):

$$
\begin{aligned}
g(x) &= f(x-\alpha) \\
\tilde{g}(k) &= \int_{-\infty}^{\infty} f(x-\alpha)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-\mathrm{i}k(y+\alpha)}\,\mathrm{d}y \\
&= \mathrm{e}^{-\mathrm{i}k\alpha}\tilde{f}(k)
\end{aligned}
$$

Property (6):

$$
\begin{aligned}
g(x) &= f'(x) \\
\tilde{g}(k) &= \int_{-\infty}^{\infty} f'(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \left[f(x)\,\mathrm{e}^{-\mathrm{i}kx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-\mathrm{i}k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \mathrm{i}k\tilde{f}(k)
\end{aligned}
$$

The integrated part vanishes because $f(x)$ must tend to zero as $x \to \pm\infty$ in order to possess a Fourier transform.

Property (7):

$$\begin{aligned}
g(x) &= x f(x) \\
\tilde{g}(k) &= \int_{-\infty}^{\infty} x f(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x \\
&= \mathrm{i} \int_{-\infty}^{\infty} f(x)(-\mathrm{i}x)\mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x \\
&= \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}k} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x \\
&= \mathrm{i}\tilde{f}'(k)
\end{aligned}$$

Property (8):

$$\begin{aligned}
g(x) &= \tilde{f}(x) \\
\tilde{g}(k) &= \int_{-\infty}^{\infty} \tilde{f}(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x \\
&= 2\pi f(-k)
\end{aligned}$$

Property (10): if $f(-x) = \pm f(x)$, i.e. f is even or odd, then

$$\begin{aligned}
\tilde{f}(-k) &= \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{+\mathrm{i}kx}\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \pm f(-x)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}x \\
&= \pm \int_{-\infty}^{\infty} f(y)\, \mathrm{e}^{-\mathrm{i}ky}\, \mathrm{d}y \\
&= \pm\tilde{f}(k)
\end{aligned}$$

4.5 The delta function and the Fourier transform

Consider the Gaussian function of example (3):

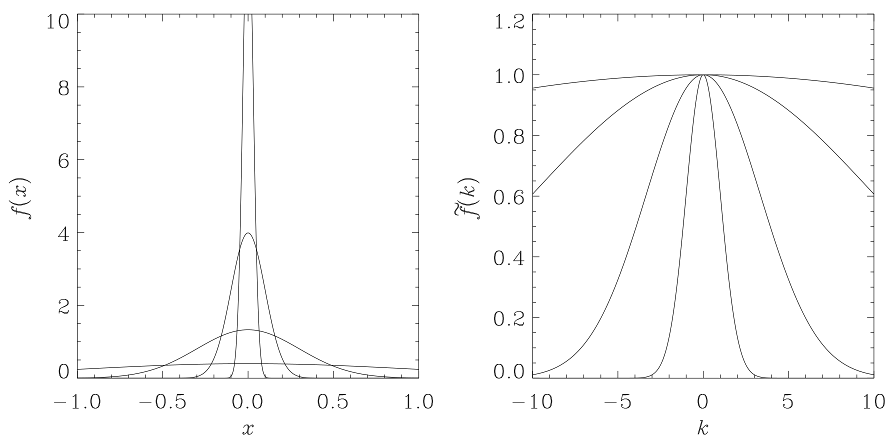
$$f(x) = (2\pi\sigma_x^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma_x^2}\right)$$

$$\tilde{f}(k) = \exp\left(-\frac{\sigma_x^2 k^2}{2}\right)$$

The Gaussian is normalized such that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

As the width σ_x tends to zero, the Gaussian becomes taller and narrower, but the area under the curve remains the same. The value of $f(x)$ tends to zero for any non-zero value of x. At the same time, the value of $\tilde{f}(k)$ tends to unity for any finite value of k.



In this limit we approach the Dirac delta function, $\delta(x)$.

The substitution property allows us to verify the Fourier transform of the delta function:

$$\tilde{\delta}(k) = \int_{-\infty}^{\infty} \delta(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \mathrm{e}^{-\mathrm{i}k0} = 1$$

Now formally apply the inverse transform:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k$$

Relabel the variables and rearrange (the exponent can have either sign):

$$\boxed{\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{i}kx}\,\mathrm{d}x = 2\pi\delta(k)}$$

Therefore the Fourier transform of a unit constant (1) is $2\pi\delta(k)$. Note that a constant function does not satisfy the necessary condition for the existence of a (regular) Fourier transform. But it does have a Fourier transform in the space of generalized functions.

4.6 The convolution theorem

4.6.1 Definition of convolution

The *convolution* of two functions, $h = f * g$, is defined by

$$\boxed{h(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,\mathrm{d}y}$$

Note that the sum of the arguments of f and g is the argument of h. Convolution is a symmetric operation:

$$\begin{aligned}
[g * f](x) &= \int_{-\infty}^{\infty} g(y)f(x-y)\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} f(z)g(x-z)\,\mathrm{d}z \\
&= [f * g](x)
\end{aligned}$$

4.6.2 Interpretation and examples

In statistics, a continuous random variable x (e.g. the height of a person drawn at random from the population) has a *probability distribution* (or *density*) *function* $f(x)$. The probability of x lying in the range $x_0 < x < x_0 + \delta x$ is $f(x_0)\delta x$, in the limit of small δx.

If x and y are independent random variables with distribution functions $f(x)$ and $g(y)$, then let the distribution function of their sum, $z = x + y$, be $h(z)$.

Now, for any given value of x, the probability that z lies in the range

$$z_0 < z < z_0 + \delta z$$

is just the probability that y lies in the range

$$z_0 - x < y < z_0 - x + \delta z$$

which is $g(z_0 - x)\delta z$. Therefore

$$h(z_0)\delta z = \int_{-\infty}^{\infty} f(x)g(z_0 - x)\delta z\,\mathrm{d}x$$

which implies

$$h = f * g$$

The effect of measuring, observing or processing scientific data can often be described as a convolution of the data with a certain function.

e.g. when a point source is observed by a telescope, a broadened image is seen, known as the *point spread function* of the telescope. When an extended source is observed, the image that is seen is the convolution of the source with the point spread function.

In this sense convolution corresponds to a broadening or distortion of the original data.

A point mass M at position $\boldsymbol{R}$ gives rise to a gravitational potential $\Phi(\boldsymbol{r}) = -GM/|\boldsymbol{r} - \boldsymbol{R}|$. A continuous mass density $\rho(\boldsymbol{r})$ can be thought of as a sum of infinitely many point masses $\rho(\boldsymbol{R})\,\mathrm{d}^3\boldsymbol{R}$ at positions $\boldsymbol{R}$. The resulting gravitational potential is

$$\Phi(\boldsymbol{r}) = -G\int \frac{\rho(\boldsymbol{R})}{|\boldsymbol{r} - \boldsymbol{R}|}\,\mathrm{d}^3\boldsymbol{R}$$

which is the (3D) convolution of the mass density $\rho(\boldsymbol{r})$ with the potential of a unit point charge at the origin, $-G/|\boldsymbol{r}|$.

4.6.3 The convolution theorem

The Fourier transform of a convolution is

$$\begin{aligned}
\tilde{h}(k) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(y) g(x-y)\,\mathrm{d}y \right] \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(x-y)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(z)\,\mathrm{e}^{-\mathrm{i}ky}\,\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z\,\mathrm{d}y \qquad (z = x - y) \\
&= \int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-\mathrm{i}ky}\,\mathrm{d}y \int_{-\infty}^{\infty} g(z)\,\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z \\
&= \tilde{f}(k)\tilde{g}(k)
\end{aligned}$$

Similarly, the Fourier transform of $f(x)g(x)$ is $\frac{1}{2\pi}[\tilde{f} * \tilde{g}](k)$.

This means that:

- convolution is an operation best carried out as a multiplication in the Fourier domain
- the Fourier transform of a product is a complicated object
- convolution can be undone (*deconvolution*) by a division in the Fourier domain. If g is known and $f * g$ is measured, then f can be obtained, in principle.

4.6.4 Correlation

The *correlation* of two functions, $h = f \otimes g$, is defined by

$$\boxed{h(x) = \int_{-\infty}^{\infty} [f(y)]^* g(x+y)\,\mathrm{d}y}$$

Now the argument of h is the shift between the arguments of f and g.

Correlation is a way of quantifying the relationship between two (typically oscillatory) functions. If two signals (oscillating about an average value of zero) oscillate in phase with each other, their correlation will be positive. If they are out of phase, the correlation will be negative. If they are completely unrelated, their correlation will be zero.

The Fourier transform of a correlation is

$$\begin{aligned}
\tilde{h}(k) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}[f(y)]^* g(x+y)\,\mathrm{d}y\right]\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[f(y)]^* g(x+y)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[f(y)]^* g(z)\,\mathrm{e}^{\mathrm{i}ky}\,\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z\,\mathrm{d}y \qquad (z = x+y) \\
&= \left[\int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-\mathrm{i}ky}\,\mathrm{d}y\right]^* \int_{-\infty}^{\infty} g(z)\,\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z \\
&= [\tilde{f}(k)]^*\tilde{g}(k)
\end{aligned}$$

This result (or the special case $g = f$) is the *Wiener–Khinchin theorem*.

The *autoconvolution* and *autocorrelation* of f are $f * f$ and $f \otimes f$. Their Fourier transforms are $\tilde{f}^2$ and $|\tilde{f}|^2$, respectively.

4.7 Parseval's theorem

If we apply the inverse transform to the WK theorem we find

$$\int_{-\infty}^{\infty}[f(y)]^* g(x+y)\,\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\tilde{f}(k)]^*\tilde{g}(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k$$

Now set $x = 0$ and relabel $y \mapsto x$ to obtain *Parseval's theorem*

$$\int_{-\infty}^{\infty} [f(x)]^* g(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\tilde{f}(k)]^* \tilde{g}(k)\,\mathrm{d}k$$

The special case used most frequently is when $g = f$:

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2\,\mathrm{d}k}$$

Note that division by 2π accompanies the integration with respect to k.

Parseval's theorem means that the Fourier transform is a 'unitary transformation' that preserves the 'inner product' between two functions (see later!), in the same way that a rotation preserves lengths and angles.

Alternative derivation using the delta function:

$$\begin{aligned}
&\int_{-\infty}^{\infty} [f(x)]^* g(x)\,\mathrm{d}x \\
&\quad = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k\right]^* \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(k')\,\mathrm{e}^{\mathrm{i}k'x}\,\mathrm{d}k'\,\mathrm{d}x \\
&\quad = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\tilde{f}(k)]^* \tilde{g}(k')\,\mathrm{e}^{\mathrm{i}(k'-k)x}\,\mathrm{d}x\,\mathrm{d}k'\,\mathrm{d}k \\
&\quad = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\tilde{f}(k)]^* \tilde{g}(k')\,2\pi\delta(k'-k)\,\mathrm{d}k'\,\mathrm{d}k \\
&\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\tilde{f}(k)]^* \tilde{g}(k)\,\mathrm{d}k
\end{aligned}$$

4.8 Power spectra

The quantity

$$\Phi(k) = |\tilde{f}(k)|^2$$

appearing in the Wiener–Khinchin theorem and Parseval's theorem is the *(power) spectrum* or *(power) spectral density* of the function $f(x)$. The WK theorem states that the FT of the autocorrelation function is the power spectrum.

This concept is often used to quantify the spectral content (as a function of angular frequency ω) of a signal $f(t)$.

The spectrum of a perfectly periodic signal consists of a series of delta functions at the principal frequency and its harmonics, if present. Its autocorrelation function does not decay as $t \to \infty$.

White noise is an ideal random signal with autocorrelation function proportional to $\delta(t)$: the signal is perfectly decorrelated. It therefore has a flat spectrum ($\Phi =$ constant).

Less idealized signals may have spectra that are peaked at certain frequencies but also contain a general noise component.