Edge Detection from Spectral Phase Data

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Barrett Thesis Defense
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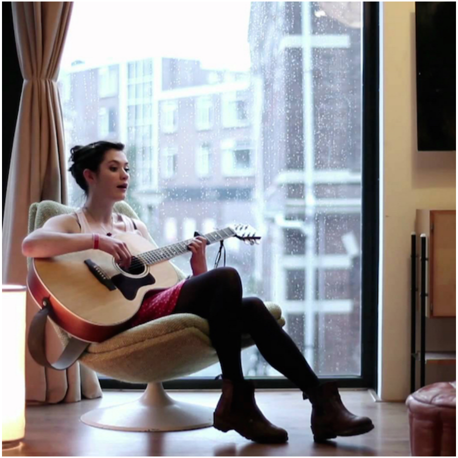
Overview

What is Edge Detection?



Example: Edge Detection on an image

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Applications of Edge Detection

Primary applications:

- Magnetic Resonance Imaging (MRI)

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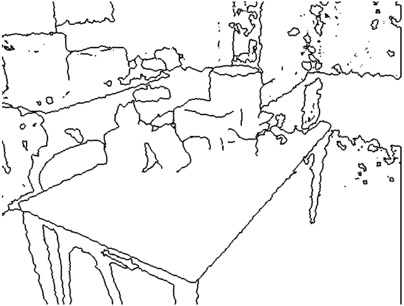

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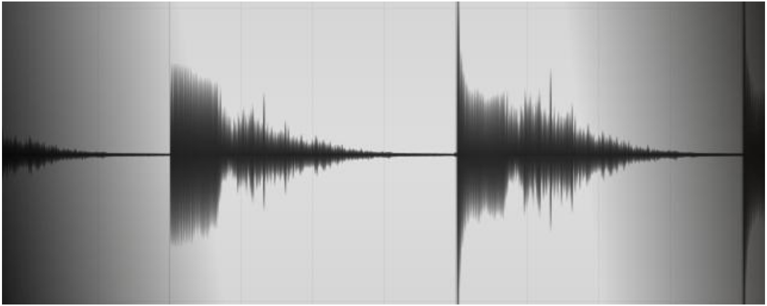
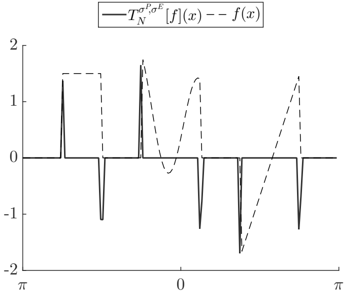


Primary applications:

- Magnetic Resonance Imaging (MRI)
- Synthetic Aperture Radar (SAR)
- Computer/machine vision (object tracking, stereo matching)
- Transients of a signal (triggering)
- Discontinuities in functions (numerical accuracy)



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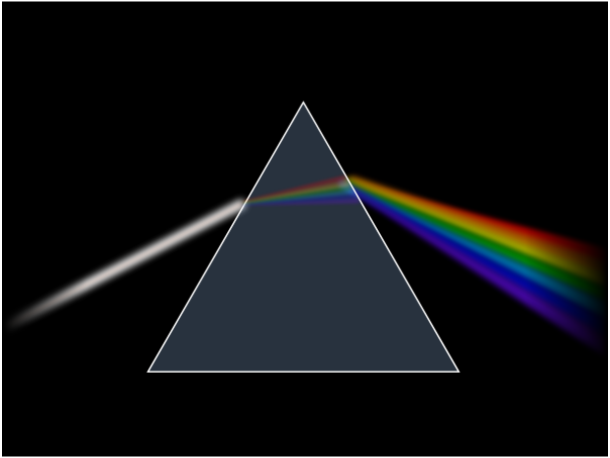

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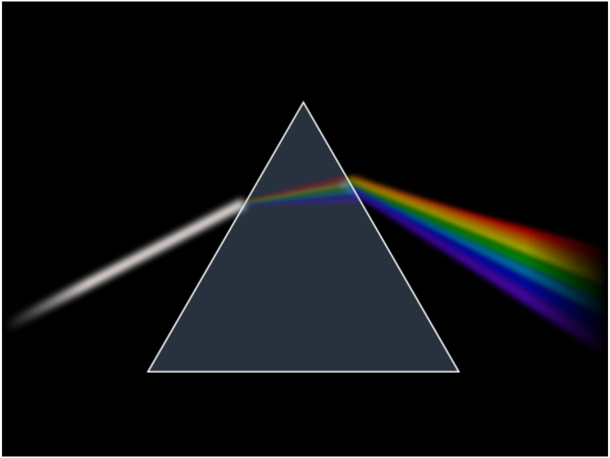
Physical



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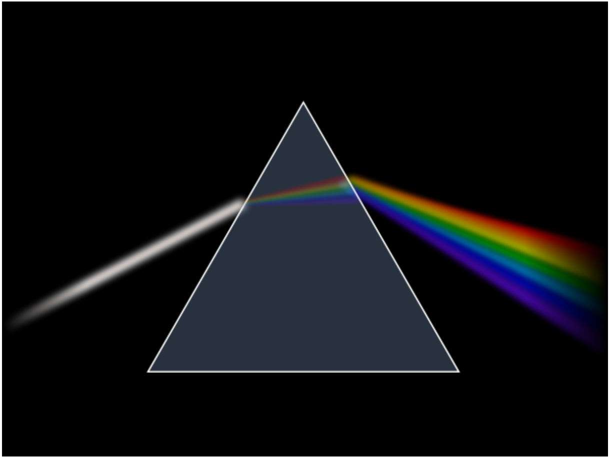

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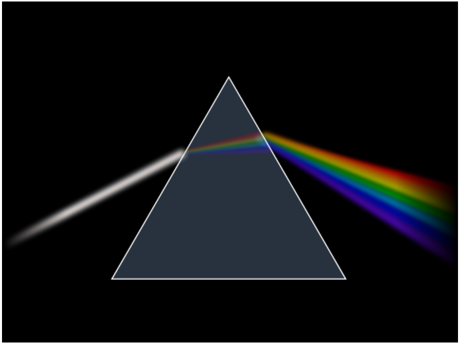

- "Components" of a signal

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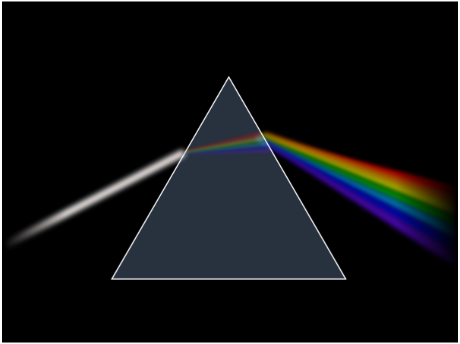

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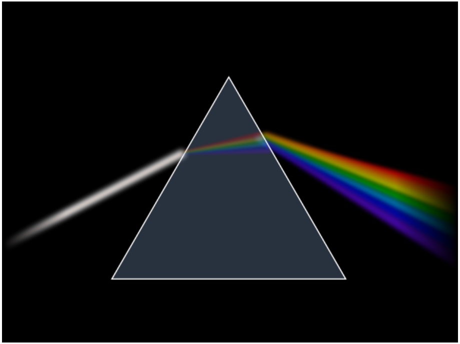

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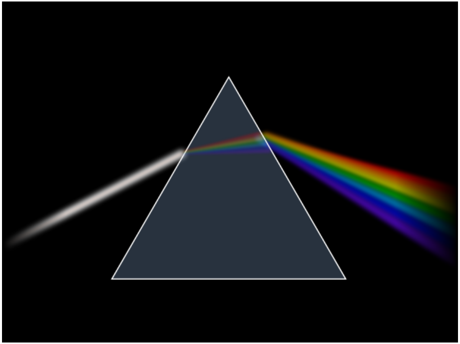

- "Components" of a signal
- Amplitude vs. frequency
- Redundancy
- Collected in telemetry, microscopy, radar imaging, medical imaging, etc.

Fourier Analysis and Synthesis

A Fourier series is a harmonic expansion of a periodic function

Fourier Series (Synthesis)

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx} \qquad S_N f(x) = \sum_{k=-N}^{N} \hat{f}_k e^{ikx}$$

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where the **Fourier coefficients** $\hat{f}_k$ are given by the Fourier transform of f.

Fourier Transform (Analysis)

$$\hat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$$

Convolution

We can also express $S_N f(x)$ via convolution

Convolution with the Dirichlet Kernel

$$(f * D_N)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) D_N(x-y)\, dy = S_N f(x)$$

where D_N is the N-th Dirichlet Kernel, given by

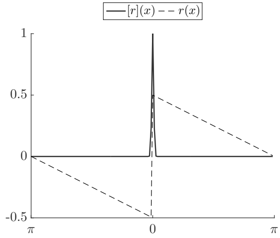
Dirichlet Kernel

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx}$$

Concentration Factor Method

Define the **jump function** of f, $[f](x)$, as the difference between the right and left hand limits of the function f at every point x; viz.,

$$[f](x) = f(x^+) - f(x^-).$$



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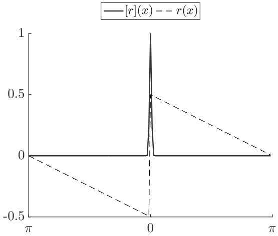

There exists a relationship between these limits and the Fourier partial sum approximation $S_N f(x)$:

Dirichlet's Theorem (1824)

$$(f * D_N)(x) = S_N f(x) = \sum_{k=-N}^{N} \hat{f}_k e^{ikx} \longrightarrow \frac{1}{2}\big(f(x^+) + f(x^-)\big)$$

Concentration Edge Detection Method (cont.)

The edge detection method developed by Gelb and Tadmor (1999) introduces a function $\sigma(k)$, called a **concentration factor**, which modifies the Dirichlet kernel to *concentrate* the partial sum along the singular support of the underlying function.

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Concentration Factor Jump Detector

$$(f * K_N)(x) = S_N^{\sigma}[f](x) = \sum_{0<|k|\le N} i\,\mathrm{sgn}(k)\sigma(k)\hat{f}_k e^{ikx} \longrightarrow [f](x)$$

Analytical Concentration Factors

Examples of some analytical concentration factors:

Factor	Expression	Remarks
Trigonometric	$\sigma(\eta) = \frac{\pi \sin(\pi\eta)}{\mathrm{Si}(\pi)}$	$\mathrm{Si}(\pi) = \int_0^\pi \frac{\sin(x)}{x}\,dx$
Polynomial	$\sigma(\eta) = p\pi\eta^p$	p is the order of the factor
Exponential	$\sigma(\eta) = C\eta e^{\frac{1}{\alpha\eta(\eta-1)}}$	α is the order
		C is a normalizing constant
		$\frac{\pi}{C} = \int_{\frac{1}{N}}^{1-\frac{1}{N}} \exp\left(\frac{1}{\alpha\tau(\tau-1)}\right) d\tau$

Jump Reconstruction from Fourier Data

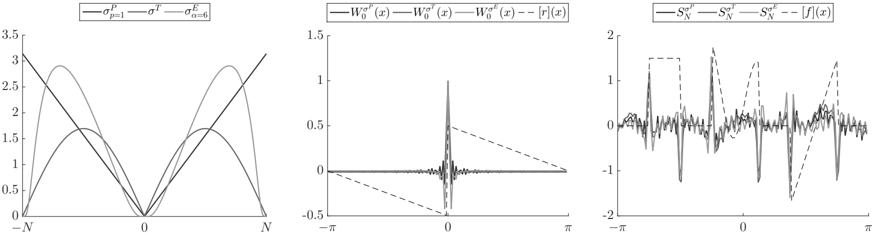

Figure: Polynomial, trigonometric, and exponential concentration factors $\sigma(k)$, jump responses $S_N^\sigma[r](x)$, jump approximations $S_N^\sigma[f](x)$.

Jump Reconstruction from Fourier Data

Figure: Polynomial concentration factor jump approximation on an image.

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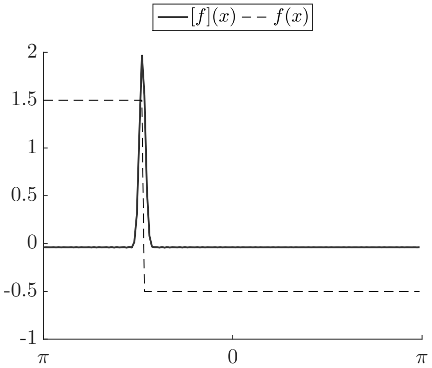
Goal: Find σ so that $S_N^\sigma[f](x) \approx [f](x)$



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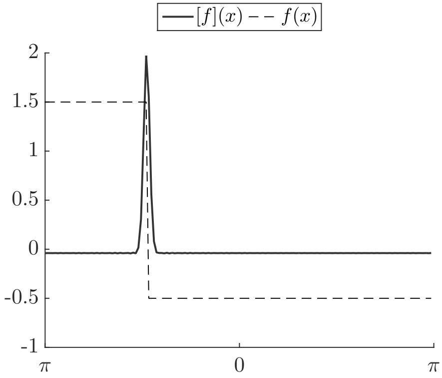

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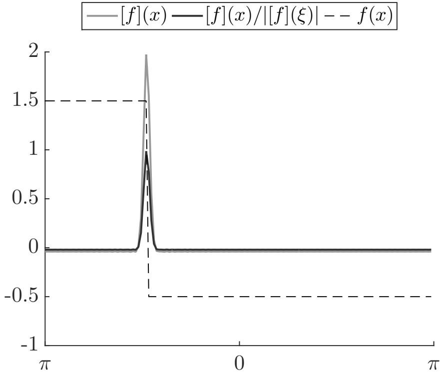
1. Take jump function $[f](x)$
2. Normalize the height



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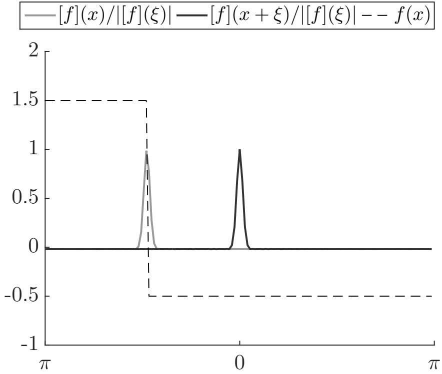
3. Shift by ξ



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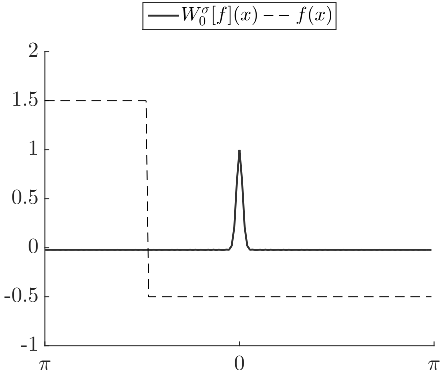
1. Take jump function $[f](x)$
2. Normalize the height
3. Shift by ξ
4. Find σ for new jump function



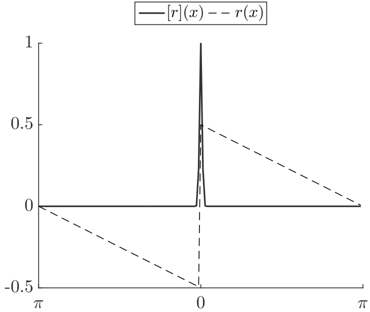
Designed Concentration Factors

Consider the ramp $r(x)$ on $[-\pi, \pi)$:

$$r(x) = \begin{cases} \frac{1}{2\pi}(-x-\pi) & \text{if } x < 0, \\ \frac{1}{2\pi}(-x+\pi) & \text{if } x \geq 0. \end{cases}$$

$$[r](x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

$$\hat{r}_k = \begin{cases} \frac{1}{2\pi i k} & \text{if } k \neq 0, \\ 0 & \text{if } k = 0. \end{cases}$$



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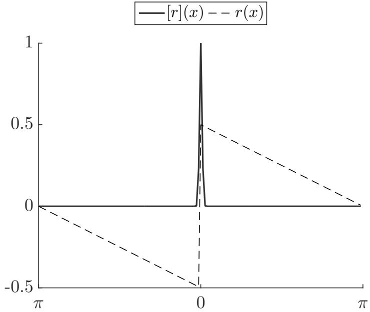

Jump Response

$$W_0^\sigma(x) := S_N^\sigma[r](x) = \sum_{0<|k|\leq N} i\,\mathrm{sgn}(k)\sigma(k)\hat{r}_k e^{ikx} \approx [r](x)$$

Designed Concentration Factors (cont.)

The Fourier coefficients of a piecewise smooth function f periodic on $[-\pi, \pi)$ with a single jump discontinuity at $x = \xi \in (-\pi, \pi)$ are given by

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Concentration Factor Jump Detector

$$S_N^\sigma[f](x) = \sum_{0<|k|\le N} i\,\mathrm{sgn}(k)\sigma(k)\hat{f}_k e^{ikx} \approx \sum_{0<|k|\le N} i\,\mathrm{sgn}(k)\sigma(k)\hat{r}_k[f](\xi) e^{ik(x-\xi)}$$

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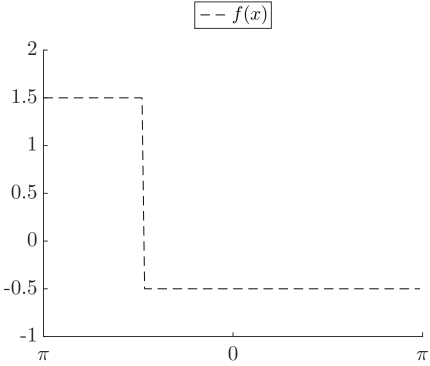

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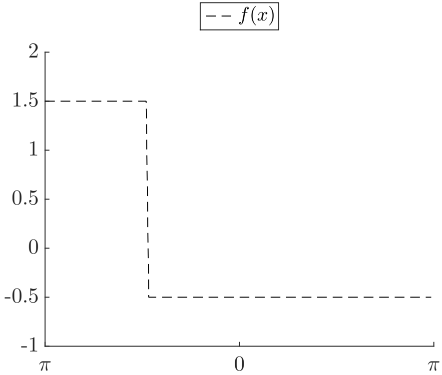

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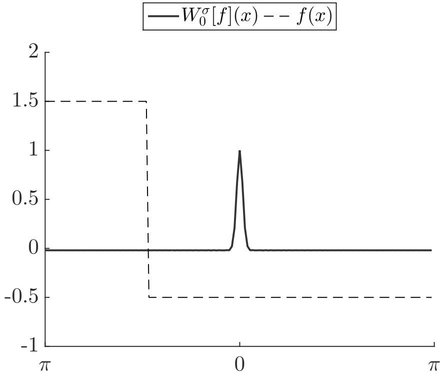

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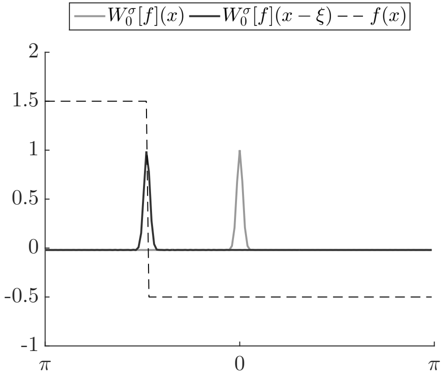

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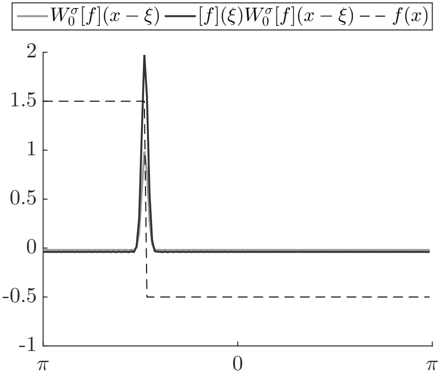
4. Scale by $[f](\xi)$



Design Method Example

Use optimization to solve:

$$\begin{aligned}\min_{\sigma} \quad & \left\| W_0^{\sigma}(x) - [r](x) \right\|_2 + \lambda \left\| W_0^{\sigma}(x) \right\|_1 \\ \text{subject to} \quad & W_0^{\sigma}(x)\Big|_{x=0} = 1.\end{aligned}$$

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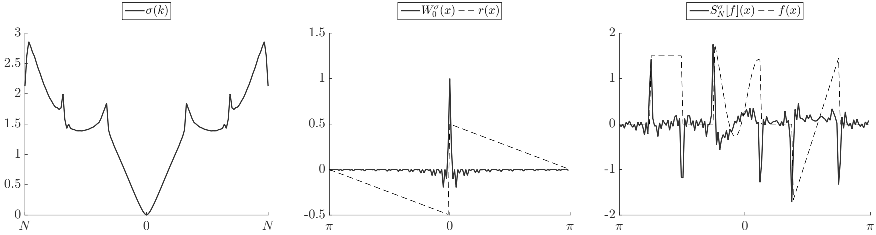

Figure: Solution σ, $W_0^{\sigma}(x)$, and $S_N^{\sigma}[f](x)$, with $\lambda = 0.5$, $N = 64$.

Design Method Example

Figure: Solution σ with $\lambda = 0.5$ on an image, $N = 300$.

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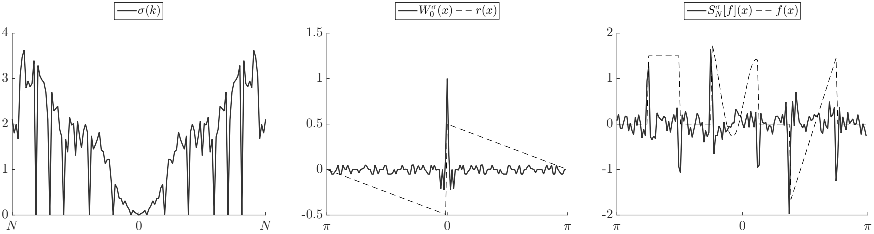

Figure: Solution σ, $W_0^{\sigma}(x)$, and $S_N^{\sigma}[f](x)$, with $N = 64$.

Intermittent Data Example

Figure: Solution σ with $K = \{k : 100 \leq |k| \leq 150\}$, on an image, $N = 300$.

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Fourier Series

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx} \qquad \hat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx.$$

The Fourier coefficients $\hat{f}_k$ are complex numbers, i.e., $\hat{f}_k = \alpha_k e^{i\theta_k}$, where α_k is the **amplitude** and $e^{i\theta_k}$ is the **phase**.

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where α_k is the **amplitude** and $e^{i\theta_k}$ is the **phase**.

$$\hat{f}_k \approx \frac{[f](\xi)}{2\pi ik} e^{-ik\xi} \approx \frac{[f](\xi)}{2\pi k} e^{-i(k\xi+\pi/2)}$$

Spectral Phase

Figure: Two grayscale images

Spectral Phase

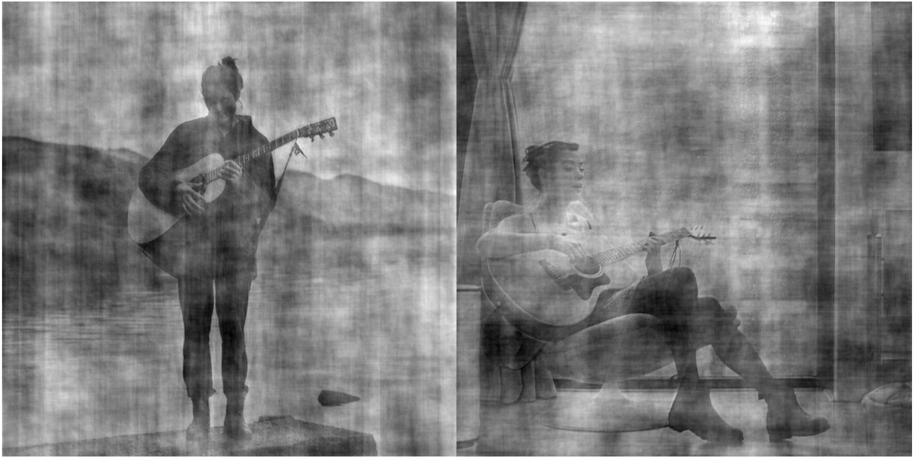

Figure: Images with spectral magnitude swapped

Jump Reconstruction from Spectral Phase Data

The spectral phase of the ramp are

$$\tilde{r}_k := \frac{\hat{r}_k}{|\hat{r}_k|}.$$

Let

$$W_0^{\tilde{\sigma}}(x) := \sum_{0<|k|\leq N} i\,\mathrm{sgn}(k)\tilde{\sigma}(k)\tilde{r}_k e^{ikx}.$$

If $\tilde{\sigma}(k) = \sigma(k)|\hat{r}_k|$, then

$$W_0^{\tilde{\sigma}}(x) = \sum_{0<|k|\leq N} i\,\mathrm{sgn}(k)\tilde{\sigma}(k)\tilde{r}_k e^{ikx} = \sum_{0<|k|\leq N} i\,\mathrm{sgn}(k)\sigma(k)|\hat{r}_k|\frac{\hat{r}_k}{|\hat{r}_k|}e^{ikx} = W_0^{\sigma}(x)$$

Spectral Phase Data Example

Suppose $\hat{f}_k/|\hat{f}_k|$ is known.

$$\min_{\tilde{\sigma}} \quad \left\| W_0^{\tilde{\sigma}}(x) - [r]_\epsilon(x) \right\|_2 + \lambda \left\| W_0^{\tilde{\sigma}}(x) \right\|_1$$

$$\text{subject to} \quad W_0^{\sigma}(x)\Big|_{x=0} = 1.$$

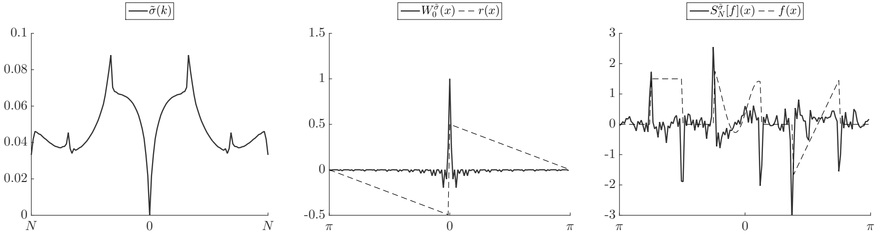

Figure: Solution $\tilde{\sigma}$, $W_0^{\tilde{\sigma}}(x)$, and $S_N^{\tilde{\sigma}}[f](x)$ from Fourier phase data $\hat{f}_k/|\hat{f}_k|$, with $\lambda = 0.5$, $N = 64$.

Spectral Phase Data Example

Figure: Solution $\tilde{\sigma}$ with $\lambda = 0.5$ on an image, $N = 300$.

Post-Processing Ideas

How can we do better?

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Post-Processing Ideas

How can we do better? Combine multiple jump approximations into a detector and then separate scales.

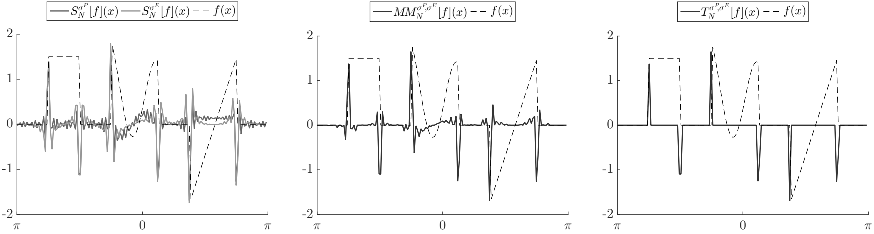

Figure: Combining jump approximations and thresholding for a robust detector.

Minmod Operator

Remove spurious oscillations via minmod operator.

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Can be overloaded to n-tuples by composition. The detector built on jump approximations from concentration factors $\sigma_1,\ldots,\sigma_n$ then is

Minmod Edge Detector

$$MM_N^{\sigma_1,\ldots,\sigma_n}[f](x) := \mathrm{minmod}(S_N^{\sigma_1}[f],\ldots,S_N^{\sigma_n}[f])(x).$$

Choosing Multiple Concentration Factors

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$$C_{i,j}^{p,q} := \operatorname{cov}\left(S_N^{\sigma_p}(x_i), S_N^{\sigma_q}(x_j)\right) = \sum_{0<|k|\leq N} \sigma_p(k)\sigma_q(k)e^{ik(x_i-x_j)}.$$

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Covariance is 0 if $\langle \sigma_p, \sigma_q \rangle = 0$.

Thus if σ_p is given, we can optimize for σ_q with the constraint $\langle \sigma_p, \sigma_q \rangle = 0$ to get as different as possible responses.

Minmod Example

Let $\sigma_1 = \sigma^P_{p=1}$.

$$\min_{\sigma_2} \quad \left\| W_0^{\sigma_2}(x) \right\|_2$$

$$\text{subject to} \quad W_0^{\sigma_2}(x)\Big|_{x=0} = 1, \quad \langle \sigma_1, \sigma_2 \rangle = 0.$$

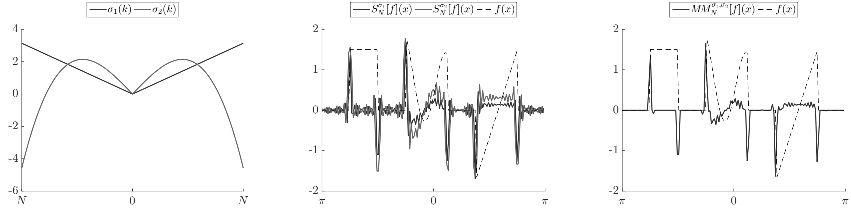

Figure: Solution σ_2 and concentration factor $\sigma_1 = \sigma^P_{p=1}$, jump approximations $S_N^{\sigma_1}[f](x)$ and $S_N^{\sigma_2}[f](x)$ and minmod detector $MM_N^{\sigma_1,\sigma_2}[f](x)$, $N = 64$.

Separation of Scales and Thresholding

Let f have discontinuities $\{\xi_j : j = 1, \dots, n\}$ and suppose $S_N^\sigma[f](x) \approx [f](x)$. Choose $q > 1$ and compute

Separation of Scales

$$E_{q,N}(x) := N^{q/2}(S_N^\sigma[f](x))^q \approx \begin{cases} N^{q/2}([f](\xi_j))^q & x = \xi_j, \\ O(N^{-q/2}) & x \neq \xi_j. \end{cases}$$

Compare to some $O(1)$ threshold ϑ.

Enhanced Detector

$$T_N^\sigma[f](x) := \begin{cases} S_N^\sigma[f](x) & |E_{q,N}(x)| > \vartheta, \\ 0 & |E_{q,N}(x)| \leq \vartheta. \end{cases}$$

Enhanced Detector Example

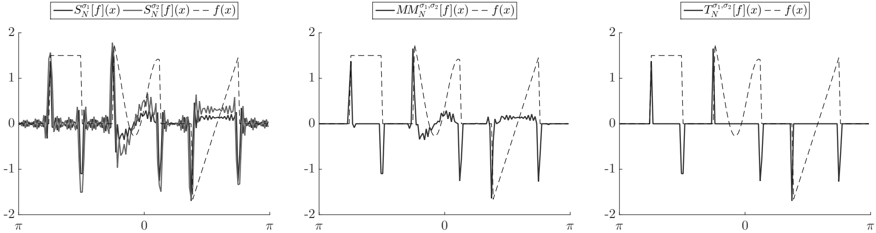

Figure: Combining jump approximations and thresholding for a robust detector.

Putting it all together

Suppose the obtained data has the form

$$\tilde{g}_k = \frac{\hat{f}_k + \nu_k}{|\hat{f}_k + \nu_k|}, \qquad \nu_k \sim \text{AWGN}, \qquad \text{SNR} = 12\text{dB}$$

with data unusable on $K = \{k : 15 \leq |k| \leq 20\}$.

$$\begin{aligned}
\min_{\tilde{\sigma}_1} \quad & \left\| W_0^{\tilde{\sigma}_1}(x) \right\|_2 \\
\text{subject to} \quad & W_0^{\tilde{\sigma}_1}(x)\Big|_{x=0} = 1, \quad \tilde{\sigma}_1[K] = 0, \quad \left| W_0^{\tilde{\sigma}_1}(x) \right|_{|x| \geq 0.2} \leq 10^{-2}, \\
\min_{\tilde{\sigma}_2} \quad & \left\| W_0^{\tilde{\sigma}_2}(x) \right\|_2 \\
\text{subject to} \quad & W_0^{\tilde{\sigma}_2}(x)\Big|_{x=0} = 1, \quad \tilde{\sigma}_2[K] = 0, \quad \langle \tilde{\sigma}_1, \tilde{\sigma}_2 \rangle = 0.
\end{aligned}$$

Putting it all together

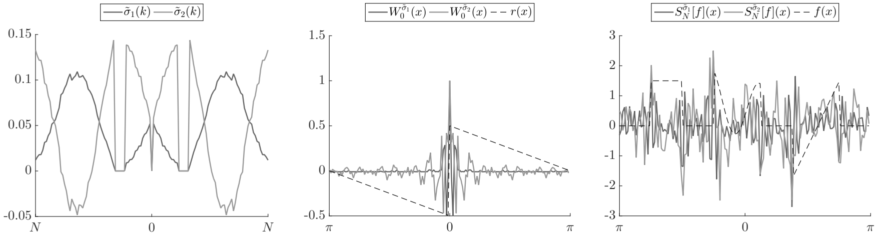

Figure: Solutions $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ with jump responses and jump approximations from noisy Fourier phase data $\tilde{g}_k$, with $\tilde{g}_k = 0$ for $k \in K$, $N = 64$.

Putting it all together

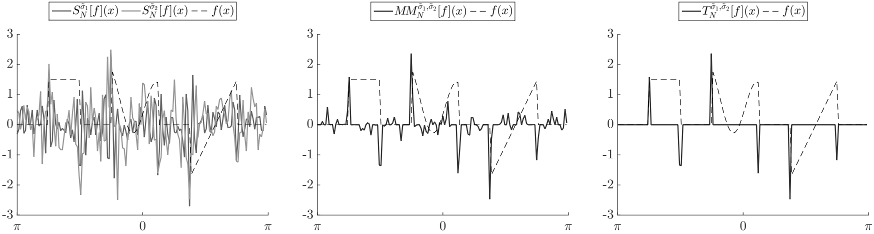

Figure: Jump approximations, minmod detector, and enhanced detector, $N = 64$.

Comparison to analytic concentration factors

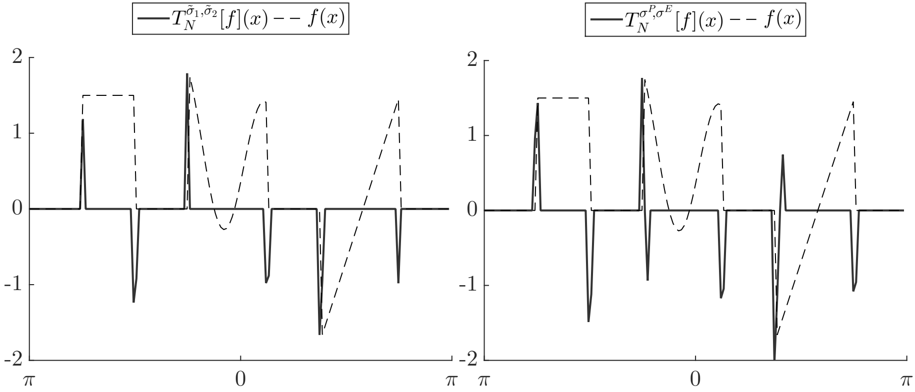

Figure: $T_N^{\sigma_1,\sigma_2}[f](x)$ vs $T_N^{\sigma^P,\sigma^E}[f](x)$, $q = 2$, $\vartheta = 30$, $N = 64$.

Examples on Images

Figure: Two input images.

Examples on Images

Figure: Input converted to grayscale.

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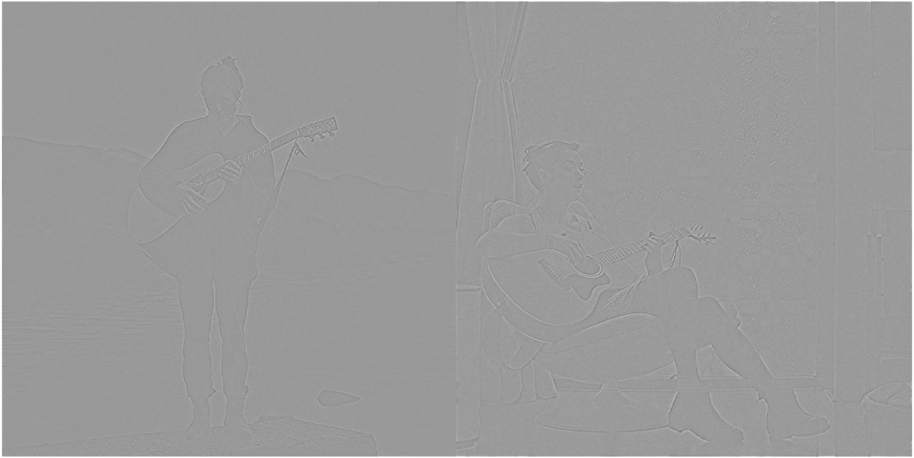

Figure: Removing the magnitude and band K.

Examples on Images

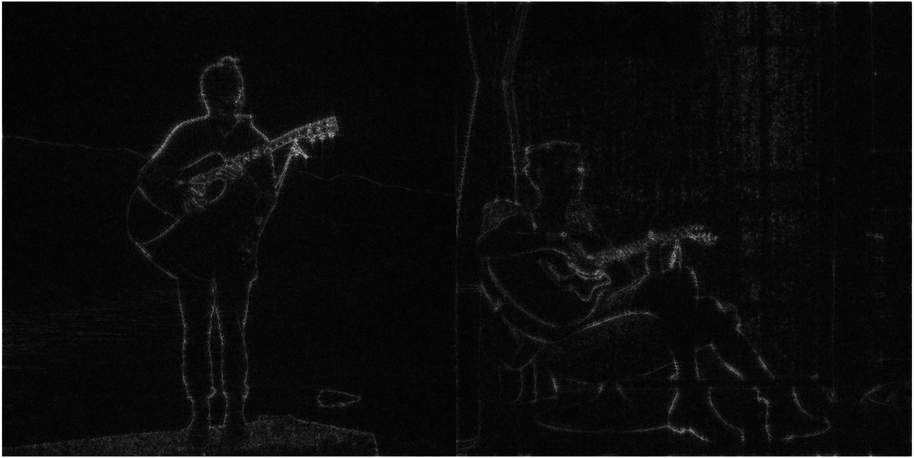

Figure: $T_N^{\sigma_1,\sigma_2}[f](x)$ on images from spectral phase data with $q = 5$, $\vartheta = 1$, $K = \{k : 120 \leq |k| \leq 150\}$, $N = 300$.

Concluding Remarks

This research originally set out to explore the possibilities of detecting edges from spectral phase data. The concentration factor method gives us some intuition and insight into the relationships between features in a signal and its spectral phase. With the understanding of the concentration factor design model, it was seen that a formulation could be posed within the framework to realize an edge detector from phase only data.

With a detector constructed, post-processing schemes relying on the difference of multiple edge approximations yielded an enhanced edge detector, which is shown to be effective even under noisy, intermittent phase data. All of the work was then extended to two dimensions, where the processes were shown to perform well on image input.

Future Work

Computational possibilities:

- Statistical combinations of multiple (orthogonal) concentration factors
- Non-uniform sampling methods

Analytical work ahead:

- Investigate quantity of data necessary for accurate jump height
- Statistical analysis for noisy phase data

Other directions:

- Application specific priors
- Application to radar data
- Corners from edges

Acknowledgements

This work was started over the summer of 2015 as part of ASU's MCTP program. I would like to thank the NSF for their gracious support of the program, and Dr. Kostelich for his dedication in continuing this program at ASU.

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And you!