Power series

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How to define a function

- By mapping rules (e.g. elementary function)
- By equations (e.g. implicit function)
- By Integral (e.g. Gamma function)
- …

Series with function terms

$$S(x) = \sum_{k=0}^{+\infty} u_n(x)$$

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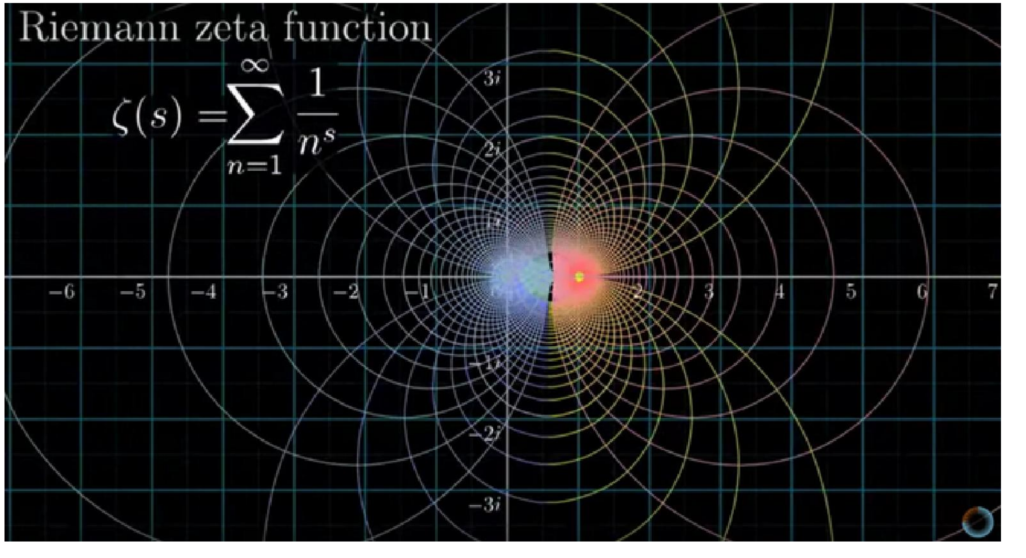


Taylor Polynomials in x

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

If f has $n+1$ continuous derivatives on an open interval I that contains 0, then for each $x \in I$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

with $R_n(x) = \dfrac{1}{n!}\displaystyle\int_0^x f^{(n+1)}(t)(x-t)^n\,dt$. We call $R_n(x)$ the *remainder*.

Taylor series in x

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.}$$

Taylor series in *x*

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad x \in (-1, 1)$$

Power series and its convergence

A power series $\sum a_k x^k$ is said to converge

(i) at c if $\sum a_k c^k$ converges;

(ii) on the set S if $\sum a_k x^k$ converges at each $x \in S$.

Abel's Theorem

If $\sum a_k x^k$ converges at $c \neq 0$, it converges absolutely at all x with $|x| < |c|$.
If $\sum a_k x^k$ diverges at d, then it diverges at all x with $|x| > |d|$.

Corollary of Abel's Theorem

Case 1. *The series converges only at* $x = 0$. This is what happens with

$$\sum k^k x^k .$$

For $x \neq 0$, $k^k x^k \not\to 0$, and so the series cannot converge.

Case 2. *The series converges absolutely at all real numbers* x. This is what happens with the exponential series

$$\sum \frac{x^k}{k!}.$$

Case 3. *There exists a positive number* r *such that the series converges absolutely for* $|x| < r$ *and diverges for* $|x| > r$. This is what happens with the geometric series

$$\sum x^k .$$

Here there is absolute convergence for $|x| < 1$ and divergence for $|x| > 1$.

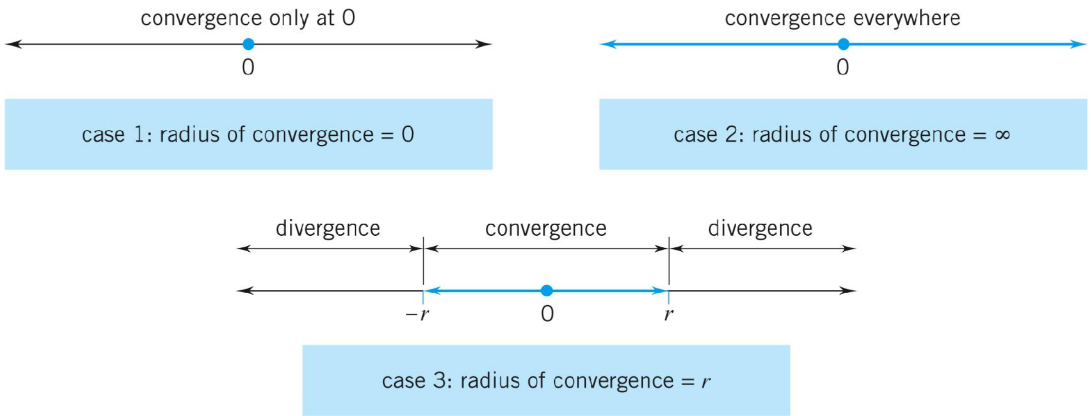
Radius of convergence

Associated with each case is a *radius of convergence*:

In Case 1, we say that the radius of convergence is 0.

In Case 2, we say that the radius of convergence is ∞.

In Case 3, we say that the radius of convergence is r.



Radius of convergence

- For $\sum_{n=0}^{+\infty} a_n x^n, (a_n \neq 0)$

$$r = \begin{cases} 0, & \rho = +\infty \\ +\infty, & \rho = 0 \\ 1/\rho, & \rho \neq 0 \end{cases} \quad , \quad \rho = \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Radius of convergence

- For $\sum_{n=0}^{+\infty} a_n x^n, (a_n \neq 0)$

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- Please think about $\sum_{n=0}^{+\infty} a_n x^n, (a_n \neq 0 \text{ not always the case})$

Domain of convergence

- Step 1: calculate the radius r of convergence
- Step 2: test the convergence of the series at $x=r$ and $x=-r$
- Step 3: write the domain of convergence

Determine the domain of convergence of the following power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$$

Determine the domain of convergence of the following power series

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2^n}$$

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$$\lim_{n\to\infty} \frac{x^{2n+1}}{2^{n+1}} \cdot \frac{2^n}{x^{2n-1}} = \frac{1}{2}|x|^2 .$$

Determine the domain of convergence of the following power series

$$\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}+\frac{1}{4^n}\right]x^n$$

Taylor Polynomials in *x-a*

$$P_n(x) = \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(x-a)^k$$

If g has $n+1$ continuous derivatives on an open interval I that contains the point a, then for each $x \in I$

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots + \frac{g^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

with

$$R_n(x) = \frac{1}{n!}\int_a^x g^{(n+1)}(t)(x-t)^n \, dt.$$

Determine the domain of convergence of the following power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{\sqrt{n}} \left(x - \frac{1}{2} \right)^n$$

Determine the domain of convergence of the following power series

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x-2}{x}\right)^n$$

Power series expansion of *f(x)*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Domain of convergence

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{for all real } x.$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{for all real } x.$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{for all real } x.$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad \text{for } -1 < x \le 1.$$

Newtonian Binomial Law

$$(1+x)^a = 1 + ax + \cdots + \frac{a(a-1)\cdots(a-n+1)}{n!}x^n + \cdots,$$

$$x \in (-1,1)$$

Can we expand $f(x)$ without using direct derivation?

The Differentiability Theorem

If

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \text{for all } x \text{ in } (-c, c),$$

then f is differentiable on $(-c, c)$ and

$$f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}(a_k x^k) \qquad \text{for all } x \text{ in } (-c, c).$$

The Differentiability Theorem

In the interior of its interval of convergence a power series defines an infinitely differentiable function the derivatives of which can be obtained by differentiating term by term:

$$\frac{d^n}{dx^n}\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=0}^{\infty} \frac{d^n}{dx^n}(a_k x^k) \qquad \text{for all } n.$$

The integrability Theorem

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges on $(-c, c)$, then

$F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$ converges on $(-c, c)$ and $\int f(x)\,dx = F(x) + C.$

$$\boxed{\int \left(\sum_{k=0}^{\infty} a_k x^k \right) dx = \left(\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} \right) + C.}$$

Expand $f(x) = \arctan x$ as a power series at 0

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$$\begin{aligned}\arctan x &= \int_0^x \frac{dx}{1+x^2} \\ &= \int_0^x [1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \cdots]dx \\ &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots, \\ & \qquad x \in (-1,1).\end{aligned}$$

Expand $f(x) = \arctan x$ as a power series at 0

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$when\ x = 1,\ \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges

$when\ x = -1,\ \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges

Expand $f(x) = \arctan x$ as a power series at 0

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots, \quad x \in [-1,1].$$

Problems in power series

- Determine domain of convergence
- Expand function as power series at $x=x_0$
- Solve sum of power series

Solving the sum of $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n(n+1)}$