Infinite series for complex variable

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Function of a complex variable

$$z = x + iy$$
$$w = f(z) = u(x, y) + iv(x, y)$$

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Examples :

$$w = a + bz + cz^2, \qquad w = A\sin(\sqrt{z})$$

$$w = \frac{a + bz}{c + dz + ez^2}, \qquad w = \sum_{n=0}^{n=+\infty} a_n z^n$$

How to plot a complex function?

https://samuelj.li/complex-function-plotter/#

Limit and continuity

- $\lim_{z\to z_0} f(z) = a + bi \Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y) = a, \lim_{(x,y)\to(x_0,y_0)} v(x,y) = b$

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Remarks:

- Uniform continuity
- Properties of continuous function on bounded and closed region

Differentiation

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Surprisingly strong constraint!!

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$$f(z) = u(x, y) + iv(x, y)$$

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \frac{df}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Cauchy-Riemann equations

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Given that $u(x,y)$ **and** $v(x,y)$ **are differentiable :**

The Cauchy - Riemann equations are a ***necessary and sufficient*** **condition for the existence of the derivative** $\frac{df}{dz}$**.**

- $f(z) = \exp(z)$

In the case of the exponential function we have

$$f(z) = \mathrm{e}^z = \mathrm{e}^x \mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x \cos y + \mathrm{i}\,\mathrm{e}^x \sin y = u + \mathrm{i}v$$

The CR equations are satisfied for all x and y:

$$\frac{\partial u}{\partial x} = \mathrm{e}^x \cos y = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = \mathrm{e}^x \sin y = -\frac{\partial u}{\partial y}$$

The derivative of the exponential function is

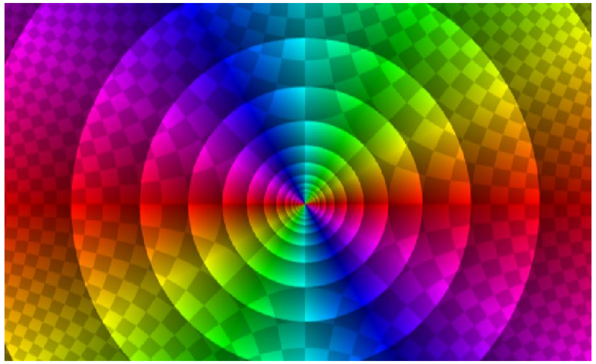
$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\,\frac{\partial v}{\partial x} = \mathrm{e}^x \cos y + \mathrm{i}\,\mathrm{e}^x \sin y = \mathrm{e}^z$$

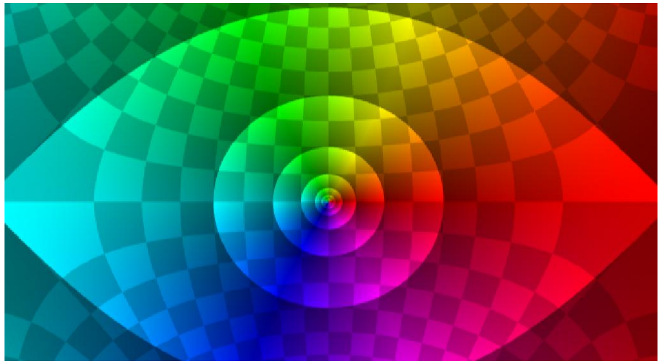
as expected.

Analytic function

$f(z)$ is said to be "analytic" at z_0 if the derivative exsits in some neighborhood of z_0;

$f(z)$ is said to be "analytic" in a domain D if the derivative exsits at each point in D.





- **The theory of complex variables largely exploits the remarkable properties of analytic functions.**

- **The terms "holomorphic", "regular", and "differentiable" are also used instead of "analytic."**

Differentiation rules

- **Replacing x by z in the usual derivations for functions of a real variable, we find practically all differentiation rules for functions of a complex variable turn out to be identical to those for real variables :**

$$\frac{d\left(f(z)\pm g(z)\right)}{dz} = f'(z)\pm g'(z)$$

$$\frac{d\left(f(z)g(z)\right)}{dz} = f'(z)g(z)+f(z)g'(z)$$

$$\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g\,f'-f\,g'}{(g)^2}$$

Examples

$$1)\ \frac{dz^n}{dz} = nz^{n-1}$$

$$2)\ \frac{de^z}{dz} = e^z$$

$$3)\ \frac{d\sin z}{dz} = \cos z, \frac{d\cos z}{dz} = -\sin z$$

$$4)\ \frac{d\sinh z}{dz} = \cosh z$$

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Multiple-valued functions require special treatment:

1) $f(z)=\ln(z)$
2) $f(z)=z^a$ (a is a real number)

Is $\overline{z^2}$ analytic on complex plane?

Singular point

Many complex functions are analytic everywhere in the complex plane except at isolated points, which are called the singular points or singularities of the function.

Power series of complex variable

$$\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots$$

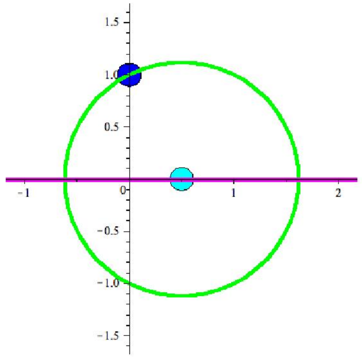
$$\sum_{n=0}^{\infty} c_n (z-a)^n = c_0 + c_1 (z-a) + c_2 (z-a)^2 + \cdots + c_n (z-a)^n + \cdots$$

Circle of convergence

- Determination of the radius of convergence

$$r=\begin{cases}0, & \rho=+\infty\\ +\infty, & \rho=0\\ 1/\rho, & \rho\neq 0\end{cases}\quad,\quad \rho=\lim_{n\to+\infty}\left|\frac{c_{n+1}}{c_n}\right|,\qquad (c_n\neq 0)$$

- The radius of convergence of the Taylor series of a function about $z=z_0$ is equal to the distance of the nearest singular point of the function from z_0.



Taylor series for an analytic function

The derivative of an analytic function is also analytic.

$f(z)$ is analytic

$\Downarrow$

$f'(z)$ is analytic

$\Downarrow$

$f''(z)$ is analytic

$\Downarrow$

$\vdots$

Hence, all derivatives of an analytic function are also analytic.

Taylor series for an analytic function

If $f(z)$ is analytic at $z = z_0$, then it has a unique Taylor series

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

analytic $\Leftrightarrow$ Taylor series expansion

Real analytic function

- f(x) is said to be analytic at x_0 if it has Taylor series at $x_{0.}$

$$f(x) \text{ is analytic } \Leftrightarrow \begin{cases} \text{infinitely differentiable} \\ \text{remainder term } R_n(x) \xrightarrow{n\to\infty} 0 \end{cases}$$

Taylor series in common use

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + z^4 + \cdots$$

Singular point

Many complex functions are analytic everywhere in the complex plane except at isolated points, which are called the singular points or singularities of the function.

$$\text{e.g. } f(z) = \frac{e^z}{z},\ \ z = 0 \text{ is a singular point.}$$

Classification of singular points

- Movable singularities, e.g.

$$f(z) = \frac{\sin z}{z}$$

- Poles, .e.g.

$$f(z) = \frac{\sin z}{z^k} \quad (k > 1)$$

- Essential singularities, e.g.

$$f(z) = e^{\frac{1}{z}}$$

Movable singularities

$$f(z) = \frac{1}{(z - z_0)^k} \sum_{n=k}^{+\infty} c_n (z - z_0)^n \quad (k \geq 1)$$

$$= \sum_{n=0}^{+\infty} c_{n+k} (z - z_0)^n$$

The order of Poles:

$$f(z)=\sum_{n=-k}^{+\infty} c_n (z-z_0)^n \quad (k \geq 1,\ \ c_{-k} \neq 0)$$

A simple pole is a pole of order 1. A double pole is one of order 2, etc

Example:

$$f(z) = \frac{2z}{(z+1)(z-\mathrm{i})^2}$$

has a simple pole at $z = -1$ and a double pole at $z = \mathrm{i}$ (as well as a simple zero at $z = 0$). The expansion about the double pole can be carried out by letting $z = \mathrm{i} + w$ and expanding in w:

$$\begin{aligned}
f(z) &= \frac{2(\mathrm{i}+w)}{(\mathrm{i}+w+1)w^2} \\
&= \frac{2\mathrm{i}(1-\mathrm{i}w)}{(\mathrm{i}+1)\left[1+\frac{1}{2}(1-\mathrm{i})w\right]w^2} \\
&= \frac{2\mathrm{i}}{(\mathrm{i}+1)w^2}(1-\mathrm{i}w)\left[1-\frac{1}{2}(1-\mathrm{i})w+O(w^2)\right] \\
&= (1+\mathrm{i})w^{-2}\left[1-\frac{1}{2}(1+\mathrm{i})w+O(w^2)\right] \\
&= (1+\mathrm{i})(z-\mathrm{i})^{-2} - \mathrm{i}(z-\mathrm{i})^{-1} + O(1) \qquad \text{as} \quad z \to \mathrm{i}
\end{aligned}$$

pp-139, 剑桥讲义

Essential singularities

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n \quad \text{(infinite number of terms with } < 0)$$

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$$\text{e.g. } e^{\frac{1}{z}} = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{-\infty} \frac{1}{n!} z^n$$

Laurent series in annulus

Suppose $f(z)$ is analytic throughout an annulus $a < |z - z_0| < b$, then it has a unique Laurent series as follows

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n$$

Example: $f(z) = \dfrac{1}{1-z} \qquad (|z| < 1 \text{ or } |z| > 1)$

$$\text{Example:} f(z)=\frac{1}{(z-1)(z-2)},(1<|z|<2 \text{ or } 2<|z|<+\infty)$$

Zeros

The zeros of $f(z)$ are the points $z = z_0$ in the complex plane where $f(z_0) = 0$. A zero is of order N if

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(N-1)}(z_0) = 0 \qquad \text{but} \quad f^{(N)}(z_0) \neq 0$$

- $f(z) = z$ has a simple zero at $z = 0$
- $f(z) = (z - \mathrm{i})^2$ has a double zero at $z = \mathrm{i}$
- $f(z) = z^2 - 1 = (z-1)(z+1)$ has simple zeros at $z = \pm 1$

Zero and Pole

- if $f(z)$ has a zero of order N at $z = z_0$, then $1/f(z)$ has a pole of order N there, and vice versa

- if $f(z)$ is analytic and non-zero at $z = z_0$ and $g(z)$ has a zero of order N there, then $f(z)/g(z)$ has a pole of order N there

Summary

- The existence of derivative is a surprisingly strong condition for complex functions
  - CR equation
  - Cauchy Theorem
  - Cauchy formula
  - Analytic (power series)

- Laurent series is a generalization of power series.