Edge Detection from Fourier Phase Data

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What is Edge Detection?

Edge detection is a sub-field of signal processing, which consists of the theory, applications, algorithms, and implementations of processing or transferring information via signals. Edge detection is the study of mathematical methods which identify points in a digital image where the image brightness changes sharply. An edge is a sudden shift, or jump, of a signal.

Example: Edge Detection on an Image

Applications of Edge Detection

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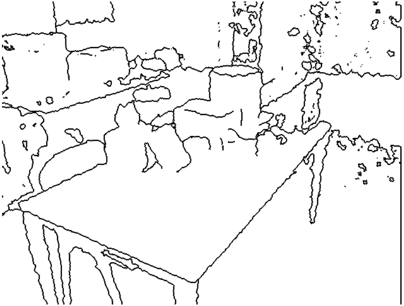

Applications of Edge Detection

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- Magnetic Resonance Imaging (MRI)
- Synthetic Aperture Radar (SAR)
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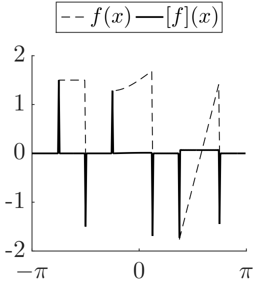
- 1-D Case: Discontinuities of a Function



Fourier Data

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Fourier Series

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}.$$

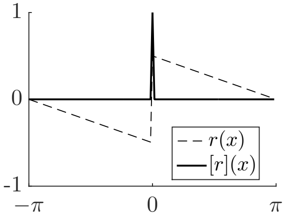
The coefficients $\hat{f}_k$ are called the **Fourier coefficients** of f. For a piecewise-smooth function periodic on $[-\pi, \pi)$, they are defined as:

$$\hat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx, \quad k = -N, \dots, N.$$

What is the Concentration Edge Detection Method?

First, define the **jump function** of f, $[f](x)$ as the difference between the right and left hand limits of the function f at every point x; i.e.,

$$[f](x) = f(x^+) - f(x^-).$$



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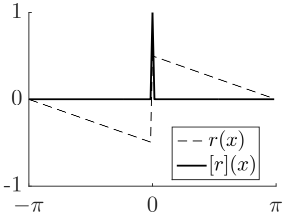

The concentration edge detection method computes an approximation to the jump function using:

Partial Sum Approximation to the Jump Function

$$S_N^\sigma[f](x) = \sum_{k=-N}^{N} i\,\mathrm{sgn}(k)\sigma_k \hat{f}_k e^{ikx} \approx [f](x).$$

Analytical Concentration Factors

The concentration factor, $\sigma(k)$, concentrates the partial sum along the edges in the function f. There are multiple analytical solutions σ; the three main types we are using are:

Factor	Expression	Remarks
Trigonometric	$\sigma(\eta) = \frac{\pi \sin(\pi\eta)}{\mathrm{Si}(\pi)}$	$\mathrm{Si}(\pi) = \int_0^{\pi} \frac{\sin(x)}{x}\, dx$
Polynomial	$\sigma(\eta) = p\pi\eta^p$	p is the order of the factor
Exponential	$\sigma(\eta) = C\eta e^{\frac{1}{\alpha\eta(\eta-1)}}$	α is the order
		C is a normalizing constant
		$C = \frac{\pi}{\int_{\frac{1}{N}}^{1-\frac{1}{N}} \exp(\frac{1}{\alpha\tau(\tau-1)})\, d\tau}$

Jump Reconstruction from Fourier Data

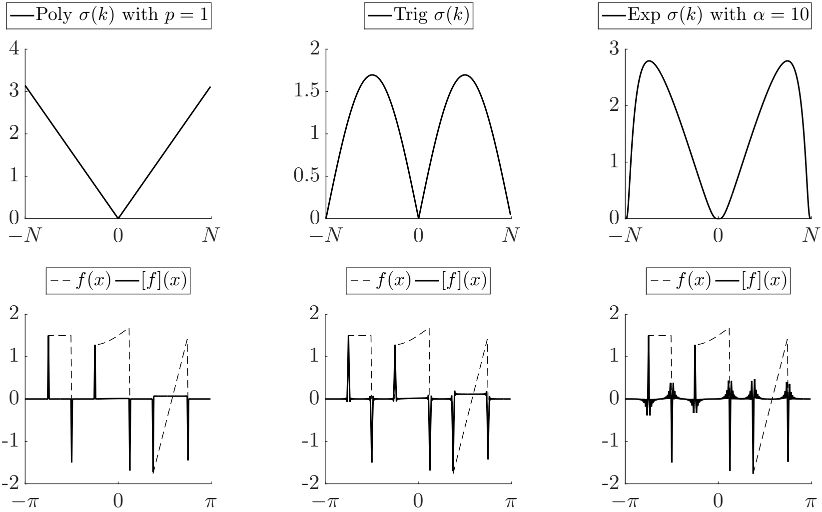

Figure: Reconstructions of $[f](x)$ with multiple different σ.

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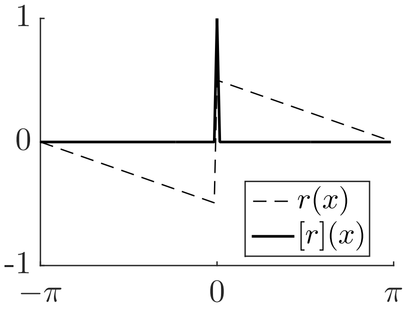
We can engineer σ for a function $g(x)$ where we know what $[g](x)$ is.

Designed Concentration Factors

Consider the ramp function $r(x)$ with the associated jump function $[r](x)$ on $[-\pi, \pi]$:



$$r(x) = \begin{cases} \frac{1}{2\pi}(-x-\pi) & \text{if } x < 0, \\ \frac{1}{2\pi}(-x+\pi) & \text{if } x \geq 0. \end{cases} \qquad [r](x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

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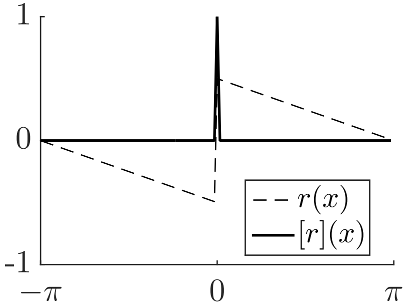

$$S_N^\sigma[r](x) = \sum_{k=-N}^{N} i\,\mathrm{sgn}(k)\sigma_k \hat{r}_k e^{ikx} \approx [r](x).$$

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We can solve the system $S\sigma = [r](x)$ for a solution σ to

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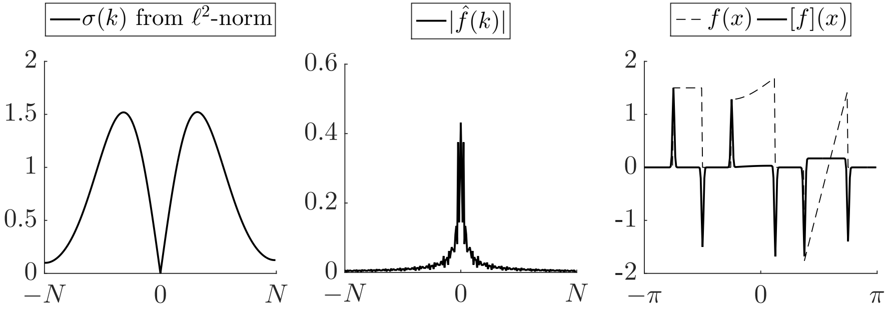

Figure: Reconstruction of $[f](x)$ from σ and $\hat{f}$

Fourier Phase Data

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The Fourier coefficients $\hat{f}_k$ are complex numbers, i.e., $\hat{f}_k = \alpha_k e^{i\theta_k}$, where α_k is the **amplitude** and $e^{i\theta_k}$ is the **phase**.

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Partial Sum Approximations of $[f](x)$

$$S_N^{\sigma}[f](x) = \sum_{k=-N}^{N} i\,\mathrm{sgn}(k)\sigma_k \hat{f}_k e^{ikx} \approx [f](x)$$

$$S_N^{\tilde{\sigma}}[f](x) = \sum_{k=-N}^{N} i\,\mathrm{sgn}(k)\tilde{\sigma}_k \tilde{f}_k e^{ikx} \approx [f](x)$$

$$\sum_{k=-N}^{N} \sigma_k \hat{f}_k = \sum_{k=-N}^{N} \tilde{\sigma}_k \tilde{f}_k$$

Jump Reconstruction from Fourier Phase Data

Yes!

Jump Reconstruction from Fourier Phase Data

Yes! We can solve the system $S\tilde{\sigma} = [r](x)$ for a solution $\tilde{\sigma}$ to

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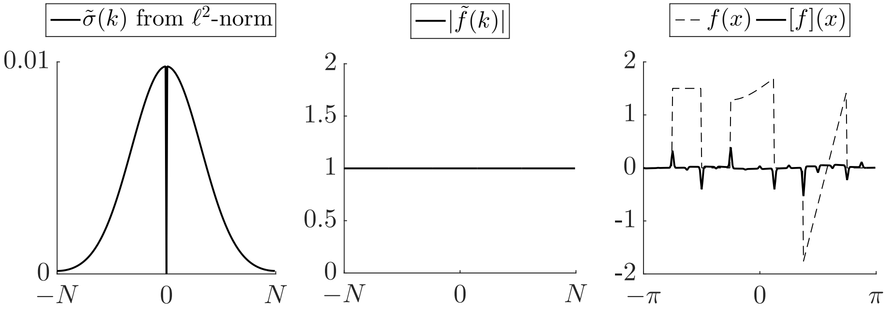

Figure: Reconstruction of $[f](x)$ from $\tilde{\sigma}$ and $\tilde{f}$

Jump Reconstruction from Fourier Phase Data

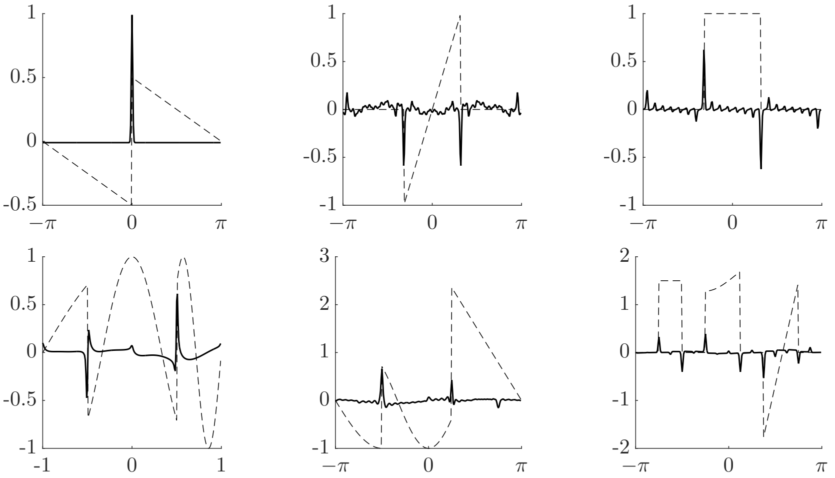

Figure: Reconstructions of multiple functions from $\tilde{\sigma}$ and $\tilde{f}$.

Future Work

Questions that remain:

- What if the phase data is noisy?
- What if the phase data is intermittent, i.e., banded?
- What if the phase data is noisy and intermittent?
- Generalizations to two dimensions

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