Fourier Transforms

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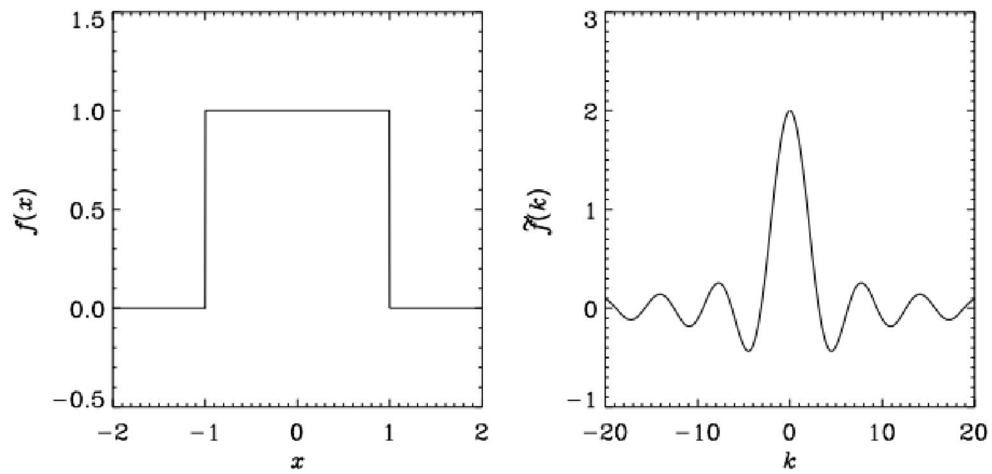
Forward Fourier transform $\tilde{f}(k) = \mathcal{F}[f(x)]$

$$\boxed{\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x}$$



Inverse Fourier transform $f(x) = \mathcal{F}^{-1}[\tilde{f}(k)]$

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k}$$

Fourier transform of a unit constant

$$\boxed{\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{i}kx}\,\mathrm{d}x = 2\pi\delta(k)}$$

Fourier transform of e^{iw_0x}

$$\boxed{\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{i}kx}\,\mathrm{d}x = 2\pi\delta(k)}$$

Fourier transform of $e^{iw_0 x}$

$$\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{i}kx}\,\mathrm{d}x = 2\pi\delta(k)$$

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} e^{iw_0 x} e^{-ikx} dx = \int_{-\infty}^{+\infty} e^{i(w_0 - k)x} dx$$

$$= 2\pi\delta(w_0 - k)$$

Fourier transform of $sinw_0x$

$$\boxed{\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{i}kx}\, \mathrm{d}x = 2\pi\delta(k)}$$

Fourier transform of $sin w_0 x$

$$\boxed{\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{i}kx}\,\mathrm{d}x = 2\pi\delta(k)}$$

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} \sin(w_0 x) e^{-ikx} dx = \int_{-\infty}^{+\infty} \frac{e^{iw_0 x} - e^{-iw_0 x}}{2i} e^{ikx} dx$$

$$= \frac{\pi}{i}[\delta(w_0 + k) - \delta(w_0 - k)]$$

Fourier transform of period functions

$$\boxed{\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{i}kx}\, \mathrm{d}x = 2\pi\delta(k)}$$

Fourier transform of period functions

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} c_n e^{i\omega_n\theta} e^{-ikx} dx = \sum_{n=-\infty}^{+\infty} c_n \int_{-\infty}^{+\infty} e^{i(k-\omega_n)x} dx$$

$$= \sum_{n=-\infty}^{+\infty} 2\pi c_n \delta(k-\omega_n)$$

$$\boxed{\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{i}kx}\, \mathrm{d}x = 2\pi\delta(k)}$$

The convolution theorem

The *convolution* of two functions, $h = f * g$, is defined by

$$\boxed{h(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,\mathrm{d}y}$$

The convolution theorem $\tilde{h}(k) = \tilde{f}(k)\,\tilde{g}(k)$

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The convolution theorem $\tilde{h}(k) = \tilde{f}(k)\,\tilde{g}(k)$

$$
\begin{aligned}
\tilde{h}(k) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(y) g(x-y)\, \mathrm{d}y \right] \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(x-y)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(z)\, \mathrm{e}^{-\mathrm{i}ky}\, \mathrm{e}^{-\mathrm{i}kz}\, \mathrm{d}z\, \mathrm{d}y \qquad (z = x - y) \\
&= \int_{-\infty}^{\infty} f(y)\, \mathrm{e}^{-\mathrm{i}ky}\, \mathrm{d}y \int_{-\infty}^{\infty} g(z)\, \mathrm{e}^{-\mathrm{i}kz}\, \mathrm{d}z \\
&= \tilde{f}(k) \tilde{g}(k)
\end{aligned}
$$

The convolution theorem $\tilde{h}(k) = \tilde{f}(k)\,\tilde{g}(k)$

- Convolution is an operation best carried out as a multiplication in the Fourier domain.
- The FT of *f(x)g(x)* is $\frac{1}{2\pi}\left[\tilde{f} * \tilde{g}\right](k)$.
- If *g* is known and $f{*}g$ is measured, then f can be obtained in principle.

Correlation (互相关函数)

The *correlation* of two functions, $h = f \otimes g$, is defined by

$$h(x) = \int_{-\infty}^{+\infty} \overline{f(y)} g(x+y) dy$$

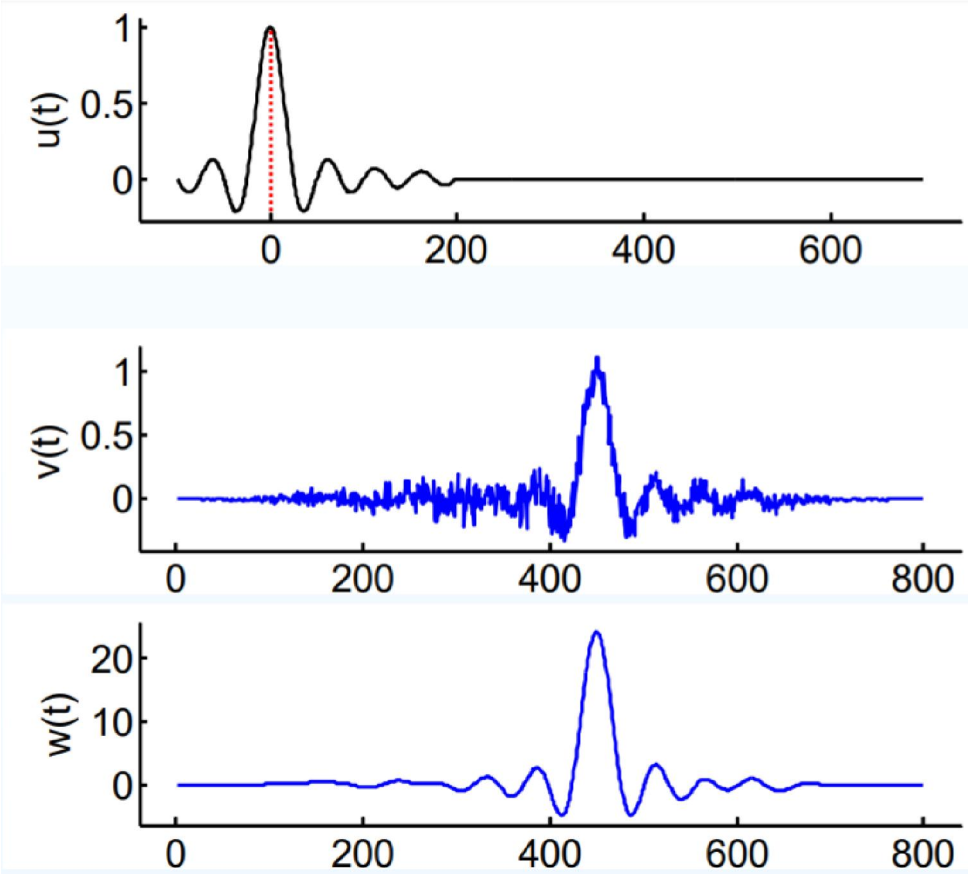
$$(= \int_{-\infty}^{+\infty} [f(y)]^{*} g(x+y) dy)$$

Correlation (互相关函数)

- (Cross-) correlation is a measure of similarity between two signals, where one signal is allowed to be time-shifted. In this sense, the correlation is not a single number, but a function of the time shift. We say, "these two signal have a certain correlation *h(Δx)* for a time shift *Δx*".
- If two signals（oscillating about an average value of zero）oscillate in phase with each other (up to a time shift), their correlation will be positive; If they are out of phase, their correlation will be negative; If they are completely unrelated, their correlation will be zero.

Correlation (互相关函数)

$$w(t) = \int_{-\infty}^{+\infty} v(x)u(t+x)dx$$



Wiener–Khinchin (WK) theorem

$$h(x) = \int_{-\infty}^{+\infty} \overline{f(y)} g(x+y) dy$$

$$\Downarrow$$

$$\tilde{h}(k) = \overline{\tilde{f}(k)} \tilde{g}(k)$$

$$h(x)=\int_{-\infty}^{+\infty}\overline{f(y)}g(x+y)dy$$

Wiener–Khinchin (WK) theorem

$$\Downarrow$$

$$\tilde{h}(k)=\overline{\tilde{f}(k)}\,\tilde{g}(k)$$

$$\begin{aligned}
\tilde{h}(k) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}[f(y)]^{*}g(x+y)\,\mathrm{d}y\right]\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[f(y)]^{*}g(x+y)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[f(y)]^{*}g(z)\,\mathrm{e}^{\mathrm{i}ky}\,\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z\,\mathrm{d}y \qquad (z=x+y) \\
&= \left[\int_{-\infty}^{\infty}f(y)\,\mathrm{e}^{-\mathrm{i}ky}\,\mathrm{d}y\right]^{*}\int_{-\infty}^{\infty}g(z)\,\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z \\
&= [\tilde{f}(k)]^{*}\tilde{g}(k)
\end{aligned}$$

Remarks

- auto-convolution
- auto-correlation

Parseval's theorem (Fourier series)

Suppose

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 \, dx = 2a_0^2 + \sum_{k=1}^{+\infty} (a_k^2 + b_k^2).$$

Parseval's theorem (complex form of Fourier series)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx,$$

Parseval's theorem (Fourier transform)

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, \mathrm{d}k}$$

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, \mathrm{d}k}$$

If we apply the inverse transform to the WK theorem we find

$$\int_{-\infty}^{\infty} [f(y)]^* g(x+y) \, \mathrm{d}y = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\tilde{f}(k)]^* \tilde{g}(k) \, \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k$$

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, \mathrm{d}k}$$

If we apply the inverse transform to the WK theorem we find

$$\int_{-\infty}^{\infty} [f(y)]^* g(x+y) \, \mathrm{d}y = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\tilde{f}(k)]^* \tilde{g}(k) \, \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k$$

Now set $x = 0$ and relabel $y \mapsto x$ to obtain *Parseval's theorem*

$$\int_{-\infty}^{\infty} [f(x)]^* g(x) \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\tilde{f}(k)]^* \tilde{g}(k) \, \mathrm{d}k$$

The special case used most frequently is when $g = f$:

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, \mathrm{d}k}$$

Essential of Parseval’s theorem

- More generally, Parseval‘s identity holds in any inner-product space:

$$\sum_k |\langle x, e_k \rangle|^2 = \|x\|^2$$

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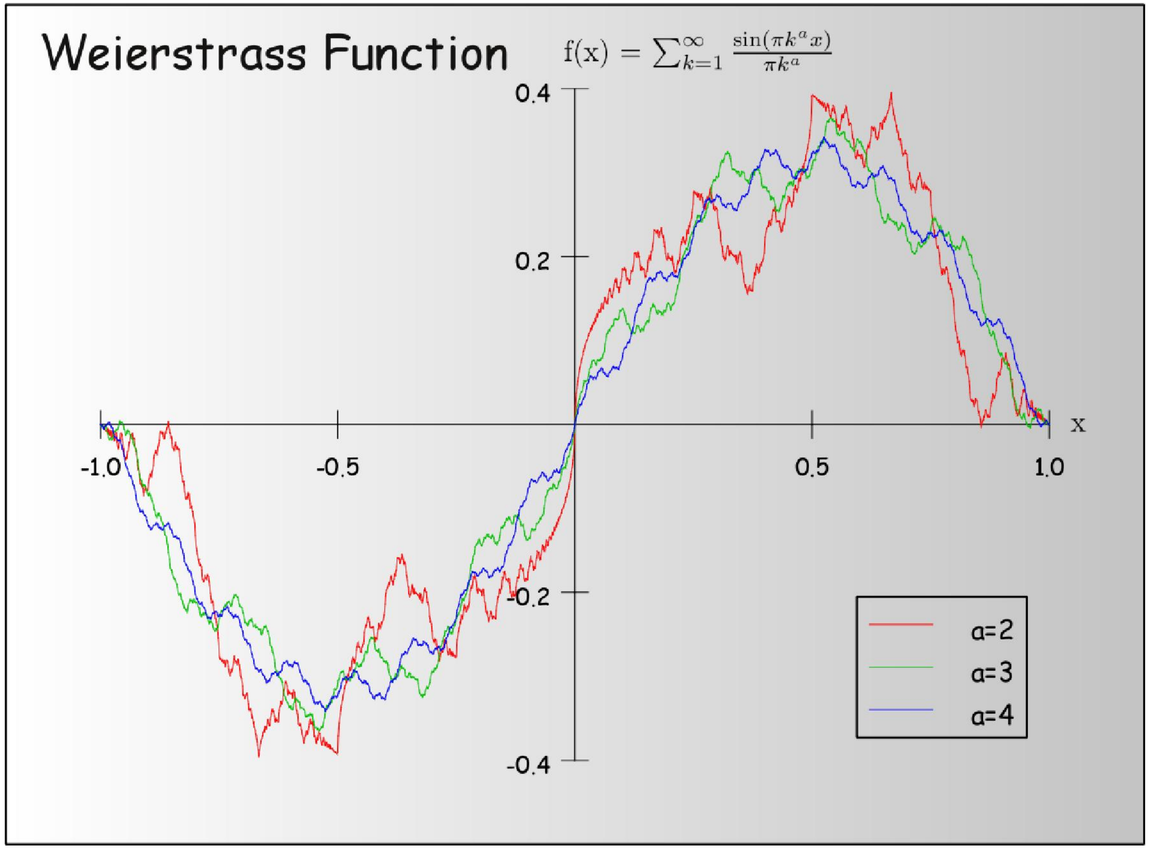
$$\sum_k |\langle x, e_k \rangle|^2 = \|x\|^2$$

The Parseval’s theorem is a generalization of the Pythagorean theorem!!

Power spectra $\Phi(k) = |\tilde{f}(k)|^2$

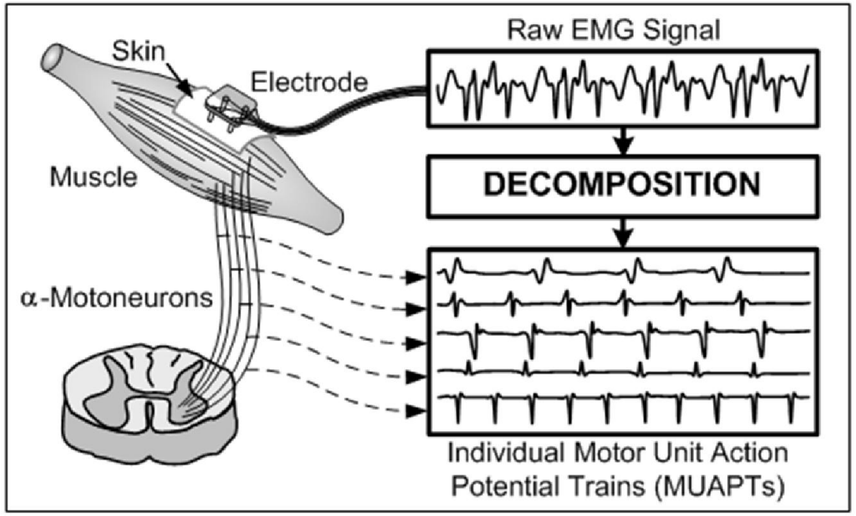
Summary of infinite series section

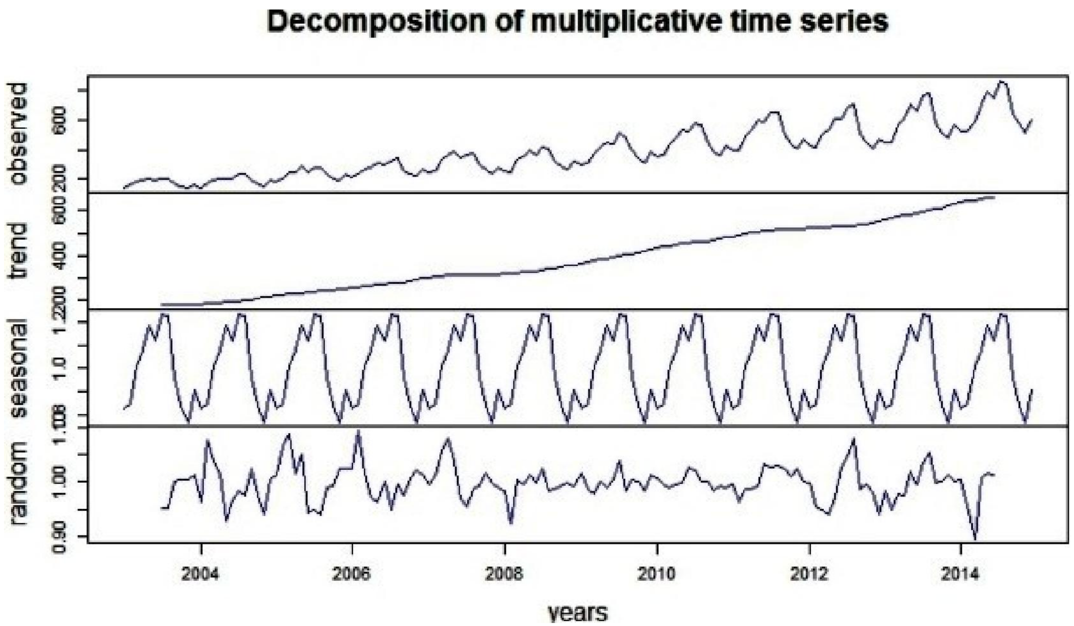
- Infinite series greatly enrich the concept of function



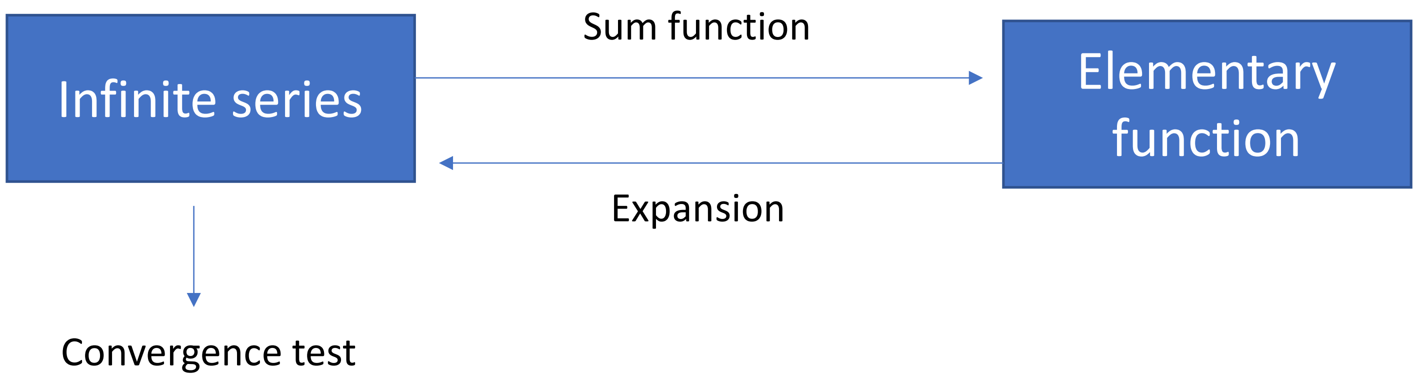
Summary of infinite series section

- “Breaking up the whole into parts” (化整为零)
- “Feature representation”（特征表示）





Summary of infinite series section



Summary of infinite series section