Introduction to infinite series

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Partial sums

$$s_0 = a_0 = \sum_{k=0}^{0} a_k\,,$$

$$s_1 = a_0 + a_1 = \sum_{k=0}^{1} a_k\,,$$

$$s_2 = a_0 + a_1 + a_2 = \sum_{k=0}^{2} a_k\,,$$

$$s_3 = a_0 + a_1 + a_2 + a_3 = \sum_{k=0}^{3} a_k\,,$$

$$s_n = a_0 + a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=0}^{n} a_k$$

and so on.

Infinite series, convergence and divergence

If, as $n \to \infty$, the sequence of partial sums

$$s_n = \sum_{k=0}^{n} a_k$$

tends to a finite limit L, we write

$$\sum_{k=0}^{\infty} a_k = L$$

and say that

$$\text{the series} \qquad \sum_{k=0}^{\infty} a_k \qquad \textit{converges to } L.$$

We call L the *sum* of the series. If the sequence of partial sums diverges, we say that

$$\text{the series} \qquad \sum_{k=0}^{\infty} a_k \qquad \textit{diverges.}$$

Bounded and unbounded divergence

$$\sum_{k=0}^{\infty}(-1)^k \quad \text{and} \quad \sum_{k=0}^{\infty}2^k$$

Example: The Geometric Series

- The *geometric progression* $1, x, x^2, x^3, \ldots$ gives rise to the numbers $1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3, \ldots$.
- These numbers are the partial sums of what is called the *geometric series*:

$$\sum_{k=0}^{\infty} x^k$$

For the Geometric Series $\sum_{k=0}^{\infty} x^k$

(i) If $|x| < 1$, then $\sum_{k=0}^{\infty} x^k = \dfrac{1}{1-x}$.

(ii) If $|x| \geq 1$, then $\sum_{k=0}^{\infty} x^k$ diverges.

Properties of infinite series

1. If $\sum_{k=0}^{\infty} a_k$ converges and $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty}(a_k + b_k)$ converges.

   Moreover, if $\sum_{k=0}^{\infty} a_k = L$ and $\sum_{k=0}^{\infty} b_k = M$, then $\sum_{k=0}^{\infty}(a_k + b_k) = L + M$.

2. If $\sum_{k=0}^{\infty} a_k$ converges, then $\sum_{k=0}^{\infty} \alpha\, a_k$ converges for each real number α.

   Moreover, if $\sum_{k=0}^{\infty} a_k = L$, then $\sum_{k=0}^{\infty} \alpha\, a_k = \alpha\, L$.

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   Moreover, if $\sum_{k=0}^{\infty} a_k = L$, then $\sum_{k=0}^{\infty} \alpha\, a_k = \alpha\, L$.

$$\sum_{k=0}^{\infty} x^k \longrightarrow \sum_{k=0}^{\infty} a x^k$$

Properties of infinite series

The *kth term* of a convergent series tends to 0; namely,

$$\text{if} \quad \sum_{k=0}^{\infty} a_k \quad \text{converges,} \quad \text{then} \quad a_k \to 0 \quad \text{as} \quad k \to \infty.$$

$$\text{if} \quad a_k \not\to 0 \quad \text{then} \quad \sum_{k=0}^{\infty} a_k \quad \text{diverges.}$$

Example $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin(1/n)\cos(2/n)}{1/n}$

Positive (nonnegative) series

- Series with positive (nonnegative) terms

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- Basic Theorem:

A series with nonnegative terms converges iff the sequence of partial sums is bounded.

(I) Integral Test

If f is continuous, positive, and decreasing on $[1, \infty)$, then

$$\sum_{k=1}^{\infty} f(k) \quad \text{converges} \quad \text{iff} \quad \int_1^{\infty} f(x)\,dx \quad \text{converges.}$$

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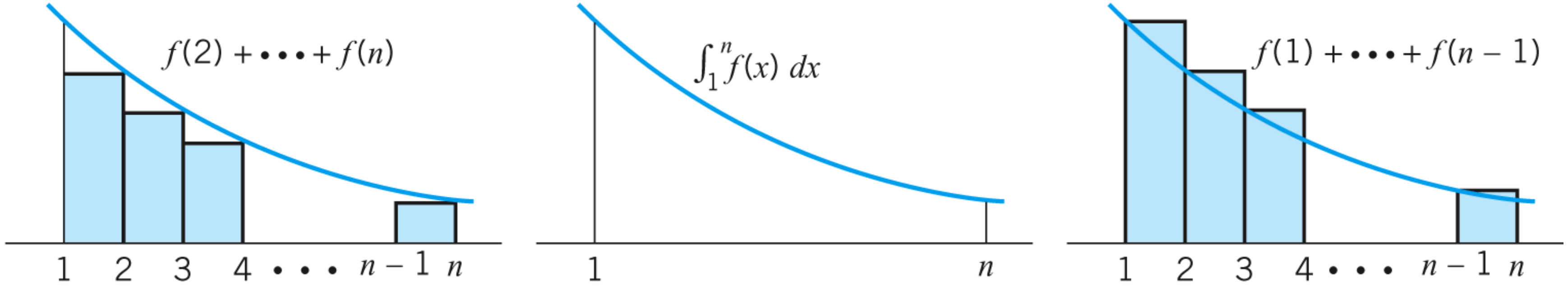

Two special series

- Harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad \text{diverges.}$$

- p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \qquad \text{converges} \qquad \text{iff} \quad p > 1.$$

(II) Basic comparison test

Suppose that $\sum a_k$ and $\sum b_k$ are series with nonnegative terms and

$$\sum a_k \leq \sum b_k \qquad \text{for all } k \text{ sufficiently large.}$$

(i) If $\sum b_k$ converges, then $\sum a_k$ converges

(ii) If $\sum a_k$ diverges, then $\sum b_k$ diverges.

Examples

$$\sum \frac{1}{2k^3+1}$$

Examples

$$\sum \frac{k^3}{k^5 + 5k^4 + 7}$$

The limit comparison test

Let $\sum a_k$ and $\sum b_k$ be series with *positive terms*. If $a_k/b_k \to L$, and L is *positive*, then

$$\sum a_k \quad \text{converges} \quad \text{iff} \quad \sum b_k \quad \text{converges.}$$

Examples

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$$

Examples

$$\sum_{n=1}^{\infty} \sqrt{n+1}\left(1-\cos\frac{\pi}{n}\right)$$

Exercises

$$\sum_{n=1}^{\infty}\left(\frac{\pi}{n}-\sin\frac{\pi}{n}\right)$$

(III) The ratio test

Let $\sum a_k$ be a series with positive terms and suppose that

$$\frac{a_{k+1}}{a_k} \to \lambda.$$

(i) If $\lambda < 1$, then $\sum a_k$ converges.

(ii) If $\lambda > 1$, then $\sum a_k$ diverges.

(iii) If $\lambda = 1$, then the test is inconclusive. The series may converge; it may diverge.

(IV) The root test

Let $\sum a_k$ be a series with nonnegative terms, and suppose that

$$(a_k)^{1/k} \to \rho.$$

(i) If $\rho < 1$, then $\sum a_k$ converges.

(ii) If $\rho > 1$, then $\sum a_k$ diverges.

(iii) If $\rho = 1$, then the test is inconclusive. The series may converge; it may diverge.

Examples

$$\sum \frac{1}{(\ln k)^k}$$

$$\sum \frac{k}{10^k}$$

Examples

$$\sum \frac{1}{(\ln k)^k}$$

$$(a_k)^{1/k} = \frac{1}{\ln k} \to 0$$

$$\sum \frac{k}{10^k}$$

Examples

$$\sum \frac{1}{(\ln k)^k}$$

$$\sum \frac{k}{10^k}$$

$$\frac{a_{k+1}}{a_k} = \frac{k+1}{10^{k+1}} \cdot \frac{10^k}{k} = \frac{1}{10}\frac{k+1}{k} \to \frac{1}{10}$$

Exercises $\sum_{n=1}^{\infty} \frac{n}{e^{n}-1}$

Exercises $\sum_{n=1}^{\infty} \frac{n}{e^n - 1}$

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{e^n - 1}}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{e^n}{e^n - 1}} . \sqrt[n]{\frac{n}{e^n}}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{e^n}{e^n - 1}} . \frac{\sqrt[n]{n}}{e} = 1 . \frac{1}{e} = \frac{1}{e} < 1$$

Alternating Series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$

$$a_0-a_1+a_2-a_3+\cdots=\sum_{k=0}^{\infty}(-1)^k a_k \qquad (a_k \text{ is positive})$$

Leibniz Theorem

Let $a_0, a_1, a_2,$ be a decreasing sequence of positive numbers. The series

$$a_0 - a_1 + a_2 - a_3 + \cdots = \sum_{k=0}^{\infty} (-1)^k a_k \quad \text{converges} \quad \text{iff} \quad a_k \to 0$$

$\sum_{k=0}^{\infty} (-1)^k a_k$ converges to some sum L.

The number L lies between all consecutive partial sums, s_n, s_{n+1}. From this it follows that s_n approximates L to within a_{n+1} :

$$|s_n - L| < a_{n+1}.$$

Examples $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$

Examples $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$

Absolute and conditional convergence

If $\sum |a_k|$ converges, then $\sum a_k$ converges.

- Series Σa_k for which $\Sigma|a_k|$ converges are called *absolutely convergent;*
- Convergent series Σa_k for which $\Sigma|a_k|$ diverges are called *conditionally convergent.*

Given Σa_k

- Step 1: Check if $\Sigma|a_k|$ converges or not; If yes, say that "Σa_k is absolutely convergent"; If no, continue to Step 2.

- Step 2: Check if Σa_k converge or not; If yes, say that "Σa_k is conditionally convergent"; If no, say that "say that "Σa_k is divergent".

Exercises

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$