

Fourier series

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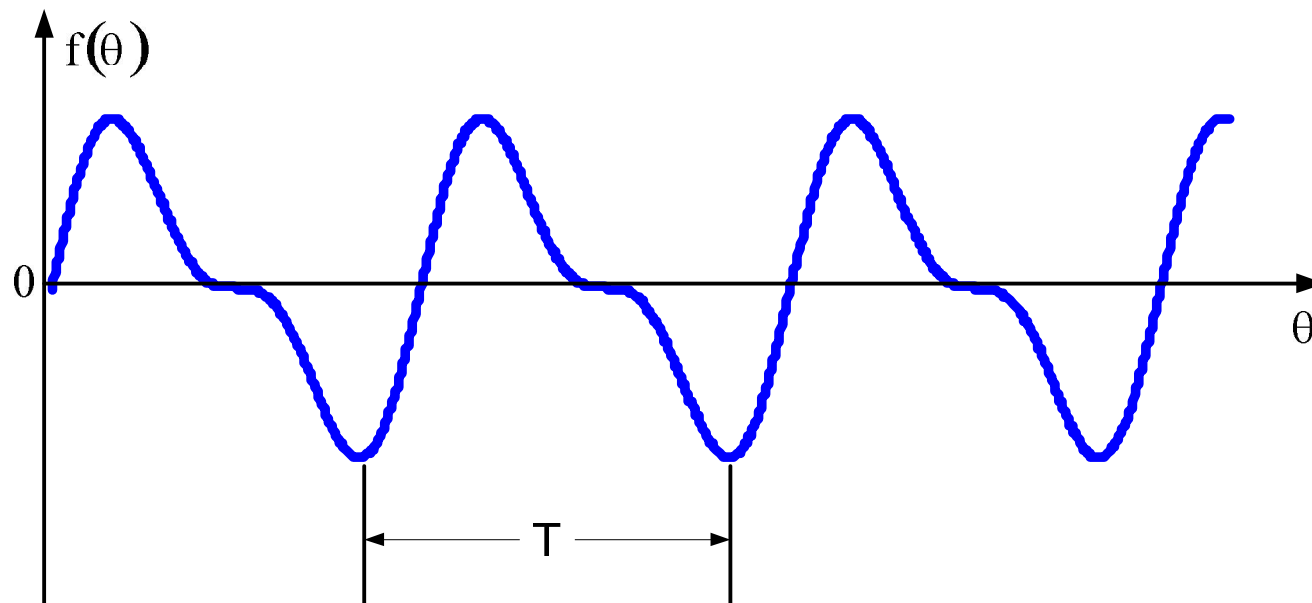
Joseph Fourier 1768-1830

$$f(t) = \frac{1}{\pi} \sum_{m=0}^{\infty} F_m \cos(mt) + \frac{1}{\pi} \sum_{m=0}^{\infty} F'_m \sin(mt)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt$$

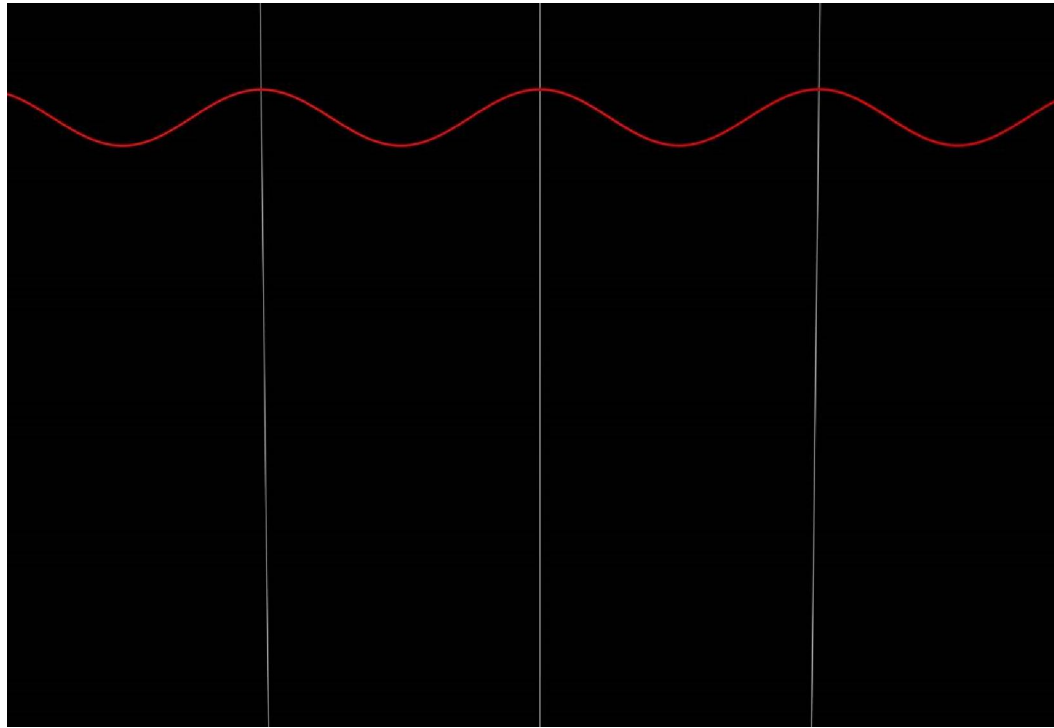
Periodic Function



Periodic Function

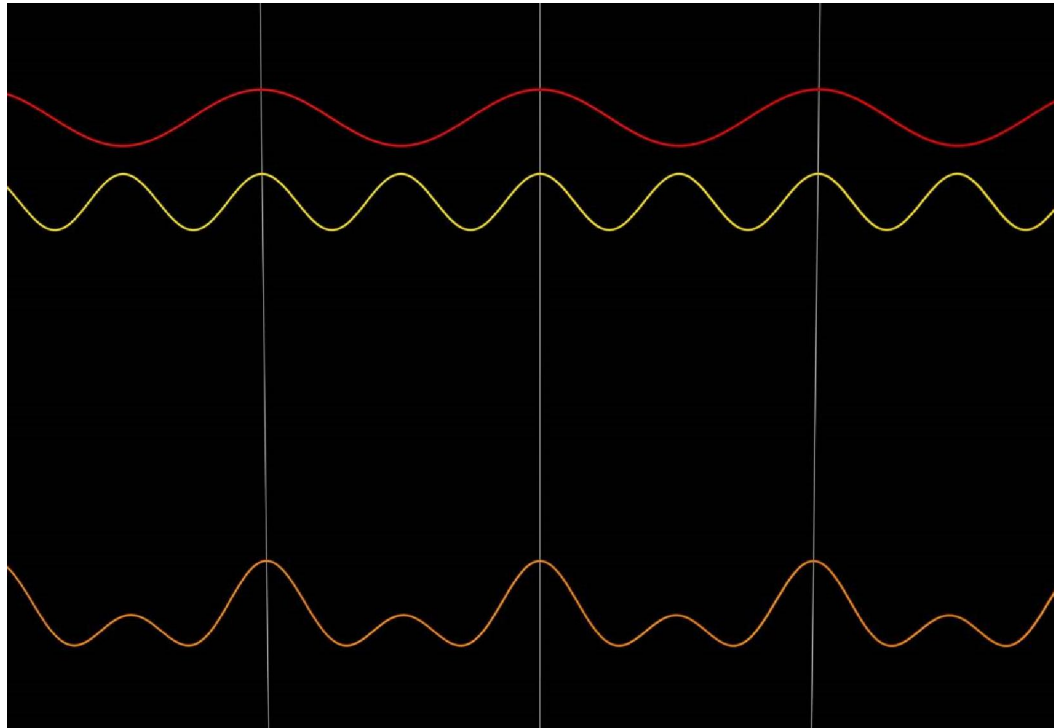
A function $f(\theta)$ is periodic
if it is defined for all real θ
and if there is some positive number,
 T such that $f(\theta + T) = f(\theta)$.

Light pulses



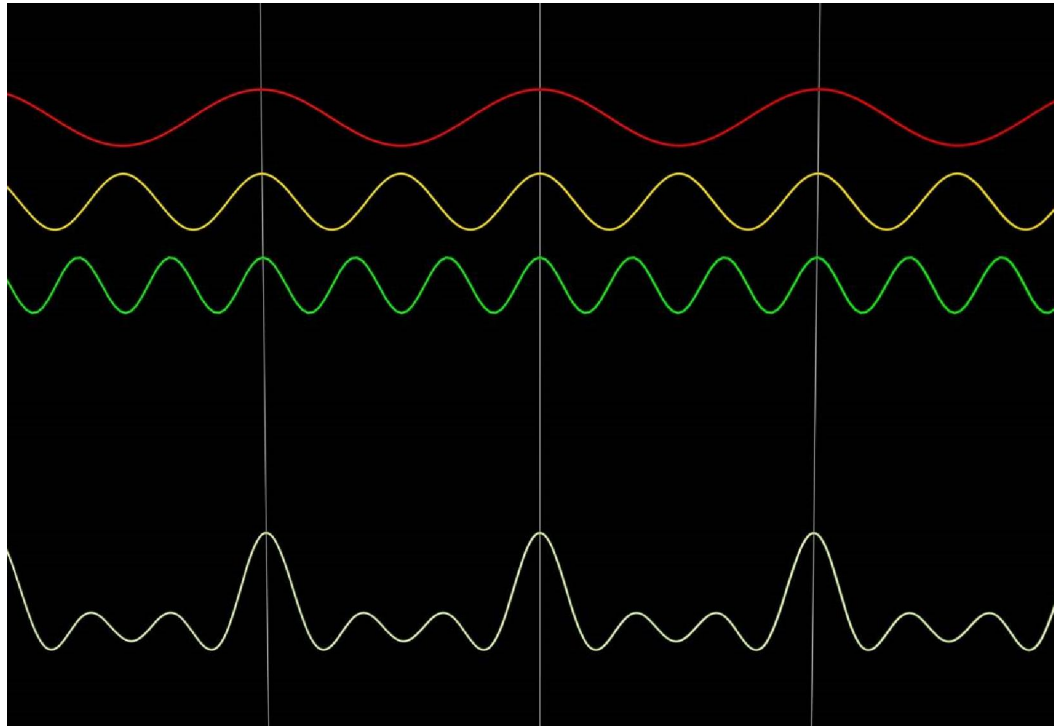
A plane wave with a single frequency

Light pulses

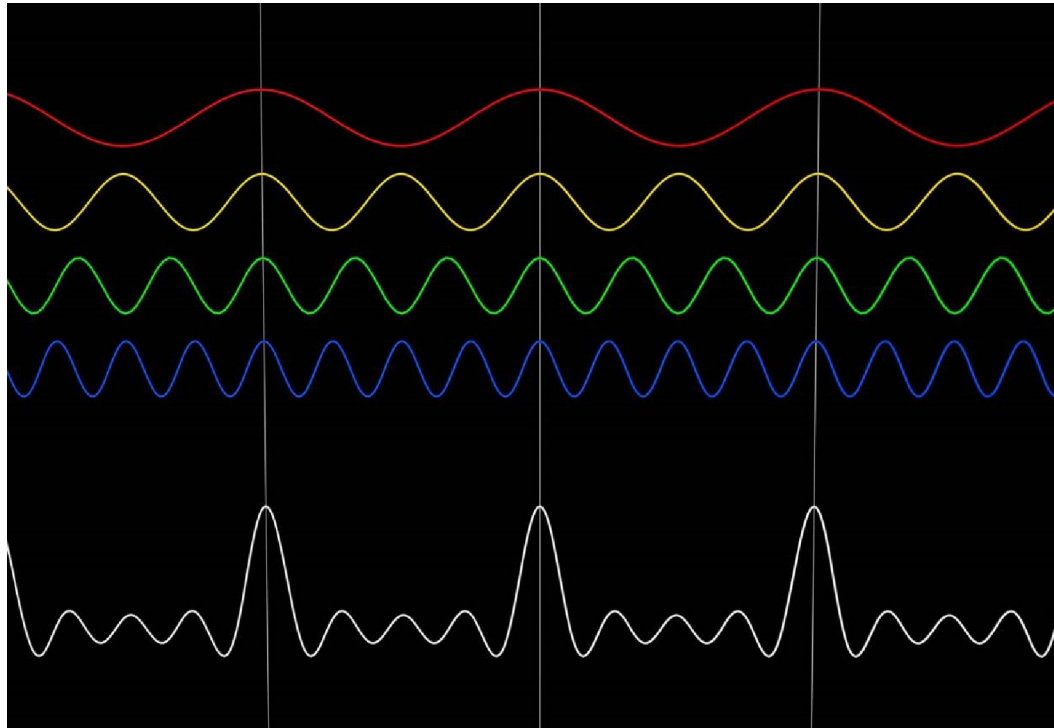


Add a second plane wave at a different frequency

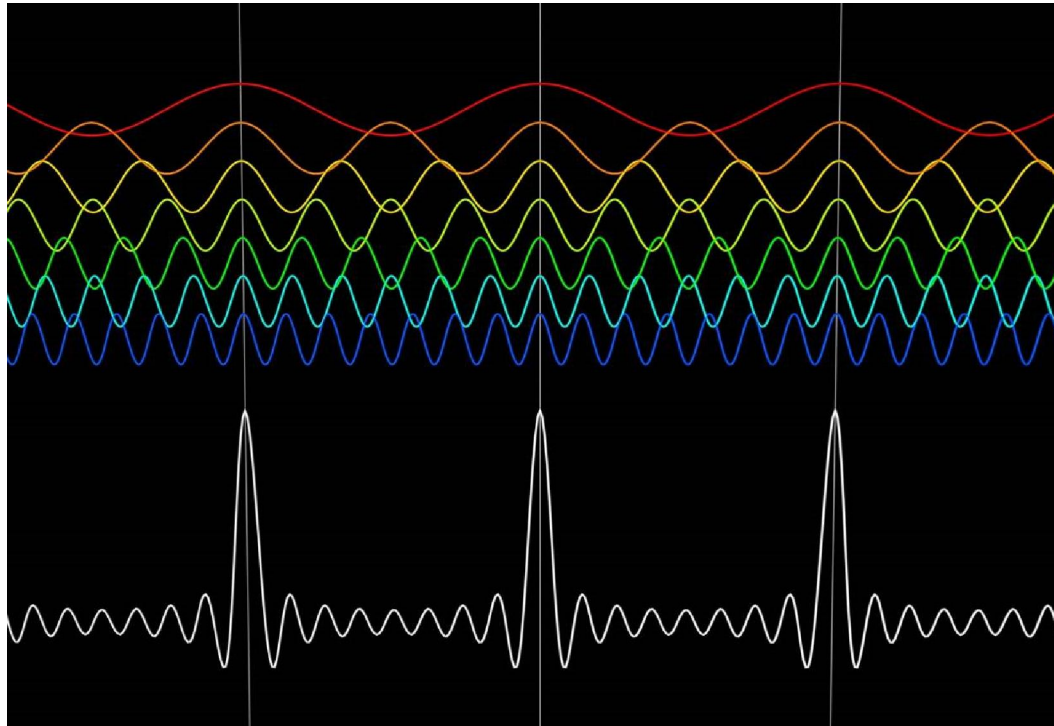
Light pulses



Light pulses



Light pulses



A large combination of plane waves can result in a short light pulse

Note that this signal is periodic, as we used a discrete set of waves

Fourier series

$f(\theta)$ be a periodic function with period 2π

If the function can be represented by a trigonometric series as:

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

then we the series is the Fourier series of f .

注意：在同济版的教材里面，用 $a_0/2$ 表示常数项，所以跟我们这里的表示略微不同，虽然没有本质区别，但计算时请大家务必留心！！

Dirichlet condition

A piecewise regular function that

1. Has a finite number of finite discontinuities and
2. Has a finite number of extrema in a period

can be expanded in a Fourier series which converges to the function at continuous points and the mean of the positive and negative limits at points of discontinuity.

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

What kind of trigonometric (series) functions are we talking about?

$\cos \theta, \cos 2\theta, \cos 3\theta \dots$ and

$\sin \theta, \sin 2\theta, \sin 3\theta \dots$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta d\theta$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)\theta d\theta + \frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)\theta d\theta$$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta d\theta = 0 \quad \text{if } n \neq m$$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta d\theta = \pi \quad \text{if } n = m$$

$$\begin{aligned}
& \int_{-\pi}^{\pi} \sin n\theta \sin m\theta d\theta \\
&= \frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)\theta d\theta - \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)\theta d\theta
\end{aligned}$$

$$\int_{-\pi}^{\pi} \sin n\theta \sin m\theta d\theta = 0 \quad \text{if } n \neq m$$

$$\int_{-\pi}^{\pi} \sin n\theta \sin m\theta d\theta = \pi \quad \text{if } n = m$$

$$\begin{aligned}
 & \int_{-\pi}^{\pi} \sin n\theta \cos m\theta d\theta \\
 &= \frac{1}{2} \int_{-\pi}^{\pi} \sin(n+m)\theta d\theta + \frac{1}{2} \int_{-\pi}^{\pi} \sin(n-m)\theta d\theta
 \end{aligned}$$

$$\int_{-\pi}^{\pi} \sin n\theta \cos m\theta d\theta = 0$$

for all values of m.

Orthogonality of functions

The scalar product of two functions $f(x)$ and $g(x)$ defined on the same interval $[a, b]$ can be defined by

$$\int_a^b f(x)g(x)dx$$

The functions $f(x)$ and $g(x)$ are said to be orthogonal on the interval $[a, b]$ if

$$\int_a^b f(x)g(x)dx=0$$

Determine a_0

$$\int_{-\pi}^{\pi} f(\theta) d\theta = \int_{-\pi}^{\pi} a_0 d\theta + \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} a_n \cos n\theta \right) d\theta$$
$$+ \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} b_n \sin n\theta \right) d\theta$$

$$\int_{-\pi}^{\pi} f(\theta) d\theta = 2\pi a_0 + 0 + 0$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

Determine a_n

$$\int_{-\pi}^{\pi} f(\theta) \cos m\theta d\theta$$

$$= \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \right] \cos m\theta d\theta$$

Let us do the integration on the right-hand-side one term at a time.

First term,

$$\int_{-\pi}^{\pi} a_0 \cos m\theta d\theta = 0$$

Second term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos n\theta \cos m\theta d\theta$$

Second term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos n\theta \cos m\theta d\theta = a_m \pi$$

Third term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin n\theta \cos m\theta d\theta = 0$$

Therefore,

$$\int_{-\pi}^{\pi} f(\theta) \cos m\theta d\theta = a_m \pi$$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos m\theta d\theta \quad m = 1, 2, \dots$$

Determine b_n

$$\int_{-\pi}^{\pi} f(\theta) \sin m\theta \, d\theta$$

$$= \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \right] \sin m\theta \, d\theta$$

The coefficients are:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos m\theta d\theta \quad m = 1, 2, \dots$$

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin m\theta d\theta \quad m = 1, 2, \dots$$

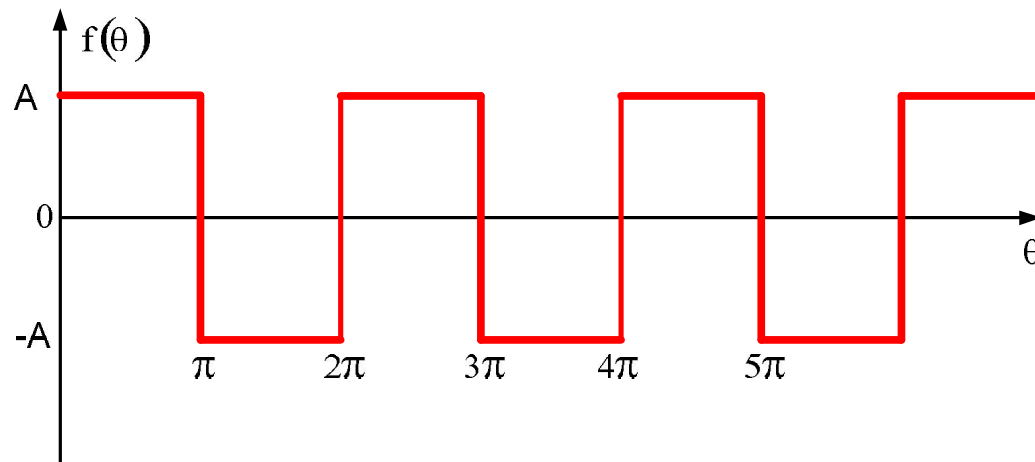
We can write n in place of m :

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta \quad n = 1, 2, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta \quad n = 1, 2, \dots$$

Example 1. Find the Fourier series of the following periodic function.



$$\begin{aligned} f(\theta) &= A \quad \text{when} \quad 0 < \theta < \pi \\ &= -A \quad \text{when} \quad \pi < \theta < 2\pi \end{aligned}$$

$$f(\theta + 2\pi) = f(\theta)$$

$$\begin{aligned}a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \\&= \frac{1}{2\pi} \left[\int_0^{\pi} f(\theta) d\theta + \int_{\pi}^{2\pi} f(\theta) d\theta \right] \\&= \frac{1}{2\pi} \left[\int_0^{\pi} A d\theta + \int_{\pi}^{2\pi} -A d\theta \right] \\&= 0\end{aligned}$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta \\
 &= \frac{1}{\pi} \left[\int_0^\pi A \cos n\theta \, d\theta + \int_\pi^{2\pi} (-A) \cos n\theta \, d\theta \right] \\
 &= \frac{1}{\pi} \left[A \frac{\sin n\theta}{n} \right]_0^\pi + \frac{1}{\pi} \left[-A \frac{\sin n\theta}{n} \right]_\pi^{2\pi} = 0
 \end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta \\
&= \frac{1}{\pi} \left[\int_0^{\pi} A \sin n\theta \, d\theta + \int_{\pi}^{2\pi} (-A) \sin n\theta \, d\theta \right] \\
&= \frac{1}{\pi} \left[-A \frac{\cos n\theta}{n} \right]_0^{\pi} + \frac{1}{\pi} \left[A \frac{\cos n\theta}{n} \right]_{\pi}^{2\pi} \\
&= \frac{A}{n\pi} [-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi]
\end{aligned}$$

$$b_n = \frac{A}{n\pi} [-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi]$$

$$= \frac{A}{n\pi} [1 + 1 + 1 + 1]$$

$$= \frac{4A}{n\pi} \quad \text{when } n \text{ is odd}$$

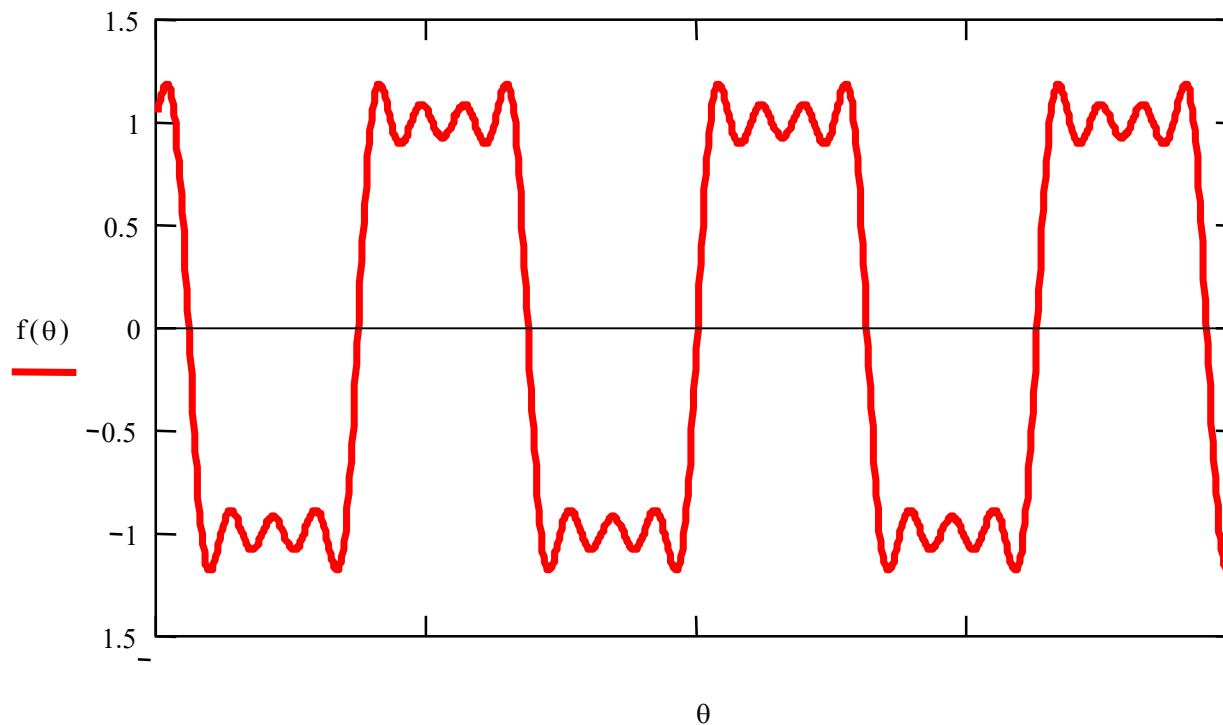
$$\begin{aligned}
 b_n &= \frac{A}{n\pi} [-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi] \\
 &= \frac{A}{n\pi} [-1 + 1 + 1 - 1] \\
 &= 0 \quad \text{when } n \text{ is even}
 \end{aligned}$$

Therefore, the corresponding Fourier series is

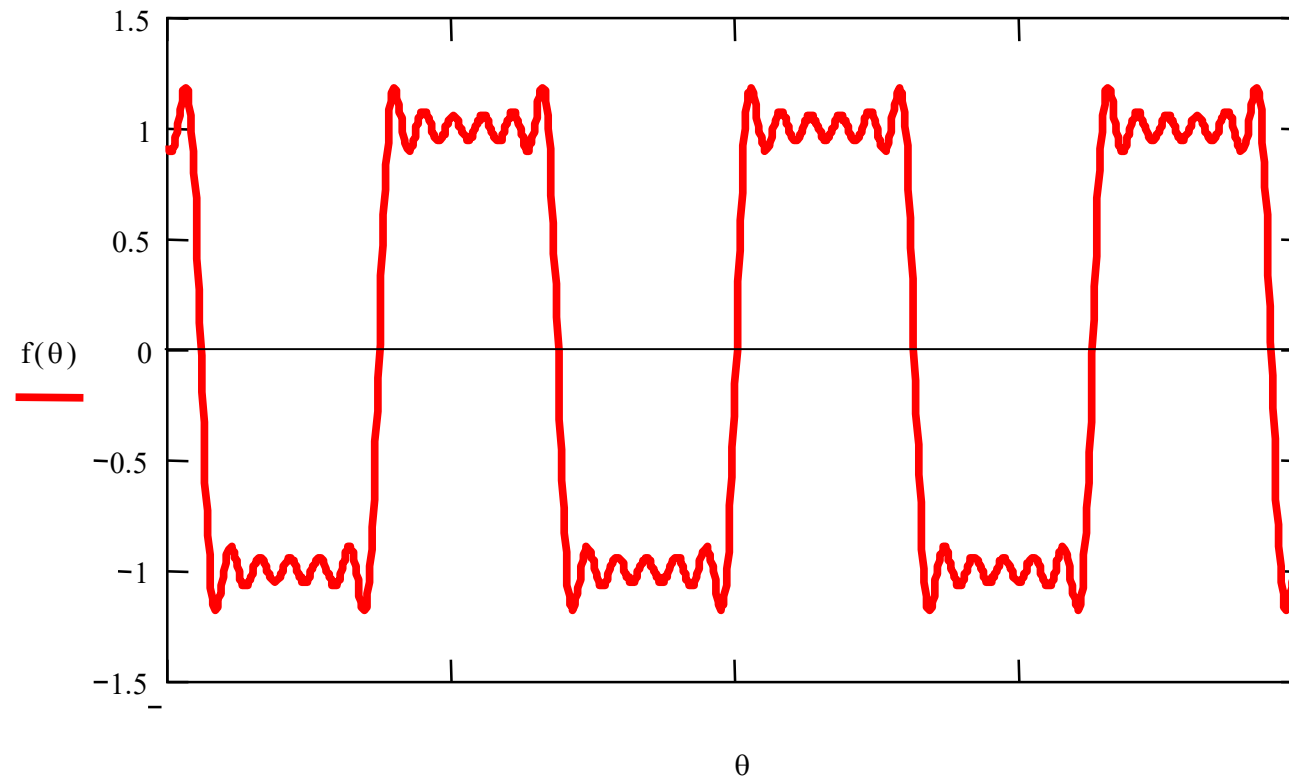
$$\frac{4A}{\pi} \left(\sin \theta + \frac{1}{3} \sin 3\theta + \frac{1}{5} \sin 5\theta + \frac{1}{7} \sin 7\theta + \dots \right)$$

In writing the Fourier series we may not be able to consider infinite number of terms for practical reasons. The question therefore, is – how many terms to consider?

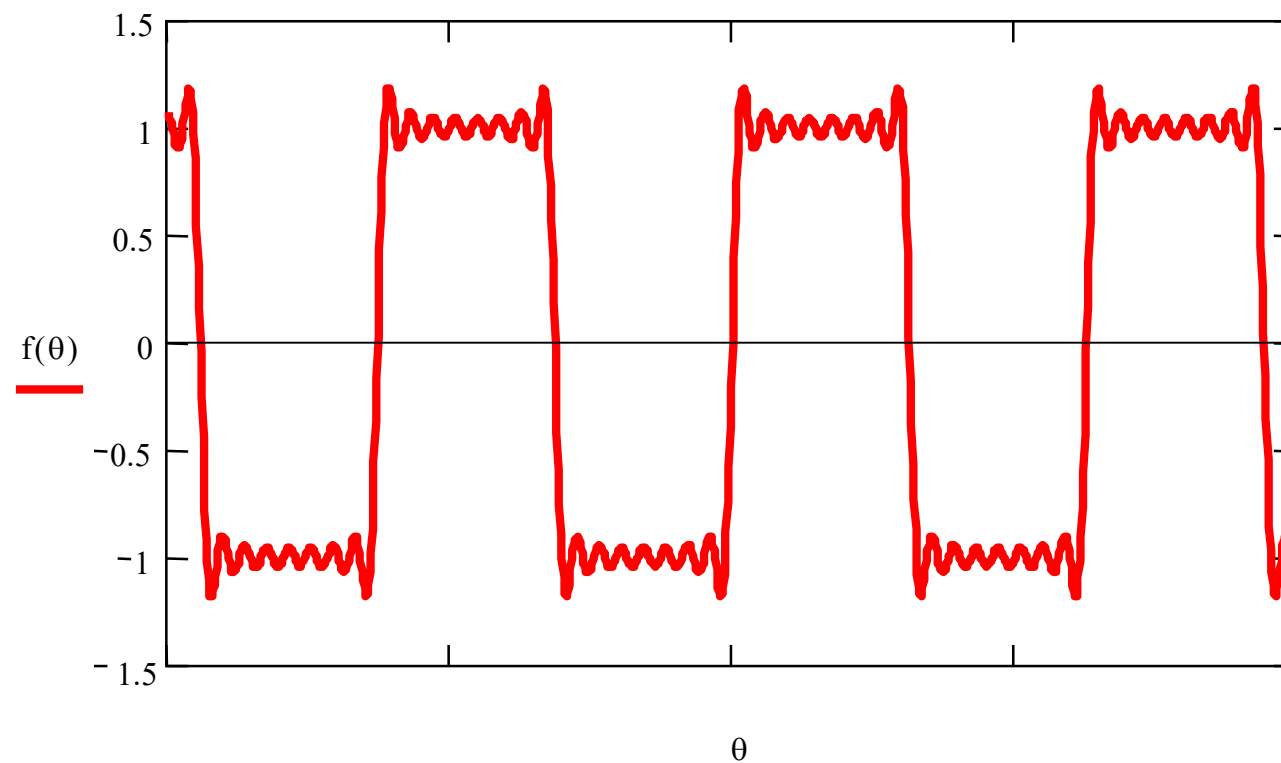
When we consider 4 terms as shown in the previous slide, the function looks like the following.



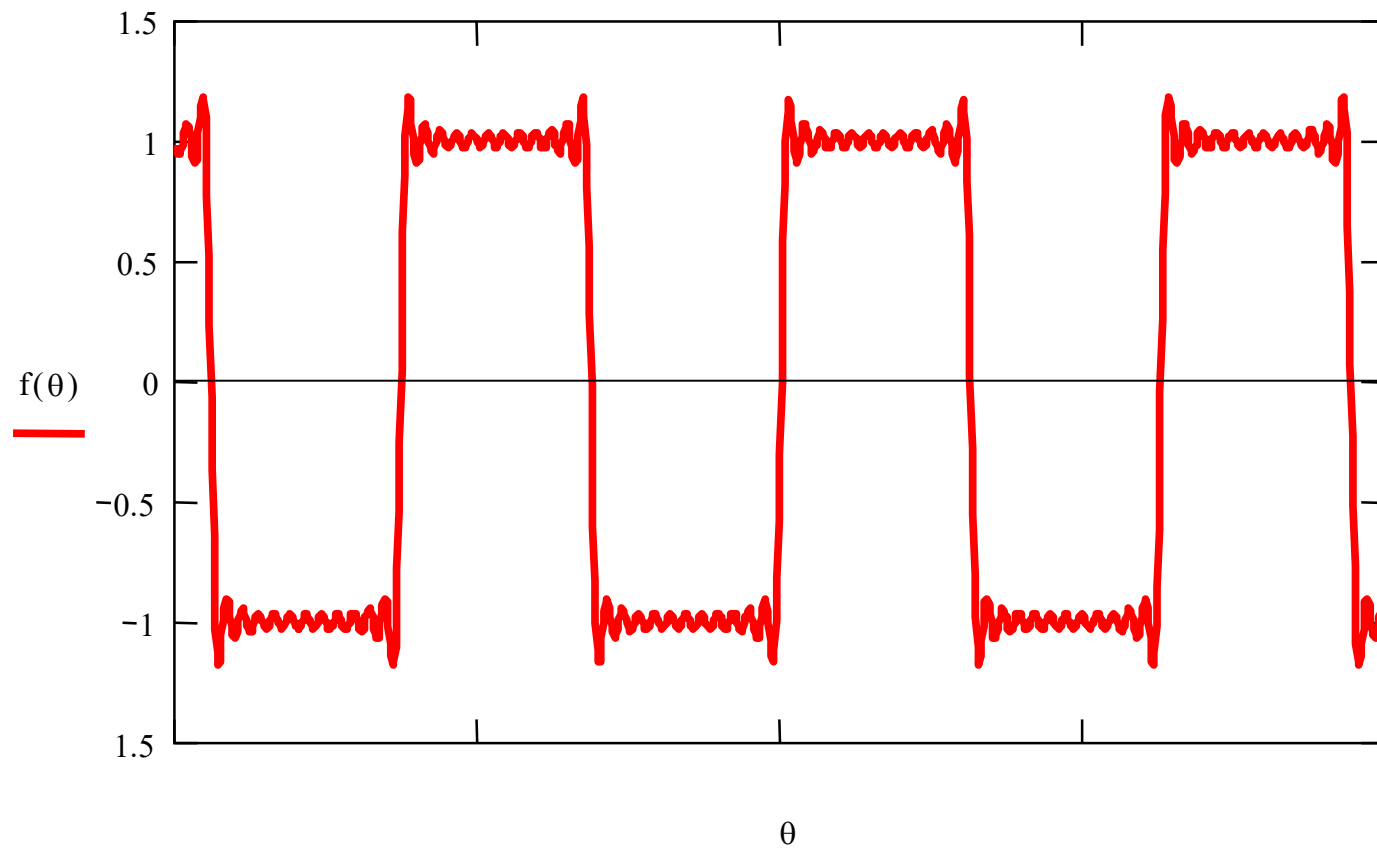
When we consider 6 terms, the function looks like the following.



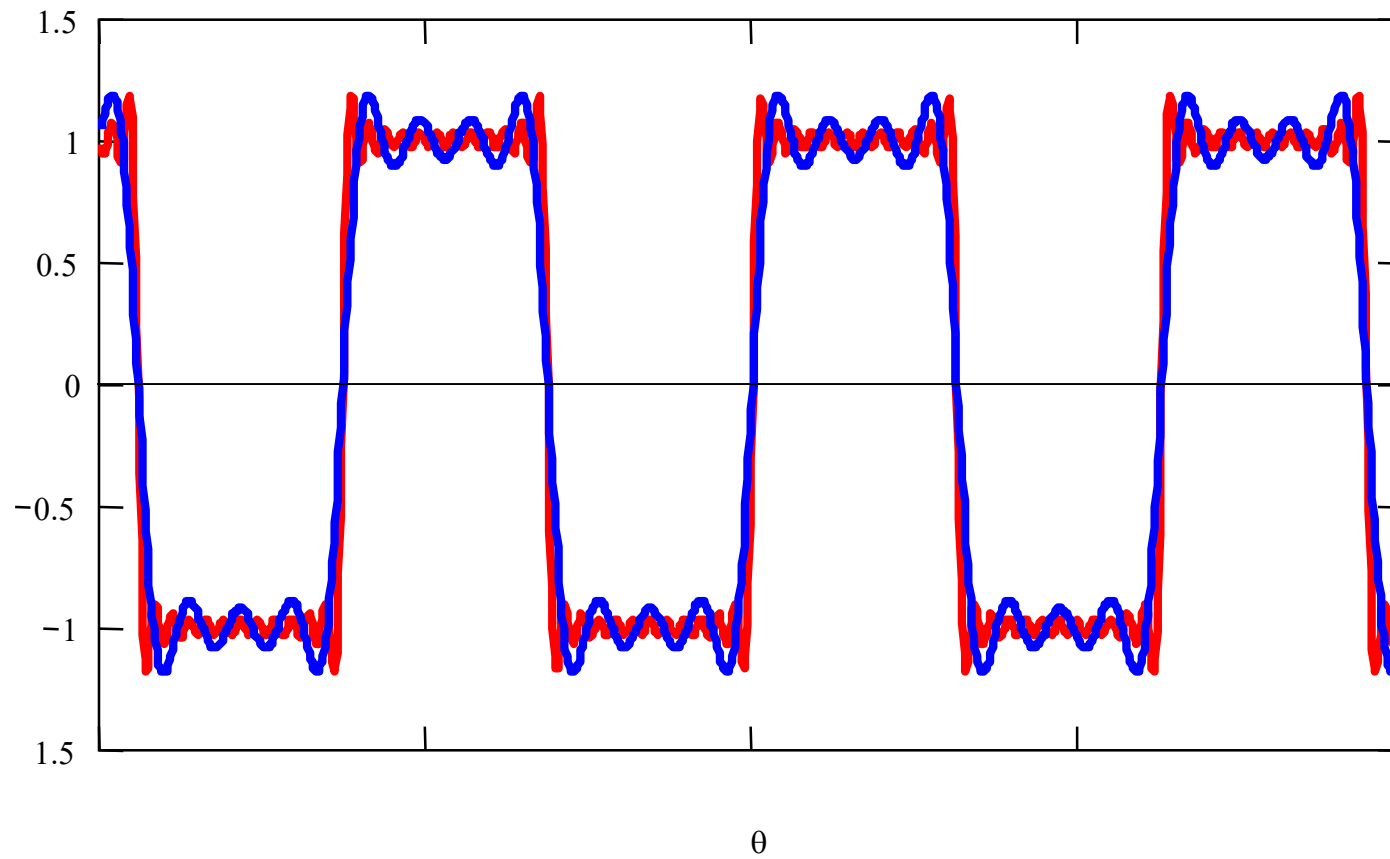
When we consider 8 terms, the function looks like the following.



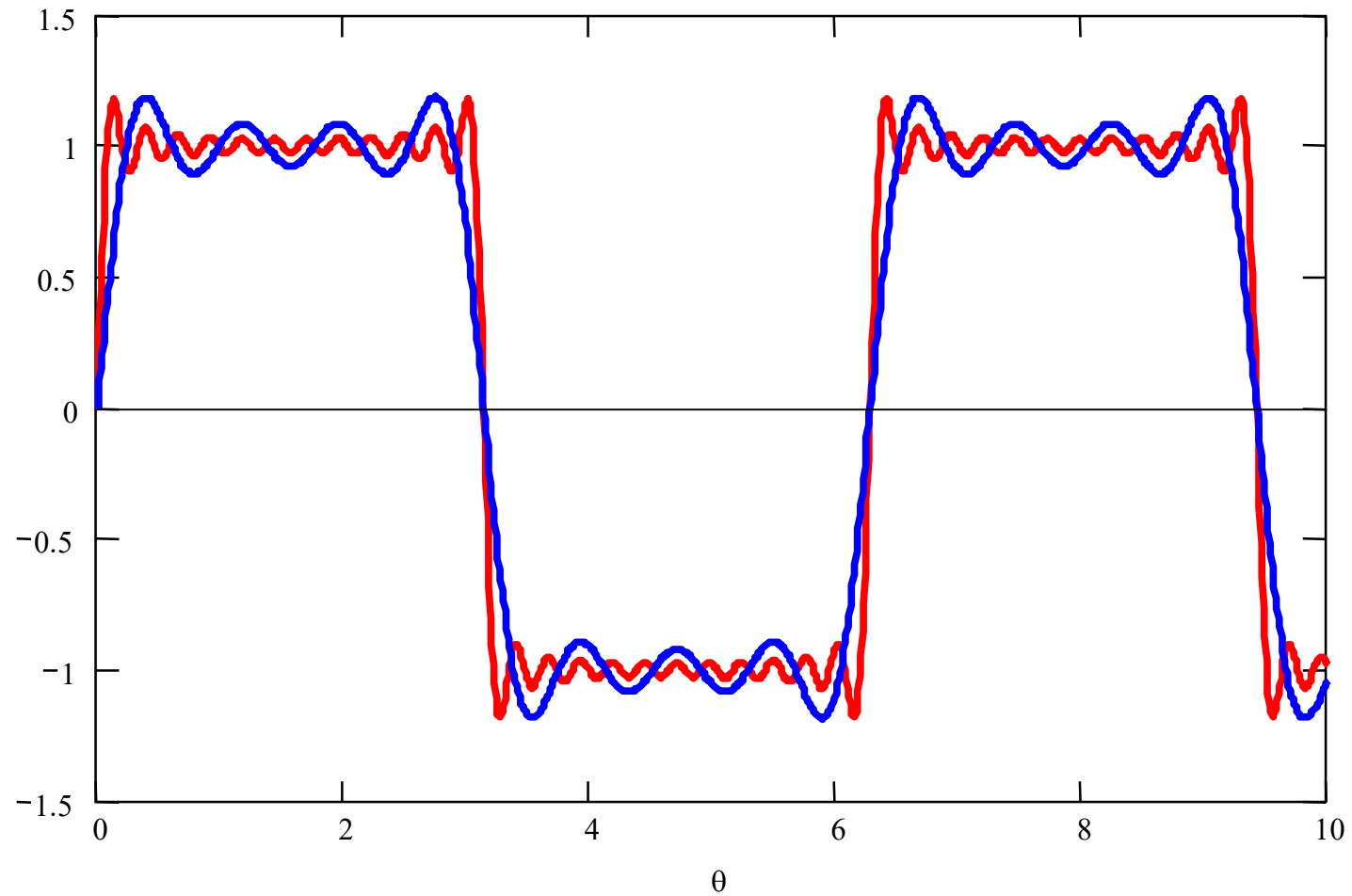
When we consider 12 terms, the function looks like the following.



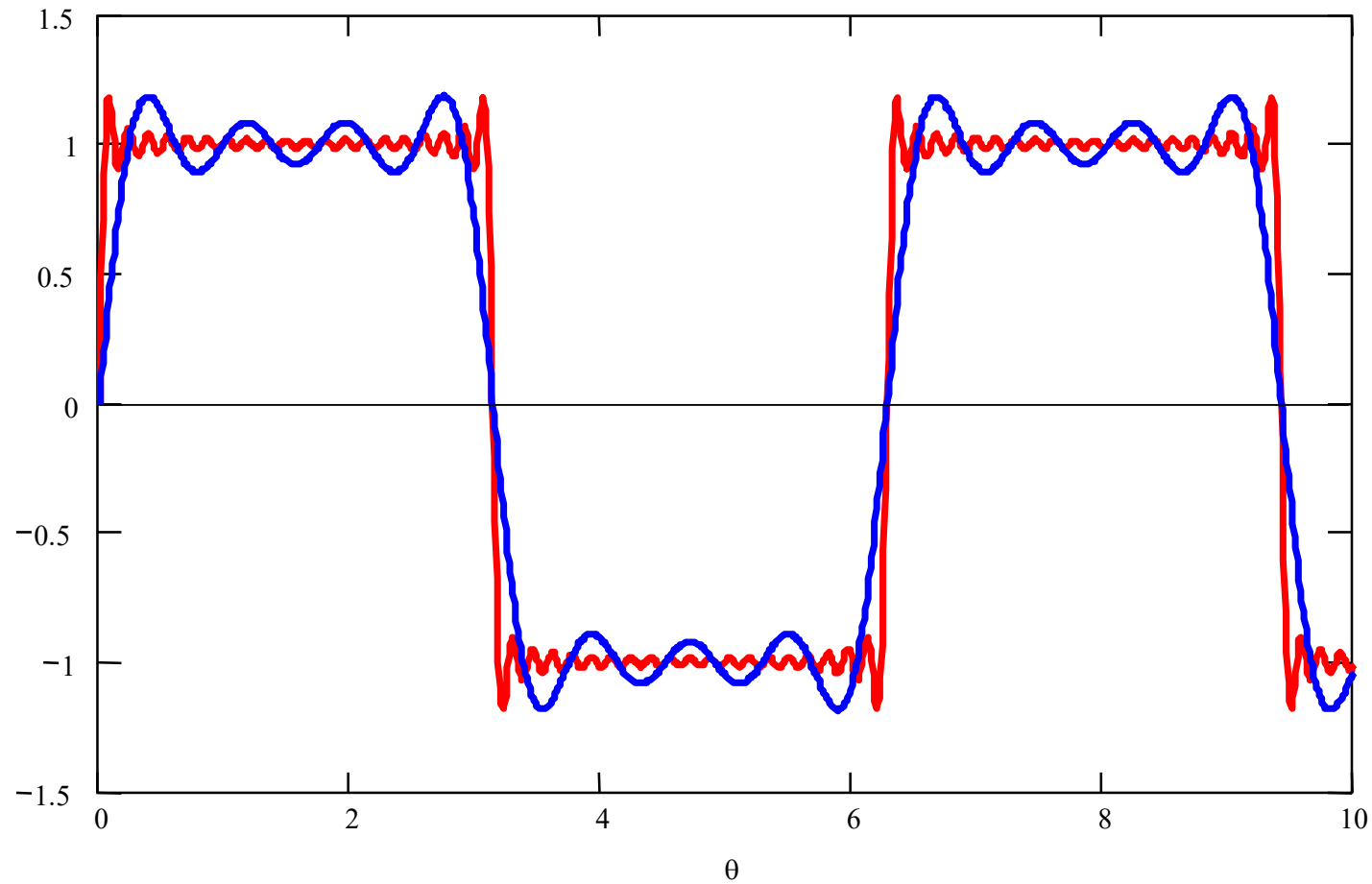
**The red curve was drawn with 12 terms and
the blue curve was drawn with 4 terms.**



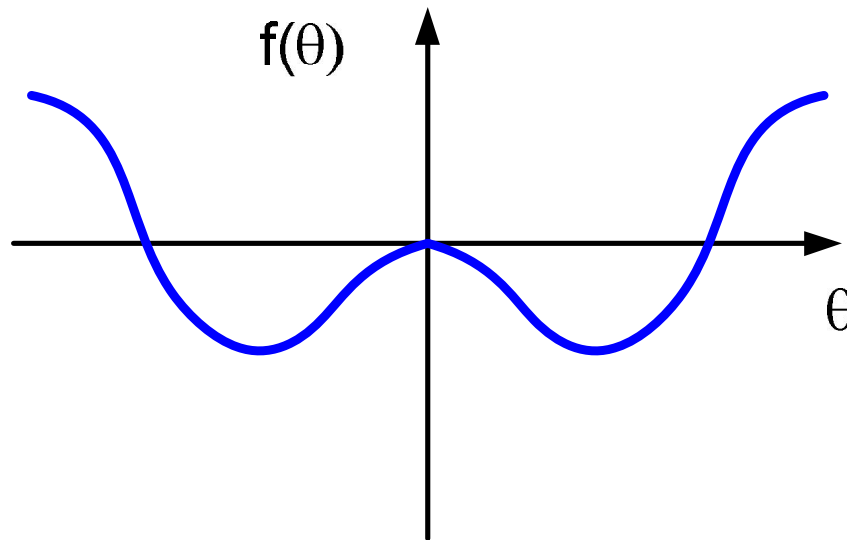
The red curve was drawn with 12 terms and the blue curve was drawn with 4 terms.



**The red curve was drawn with 20 terms and
the blue curve was drawn with 4 terms.**



Even Functions

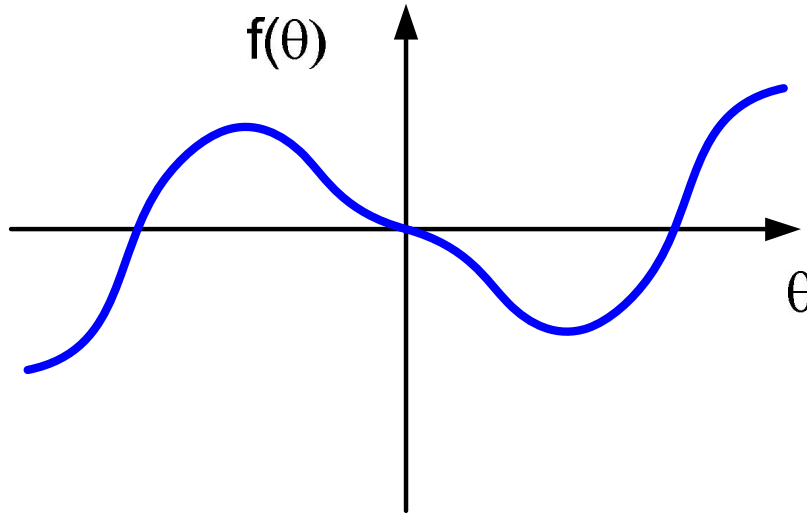


The value of the function would be the same when we walk equal distances along the X-axis in opposite directions.

Mathematically speaking -

$$f(-\theta) = f(\theta)$$

Odd Functions



The value of the function would change its sign but with the same magnitude when we walk equal distances along the X-axis in opposite directions.

Mathematically speaking -

$$f(-\theta) = -f(\theta)$$

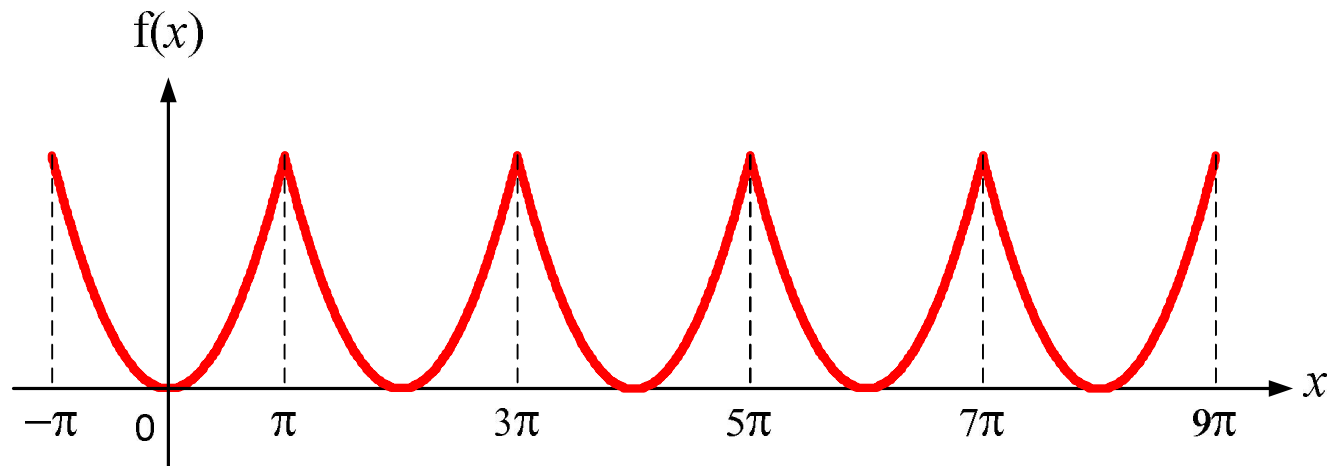
The Fourier series of an even function $f(\theta)$ is expressed in terms of a cosine series.

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$$

The Fourier series of an odd function $f(\theta)$ is expressed in terms of a sine series.

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

Example 2. Find the Fourier series of the following periodic function.



$$f(x) = x^2 \quad \text{when} \quad -\pi \leq x \leq \pi$$

$$f(\theta + 2\pi) = f(\theta)$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx \\ &= \frac{1}{2\pi} \left[\frac{x^3}{3} \right]_{x=-\pi}^{x=\pi} = \frac{\pi^2}{3} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x^2 \cos nx \, dx \right] \end{aligned}$$

Use integration by parts.

$$a_n = \frac{4}{n^2} \cos n\pi$$

$$a_n = -\frac{4}{n^2} \quad \text{when } n \text{ is odd}$$

$$a_n = \frac{4}{n^2} \quad \text{when } n \text{ is even}$$

This is an even function.

Therefore, $b_n = 0$

The corresponding Fourier series is

$$\frac{\pi^2}{3} - 4 \left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots \right)$$

Odd and Even Extensions

Expand the following function as sine series and cosine series respectively.

$$f(x) = \begin{cases} \cos x, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi \end{cases}$$

Functions Having Arbitrary Period T ?

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$



$$a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta \quad n = 1, 2, \dots$$



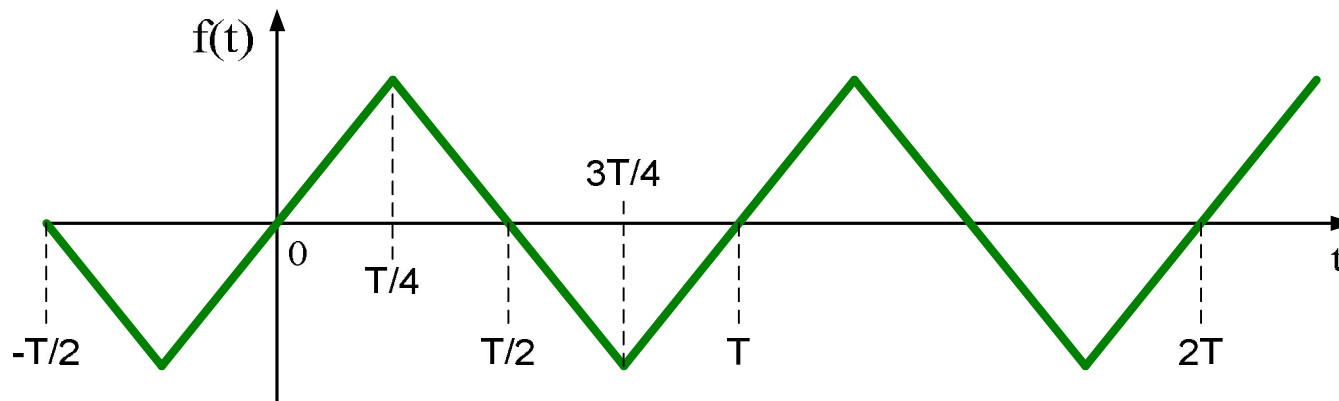
$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos\left(\frac{2\pi n}{T} t\right) dt \quad n = 1, 2, \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta \quad n = 1, 2, \dots$$



$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin\left(\frac{2\pi n}{T} t\right) dt \quad n = 1, 2, \dots$$

Example 4. Find the Fourier series of the following periodic function.



$$f(t) \begin{cases} = t & \text{when } -\frac{T}{4} \leq t \leq \frac{T}{4} \\ = -t + \frac{T}{2} & \text{when } \frac{T}{4} \leq t \leq \frac{3T}{4} \end{cases}$$

This is an odd function. Therefore, $a_n = 0$

$$\begin{aligned} b_n &= \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2\pi n}{T} t\right) dt \\ &= \frac{4}{T} \int_0^{\frac{T}{2}} f(t) \sin\left(\frac{2\pi n}{T} t\right) dt \end{aligned}$$

$$b_n = \frac{4}{T} \int_0^{\frac{T}{4}} t \sin\left(\frac{2\pi n}{T} t\right) dt \\ + \frac{4}{T} \int_{\frac{T}{4}}^{\frac{T}{2}} \left(-t + \frac{T}{2}\right) \sin\left(\frac{2\pi n}{T} t\right) dt$$

Use integration by parts.

$$b_n = \frac{4}{T} \left[2 \cdot \left(\frac{T}{2\pi n} \right)^2 \sin \left(\frac{n\pi}{2} \right) \right]$$

$$= \frac{2T}{n^2 \pi^2} \sin \left(\frac{n\pi}{2} \right)$$

$$b_n = 0 \quad \text{when } n \text{ is even.}$$

Therefore, the Fourier series is

$$\frac{2T}{\pi^2} \left[\sin\left(\frac{2\pi}{T}t\right) - \frac{1}{3^2} \sin\left(\frac{6\pi}{T}t\right) + \frac{1}{5^2} \sin\left(\frac{10\pi}{T}t\right) - \dots \right]$$

The Complex Form of Fourier Series

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

Let us utilize the Euler formulae.

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\begin{aligned}
 & a_n \cos n\theta + b_n \sin n\theta \\
 &= \left(\frac{a_n - ib_n}{2} \right) e^{in\theta} + \left(\frac{a_n + ib_n}{2} \right) e^{-in\theta}
 \end{aligned}$$

Denoting

$$c_n = \left(\frac{a_n - ib_n}{2} \right) \qquad c_{-n} = \left(\frac{a_n + ib_n}{2} \right)$$

and $c_0 = a_0$

The Fourier series can be expressed as:

$$f(\theta) = c_0 + \sum_{n=1}^{\infty} (c_n e^{in\theta} + c_{-n} e^{-in\theta})$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$

**Functions defined on entire real line
without any periodicity?**