we see

$$\hat{f}_k = \frac{[f](\xi)}{2\pi k} e^{-i(k\xi+\pi/2)} + O\left(\frac{1}{k^2}\right). \tag{3.1}$$

The magnitude and phase of these Fourier coefficients are then approximately given by

$$|\hat{f}_k| \approx \left|\frac{[f](\xi)}{2\pi k}\right| \text{ and } \varphi(\hat{f}_k) \approx -\left(k\xi + \frac{\pi}{2}\operatorname{sgn}\left(k\cdot[f](\xi)\right)\right). \tag{3.2}$$

respectively. As seen in (3.2), the jump location ξ is encoded in the phase, while the jump height $[f](\xi)$ is encoded in the magnitude of the Fourier coefficients. Note that due to Euler's identity $e^{i\pi} = -1$, the sign of the jump and the sign of k are also encoded in the phase.

The standard design approach reviewed in Section 2.1 looks for functions σ which satisfy (2.3). Suppose σ is such a concentration factor. We would like to explore the possibility of modifying this framework to develop concentration factors $\tilde{\sigma}$ which yield $[r](x)$ from (2.3) when the spectral magnitude is removed from the Fourier coefficients $\hat{r}_k$.

To that end, define

$$\tilde{r}_k := \begin{cases} \frac{\hat{r}_k}{|\hat{r}_k|} & k \neq 0, \\ 0 & k = 0. \end{cases} \tag{3.3}$$

If the jump response from full data is given by $W_0^{\sigma}(x)$, let the jump response built from phase data be defined as

$$W_0^{\tilde{\sigma}}(x) := \sum_{0<|k|\leq N} i\operatorname{sgn}(k)\tilde{\sigma}(k)\tilde{r}_k e^{ikx}. \tag{3.4}$$

If $\tilde{\sigma}(k) = \sigma(k)|\hat{r}_k|$, then

$$\sum_{0<|k|\leq N} i\operatorname{sgn}(k)\tilde{\sigma}(k)\tilde{r}_k e^{ikx} = \sum_{0<|k|\leq N} i\operatorname{sgn}(k)\sigma(k)|\hat{r}_k|\frac{\hat{r}_k}{|\hat{r}_k|} e^{ikx} = W_0^{\sigma}(x), \tag{3.5}$$

so $\tilde{\sigma}(k)$ is a concentration factor exactly giving $W_0^{\sigma}(x)$ from only the phase information $\tilde{r}_k$. As in (2.1), the Fourier coefficients of the ramp function are given by $\hat{r}(k) = 1/(2\pi i k)$ whenever $k \neq 0$ and $\hat{r}(0) = 0$. Removing the magnitude from the Fourier coefficients of the ramp function,

$$\tilde{r}_k = -i\operatorname{sgn}(k).$$

With $\tilde{r}_k$ and the concentration factor $\tilde{\sigma}$, (3.4) simplifies to

$$W_0^{\tilde{\sigma}}(x) = \sum_{0<|k|\leq N} \tilde{\sigma}(k) e^{ikx}.$$

Notice that when $\tilde{\sigma}(k) = 1$, this summation yields the Dirichlet kernel with a DC offset. If the concentration factor effectively normalizes the height, then this converges to a shifted Kronecker delta (1.6).

This concentration factor then can also yield an approximation $S_N^{\tilde{\sigma}}[f](x)$. It should be stated that the amplitude of the jumps in the partial sum approximation cannot be expected to be accurate, since we are not given amplitude data. In particular, the solutions assume that the magnitude of the Fourier coefficients are $1/(2\pi|k|)$. Since the Fourier coefficients of a piecewise-smooth function decay with $O(1/k^2)$, this assumption at least stays on the correct order of magnitude for the Fourier coefficients.

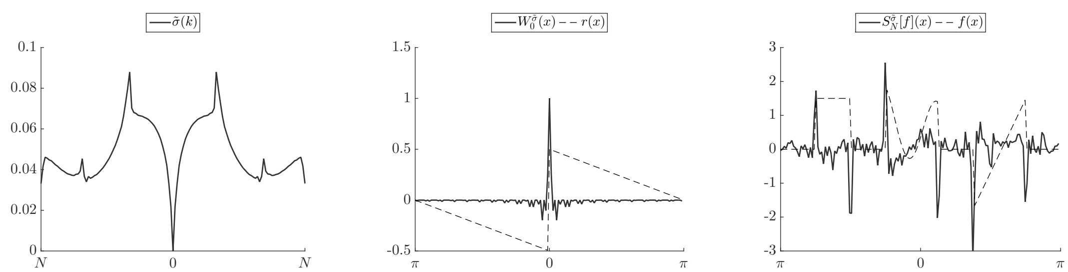
Thus, we can pose a problem formulation where only the spectral phase data $\hat{f}_k/|\hat{f}_k|$ is known, and recover the jumps $[f](x)$ of the underlying function.

Problem Formulation 5

$$\begin{aligned} \min_{\tilde{\sigma}} \quad & \left\| W_0^{\tilde{\sigma}}(x) - [r]_\epsilon(x) \right\|_2 + \lambda \left\| W_0^{\tilde{\sigma}}(x) \right\|_1 \\ \text{subject to} \quad & W_0^{\sigma}(x)\big|_{x=0} = 1. \end{aligned} \tag{3.6}$$

This gives a concentration factor $\tilde{\sigma}$ with the jump response and jump approximation shown in Figure 3.2 with $\lambda = 0.5$. Due to the missing information of the jump heights in the spectral data, only the jump locations are exact, while the jump heights are not properly scaled. Notice that that the jump response is identical to that from (2.12), the same problem formulation with full data.

Figure 3.2: Formulation 5 solution $\tilde{\sigma}$, $W_0^{\tilde{\sigma}}(x)$, and $S_N^{\tilde{\sigma}}[f](x)$ from Fourier phase data $\hat{f}_k/|\hat{f}_k|$, with $\lambda = 0.5$, $N = 64$.



It can be seen in Figure 3.2 that there are some spurious jumps in the domain. Some false positives are expected when many harmonics either share the same phase (leading to step edges) or are out of phase by exactly $\pi/2$ radians (leading to delta edges) since phase congruency will be high at regular intervals in the domain [3] when the frequencies are weighted identically. This is exacerbated in the case of banded or intermittent data, where the set of missing frequencies will contain some which either add