Fourier Transforms

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Fourier series revisited (with period *T*)

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi n}{T}\theta + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi n}{T}\theta$$

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$$f\left(t\right)=a_0+\sum_{n=1}^{\infty}a_n\cos\omega_n t+\sum_{n=1}^{\infty}b_n\sin\omega_n t=a_0+\sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}\sin(\omega_n t+\varphi_n)$$

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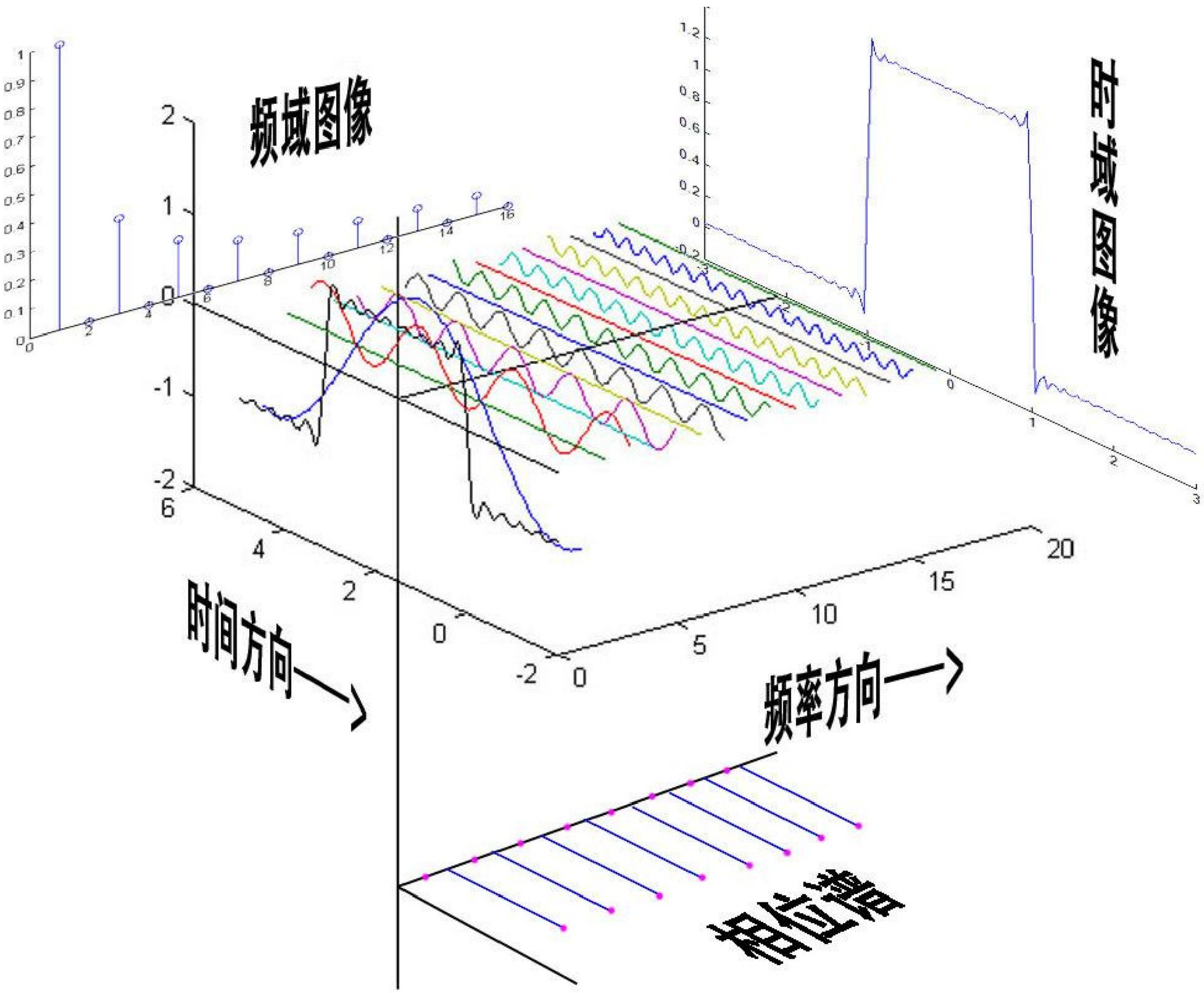
$$f(t) \Rightarrow \{a_n, b_n\} \Rightarrow \{\text{Amplitude } \sqrt{a_n^2 + b_n^2}, \text{ Phase } \varphi_n\}$$



频域图像
时域图像
时间方向—>
频率方向—>
相位谱

Complex form of Fourier series (with period T)

$$f\left(\theta\right)=c_0+\sum_{n=1}^{+\infty}\left(c_n e^{i\frac{2\pi n}{T}\theta}+c_{-n}e^{-i\frac{2\pi n}{T}\theta}\right)=\sum_{n=-\infty}^{+\infty}c_n e^{i\frac{2\pi n}{T}\theta}$$

$$f\left(\theta\right)\Rightarrow\{c_n\}$$

Orthogonality relation

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\theta)e^{-i\frac{2\pi m}{T}\theta}d\theta$$

$$=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\sum_{n=-\infty}^{+\infty}c_n e^{i\frac{2\pi n}{T}\theta}e^{-i\frac{2\pi m}{T}\theta}d\theta=\sum_{n=-\infty}^{+\infty}(c_n\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}e^{i\frac{2\pi}{T}\theta(n-m)}d\theta)$$

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$$c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\theta)e^{-i\frac{2\pi n}{T}\theta}d\theta$$

Forward Fourier transform $\tilde{f}(k) = \mathcal{F}[f(x)]$

$$\boxed{\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x}$$

Remarks

$$\boxed{\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x}$$

- The variables are often called t and ω instead of x and k.
- k could be complex variable.
- The existence of Fourier transform (FT)

  - necessary condition ($f(x) \to 0$ as $x \to \infty$)

  - sufficient condition (bounded variation, a finite number of discontinuities and absolutely integrable)
- Several different definitions of the FT.

Inverse Fourier transform $f(x) = \mathcal{F}^{-1}[\tilde{f}(k)]$

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k}$$

Example (1): top-hat function:

$$f(x) = \begin{cases} c, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{f}(k) = \int_a^b c\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{\mathrm{i}c}{k}\left(\mathrm{e}^{-\mathrm{i}kb} - \mathrm{e}^{-\mathrm{i}ka}\right)$$

e.g. if $a = -1$, $b = 1$ and $c = 1$:

$$\tilde{f}(k) = \frac{\mathrm{i}}{k}\left(\mathrm{e}^{-\mathrm{i}k} - \mathrm{e}^{\mathrm{i}k}\right) = \frac{2\sin k}{k}$$

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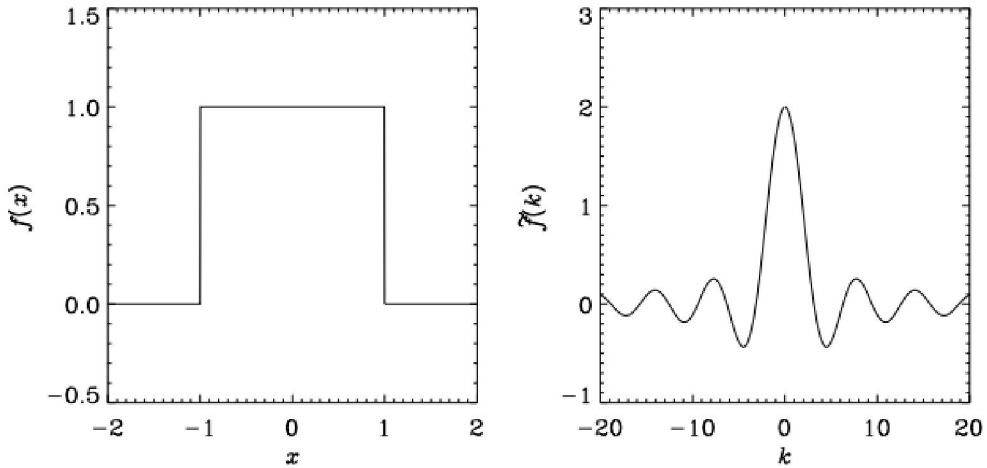

Example (2):

$$f(x) = \mathrm{e}^{-|x|}$$

$$\begin{aligned}
\tilde{f}(k) &= \int_{-\infty}^{0} \mathrm{e}^{x}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-x}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \frac{1}{1-\mathrm{i}k}\left[\mathrm{e}^{(1-\mathrm{i}k)x}\right]_{-\infty}^{0} - \frac{1}{1+\mathrm{i}k}\left[\mathrm{e}^{-(1+\mathrm{i}k)x}\right]_{0}^{\infty} \\
&= \frac{1}{1-\mathrm{i}k} + \frac{1}{1+\mathrm{i}k} \\
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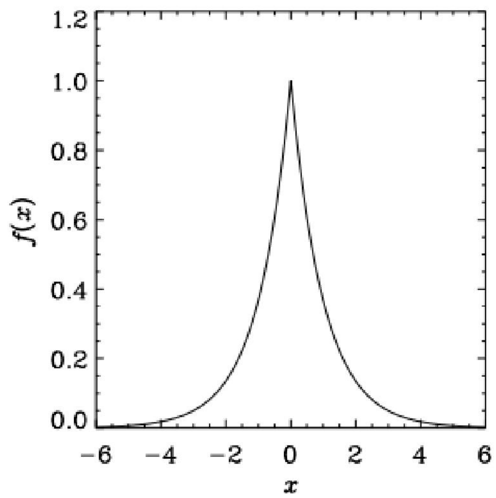

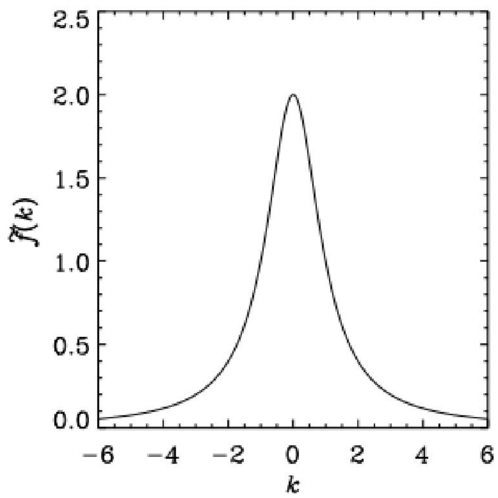

Is there any function that could be equal to its FT?

Example (3): Gaussian function (normal distribution):

$$f(x) = (2\pi\sigma_x^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma_x^2}\right)$$

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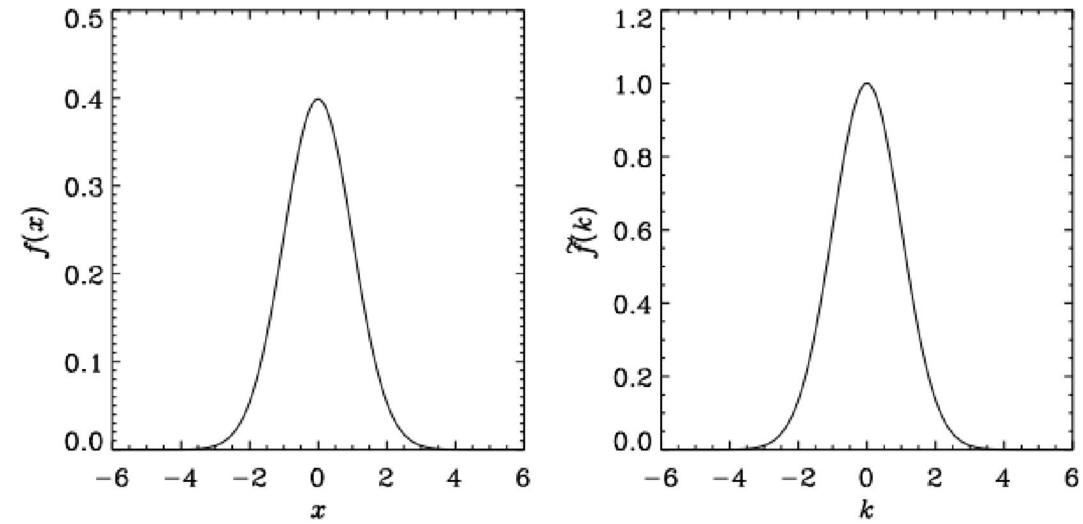

Basic properties of the Fourier transform

Linearity:

$$g(x) = \alpha f(x) \qquad \Leftrightarrow \qquad \tilde{g}(k) = \alpha \tilde{f}(k) \tag{1}$$

$$h(x) = f(x) + g(x) \qquad \Leftrightarrow \qquad \tilde{h}(k) = \tilde{f}(k) + \tilde{g}(k) \tag{2}$$

Basic properties of the Fourier transform

Rescaling (for real α):

$$g(x) = f(\alpha x) \qquad \Leftrightarrow \qquad \tilde{g}(k) = \frac{1}{|\alpha|}\tilde{f}\left(\frac{k}{\alpha}\right) \tag{3}$$

Basic properties of the Fourier transform

Shift/exponential (for real α):

$$g(x) = f(x - \alpha) \quad \Leftrightarrow \quad \tilde{g}(k) = \mathrm{e}^{-\mathrm{i}k\alpha}\tilde{f}(k) \tag{4}$$

$$g(x) = \mathrm{e}^{\mathrm{i}\alpha x} f(x) \quad \Leftrightarrow \quad \tilde{g}(k) = \tilde{f}(k - \alpha) \tag{5}$$

Basic properties of the Fourier transform

Differentiation/multiplication:

$$g(x) = f'(x) \qquad \Leftrightarrow \qquad \tilde{g}(k) = \mathrm{i}k\tilde{f}(k) \tag{6}$$

$$g(x) = xf(x) \qquad \Leftrightarrow \qquad \tilde{g}(k) = \mathrm{i}\tilde{f}'(k) \tag{7}$$

Basic properties of the Fourier transform

Duality:

$$g(x) = \tilde{f}(x) \qquad \Leftrightarrow \qquad \tilde{g}(k) = 2\pi f(-k) \tag{8}$$

Basic properties of the Fourier transform

Complex conjugation and parity inversion (for real x and k):

$$g(x) = [f(x)]^* \qquad \Leftrightarrow \qquad \tilde{g}(k) = [\tilde{f}(-k)]^* \tag{9}$$

Basic properties of the Fourier transform

Symmetry:

$$f(-x) = \pm f(x) \qquad \Leftrightarrow \qquad \tilde{f}(-k) = \pm \tilde{f}(k) \tag{10}$$

Dirac delta function

$$f(x) = (2\pi\sigma_x^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma_x^2}\right) \qquad \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

As σ_x goes to zero, what will happen?

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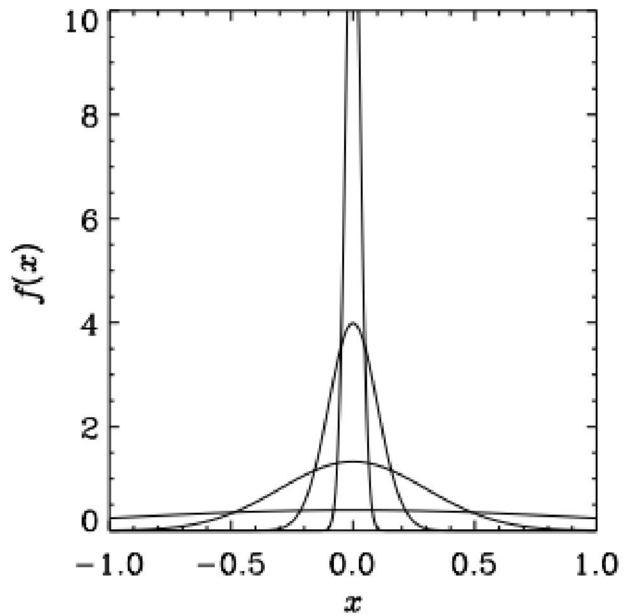

Dirac delta function

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

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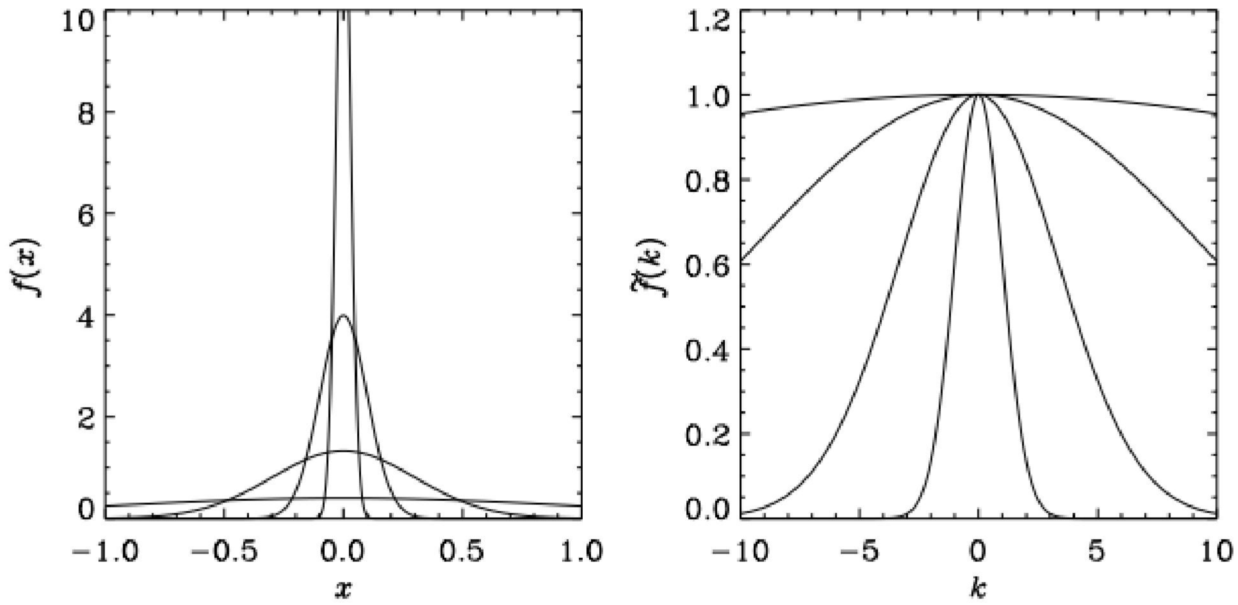
- **sifting property**

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-\mu)\, dx = f(\mu)$$

Fourier transform for Dirac delta function



$$f(x) = (2\pi\sigma_x^2)^{-1/2} \exp\left(-\frac{x^2}{2\sigma_x^2}\right)$$

$$\tilde{f}(k) = \exp\left(-\frac{\sigma_x^2 k^2}{2}\right)$$

Fourier transform for Dirac delta function

$$\tilde{\delta}(k) = \int_{-\infty}^{\infty} \delta(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x = \mathrm{e}^{-\mathrm{i}k0} = 1$$

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$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k \quad \Rightarrow \quad \boxed{\int_{-\infty}^{\infty} \mathrm{e}^{\pm\mathrm{i}kx}\, \mathrm{d}x = 2\pi\delta(k)}$$