Edge Detection from Fourier Phase Data

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Overview

1. Introduction
   - Edge Detection
   - Fourier Analysis Background
   - Concentration Factor Method
2. Recent Results
3. Current Work
   - Fourier Phase Data
   - Analysis
   - Computational Examples
4. Future Work
5. Acknowledgements

What is Edge Detection?

Example: Edge Detection on an Image

Applications of Edge Detection

Primary applications:

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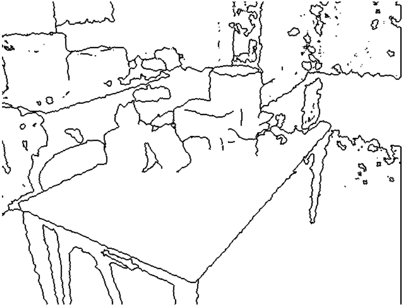
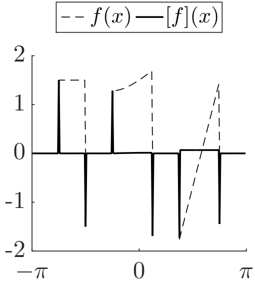


Applications of Edge Detection

Primary applications:

- Magnetic Resonance Imaging (MRI)
- Synthetic Aperture Radar (SAR)
- Computer/Machine Vision (e.g. Kinect)
- 1-D Case: Discontinuities of a Function



Fourier Analysis and Synthesis

A Fourier series is a harmonic expansion of a periodic function

Fourier Series (Synthesis)

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx} \qquad\qquad S_N f(x) = \sum_{k=-N}^{N} \hat{f}_k e^{ikx}$$

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where the **Fourier coefficients** $\hat{f}_k$ are given by the Fourier transform of f.

Fourier Transform (Analysis)

$$\hat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$$

Convolution

We can also express $S_N f(x)$ via convolution

Convolution with the Dirichlet Kernel

$$(f * D_N)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) D_N(x-y)\,dy = S_N f(x)$$

where D_N is the N-th Dirichlet Kernel, given by

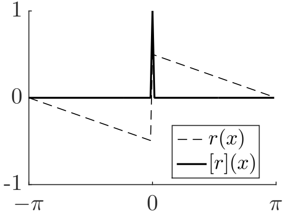
Dirichlet Kernel

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx}$$

Concentration Edge Detection Method

Define the **jump function** of f, $[f](x)$, as the difference between the right and left hand limits of the function f at every point x; viz.,

$$[f](x) = f(x^{+}) - f(x^{-}).$$



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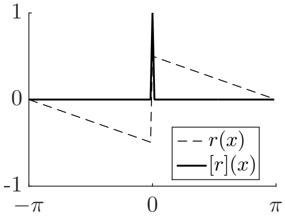

There exists a relationship between these limits and the Fourier partial sum approximation $S_N f(x)$

Dirichlet's Theorem (1824)

$$(f * D_N)(x) = S_N f(x) = \sum_{k=-N}^{N} \hat{f}_k e^{ikx} \longrightarrow \frac{1}{2}\left(f(x^+) + f(x^-)\right)$$

Concentration Edge Detection Method (cont.)

The edge detection method developed by Gelb and Tadmor (1999) introduces a function $\sigma(k)$, called a **concentration factor**, which modifies the Dirichlet kernel to *concentrate* the partial sum along the singular support of the underlying function.

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Concentration Factor Jump Detector

$$(f * K_N)(x) = S_N^\sigma[f](x) = \sum_{k=-N}^{N} \sigma(k)\hat{f}_k e^{ikx} \longrightarrow [f](x)$$

Analytical Concentration Factors

Examples of some analytical concentration factors:

Factor	Expression	Remarks
Trigonometric	$\sigma(\eta) = \frac{\pi \sin(\pi\eta)}{\mathrm{Si}(\pi)}$	$\mathrm{Si}(\pi) = \int_0^\pi \frac{\sin(x)}{x}\, dx$
Polynomial	$\sigma(\eta) = p\pi\eta^p$	p is the order of the factor
Exponential	$\sigma(\eta) = C\eta e^{\frac{1}{\alpha\eta(\eta-1)}}$	α is the order
		C is a normalizing constant
		$\frac{\pi}{C} = \int_{\frac{1}{N}}^{1-\frac{1}{N}} \exp\left(\frac{1}{\alpha\tau(\tau-1)}\right) d\tau$

Jump Reconstruction from Fourier Data

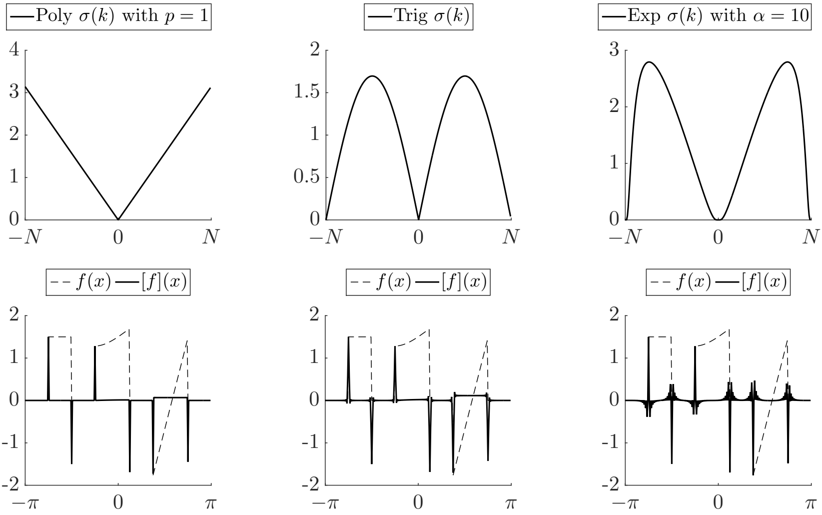

Figure: $S_N[f](x)$ with various concentration factors σ.

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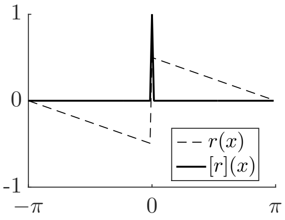
We can engineer σ for a function $g(x)$ where we know what $[g](x)$ is.

Designed Concentration Factors (cont.)

Consider the ramp function $r(x)$ with the associated jump function $[r](x)$ on the interval $[-\pi, \pi)$:

$$r(x) = \begin{cases} \frac{1}{2\pi}(-x-\pi) & \text{if } x < 0, \\ \frac{1}{2\pi}(-x+\pi) & \text{if } x \geq 0. \end{cases}$$

$$[r](x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$



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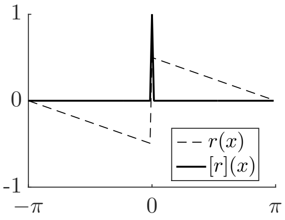

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We can solve the system $S\sigma = [r](x)$ for a solution $\sigma(k)$ to the equation

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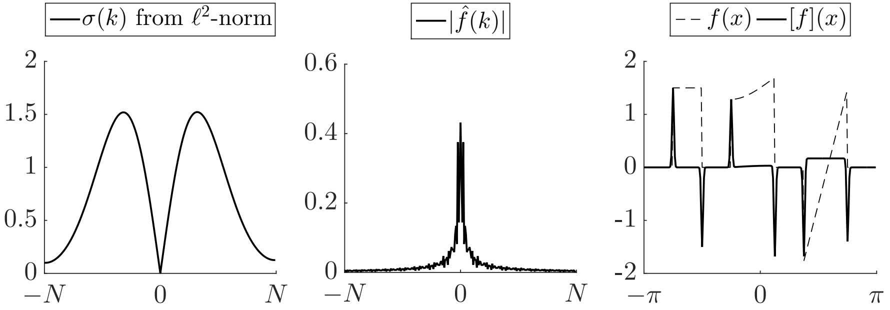

Figure: $S_N[f](x)$ from σ and $\hat{f}$

Fourier Phase Data

What is Fourier Phase Data?

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Fourier Series

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The Fourier coefficients $\hat{f}_k$ are complex numbers, i.e., $\hat{f}_k = \alpha_k e^{i\theta_k}$, where α_k is the **amplitude** and $e^{i\theta_k}$ is the **phase**.

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Is it still possible to reconstruct the jumps from $\tilde{f}_k$?

Jump Location from Fourier Phase

Suppose f is a 2π-periodic function on $[-\pi, \pi)$ with a single jump discontinuity at $x = \xi$. Integration by parts reveals a relationship Fourier coefficients of f and the jump function $[f]$.

Fourier Coefficients

$$\begin{aligned}\hat{f}_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx \\ &\approx \frac{[f](\xi)e^{-ik\xi}}{2\pi i k} + \mathcal{O}\left(\frac{1}{k^2}\right) \\ &\approx \frac{[f](\xi)}{2\pi k}e^{-i(k\xi+\pi/2)}\end{aligned}$$

Jump Reconstruction from Fourier Phase Data

We can again solve the system $S\tilde{\sigma} = [r](x)$ for a solution $\tilde{\sigma}$ to

$$S_N^{\tilde{\sigma}}[f](x) = \sum_{k=-N}^{N} \tilde{\sigma}(k)\tilde{f}_k e^{ikx} \longrightarrow [f](x)$$

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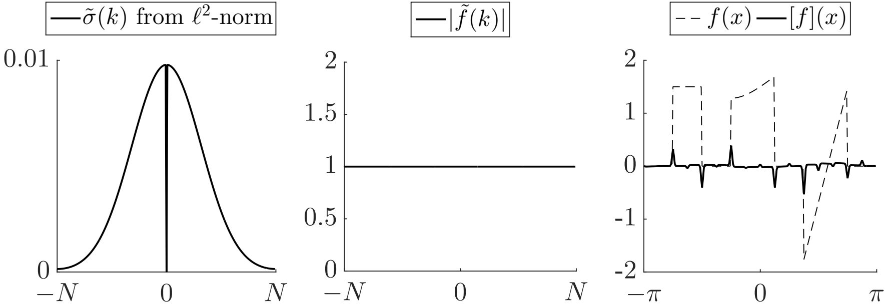

Figure: Reconstruction of $[f](x)$ from $\tilde{\sigma}$ and $\tilde{f}$

Jump Reconstruction from Fourier Phase Data

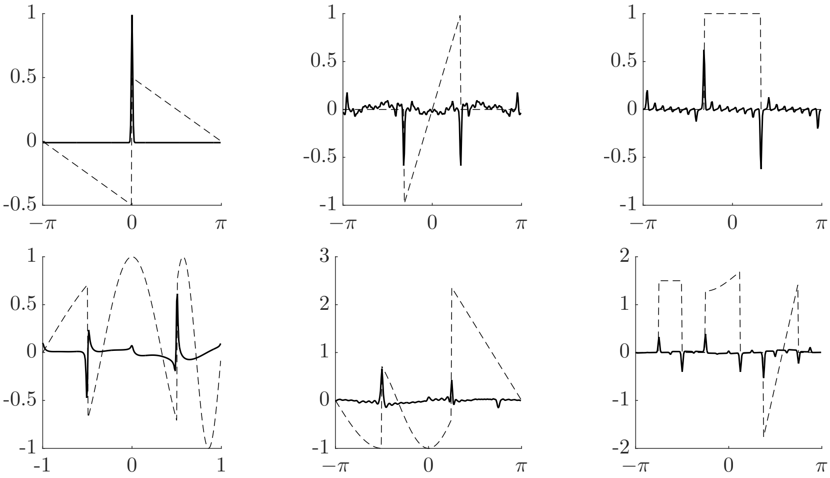

Figure: Reconstructions of multiple functions from $\tilde{\sigma}$ and $\tilde{f}$.

Noisy Data

We used the MATLAB command `awgn(data,SNR)` to generate additive white Gaussian (complex) noise onto the Fourier data.

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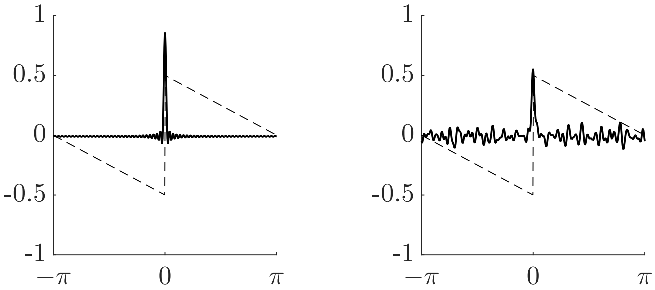

Figure: Ramp jump approximation with and without noise

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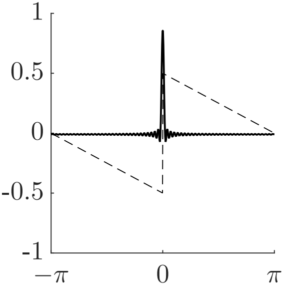
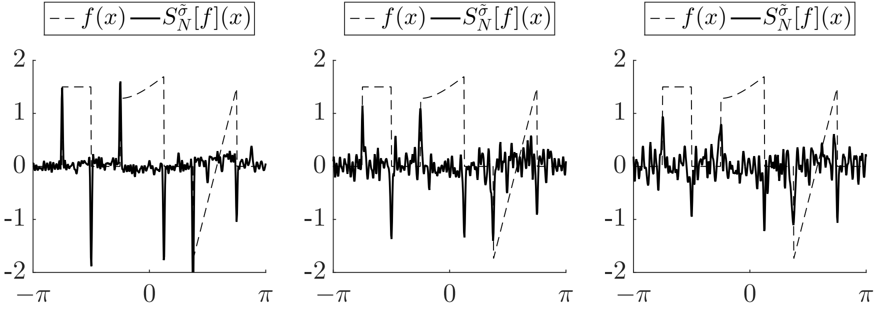


Figure: Ramp jump approximation with and without noise

We can compare how different concentration factors hold up in different levels of noise.

Noisy Phase Data Jump Approximations

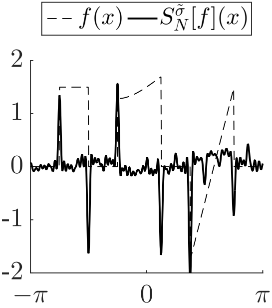


(a) SNR $= 24$ dB (b) SNR $= 12$ dB (c) SNR $= 6$ dB

Figure: Jump approximations given noisy Fourier phase data using analytical trigonometric $\tilde{\sigma}$.

Noisy, Banded Phase Data Jump Approximations

Finally, we can consider noisy, banded Fourier phase data



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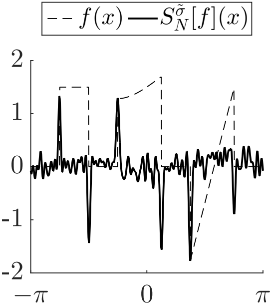
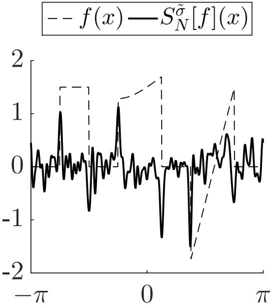


(b) SNR $= 12$ dB



(c) SNR $= 6$ dB

Figure: Jump approximations given noisy, low-frequency banded Fourier phase data using the designed $\tilde{\sigma}$ from ℓ_2-norm.

Future Work

Computational possibilities:

- Combining multiple (orthogonal) concentration factors
- Generalizations to two dimensions
- Non-uniform sampling methods

Analytical work ahead:

- Investigate quantity of data necessary for accurate jump height
- Statistical analysis for noisy phase data

Other directions:

- Application specific priors
- Comparing with real data
- Corners from edges

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- Rachel Nahon