THE JOURNAL OF SYMBOLIC LOGIC
Volume 56, Number 4, Dec. 1991

PHASE SEMANTICS AND SEQUENT CALCULUS FOR PURE NONCOMMUTATIVE CLASSICAL LINEAR PROPOSITIONAL LOGIC

V. MICHELE ABRUSCI

CONTENTS

§0. Introduction.

0.0. Aims. The linear logic introduced in [3] by J.-Y. Girard keeps one of the so-called *structural rules* of the sequent calculus: the *exchange rule.* In a one-sided sequent calculus this rule can be formulated as

$$\frac{\vdash \Gamma_1, A, B, \Gamma_2}{\vdash \Gamma_1, B, A, \Gamma_2} \quad \text{(Exchange).}$$

The exchange rule allows one to disregard the order of the assumptions and the order of the conclusions of a proof, and this means, when the proof corresponds to a logically correct program, to disregard the order in which the inputs and the outputs occur in a program.

In the linear logic introduced in [3], the exchange rule allows one to prove the commutativity of the multiplicative connectives, conjunction ($\otimes$) and disjunction ($⅋$). Due to the presence of the exchange rule in linear logic, in the phase semantics for linear logic one starts with a commutative monoid. So, the usual linear logic may be called *commutative linear logic.*

Received September 5, 1990; revised February 26, 1991.
This work was supported by the Italian Ministry for University and Scientific Research (MURST) and by the European Economic Community's Project Stimulation.

0022-4812/91/5604-0020/$05.90

The aim of the investigations underlying this paper was to see, first, what happens when we remove the exchange rule from the sequent calculus for the linear propositional logic at all, and then, how to recover the strength of the exchange rule by means of exponential connectives (in the same way as by means of the exponential connectives ! and ? we recover the strength of the weakening and contraction rules).

Linear propositional logic without the exchange rule will be called *noncommutative linear propositional logic. Pure* noncommutative linear propositional logic will mean the noncommutative linear propositional logic without exponential connectives.

It is usual to distinguish between intuitionistic and classical linear logic. *Intuitionistic* linear logic deals with sequents of the form $\Gamma \Rightarrow A$ or $\Gamma \Rightarrow$ (an arbitrary finite number of assumptions, but at most one conclusion), whereas *classical* linear logic deals with sequents of the form $\Gamma \Rightarrow \Delta$ (an arbitrary finite number of assumptions, and an arbitrary finite number of conclusions).

The sequent calculus and phase semantics of noncommutative intuitionistic linear propositional logic has been investigated in our previous paper [1]. In the present paper we investigate the sequent calculus and phase semantics of pure noncommutative classical linear propositional logic: compared with the intuitionistic case, the classical case involves serious problems both in the formulation of the sequent calculus and in phase semantics.

Several contributions and/or suggestions in the noncommutative classical linear logic can be found in [4] and [6], where a restricted form of exchange rule is allowed, i.e. the cyclic exchange rule:

$$\frac{\vdash \Gamma, A}{\vdash A, \Gamma} \quad \text{(Cyclic Exchange).}$$

Our aim is different: we want to start with the full removal of the exchange rule!

0.1. In §1 we introduce the general concept of noncommutative classical phase space.

What happens when, in the phase semantics, we start with a noncommutative monoid, instead of a commutative one as in [3]?

When we start with a noncommutative monoid $\langle M, \cdot, i\rangle$ and an arbitrary $\perp \subseteq M$, then we can define on $\mathscr{P}(M)$ the operations

$$F \multimap G = \{x \mid \forall y \in F\ y \cdot x \in G\},$$
$$G \multimapinv F = \{x \mid \forall y \in F\ x \cdot y \in G\} \neq F \multimap G,$$
$$F \cdot G = \{x \cdot y \mid x \in F \text{ and } y \in G\}$$

and we get two different closure operations on $\mathscr{P}(M)$, i.e. the mappings

$$F \in \mathscr{P}(M) \mapsto \perp \multimapinv (F \multimap \perp) = {}^{\perp}(F^{\perp}),$$
$$F \in \mathscr{P}(M) \mapsto (\perp \multimapinv F) \multimap \perp = ({}^{\perp}F)^{\perp}.$$

We are interested in the subsets $F = {}^{\perp}(F^{\perp}) = ({}^{\perp}F)^{\perp}$, the *facts* over our monoid and $\perp$: the facts are closed under $\cap$, but in general they are not closed under the operations $(-)^{\perp}$, ${}^{\perp}(-)$, $\multimap$, $\multimapinv$, ${}^{\perp}((-\cdot-)^{\perp})$, $({}^{\perp}(-\cdot-))^{\perp}$, ${}^{\perp}((-\cup-)^{\perp})$, and $({}^{\perp}(-\cup-))^{\perp}$.

If the monoid is commutative, or if $\perp$ is cyclic (i.e. $x \cdot y \in \perp$ iff $y \cdot x \in \perp$), then $F^{\perp} = {}^{\perp}F$ for every F, and thus ${}^{\perp}(F^{\perp}) = ({}^{\perp}F)^{\perp} = F^{\perp\perp}$ for every F, so that

$$F \in \mathscr{P}(M) \mapsto (F \multimap \perp) \multimap \perp = F^{\perp\perp}$$

is a closure operation on $\mathscr{P}(M)$ and satisfies

$$F^{\perp\perp} \cdot G^{\perp\perp} \subseteq (F \cdot G)^{\perp\perp}.$$

The subsets F such that $F = F^{\perp\perp}$ are called the *facts of a phase space* in [3], and they are closed under the operations $(-)^{\perp}$, $\multimap$, $\circ\!-$, $(- \cdot -)^{\perp\perp} = \otimes$, $\cap$, and $(- \cup -)^{\perp\perp} = \oplus$. But this result is obtained by assuming either the commutativity of the monoid or the cyclicity (i.e. a weak form of commutativity) of $\perp$, and thus by considering just one form of duality (the operation $F^{\perp} = {}^{\perp}F$).[1]

We find a first-order condition on $\perp$ (trivially satisfied if the monoid is commutative or if $\perp$ is cyclic), having the following facts as consequences:

(a) $F^{\perp}$ is in general different from ${}^{\perp}F$.

(b) ${}^{\perp}(F^{\perp}) = ({}^{\perp}F)^{\perp}$ for every F.

(c) The subsets F such that $F = {}^{\perp}(F^{\perp}) = ({}^{\perp}F)^{\perp}$ are closed under the operations $(-)^{\perp}$, ${}^{\perp}(-)$, $\multimap$, $\circ\!-$, ${}^{\perp}((-\cdot-)^{\perp}) = ({}^{\perp}(-\cdot-))^{\perp} = \otimes$, $\cap$, and ${}^{\perp}((-\cup-)^{\perp}) = ({}^{\perp}(-\cup-))^{\perp} = \oplus$.

The resulting structure of facts will be called a *noncommutative classical phase space*. (The structure of facts will be called a *commutative classical phase space* when the monoid is commutative, and a *cyclic classical phase space* when $\perp$ is cyclic.)

Noncommutative classical phase space are examples of an abstract structure called *quantale* and used to give models for the logic of quantum mechanics (cf. [5]).

0.2. In §2 we introduce a two-sided sequent calculus for the pure noncommutative classical linear propositional logic, PNCL, and its semantics with respect to the noncommutative classical phase spaces.

The connectives are the following ones: two linear negations (the *linear postnegation* $A^{\perp}$ and the *linear retronegation* ${}^{\perp}A$), the multiplicative connectives $\otimes$ (*times*) and $⅋$ (*par*) (the *linear postimplication* $A \multimap B$ is defined in the metalanguage as $A^{\perp} ⅋ B$, and the *linear retroimplication* $B \circ\!- A$ is defined in the metalanguage as $B ⅋ {}^{\perp}A$), and the additive connectives & (*with*) and $\oplus$ (*plus*).

The main result is the proof of the soundness and completeness of PNCL with respect to validity in every noncommutative classical phase space. The way used to prove the completeness theorem is to show how provability in PNCL leads to a noncommutative classical phase space: when we take the monoid of the finite sequences of formulas of PNCL and the set

$$\perp = \{\Gamma \mid \Gamma \Rightarrow \text{ is a provable sequent in PNCL}\},$$

then we obtain a noncommutative classical phase space, for each formula A the set

$$\mathrm{PR}(A) = \{\Gamma \mid \Gamma \Rightarrow A \text{ is a provable sequent in PNCL}\}$$

[1] The necessity of two forms of duality, $F^{\perp}$ and ${}^{\perp}F$, in fully noncommutative linear logic was pointed out in [6, p. 47].

is a fact, and the connectives of PNCL correspond to the operations of that noncommutative classical phase space.

0.3. §3 is a first attempt to deal with a class of graphs, called *noncommutative deduction nets*, associated to the proofs in the strict multiplicative fragment of PNCL. Recall that the proof nets introduced in [3] for the commutative linear logic are the graphs of formulas associated to the proofs of the one-sided sequent calculus for the strict multiplicative fragment of the commutative linear logic.

The underlying idea is that each provable sequent $\Gamma \Rightarrow \Delta$ in PNCL asserts the existence of a deduction from the assumptions Γ to the conclusions Δ, and each proof of such a sequent expresses a particular deduction from Γ to Δ; in the case of the strict multiplicative fragment, the deduction from Γ to Δ may be represented as a noncommutative deduction net, an oriented graph of occurrences of formulas where the initial formulas are (from left to right) just the sequence $\Gamma = A_1, \ldots, A_m$ and the terminal formulas are (from left to right) just the sequence $\Delta = B_1, \ldots, B_n$:

$$A_1 \cdots A_m$$

$$B_1 \cdots B_n$$

The failure of the exchange rule has the following consequences:

(a) In the deduction nets we have to consider the order of the initial and terminal formulas (whereas in commutative linear logic we disregard the order of the conclusions of a proof net).

(b) Deduction nets are *planar* graphs (whereas in commutative linear logic proof nets may be nonplanar graphs).

A feature of PNCL is that the rules expressing a *communication* between two processes, i.e. the cut rule and the binary rules for the multiplicative connectives, are subject to conditions on the contexts. The conditions are very natural in terms of the communication: they require that there is no obstacle in the communication. A typical obstacle in the communication input-output (cut rule) is when we have a process with an output A occurring between several outputs and a process with an input A occurring between several inputs:

$$A_1 \cdots A_m$$

$$B_1 \cdots B_n\ A\ C_1 \cdots C_p$$

$$D_1 \cdots D_q\ A\ E_1 \cdots E_r$$

$$F_1 \cdots F_s$$

In order to communicate we must first remove the obstacles, i.e. either the outputs occurring after the output A and the inputs occurring before the input A, or the outputs occurring before A and the inputs occurring after A. The removal of the obstacles may be made by means of both the negations of noncommutative linear logic, e.g. first we transform the above graphs as follows:

$$A_1 \cdots A_m \, (C_p)^\perp \cdots (C_1)^\perp$$

$$B_1 \cdots B_n \, A$$

$$A \, E_1 \cdots E_r$$

$$(D_q)^\perp \cdots (D_1)^\perp F_1 \cdots F_s$$

and then we communicate as follows:

$$A_1 \cdots A_m \, (C_p)^\perp \cdots (C_1)^\perp$$

$$B_1 \cdots B_n \, A \, E_1 \cdots E_r$$

$$(D_q)^\perp \cdots (D_1)^\perp F_1 \cdots F_s.$$

We may use the slogan: commutative linear logic does not consider the existence of the obstacles in the communication, noncommutative linear logic considers seriously the obstacles in the communication!

0.4. §4 deals with two formulations of the one-sided sequent calculus for the pure noncommutative classical linear propositional logic, SPNCL and SPNCL′, based on a language in which both the negations are defined in the metalanguage by means of De Morgan laws.

SPNCL and SPNCL′ are semantically equivalent to PNCL.

In the calculus SPNCL we have conditions on the cut rule and binary multiplicative rules, whereas in SPNCL′ these conditions are removed by means of two very interesting rules (suggested by J.-Y. Girard in a private communication):

$$\frac{\vdash \Gamma, A}{\vdash A^{\perp\perp}, \Gamma} \; ((-)^{\perp\perp}) \qquad\qquad \frac{\vdash A, \Gamma}{\vdash \Gamma, {}^{\perp\perp}A} \; ({}^{\perp\perp}(-)),$$

allowing an unexpected form of exchange (note that $({}^{\perp\perp}A)^{\perp\perp} = A = {}^{\perp\perp}(A^{\perp\perp})$).

To the proofs in the strict multiplicative fragment of SPNCL we associate planar graphs called *noncommutative proof nets*, and to the proofs in the strict multiplicative fragment of SPNCL′ we associate planar graphs called *noncommutative enriched proof nets.*

0.5. §5 and perspectives. §5 is a short summary of our paper [2], in which we prove the cut-elimination theorem for SPNCL and SPNCL′, and discuss the failure of cut-elimination for PNCL.

At present there are some interesting attempts (e.g. by Girard, private communication) to find a semantics of proofs for pure noncommutative (intuitionistic or classical) linear logic.

§1. Noncommutative classical phase spaces.

1.1. Definition. Let $\langle M,\cdot,i\rangle$ be a monoid with unit. We consider the following binary operations on $\mathscr{P}(M)$ (=the power set of M). For every $F,G \in \mathscr{P}(M)$:

linear postimplication	$F \multimap G = \{x \mid \forall y \in F\ y \cdot x \in G\}$,
linear retroimplication	$G \circ\!- F = \{x \mid \forall y \in F\ x \cdot y \in G\}$,
composition	$F \cdot G = \{x \cdot y \mid x \in F \text{ and } y \in G\}$,
intersection	$F \cap G = \{x \mid x \in F \text{ and } x \in G\}$,
union	$F \cup G = \{x \mid x \in F \text{ or } x \in G\}$.

Note that if $\langle M,\cdot,i\rangle$ is a *commutative* monoid with unit, then, for every $F,G \in \mathscr{P}(M)$, $F \multimap G = G \circ\!- F$.

1.2. Lemma. *Let $\langle M,\cdot,i\rangle$ be a monoid with unit.*

(i) *$\langle \mathscr{P}(M),\cdot,\{i\}\rangle$ is a monoid with unit $\{i\}$.*

(ii) *$\forall F,G,H \in \mathscr{P}(M)$: $F \cdot G \subseteq H$ iff $G \subseteq F \multimap H$ iff $F \subseteq H \circ\!- G$.*

(iii) *$\forall F,G' \in \mathscr{P}(M)$: $F \cdot (F \multimap G) \subseteq G$ and $(G \circ\!- F) \cdot F \subseteq G$.*

(iv) *$\forall F,H \in \mathscr{P}(M)$: $F \subseteq (H \circ\!- F) \multimap H$ and $F \subseteq H \circ\!- (F \multimap H)$.*

(v) *$\forall F,G,H \in \mathscr{P}(M)$: if $F \subseteq G$, then $G \multimap H \subseteq F \multimap H$ and $H \circ\!- G \subseteq H \circ\!- F$.*

(vi) *$\forall F,H \in \mathscr{P}(M)$: $H \circ\!- ((H \circ\!- F) \multimap H) \subseteq H \circ\!- F$ and $(H \circ\!- (F \multimap H)) \multimap H \subseteq F \multimap H$.*

(vii) *$\forall F,G,H \in \mathscr{P}(M)$: $F \multimap G \subseteq (F \multimap H) \circ\!- (G \multimap H)$ and $G \circ\!- F \subseteq (H \circ\!- G) \multimap (H \circ\!- F)$.*

(viii) *$\forall F,G,H \in \mathscr{P}(M)$: $(F \cdot G) \multimap H = G \multimap (F \multimap H)$.*

(ix) *$\forall F,G,H \in \mathscr{P}(M)$: $H \circ\!- (F \cdot G) = (H \circ\!- G) \circ\!- F$.*

□ Left to the reader. □

1.3. Definition. Let $\langle M,\cdot,i\rangle$ be a monoid with unit and $\perp \subseteq M$. We define the following operations on $\mathscr{P}(M)$:

linear post-$\perp$-negation	$F^\perp = F \multimap \perp$,
linear retro-$\perp$-negation	${}^\perp F = \perp \circ\!- F$.

We define the following operations on $\mathscr{P}(M)$:

linear post-retro-$\perp$-negation	$\phi_\perp(F) = ({}^\perp F)^\perp$,
linear retro-post-$\perp$-negation	$\psi_\perp(F) = {}^\perp(F^\perp)$.

With regard to this definition, we make two remarks.

(i) If $\langle M,\cdot,i\rangle$ is a commutative monoid with unit, then for every $F \in \mathscr{P}(M)$ we have $F^\perp = {}^\perp F$ (i.e. the linear negation of F, considered in [3]) and $({}^\perp F)^\perp = {}^\perp(F^\perp)$.

(ii) The same result is obtained if $\langle M,\cdot,i\rangle$ is noncommutative and $\perp$ is cyclic (cf. [6]), i.e.

$$\forall x \in M\ \forall y \in M\ x \cdot y \in \perp \text{ iff } y \cdot x \in \perp.$$

1.4. PROPOSITION. *Let $\langle M,\cdot,i\rangle$ be a monoid with unit and $\perp \subseteq M$. The operations $\phi_\perp$ and $\psi_\perp$ on $\mathscr{P}(M)$ are closure operations on $\mathscr{P}(M)$, i.e. (we omit subscripts $\perp$):*

(i) $\forall F \in \mathscr{P}(M)$: $F \subseteq \phi(F)$ *and* $F \subseteq \psi(F)$.

(ii) $\forall F,G \in \mathscr{P}(M)$: *if* $F \subseteq G$, *then* $\phi(F) \subseteq \phi(G)$ *and* $\psi(F) \subseteq \psi(G)$.

(iii) $\forall F \in \mathscr{P}(M)$: $\phi(\phi(F)) \subseteq \phi(F)$ *and* $\psi(\psi(F)) \subseteq \psi(F)$.

□(i) is a particular case of 1.2(iv), (ii) follows from 1.2(v), and (iii) is a particular case of 1.2(vi). □

1.5. DEFINITION. Let $\langle M,\cdot,i\rangle$ be a monoid with unit and $\perp \subseteq M$. Let $F \in \mathscr{P}(M)$.

(i) F is a *$\phi_\perp$-fixpoint* iff $F = \phi_\perp(F)$.

(ii) F is a *$\psi_\perp$-fixpoint* iff $F = \psi_\perp(F)$.

(iii) F is a *$\perp$-fact* iff F is a $\phi_\perp$-fixpoint and a $\psi_\perp$-fixpoint.

We mention that if $\langle M,\cdot,i\rangle$ is a commutative monoid with unity or $\perp$ is cyclic, then the $\perp$-facts are exactly the facts of a phase space considered in [3].

1.6. PROPOSITION. *Let $\langle M,\cdot,i\rangle$ be a monoid with unit and $\perp \subseteq M$. (We omit subscripts $\perp$.)*

(i) $\perp$ *and* M *are* $\perp$*-facts.*

(ii) *Let* $F \subseteq M$; *then* $F = \phi(F)$ *iff*

(C1) $\qquad \forall H \in \mathscr{P}(M)\ \forall x \in M$, *if* $\forall y \in H\ x \cdot y \in F$, *then* $\forall y \in \phi(H)\ x \cdot y \in F$.

(iii) *Let* $F \subseteq M$; *then* $F = \psi(F)$ *iff*

(C2) $\qquad \forall H \in \mathscr{P}(M)\ \forall x \in M$, *if* $\forall y \in H\ y \cdot x \in F$, *then* $\forall y \in \psi(H)\ y \cdot x \in F$.

□(i) Left to the reader. □

□(ii) Let $F = \phi(F)$. Suppose that $\forall y \in H\ \ x \cdot y \in F$, and take $v \in \phi(H)$. Since $\forall y \in H\ \ x \cdot y \in F$, we have that $\forall w \in {}^\perp F\ \ \forall y \in H\ \ w \cdot x \cdot y \in \perp$, i.e. that $\forall w \in {}^\perp F$ $w \cdot x \in {}^\perp H$; therefore, $\forall w \in {}^\perp F\ \ w \cdot x \cdot v \in \perp$, i.e. $x \cdot v \in \phi(F)$, but $\phi(F) = F$. Let F satisfy (C1): we prove that $\phi(F) \subseteq F$. This follows from (i), if you consider the trivial statement: $\forall y \in F\ i \cdot y = y \in F$. □

□(iii) Let $F = \psi(F)$. Suppose that $\forall y \in H\ \ y \cdot x \in F$, and take $v \in \psi(H)$. Since $\forall y \in H\ \ y \cdot x \in F$, we have that $\forall w \in F^\perp\ \ \forall y \in H\ \ y \cdot x \cdot w \in \perp$, i.e. that $\forall w \in F^\perp$ $x \cdot w \in H^\perp$; therefore, $\forall w \in F^\perp\ v \cdot x \cdot w \in \perp$, i.e. $v \cdot x \in {}^\perp(F^\perp)$, but ${}^\perp(F^\perp) = F$. Let F satisfy (C2): we prove that $\psi(F) \subseteq F$. This follows from (ii), if you consider the trivial statement: $\forall y \in F\ y \cdot i = y \in F$. □

1.7. LEMMA. *Let $\langle M,\cdot,i\rangle$ be a monoid with unit and $\perp \subseteq M$. (We omit subscripts $\perp$.)*

(i) $\forall F,G \in \mathscr{P}(M)$: $\phi(F \multimap G) \subseteq F \multimap \phi(G)$ *and* $\psi(G \circ\!- F) \subseteq \psi(G) \circ\!- F$.

(ii) $\forall F,G \in \mathscr{P}(M)$: $\phi(F \cap G) \subseteq \phi(F) \cap \phi(G)$ *and* $\psi(F \cap G) \subseteq \psi(F) \cap \psi(G)$.

(iii) $\forall F,G \in \mathscr{P}(M)$: $F \cdot \phi(G) \subseteq \phi(F \cdot G)$ *and* $\psi(F) \cdot G \subseteq \psi(F \cdot G)$.

(iv) $\forall F,G \in \mathscr{P}(M)$: $\phi(F) \cup \phi(G) \subseteq \phi(F \cup G)$ *and* $\psi(F) \cup \psi(G) \subseteq \psi(F \cup G)$.

(v) *If* $F = \psi(F)$ *and* $G = \psi(G)$, *then* $F \multimap G = F^\perp \circ\!- G^\perp$.

(vi) *If* $F = \phi(F)$ *and* $G = \phi(G)$, *then* $G \circ\!- F = {}^\perp G \multimap {}^\perp F$.

□(i) Let $x \in \phi(F \multimap G)$, $y \in F$ and $z \in {}^\perp G$; we show that $z \cdot y \cdot x \in \perp$. First, we get that $z \cdot y \in {}^\perp(F \multimap G)$: indeed, if $w \in F \multimap G$, then $y \cdot w \in G$, so that $z \cdot y \cdot w \in \perp$. Thus, $z \cdot y \cdot x \in \perp$. Let $x \in \psi(G \circ\!- F)$, $y \in F$ and $z \in G^\perp$; we show that $x \cdot y \cdot z \in \perp$. First we get that $y \cdot z \in (G \circ\!- F)^\perp$: indeed, if $w \in G \circ\!- F$, then $w \cdot y \in G$, so that $w \cdot y \cdot x \in \perp$. Thus, $x \cdot y \cdot z \in \perp$. □

□(ii) Let $x \in \phi(F \cap G)$, $y \in {}^{\perp}F$ and $z \in {}^{\perp}G$. Since $F \cap G \subseteq F$, we have ${}^{\perp}F \subseteq {}^{\perp}(F \cap G)$ and so $y \cdot x \in \perp$. Since $F \cap G \subseteq G$, we have ${}^{\perp}G \subseteq {}^{\perp}(F \cap G)$ and so $z \cdot x \in \perp$. Let $x \in \psi(F \cap G)$, $y \in F^{\perp}$ and $z \in G^{\perp}$. Since $F \cap G \subseteq F$, we have $F^{\perp} \subseteq (F \cap G)^{\perp}$, so that $x \cdot y \in \perp$. Since $F \cap G \subseteq G$, we have $G^{\perp} \subseteq (F \cap G)^{\perp}$, so that $x \cdot z \in \perp$. □

□(iii) Let $x \in F$, $y \in \psi(G)$ and $z \in {}^{\perp}(F \cdot G)$. So, $\forall v \in F\ \forall w \in G\ z \cdot v \cdot w \in \perp$, and this gives (since $\perp$ is a fact) that $\forall v \in F\ \forall w \in \phi(G)\ z \cdot v \cdot w \in \perp$, and in particular $z \cdot x \cdot y \in \perp$. Let $x \in \psi(F)$, $y \in G$ and $z \in (F \cdot G)^{\perp}$. So, $\forall v \in F\ \forall w \in G\ v \cdot w \cdot z \in \perp$, and this gives (since $\perp$ is a fact) that $\forall v \in \psi(F)\ \forall w \in G\ v \cdot w \cdot z \in \perp$, and in particular $x \cdot y \cdot z \in \perp$. □

□(iv) Exercise. □

□(v) By 1.2(vii), we have $F \multimap G \subseteq F^{\perp} \multimapinv G^{\perp}$ and $F^{\perp} \multimapinv G^{\perp} \subseteq {}^{\perp}(F^{\perp}) \multimap {}^{\perp}(G^{\perp})$, i.e. $F^{\perp} \multimapinv G^{\perp} \subseteq F \multimap G$. □

□(vi) Analogous to (v). □

1.8. Remark. Let $\langle M,\cdot,i\rangle$ be a monoid with unit and $\perp \subseteq M$.

An immediate consequence of 1.7(ii) is that the $\phi_{\perp}$-fixpoints are closed under $\cap$, the $\psi_{\perp}$-fixpoints are closed under $\cap$, and the $\perp$-facts are closed under $\cap$.

An immediate consequence of 1.7(i) is that the $\phi_{\perp}$-fixpoints are closed under $\multimap$ and $(-)^{\perp}$, and the $\psi_{\perp}$-fixpoints are closed under $\multimapinv$ and ${}^{\perp}(-)$. But in general the $\phi_{\perp}$-fixpoints are not closed under $\multimapinv$ and ${}^{\perp}(-)$, i.e. we cannot prove (even if F and G are $\phi_{\perp}$-fixpoints)

$$\phi_{\perp}(G \multimapinv F) \subseteq G \multimapinv F \quad \text{and} \quad \phi_{\perp}({}^{\perp}F) \subseteq {}^{\perp}F \quad (\text{i.e. } ({}^{\perp\perp}F)^{\perp} \subseteq {}^{\perp}F).$$

Analogously, in general the $\psi_{\perp}$-fixpoints are not closed under $\multimap$ and $(-)^{\perp}$, i.e. we cannot prove (even if F and G are $\psi_{\perp}$-fixpoints)

$$\psi_{\perp}(G \multimap F) \subseteq G \multimap F \quad \text{and} \quad \psi_{\perp}(F^{\perp}) \subseteq F^{\perp} \quad (\text{i.e } {}^{\perp}(F^{\perp\perp}) \subseteq F^{\perp}).$$

Therefore, in general the $\perp$-facts are not closed under $\multimap$, $\multimapinv$, $(-)^{\perp}$, and ${}^{\perp}(-)$.

As a consequence of 1.7(iv), we have that the $\phi_{\perp}$-fixpoints are closed under the operation $F,G \mapsto \phi_{\perp}(F \cup G)$ (a commutative, associative operation, with neuter element $\phi_{\perp}(\varnothing)$), and that the $\psi_{\perp}$-fixpoints are closed under the operation $F,G \mapsto \psi_{\perp}(F \cup G)$ (a commutative, associative operation, with neuter element $\psi_{\perp}(\varnothing)$).

As a consequence of 1.7(iii) we have that the $\phi_{\perp}$-fixpoints are closed under the operation $F,G \mapsto \phi_{\perp}(F \cdot G)$, but this operation is not associative (whereas it has the neuter element $\phi_{\perp}(\{i\})$), and that the $\psi_{\perp}$-fixpoints are closed under the operation $F,G \mapsto \psi_{\perp}(F \cdot G)$, but this operation is not associative (whereas it has the neuter element $\psi_{\perp}(\{i\})$).

We remark that, if F and G are $\perp$-facts, then ${}^{\perp}F \multimap G = F \multimapinv G^{\perp}$.

1.9. Definition. Let $\langle M,\cdot,i\rangle$ be a monoid with unity and $\perp \subseteq M$. We say that $\perp$ is *suitable* iff, for every $z \in M$,

(i) $({}^{\perp\perp}\{z\})^{\perp} \subseteq {}^{\perp}\{z\}$ (i.e $\phi({}^{\perp}\{z\}) \subseteq {}^{\perp}\{z\}$), and

(ii) ${}^{\perp}(\{z\}^{\perp\perp}) \subseteq \{z\}^{\perp}$ (i.e. $\psi(\{z\}^{\perp}) \subseteq \{z\}^{\perp}$).

Note that if $\langle M,\cdot,i\rangle$ is commutative or $\perp$ is cyclic, then $\perp$ is trivially suitable.

1.10. Proposition. *Let $\langle M,\cdot,i\rangle$ be a monoid with unity and $\perp \subseteq M$. Then $\perp$ is suitable iff*

$$\text{(A)} \quad \forall H \in \mathscr{P}(M)\ \forall x,z \in M\text{: } \textit{if } \forall y \in H\ x \cdot y \cdot z \in \perp, \quad \textit{then } \forall y \in \phi_{\perp}(H) \cup \psi_{\perp}(H)\ x \cdot y \cdot z \in \perp.$$

□ (We omit subscripts $\perp$). Let $\perp$ be suitable and suppose that $\forall y \in H\ x \cdot y \cdot z \in \perp$.

First, we prove that $\forall y \in \phi(H)\ x \cdot y \cdot z \in \perp$. From the hypothesis that $\forall y \in H\ x \cdot y \cdot z \in \perp$, we get:

$\forall y \in H\ x \cdot y \in \perp \multimapinv \{z\} = {}^{\perp}\{z\}$, by the definition of $\multimapinv$;

$\forall v \in {}^{\perp\perp}\{z\}\ \forall y \in H\ v \cdot x \cdot y \in \perp$, from above by the definition of $\multimapinv$;

$\forall v \in {}^{\perp\perp}\{z\}\ \forall y \in \phi(H)\ v \cdot x \cdot y \in \perp$, from above since $\perp$ is a $\perp$-fact;

$\forall y \in \phi(H)\ x \cdot y \in ({}^{\perp\perp}\{z\})^{\perp}$, from above by the definition of $\multimap$;

$\forall y \in \phi(H)\ x \cdot y \in {}^{\perp}\{z\}$, from above since $\perp$ is suitable;

$\forall y \in \phi(H)\ x \cdot y \cdot z \in \perp$, from above by the definition of $\multimapinv$.

Similarly, we prove that $\forall y \in \psi(H)\ x \cdot y \cdot z \in \perp$.

Let $\perp$ satisfy (A) and $z \in M$. From the trivial statement $\forall y \in {}^{\perp}\{z\}\ y \cdot z \in \perp$, we get by (A) $\forall y \in \phi({}^{\perp}\{z\})\ y \cdot z \in \perp$, i.e. $\forall y \in \phi({}^{\perp}\{z\})\ y \in {}^{\perp}\{z\}$. From the trivial statement $\forall y \in \{z\}^{\perp}\ z \cdot y \in \perp$, we get by (A) that $\forall y \in \psi(\{z\}^{\perp})\ z \cdot y \in \perp$, i.e. $\forall y \in \psi(\{z\}^{\perp})\ y \in \{z\}^{\perp}$. □

1.11. PROPOSITION. *Let $\langle M,\cdot,i\rangle$ be a monoid with unit and $\perp \subseteq M$. (We omit subscripts $\perp$.) If $\perp$ is suitable, then*:

(i) $\forall F \in \mathscr{P}(M)$: $\phi_\perp(F) = \psi_\perp(F)$, *and*

(ii) $\forall F,G \in \mathscr{P}(M)$: $\phi_\perp(F) \cdot \phi_\perp(G) \subseteq \phi_\perp(F \cdot G)$ *and* $\psi_\perp(F) \cdot \psi_\perp(G) \subseteq \psi_\perp(F \cdot G)$.

□(i) Let $\perp$ be suitable. From the trivial statement $\forall y \in F\ \forall z \in F^{\perp}\ y \cdot z \in \perp$, we get (since $\perp$ is suitable) that $\forall y \in \phi_\perp(F)\ \forall z \in F^{\perp}\ y \cdot z \in \perp$, and this means that $\forall y \in \phi_\perp(F)\ y \in \psi_\perp(F)$. From the trivial statement $\forall y \in F\ \forall z \in {}^{\perp}F\ z \cdot y \in \perp$, we get (since $\perp$ is suitable) that $\forall y \in \psi_\perp(F)\ \forall z \in {}^{\perp}F\ z \cdot y \in \perp$, and this means that $\forall y \in \psi_\perp(F)\ y \in \phi_\perp(F)$. □

□(ii) Let $x \in \phi_\perp(F)$, $y \in \phi_\perp(G)$ and $z \in {}^{\perp}(F \cdot G)$. So, $\forall v \in F\ \forall w \in G\ z \cdot v \cdot w \in \perp$, and this gives (since $\perp$ is a $\perp$-fact) that $\forall v \in F\ \forall w \in \phi_\perp(G)\ z \cdot v \cdot w \in \perp$, and then (since $\perp$ is suitable) $\forall v \in \phi_\perp(F)\ \forall w \in \phi_\perp(G)\ z \cdot v \cdot w \in \perp$; in particular, $z \cdot x \cdot y \in \perp$. Let $x \in \psi_\perp(F)$, $y \in \psi_\perp(G)$ and $z \in (F \cdot G)^{\perp}$. So, $\forall v \in F\ \forall w \in G\ v \cdot w \cdot z \in \perp$, and this gives (since $\perp$ is a $\perp$-fact) that $\forall v \in \psi_\perp(F)\ \forall w \in G\ v \cdot w \cdot z \in \perp$, and then (since $\perp$ is suitable) $\forall v \in \psi_\perp(F)\ \forall w \in \psi_\perp(G)\ v \cdot w \cdot z \in \perp$; in particular, $x \cdot y \cdot z \in \perp$. □

1.12. DEFINITION. Let $\langle M,\cdot,i\rangle$ be a monoid with unit and $\perp \subseteq M$. If $\perp$ is suitable, then we define the following operations on $\mathscr{P}(M)$:

$\perp$-*tensorialization* (*times*)	$F \otimes_\perp G = \phi_\perp(F \cdot G) = \psi_\perp(F \cdot G)$,
$\perp$-*sum* (*plus*)	$F \oplus_\perp G = \phi_\perp(F \cup G) = \psi_\perp(F \cup G)$,
$\perp$-*parallelization* (*par*)	$F \mathbin{⅋}_\perp G = {}^{\perp}F \multimap G = F \multimapinv G^{\perp}$,

and the following elements of $\mathscr{P}(M)$:

$\perp$-*one*	$1_\perp = \phi_\perp(\{i\}) = \psi_\perp(\{i\})$,
$\perp$-*zero*	$0_\perp = \phi_\perp(\varnothing) = \psi_\perp(\varnothing)$.

1.13. PROPOSITION. *Let $\langle M,\cdot,i\rangle$ be a monoid with unity and $\perp \subseteq M$. If $\perp$ is suitable, then*:

(0) $1_\perp$ *and* $0_\perp$ *are* $\perp$-*facts.*

(i) $\forall F \in \mathscr{P}(M)$: $F^{\perp}$ *and* ${}^{\perp}F$ *are* $\perp$-*facts.*

(ii) *If G is a $\perp$-fact, then $\forall F \in \mathscr{P}(M)$ $F \multimap G$ and $G \multimapinv F$ are $\perp$-facts.*

(iii) *If F and G are $\perp$-facts, then $F \otimes_\perp G$, $F \mathbin{⅋}_\perp G$, $F \cap G$ and $F \oplus_\perp G$ are $\perp$-facts.*

□ Immediate from the hypothesis that $\perp$ is suitable, by the above definitions and propositions. □

1.14. Definition. Let $\langle M,\cdot,i\rangle$ be a monoid with unit and $\perp \subseteq M$. If $\perp$ is suitable and $\mathscr{A}$ is the set of all the facts over $\perp$, then the structure

$$\langle \mathscr{A},\subseteq,\perp,1_\perp,0_\perp,M,(-)^\perp,{}^\perp(-),\otimes_\perp,⅋_\perp,\cap,\oplus_\perp\rangle$$

is called a *noncommutative classical phase space*.

With regard to this definition, we make two remarks.

(i) If $\langle M,\cdot,i\rangle$ is commutative, and thus $\perp$ is trivially suitable and $(-)^\perp = {}^\perp(-)$, then $\langle \mathscr{A},\subseteq,\perp,1_\perp,0_\perp,M,(-)^\perp,{}^\perp(-),\otimes_\perp,⅋_\perp,\cap,\oplus_\perp\rangle$ is exactly the structure of facts of a phase space considered in [3], and we call it a *commutative classical phase space.*

(ii) If $\perp$ is cyclic, and thus $\perp$ is trivially suitable and $(-)^\perp = {}^\perp(-)$, then $\langle \mathscr{A},\subseteq,\perp,1_\perp,0_\perp,M,(-)^\perp,{}^\perp(-),\otimes_\perp,⅋_\perp,\cap,\oplus_\perp\rangle$ is exactly the structure of facts of a phase space considered in [6], and we call it a *cyclic classical phase space.*

1.15. Proposition. *Let $\langle \mathscr{A},\subseteq,\perp,1_\perp,0_\perp,M,(-)^\perp,{}^\perp(-),\otimes_\perp,⅋_\perp,\cap,\oplus_\perp\rangle$ be a noncommutative classical phase space (where $\mathscr{A} \subseteq \mathscr{P}(M)$, $\langle M,\cdot,i\rangle$ is a noncommutative monoid with unit, and $\perp \subseteq M$). Then (we omit subscript $\perp$):*

***De Morgan laws*:**

(i) $\forall F,G \in \mathscr{A}$: $F \multimap G = F^\perp ⅋ G$ *and* $G \multimapinv F = G ⅋ {}^\perp F$;

(ii) $\forall F,G \in \mathscr{A}$: $(F \otimes G)^\perp = G^\perp ⅋ F^\perp$;

(iii) $\forall F,G \in \mathscr{A}$: ${(F \otimes G)}^\perp = {}^\perp G ⅋ {}^\perp F$;

(iv) $\forall F,G \in \mathscr{A}$: $F \otimes G = ({}^\perp G ⅋ {}^\perp F)^\perp = {}^\perp(G^\perp ⅋ F^\perp)$;

(v) $\forall F,G \in \mathscr{A}$: $(F ⅋ G)^\perp = G^\perp \otimes F^\perp$;

(vi) $\forall F,G \in \mathscr{A}$: ${}^\perp(F ⅋ G) = {}^\perp G \otimes {}^\perp F$;

(viii) $\forall F,G \in \mathscr{A}$: $F ⅋ G = ({}^\perp G \otimes {}^\perp F)^\perp = {}^\perp(G^\perp \otimes F^\perp)$.

***Neutral elements*:**

(viii) $\forall F \in \mathscr{A}$: $F \otimes 1 = 1 \otimes F = F$ *and* $F \cap M = M \cap F = F$;

(ix) $\forall F \in \mathscr{A}$: $F ⅋ \perp = \perp ⅋ F = F$ *and* $F \oplus 0 = 0 \oplus F = F$.

***Associativity of $\otimes$ and ⅋*:**

(x) $\forall F,G,H \in \mathscr{A}$: $(F \otimes G) \otimes H = F \otimes (G \otimes H)$;

(xi) $\forall F,G,H \in \mathscr{A}$: $(F ⅋ G) ⅋ H = F ⅋ (G ⅋ H)$.

***Negation rules*:**

(xii) $\forall F,G,H \in \mathscr{A}$: *if* $F \otimes G \subseteq H$, *then* $G \subseteq F^\perp ⅋ H$ *and* $F \subseteq H ⅋ {}^\perp G$;

(xiii) $\forall F,G,H \in \mathscr{A}$: *if* $H \subseteq F ⅋ G$, *then* ${}^\perp F \otimes H \subseteq G$ *and* $H \otimes G^\perp \subseteq F$.

***Cut rules*:**

(xiv) $\forall F,C_1,C_2,D_1,D_2 \in \mathscr{A}$: *if* $C_1 \subseteq D_1 ⅋ F$ *and* $F \otimes C_2 \subseteq D_2$, *then* $C_1 \otimes C_2 \subseteq D_1 ⅋ D_2$;

(xv) $\forall F,C_1,C_2,D_1,D_2 \in \mathscr{A}$: *if* $C_1 \otimes F \subseteq D_1$ *and* $C_2 \subseteq F ⅋ D_2$, *then* $C_1 \otimes C_2 \subseteq D_1 ⅋ D_2$;

(xvi) $\forall F,C,D_1,D_2,D \in \mathscr{A}$: *if* $C \subseteq D_1 ⅋ F ⅋ D_2$ *and* $F \subseteq D$, *then* $C \subseteq D_1 ⅋ D ⅋ D_2$;

(xvii) $\forall F,C_1,C_2,C,D \in \mathscr{A}$: *if* $C_1 \otimes F \otimes C_2 \subseteq D$ *and* $C \subseteq F$, *then* $C_1 \otimes C \otimes C_2 \subseteq D$;

***$\otimes$-rules*:**

(xviii) $\forall F,G,D_1,D_2,C_1,C_2 \in \mathscr{A}$: *if* $C_1 \subseteq D_1 ⅋ F$ *and* $C_2 \subseteq G ⅋ D_2$, *then* $C_1 \otimes C_2 \subseteq D_1 ⅋ (F \otimes G) ⅋ D_2$;

(xix) $\forall F,G,D_1,D_2,D_3,C \in \mathscr{A}$: *if* $1 \subseteq D_1 \parr F$ *and* $C \subseteq D_2 \parr G \parr D_3$, then $C \subseteq D_2 \parr D_1 \parr (F \otimes G) \parr D_3$;

(xx) $\forall F,G,D_1,D_2,D_3,C \in \mathscr{A}$: *if* $C \subseteq D_1 \parr F \parr D_2$ *and* $1 \subseteq G \parr D_3$, *then* $C \subseteq D_1 \parr (F \otimes G) \parr D_3 \parr D_2$.

$\parr$-rules:

(xxi) $\forall F,G,D_1,D_2,C_1,C_2 \in \mathscr{A}$: *if* $C_1 \otimes F \subseteq D_1$ *and* $G \otimes C_2 \subseteq D_2$, *then* $C_1 \otimes (F \parr G) \otimes C_2 \subseteq D_1 \parr D_2$;

(xxii) $\forall F,G,D,C_1,C_2,C_3 \in \mathscr{A}$: *if* $C_1 \otimes F \subseteq \bot$ *and* $C_2 \otimes G \otimes C_3 \subseteq D$, *then* $C_2 \otimes C_1 \otimes (F \parr G) \otimes C_3 \subseteq D$;

(xxiii) $\forall F,G,D,C_1,C_2,C_3 \in \mathscr{A}$: *if* $C_1 \otimes F \otimes C_2 \subseteq D$ *and* $G \otimes C_3 \subseteq \bot$, *then* $C_1 \otimes (F \parr G) \otimes C_3 \otimes C_2 \subseteq D$.

M-rule:

(xxiv) $\forall C,C_1,C_2 \in \mathscr{A}$: $C \subseteq C_1 \parr M \parr C_2$.

$\cap$-rules:

(xxv) $\forall F,G,C,D_1,D_2 \in \mathscr{A}$: *if* $C \subseteq D_1 \parr F \parr D_2$ *and* $C \subseteq D_1 \parr G \parr D_2$, *then* $C \subseteq D_1 \parr (F \cap G) \parr D_2$;

(xxvi) $\forall F,G,C_1,C_2,D \in \mathscr{A}$: *if* $C_1 \otimes F \otimes C_2 \subseteq D$ *or* $C_1 \otimes G \otimes C_2 \subseteq D$, *then* $C_1 \otimes (F \cap G) \otimes C_2 \subseteq D$.

$\oplus$-rules:

(xxvii) $\forall F,G,C,D_1,D_2 \in \mathscr{A}$: *if* $C \subseteq D_1 \parr F \parr D_2$ *or* $C \subseteq D_1 \parr G \parr D_2$, *then* $C \subseteq D_1 \parr (F \oplus G) \parr D_2$;

(xxviii) $\forall F,G,C_1,C_2,D \in \mathscr{A}$: *if* $C_1 \otimes F \otimes C_2 \subseteq D$ *and* $C_1 \otimes G \otimes C_2 \subseteq D$, *then* $C_1 \otimes (F \cap G) \otimes C_2 \subseteq D$.

□ Left to the reader. □

1.16. PROPOSITION. *Let $\langle M,\cdot,i\rangle$ be a monoid with unit and $\bot \subseteq M$. If $\bot$ is suitable and $\mathscr{A}$ is the set of all the facts over $\bot$, then the structure*

$$\langle \mathscr{A}, 1_\bot, \otimes_\bot, \oplus_\bot \rangle$$

is a unital quantale, i.e.

(i) *$\langle \mathscr{A}, \oplus_\bot\rangle$ is a complete lattice,*

(ii) *$\otimes_\bot$ is an associative operation which distributes on both sides over arbitrary $\oplus_\bot$, and*

(iii) *$1_\bot \otimes_\bot F = F \otimes_\bot 1_\bot = F$, for every $F \in \mathscr{A}$.*

□ Follows from 1.15. □

§2. Two-sided sequent calculus: soundness and completeness.

2.1. DEFINITION. The language $\mathscr{L}$(PNCL) of the two-sided sequent calculus for pure noncommutative classical linear propositional logic is defined as follows.

(i) The *alphabet* of $\mathscr{L}$(PNCL) consists of the following symbols: denumerably many propositional variables, denoted by $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}, \ldots$; the propositional constants $\mathbf{1}$ and $\boldsymbol{T}$; the unary connectives $(-)^\bot$ and ${}^\bot(-)$; the binary connectives $\otimes, \parr$, & and $\oplus$; the sequent symbol $\Rightarrow$; and the usual auxiliary symbols.

(ii) The *formulas* of $\mathscr{L}$(PNCL) are defined inductively as follows:

(a) Each propositional variable is a formula.

(b) $\mathbf{1}$ and $\boldsymbol{T}$ are formulas.

(c) If A is a formula, then $A^\bot$ and ${}^\bot A$ are formulas.

(d) If A and B are formulas, then $A \otimes B$, $A \parr B$, $A \oplus B$ and $A \,\&\, B$ are formulas.

(e) Nothing else is a formula.

(iii) The *sequents* of $\mathscr{L}$(PNCL) are defined as follows: $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sequences of formulas of the language $\mathscr{L}$(PNCL).

(iv) For every finite sequence Γ of formulas of $\mathscr{L}$(PNCL), we define $(\Gamma)^\perp$, ${}^\perp(\Gamma)$, $\otimes(\Gamma)$, $\parr(\Gamma)$ and $\mathrm{op}(\Gamma)$ as follows, by induction. If Γ is the empty sequence, then $(\Gamma)^\perp = \otimes(\Gamma) = \parr(\Gamma) = \mathrm{op}(\Gamma) =$ the empty sequence. Otherwise, we put

$$(A,\Gamma)^\perp = A^\perp,(\Gamma)^\perp, \qquad {}^\perp(A,\Gamma) = {}^\perp A,{}^\perp(\Gamma),$$
$$\otimes(A,\Gamma) = A \otimes (\otimes(\Gamma)), \qquad \parr(A,\Gamma) = A \parr (\parr(\Gamma)),$$
$$\mathrm{op}(A,\Gamma) = \mathrm{op}(\Gamma),A.$$

2.2. DEFINITION. Let $\mathscr{S} = \langle \mathscr{A}, \subseteq, \perp, 1_\perp, 0_\perp, M, (-)^\perp, {}^\perp(-), \otimes_\perp, \parr_\perp, \cap, \oplus_\perp \rangle$ be a noncommutative classical phase space, and σ an interpretation in $\mathscr{S}$ (i.e. a function from the variables of $\mathscr{L}$(PNCL) into $\mathscr{A}$).

(i) For every formula A of $\mathscr{L}$(PNCL), we define $A^{\mathrm{PNCL},\mathscr{S},\sigma} \in \mathscr{A}$:

$$\boldsymbol{a}^{\mathrm{PNCL},\mathscr{S},\sigma} = \sigma(\boldsymbol{a}), \text{ for every variable } \boldsymbol{a},$$
$$\mathbf{1}^{\mathrm{PNCL},\mathscr{S},\sigma} = 1_\perp,$$
$$\boldsymbol{T}^{\mathrm{PNCL},\mathscr{S},\sigma} = M,$$
$$(A^\perp)^{\mathrm{PNCL},\mathscr{S},\sigma} = (A^{\mathrm{PNCL},\mathscr{S},\sigma})^\perp,$$
$$({}^\perp A)^{\mathrm{PNCL},\mathscr{S},\sigma} = {}^\perp(A^{\mathrm{PNCL},\mathscr{S},\sigma}),$$
$$(A \otimes B)^{\mathrm{PNCL},\mathscr{S},\sigma} = A^{\mathrm{PNCL},\mathscr{S},\sigma} \otimes_\perp B^{\mathrm{PNCL},\mathscr{S},\sigma},$$
$$(A \parr B)^{\mathrm{PNCL},\mathscr{S},\sigma} = A^{\mathrm{PNCL},\mathscr{S},\sigma} \parr_\perp B^{\mathrm{PNCL},\mathscr{S},\sigma},$$
$$(A \,\&\, B)^{\mathrm{PNCL},\mathscr{S},\sigma} = A^{\mathrm{PNCL},\mathscr{S},\sigma} \cap B^{\mathrm{PNCL},\mathscr{S},\sigma},$$
$$(A \oplus B)^{\mathrm{PNCL},\mathscr{S},\sigma} = A^{\mathrm{PNCL},\mathscr{S},\sigma} \oplus_\perp B^{\mathrm{PNCL},\mathscr{S},\sigma}.$$

(ii) For every sequent $\Gamma \Rightarrow \Delta$ of $\mathscr{L}$(PNCL), we define $(\Gamma \Rightarrow \Delta)^{\mathrm{PNCL},\mathscr{S},\sigma}$:

$$(A_1,\ldots,A_m \Rightarrow B_1,\ldots,B_n)^{\mathrm{PNCL},\mathscr{S},\sigma} \text{ is } (A_1 \otimes \cdots \otimes A_m)^{\mathrm{PNCL},\mathscr{S},\sigma} \subseteq (B_1 \parr \cdots \parr B_n)^{\mathrm{PNCL},\mathscr{S},\sigma};$$
$$(A_1,\ldots,A_m \Rightarrow)^{\mathrm{PNCL},\mathscr{S},\sigma} \text{ is } (A_1 \otimes \cdots \otimes A_m)^{\mathrm{PNCL},\mathscr{S},\sigma} \subseteq \perp;$$
$$(\Rightarrow B_1,\ldots,B_n)^{\mathrm{PNCL},\mathscr{S},\sigma} \text{ is } 1_\perp \subseteq (B_1 \parr \cdots \parr B_n)^{\mathrm{PNCL},\mathscr{S},\sigma}.$$

2.3. DEFINITION. Let $\Gamma \Rightarrow \Delta$ be a sequent of $\mathscr{L}$(PNCL) and $\mathscr{S}$ a noncommutative classical phase space. $\Gamma \Rightarrow \Delta$ is PNCL-*valid in* $\mathscr{S}$ iff $(\Gamma \Rightarrow \Delta)^{\mathrm{PNCL},\mathscr{S},\sigma}$ holds for every interpretation σ in $\mathscr{S}$.

2.4. DEFINITION. The sequent calculus PNCL for classical linear propositional logic is given by the following rules concerning the sequents of $\mathscr{L}$(PNCL).

(i) Basic rules. Identity rule:

$$\frac{}{A \Rightarrow A}\ (\mathrm{id}).$$

Cut rule:

$$\frac{\Gamma \Rightarrow \Delta_1,A,\Delta_2 \qquad \Gamma_1,A,\Gamma_2 \Rightarrow \Delta}{\Gamma_1,\Gamma,\Gamma_2 \Rightarrow \Delta_1,\Delta,\Delta_2}\ (\mathrm{cut}),$$

if $\Delta_1 = \Gamma_2 = \varnothing$, or $\Delta_2 = \Gamma_1 = \varnothing$, or $\Gamma_1 = \Gamma_2 = \varnothing$, or $\Delta_1 = \Delta_2 = \varnothing$.

(ii) $(-)^{\perp}$-rules:

$$\frac{A,\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A^{\perp},\Delta}\ ((-)^{\perp},\mathrm{R}), \qquad \frac{\Gamma \Rightarrow \Delta,A}{\Gamma,A^{\perp} \Rightarrow \Delta}\ ((-)^{\perp},\mathrm{L}).$$

(iii) ${}^{\perp}(-)$-rules:

$$\frac{\Gamma,A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,{}^{\perp}A}\ ({}^{\perp}(-),\mathrm{R}), \qquad \frac{\Gamma \Rightarrow A,\Delta}{{}^{\perp}A,\Gamma \Rightarrow \Delta}\ ({}^{\perp}(-),\mathrm{L}).$$

(iv) **1**-rules:

$$\frac{}{\Rightarrow \mathbf{1}}\ (\mathbf{1},\mathrm{R}), \qquad \frac{\Gamma_1,\Gamma_2 \Rightarrow \Delta}{\Gamma_1,\mathbf{1},\Gamma_2 \Rightarrow \Delta}\ (\mathbf{1},\mathrm{L}).$$

(v) ***T***-rule:

$$\frac{}{\Gamma \Rightarrow \Delta_1,\boldsymbol{T},\Delta_2}\ (\boldsymbol{T}).$$

(vi) $\otimes$-rules:

$$\frac{\Gamma_1 \Rightarrow \Delta_1,A,\Delta_2 \qquad \Gamma_2 \Rightarrow \Delta_3,B,\Delta_4}{\Gamma_1,\Gamma_2 \Rightarrow \Delta_3,\Delta_1,(A \otimes B),\Delta_4,\Delta_2}\ (\otimes,\mathrm{R}),$$

if $\Delta_2 = \Delta_3 = \varnothing$, or $\Delta_2 = \Gamma_1 = \varnothing$, or $\Delta_3 = \Gamma_2 = \varnothing$;

$$\frac{\Gamma_1,A,B,\Gamma_2 \Rightarrow \Delta}{\Gamma_1,(A \otimes B),\Gamma_2 \Rightarrow \Delta}\ (\otimes,\mathrm{L}).$$

(vii) $\parr$ -rules:

$$\frac{\Gamma \Rightarrow \Delta_1,A,B,\Delta_2}{\Gamma \Rightarrow \Delta_1(A \parr B),\Delta_2}\ (\parr,\mathrm{L});$$

$$\frac{\Gamma_1,A,\Gamma_2 \Rightarrow \Delta_1 \qquad \Gamma_3,B,\Gamma_4 \Rightarrow \Delta_2}{\Gamma_3,\Gamma_1,(A \parr B),\Gamma_4,\Gamma_2 \Rightarrow \Delta_1,\Delta_2}\ (\parr,\mathrm{L}),$$

if $\Gamma_2 = \Gamma_3 = \varnothing$, or $\Gamma_2 = \Delta_1 = \varnothing$, or $\Gamma_3 = \Delta_2 = \varnothing$.

(viii) &-rules:

$$\frac{\Gamma \Rightarrow \Delta_1,A,\Delta_2 \qquad \Gamma \Rightarrow \Delta_1,B,\Delta_2}{\Gamma \Rightarrow \Delta_1,(A \mathbin{\&} B),\Delta_2}\ (\&,\mathrm{R}),$$

$$\frac{\Gamma_1,A,\Gamma_2 \Rightarrow \Delta}{\Gamma_1,(A \mathbin{\&} B),\Gamma_2 \Rightarrow \Delta}\ (\&,\mathrm{L1}), \qquad \frac{\Gamma_1,B,\Gamma_2 \Rightarrow \Delta}{\Gamma_1,(A \mathbin{\&} B),\Gamma_2 \Rightarrow \Delta}\ (\&,\mathrm{L2}).$$

(ix) $\oplus$-rules:

$$\frac{\Gamma \Rightarrow \Delta_1,A,\Delta_2}{\Gamma \Rightarrow \Delta_1,(A \oplus B),\Delta_2}\ (\oplus,\mathrm{R1}), \qquad \frac{\Gamma \Rightarrow \Delta_1,B,\Delta_2}{\Gamma \Rightarrow \Delta_1,(A \oplus B),\Delta_2}\ (\oplus,\mathrm{R2}),$$

$$\frac{\Gamma_1,A,\Gamma_2 \Rightarrow \Delta \qquad \Gamma_1,B,\Gamma_2 \Rightarrow \Delta}{\Gamma_1,(A \oplus B),\Gamma_2 \Rightarrow \Delta}\ (\oplus,\mathrm{L}).$$

A *proof* of a sequent $\Gamma \Rightarrow \Delta$ in PNCL is defined as usual. A sequent $\Gamma \Rightarrow \Delta$ is *provable* in PNCL iff there is a proof of $\Gamma \Rightarrow \Delta$ in PNCL.

The *strict multiplicative fragment* of PNCL is obtained by restricting PNCL to the following rules: basic rules, $(-)^{\perp}$-rules, ${}^{\perp}(-)$-rules, $\otimes$-rules and ⅋-rules.

2.5. REMARKS. (i) The $(-)^{\perp}$-rules and ${}^{\perp}(-)$-rules allow us to prove the following sequents:

$$\pi_1 \quad \dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,(\mathrm{id})}{\Rightarrow A,{}^{\perp}A}\,({}^{\perp}(-),\mathrm{R})}{({}^{\perp}A)^{\perp} \Rightarrow A}\,((-)^{\perp},\mathrm{L}), \qquad \pi_2 \quad \dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,(\mathrm{id})}{{}^{\perp}A,A \Rightarrow}\,({}^{\perp}(-),\mathrm{L})}{A \Rightarrow ({}^{\perp}A)^{\perp}}\,((-)^{\perp},\mathrm{R}),$$

$$\pi_3 \quad \dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,(\mathrm{id})}{\Rightarrow A^{\perp},A}\,((-)^{\perp},\mathrm{R})}{{}^{\perp}(A^{\perp}) \Rightarrow A}\,({}^{\perp}(-),\mathrm{L}), \qquad \pi_4 \quad \dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,(\mathrm{id})}{A,A^{\perp} \Rightarrow}\,((-)^{\perp},\mathrm{L})}{A \Rightarrow {}^{\perp}(A^{\perp})}\,({}^{\perp}(-),\mathrm{R}).$$

(ii) A sequent $\Gamma \Rightarrow \Delta$ is provable in PNCL iff $\Rightarrow \mathrm{op}(\Gamma^{\perp}), \Delta$ is provable in PNCL. We argue by induction on the length of the sequence Γ. If Γ is the empty sequence, then there is nothing to prove. Let $\Gamma = A,\Gamma'$. If $A,\Gamma' \Rightarrow \Delta$ is provable, then $\Gamma' \Rightarrow A^{\perp},\Delta$ is provable by $((-)^{\perp},\mathrm{R})$, and so (by the induction hypothesis) $\Rightarrow \mathrm{op}((\Gamma')^{\perp}),A^{\perp},\Delta$ is provable, i.e. $\Rightarrow \mathrm{op}((A,\Gamma')^{\perp}),\Delta$. If $\Rightarrow \mathrm{op}((A,\Gamma')^{\perp}),\Delta$, i.e. $\Rightarrow \mathrm{op}((\Gamma')^{\perp}),A^{\perp},\Delta$, is provable, then (by the induction hypothesis) $\Gamma' \Rightarrow A^{\perp},\Delta$ is provable, and then the sequent ${}^{\perp}(A^{\perp}),\Gamma' \Rightarrow \Delta$ is provable by $({}^{\perp}(-),\mathrm{L})$, and finally $A,\Gamma' \Rightarrow \Delta$ is provable by (cut) with the provable sequent $A \Rightarrow {}^{\perp}(A^{\perp})$.

(iii) We show that $\Rightarrow A,\Delta$ is provable in PNCL iff $\Rightarrow \Delta,{}^{\perp\perp}A$ is provable in PNCL. Let $\Rightarrow A,\Delta$ be provable. Then by using $({}^{\perp}(-),\mathrm{L})$ we get ${}^{\perp}A \Rightarrow \Delta$, and then by using $({}^{\perp}(-),\mathrm{R})$ we get $\Rightarrow \Delta,{}^{\perp\perp}A$.

From $\Rightarrow \Delta,{}^{\perp\perp}A$, by using $((-)^{\perp},\mathrm{L})$ we get $({}^{\perp\perp}A)^{\perp} \Rightarrow \Delta$, and then by using $((-)^{\perp},\mathrm{R})$ we get $\Rightarrow ({}^{\perp\perp}A)^{\perp\perp},\Delta$; now, since $({}^{\perp\perp}A)^{\perp\perp} \Rightarrow A$ is provable in PNCL, by (cut) we get a proof of $\Rightarrow A,\Delta$.

(iv) Analogously, we can show that $\Rightarrow \Delta,A$ is provable in PNCL iff $\Rightarrow A^{\perp\perp},\Delta$ is provable in PNCL.

(v) The meaning of (iii) and (iv) is that, if there is no assumption, the first conclusion A can be put as the last conclusion in the form ${}^{\perp\perp}A$, and the last conclusion A can be put as the first conclusion in the form $A^{\perp\perp}$. We deal with an unexpected form of *exchange*....

We show the proof of the sequent $({}^{\perp\perp}A)^{\perp\perp} \Rightarrow A$, used to prove (iv) above: the reader will remark that the proof uses the cut rule, and will be tempted to find another cut-free proof...:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,(\mathrm{id})}{\Rightarrow A,{}^{\perp}A}\,({}^{\perp}(-),\mathrm{R}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,(\mathrm{id})}{{}^{\perp}A,A \Rightarrow}\,({}^{\perp}(-),\mathrm{L})}{{}^{\perp}A \Rightarrow {}^{\perp}A}\,({}^{\perp}(-),\mathrm{R})}{{}^{\perp\perp}A,{}^{\perp}A \Rightarrow}\,({}^{\perp}(-),\mathrm{L})}{{}^{\perp}A \Rightarrow ({}^{\perp\perp}A)^{\perp}}\,((-)^{\perp},\mathrm{R})}{\Rightarrow A,({}^{\perp\perp}A)^{\perp}}\,(\mathrm{cut})}{({}^{\perp\perp}A)^{\perp\perp} \Rightarrow A}\,((-)^{\perp},\mathrm{L})$$

The reader is advised to construct a similar proof (using the cut rule) for the sequents ${}^{\perp\perp}(A^{\perp\perp}) \Rightarrow A$, $({}^{\perp\perp}(({}^{\perp\perp}A)^{\perp\perp})^{\perp\perp}) \Rightarrow A$, etc.

(vi) The rules (cut), ($\otimes$,R) and ($\wp$,L) are rules *subject to conditions on the contexts.* The ground of the conditions lies in the semantics (cf. 1.15 and 2.6). The conditions are trivially satisfied if we allow the exchange rule.

For more comments on the conditions, see the Introduction and Remark 3.4 at the end of §3.

2.6. THEOREM. *Let $\Gamma \Rightarrow \Delta$ be a sequent of* $\mathscr{L}$(PNCL). *If $\Gamma \Rightarrow \Delta$ is provable in* PNCL, *then $\Gamma \Rightarrow \Delta$ is* PNCL-*valid in every noncommutative classical phase space.*

□ Let $\mathscr{S} = \langle \mathscr{A}, \subseteq, \perp, 1_\perp, 0_\perp, M, (-)^\perp, {}^\perp(-), \otimes_\perp, \wp_\perp, \cap, \oplus_\perp \rangle$ be a noncommutative classical phase space. By induction on the proof of the sequent in PNCL, we verify that the sequent is PNCL-valid in $\mathscr{S}$.

(i) The last rule is (id): trivial.

(ii) The last rule is (cut): use the induction hypothesis and Proposition 1.15(xiv)–(xvii).

(iii) The last rule is ($(-)^\perp$,R) or ($(-)^\perp$,L) or (${}^\perp(-)$,R) or (${}^\perp(-)$,L): use the induction hypothesis and Proposition 1.15(xii)–(xiii).

(iv) The last rule is (**1**,R): trivial.

(v) The last rule is (**1**,L): use the induction hypothesis and Proposition 1.15(viii).

(vi) The last rule is (***T***): by Proposition 1.15(xxiv).

(vii) The last rule is ($\otimes$,R): use the induction hypothesis and Proposition 1.15(xviii)–(xx).

(viii) The last rule is ($\otimes$,L): use the induction hypothesis and Proposition 1.15(x).

(ix) The last rule is ($\wp$,R): use the induction hypothesis and Proposition 1.15(xi).

(x) The last rule is ($\wp$,L): use the induction hypothesis and Proposition 1.15(xxi)–(xxiii).

(xi) The last rule is (&,R): use the induction hypothesis and Proposition 1.15(xxv).

(xii) The last rule is (&,L1) or (&,L2): use the induction hypothesis and Proposition 1.15(xxvi).

(xiii) The last rule is ($\oplus$,R1) or ($\oplus$,R2): use the induction hypothesis and Proposition 1.15(xxvii).

(xiv) The last rule is ($\oplus$,L): use the induction hypothesis and Proposition 1.15(xxviii). □

2.7. DEFINITION. Let us consider the monoid with unity $\langle \mathbb{T}, *, \bullet \rangle$ where $\mathbb{T}$ is the set of all the finite sequences of formulas of $\mathscr{L}$(PNCL), $\bullet$ is the empty sequence and $*$ is the concatenation of finite sequences.

Let us consider the following $\perp \in \mathscr{P}(\mathbb{T})$:

$$\perp = \{\Gamma \mid \Gamma \Rightarrow \quad \text{is provable in PNCL}\}.$$

For every formula A of $\mathscr{L}$(PNCL), we define

$$\mathrm{PR}(A) = \{\Gamma \mid \Gamma \Rightarrow A \text{ is provable in PNCL}\}$$

(the *set of the provability of* A).

Finally, we define the *set of the provability sets*

$$\mathbb{PR} = \{F \mid F \in \mathscr{P}(\mathbb{T}) \text{ and } F = \mathrm{PR}(A) \text{ for some formula } A \text{ of } \mathscr{L}(\text{PNCL})\}.$$

2.8. PROPOSITION. *Consider* $\langle \mathbb{T},*,\bullet\rangle$ *and* $\perp$ *as in Definition* 2.7.
(i) *For every formula* A *of* $\mathscr{L}$(PNCL), $A \in \mathrm{PR}(A)$.
(ii) *The sequent* $A \Rightarrow B$ *is provable in* PNCL *iff* $\mathrm{PR}(A) \subseteq \mathrm{PR}(B)$.
(iii) *For every formula* A *of* $\mathscr{L}$(PNCL),

$$\mathrm{PR}(A^{\perp}) = \mathrm{PR}(A) \multimap \perp \quad (i.e. = {}^{\perp}(\mathrm{PR}(A))).$$

(iv) *For every formula* A *of* $\mathscr{L}$(PNCL),

$$\mathrm{PR}({}^{\perp}A) = \perp \circ\!\!- \mathrm{PR}(A) \quad (\text{i.e.} = {}^{\perp}(\mathrm{PR}(A))).$$

(v) *For every* $\Gamma \in \mathbb{T}$,

$$\{\Gamma\}^{\perp} = \mathrm{PR}((\otimes(\Gamma))^{\perp}).$$

(vi) *For every* $\Gamma \in \mathbb{T}$,

$$^{\perp}\{\Gamma\} = \mathrm{PR}({}^{\perp}(\otimes(\Gamma))).$$

(vii) $\perp$ *is suitable.*
(viii) *For every formula* A *of* $\mathscr{L}$(PNCL), $\mathrm{PR}(A)$ *is a* $\perp$*-fact.*
□(i) This follows since for every formula A we have

$$\frac{}{A \Rightarrow A}\,(\mathrm{id}). \qquad \square$$

□(ii) Let $A \Rightarrow B$ be provable in PNCL. If $\Gamma \in \mathrm{PR}(A)$, then $\Gamma \in \mathrm{PR}(B)$ because

$$\frac{\dfrac{\vdots}{\Gamma \Rightarrow A} \quad \dfrac{\vdots}{A \Rightarrow B}}{\Gamma \Rightarrow B}\,(\mathrm{cut})$$

Conversely, let $\mathrm{PR}(A) \subseteq \mathrm{PR}(B)$: because $A \in \mathrm{PR}(A)$ by (i) above, then $A \in \mathrm{PR}(B)$, i.e. $A \Rightarrow B$ is provable in PNCL. □

□(iii) Let $\Gamma \in \mathrm{PR}(\neg A)$. If $\Delta \in \mathrm{PR}(A)$, then $\Delta * \Gamma \in \perp$ because

$$\frac{\dfrac{\vdots}{\Gamma \Rightarrow A^{\perp}} \quad \dfrac{\dfrac{\vdots}{\Delta \Rightarrow A}}{\Delta, A^{\perp} \Rightarrow}\,((-)^{\perp},\mathrm{L})}{\Delta,\Gamma \Rightarrow}\,(\mathrm{cut}).$$

Let $\Gamma \in \mathrm{PR}(A) \multimap \perp$. From the fact that $A \in \mathrm{PR}(A)$ by (i), we have $A * \Gamma \in \perp$, so that we get $\Gamma \in \mathrm{PR}(A^{\perp})$ because

$$\frac{\dfrac{\vdots}{A,\Gamma \Rightarrow}}{\Gamma \Rightarrow A^{\perp}}\,((-)^{\perp},\mathrm{R}). \qquad \square$$

□ (iv) Let $\Gamma \in \mathrm{PR}({}^{\perp}A)$. If $\Delta \in \mathrm{PR}(A)$, then $\Gamma * \Delta \in \perp$ because

$$\frac{\dfrac{\vdots}{\Gamma \Rightarrow {}^{\perp}A} \quad \dfrac{\dfrac{\vdots}{\Delta \Rightarrow A}}{{}^{\perp}A,\Delta \Rightarrow}\,({}^{\perp}(-),\mathrm{L})}{\Gamma,\Delta \Rightarrow}\,(\mathrm{cut}).$$

Let $\Gamma \in \bot \circ\!- \mathrm{PR}(A)$. From the fact that $A \in \mathrm{PR}(A)$ by (i), we have $\Gamma * A \in \bot$, so that we get $\Gamma \in \mathrm{PR}({}^{\bot}A)$ because

$$\dfrac{\dfrac{\vdots}{\Gamma, A \Rightarrow}}{\Gamma \Rightarrow {}^{\bot}A}\,({}^{\bot}(-),\mathrm{R}). \qquad \square$$

□(v) Let $\Delta \in \{\Gamma\} -\!\circ \bot$. This means $\Gamma * \Delta \in \bot$, and then we get $\Delta \in \mathrm{PR}((\otimes(\Gamma))^{\bot})$ since

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \Delta \Rightarrow}}{\otimes(\Gamma), \Delta \Rightarrow}\,(\otimes,\mathrm{L})\text{ several times}}{\Delta \Rightarrow (\otimes(\Gamma))^{\bot}}\,((-)^{\bot},\mathrm{R}).$$

Vice versa, let $\Delta \in \mathrm{PR}((\otimes(\Gamma))^{\bot})$. Note that the sequent $\Gamma \Rightarrow {}^{\bot}((\otimes(\Gamma))^{\bot})$ is provable, and then we get that $\Delta \in \{\Gamma\} -\!\circ \bot$ since

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow {}^{\bot}((\otimes(\Gamma))^{\bot})} \quad \dfrac{\dfrac{\vdots}{\Delta \Rightarrow (\otimes(\Gamma))^{\bot}}}{{}^{\bot}((\otimes(\Gamma))^{\bot}), \Delta \Rightarrow}\,({}^{\bot}(-),\mathrm{L})}{\Gamma, \Delta \Rightarrow}\,(\mathrm{cut}). \qquad \square$$

□(vi) Let $\Delta \in \bot \circ\!- \{\Gamma\}$. This means that $\Delta * \Gamma \in \bot$, and then we get $\Delta \in \mathrm{PR}({}^{\bot}(\otimes(\Gamma)))$ since

$$\dfrac{\dfrac{\dfrac{\vdots}{\Delta, \Gamma \Rightarrow}}{\Delta, \otimes(\Gamma) \Rightarrow}\,(\otimes,\mathrm{L})\text{ several times}}{\Delta \Rightarrow {}^{\bot}(\otimes(\Gamma))}\,({}^{\bot}(-),\mathrm{R}).$$

Vice versa, let $\Delta \in \mathrm{PR}({}^{\bot}(\otimes(\Gamma)))$. Note that the sequent $\Gamma \Rightarrow ({}^{\bot}(\otimes(\Gamma)))^{\bot}$ is provable, and then we get that $\Delta \in \bot \circ\!- \{\Gamma\}$ since

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow ({}^{\bot}(\otimes(\Gamma)))^{\bot}} \quad \dfrac{\dfrac{\vdots}{\Delta \Rightarrow {}^{\bot}(\otimes(\Gamma))}}{({}^{\bot}(\otimes(\Gamma)))^{\bot}, \Delta \Rightarrow}\,((-)^{\bot},\mathrm{L})}{\Delta, \Gamma \Rightarrow}\,(\mathrm{cut}). \qquad \square$$

□(vii) Let $\Gamma \in \mathbb{T}$. We have $({}^{\bot\bot}\{\Gamma\})^{\bot} = \mathrm{PR}(({}^{\bot\bot}(\otimes(\Gamma)))^{\bot})$ by (vi), (iii) and (iv) above; ${}^{\bot}\{\Gamma\} = \mathrm{PR}({}^{\bot}(\otimes(\Gamma)))$ by (vi) above; but $({}^{\bot\bot}(\otimes(\Gamma)))^{\bot} \Rightarrow {}^{\bot}(\otimes(\Gamma))$ is provable in PNCL, so that by (ii) we get $({}^{\bot\bot}\{\Gamma\})^{\bot} \subseteq {}^{\bot}\{\Gamma\}$. Also ${}^{\bot}(\{\Gamma\}^{\bot\bot}) = \mathrm{PR}({}^{\bot}((\otimes(\Gamma))^{\bot\bot}))$ by (v), (iii) and (ii) above; $\{\Gamma\}^{\bot} = \mathrm{PR}((\otimes(\Gamma))^{\bot})$ by (v) above; but ${}^{\bot}((\otimes(\Gamma))^{\bot\bot}) \Rightarrow (\otimes(\Gamma))^{\bot}$ is a provable sequent in PNCL, so that by (ii) we get ${}^{\bot}(\{\Gamma\}^{\bot\bot}) \subseteq \{\Gamma\}^{\bot}$. □

□(viii) Let A be a formula of $\mathscr{L}$(PNCL). By (iii) and (iv) above, ${}^{\bot}((\mathrm{PR}(A))^{\bot}) = \mathrm{PR}({}^{\bot}(A^{\bot}))$ and $({}^{\bot}(\mathrm{PR}(A)))^{\bot} = \mathrm{PR}(({}^{\bot}A)^{\bot})$. Because the following sequents are provable in PNCL:

$${}^{\bot}(A^{\bot}) \Rightarrow A, \qquad A \Rightarrow {}^{\bot}(A^{\bot}), \qquad ({}^{\bot}A)^{\bot} \Rightarrow A, \qquad A \Rightarrow ({}^{\bot}A)^{\bot},$$

we get by (ii) that

$$^{\perp}((\mathrm{PR}(A))^{\perp}) = (^{\perp}(\mathrm{PR}(A)))^{\perp} = \mathrm{PR}(A). \qquad \square$$

2.9. Proposition. *Consider* $\langle \mathbb{T},*,\bullet\rangle$, $\perp$ *and* $\mathbb{PR}$ *as in Definition* 2.7. *Let* $\mathscr{A}$ *be the class of all the* $\perp$*-facts. Consider the noncommutative classical phase space*

$$\langle \mathscr{A},\subseteq,1_{\perp},\perp,\mathbb{T},0_{\perp},(-)^{\perp},{}^{\perp}(-),\otimes_{\perp},\parr_{\perp},\cap,\oplus_{\perp}\rangle.$$

Then the following assertions are true:
(i) $\mathbb{PR} \subseteq \mathscr{A}$.
(ii) $\mathrm{PR}(\mathbf{1}) = 1_{\perp}$.
(iii) $\mathrm{PR}(\boldsymbol{T}) = \mathbb{T}$.
(iv) $\mathrm{PR}(A \otimes B) = \mathrm{PR}(A) \otimes_{\perp} \mathrm{PR}(B)$, *for all formulas* A *and* B *of* $\mathscr{L}$(PNCL).
(v) $\mathrm{PR}(A \parr B) = \mathrm{PR}(A) \parr_{\perp} \mathrm{PR}(B)$, *for all formulas* A *and* B *of* $\mathscr{L}$(PNCL).
(vi) $\mathrm{PR}(A \,\&\, B) = \mathrm{PR}(A) \cap \mathrm{PR}(B)$, *for all formulas* A *and* B *of* $\mathscr{L}$(PNCL).
(vii) $\mathrm{PR}(A \oplus B) = (\mathrm{PR}(A) \oplus_{\perp} \mathrm{PR}(B)$, *for all formulas* A *and* B *of* $\mathscr{L}$(PNCL).

□(i) By 2.8(viii). □

□(ii) We prove $\mathrm{PR}(\mathbf{1}) = \perp \multimap \perp$ (note that $\perp \multimap \perp = \perp \multimap(\{\bullet\} \multimapinv \perp) = 1_{\perp}$). Let $\Gamma \in \mathrm{PR}(\mathbf{1})$. If $\Delta \in \perp$, then $\Gamma * \Delta \in \perp$ and $\Delta * \Gamma \in \perp$ because

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow \mathbf{1}} \qquad \dfrac{\dfrac{\vdots}{\Delta \Rightarrow}}{\mathbf{1},\Delta \Rightarrow}\,(\mathbf{1},\mathrm{L})}{\Gamma,\Delta \Rightarrow}\,(\text{cut}), \qquad \dfrac{\dfrac{\vdots}{\Gamma \Rightarrow \mathbf{1}} \qquad \dfrac{\dfrac{\vdots}{\Delta \Rightarrow}}{\Delta,\mathbf{1} \Rightarrow}\,(\mathbf{1},\mathrm{L})}{\Delta,\Gamma \Rightarrow}\,(\text{cut}).$$

Let $\Gamma \in \perp \multimap \perp$: so, in particular, $^{\perp}\mathbf{1} * \Gamma \in \perp$ because $^{\perp}\mathbf{1} \in \perp$. We have

$$\dfrac{\dfrac{}{\Rightarrow \mathbf{1}}\,(\mathbf{1},\mathrm{R})}{^{\perp}\mathbf{1} \Rightarrow}\,(^{\perp}(-),\mathrm{L});$$

but then $\Gamma \in \mathrm{PR}(\mathbf{1})$ because

$$\dfrac{\dfrac{\dfrac{}{\mathbf{1} \Rightarrow \mathbf{1}}\,(\text{id})}{\Rightarrow {}^{\perp}\mathbf{1},\mathbf{1}}\,(^{\perp}(-),\mathrm{R}) \qquad \dfrac{\vdots}{^{\perp}\mathbf{1},\Gamma \Rightarrow}}{\Gamma \Rightarrow \mathbf{1}}\,(\text{cut}). \qquad \square$$

□(iii) Immediate, because for every $\Gamma \in \mathbb{T}$

$$\dfrac{}{\Gamma \Rightarrow \boldsymbol{T}}\,(\boldsymbol{T}). \qquad \square$$

□(iv) Let $\Gamma \in \mathrm{PR}(A \otimes B)$. Suppose $\Delta \in \mathrm{PR}(A) \cdot \mathrm{PR}(B) \multimap \perp$, i.e. for every $\Theta_1 \in \mathrm{PR}(A)$ and $\Theta_2 \in \mathrm{PR}(B)$ we have $\Theta_1 * \Theta_2 * \Delta \in \perp$. But, by 2.8(i), $A \in \mathrm{PR}(A)$ and $B \in \mathrm{PR}(B)$, so that $A * B * \Delta \in \perp$. Then $\Gamma * \Delta \in \perp$ because

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow A \otimes B} \qquad \dfrac{\dfrac{\vdots}{A,B,\Delta \Rightarrow}}{(A \otimes B),\Delta \Rightarrow}\,(\otimes,\mathrm{L})}{\Gamma,\Delta \Rightarrow}\,(\text{cut}).$$

So, $\Gamma \in \perp \multimapinv (\mathrm{PR}(A) \cdot \mathrm{PR}(B) \multimap \perp)$.

Let $\Gamma \in \perp \multimapinv (\mathrm{PR}(A) \cdot \mathrm{PR}(B) \multimap \perp)$. We have $(A \otimes B)^{\perp} \in \mathrm{PR}(A) \cdot \mathrm{PR}(B) \multimap \perp$, because if $\Gamma_1 \in \mathrm{PR}(A)$ and $\Gamma_2 \in \mathrm{PR}(B)$ then $\Gamma_1 * \Gamma_2 * (A \otimes B)^{\perp} \in \perp$ and

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \Rightarrow A} \quad \dfrac{\vdots}{\Gamma_2 \Rightarrow B}}{\Gamma_1, \Gamma_2 \Rightarrow A \otimes B}\,(\otimes,\mathrm{R}) \qquad \dfrac{A \otimes B \Rightarrow A \otimes B}{A \otimes B, (A \otimes B)^{\perp} \Rightarrow}\,((-)^{\perp},\mathrm{L})}{\Gamma_1, \Gamma_2, (A \otimes B)^{\perp} \Rightarrow}\,(\mathrm{cut}).$$

So, $\Gamma * (A \otimes B)^{\perp} \in \perp$, and therefore $\Gamma \in \mathrm{PR}(A \otimes B)$ because

$$\dfrac{\dfrac{\Gamma, (A \otimes B)^{\perp} \Rightarrow}{\Gamma \Rightarrow {}^{\perp}((A \otimes B)^{\perp})}\,({}^{\perp}(-),\mathrm{R}) \qquad \dfrac{\vdots}{{}^{\perp}((A \otimes B)^{\perp}) \Rightarrow A \otimes B}}{\Gamma \Rightarrow A \otimes B}\,(\mathrm{cut}). \qquad \square$$

$\square$(v) Let $\Gamma \in \mathrm{PR}(A \parr B)$. If $\Delta \in \perp \multimapinv \mathrm{PR}(A)$, since $A \in \mathrm{PR}(A)$ we have $\Delta * A \in \perp$ and therefore $\Delta * \Gamma \in \mathrm{PR}(B)$ because

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow A \parr B} \qquad \dfrac{\dfrac{\vdots}{\Delta, A \Rightarrow} \quad \dfrac{}{B \Rightarrow B}\,(\mathrm{id})}{\Delta, A \parr B \Rightarrow B}\,(\parr,\mathrm{L})}{\Delta, \Gamma \Rightarrow B}\,(\mathrm{cut}).$$

Let $\Gamma \in (\perp \multimapinv \mathrm{PR}(A)) \multimap \mathrm{PR}(B)$. Now, ${}^{\perp}A \in \perp \multimapinv \mathrm{PR}(A)$ and thus ${}^{\perp}A * \Gamma \in \mathrm{PR}(B)$; therefore $\Gamma \in \mathrm{PR}(A \parr B)$ because

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{{}^{\perp}A, \Gamma \Rightarrow B}}{\Gamma \Rightarrow ({}^{\perp}A)^{\perp}, B}\,((-)^{\perp},\mathrm{R}) \qquad \dfrac{\vdots}{({}^{\perp}A)^{\perp} \Rightarrow A}}{\Gamma \Rightarrow A, B}\,(\mathrm{cut})}{\Gamma \Rightarrow A \parr B}\,(\parr,\mathrm{R}). \qquad \square$$

$\square$(vi) Let $\Gamma \in \mathrm{PR}(A \,\&\, B)$. Then $\Gamma \in \mathrm{PR}(A)$ and $\Gamma \in \mathrm{PR}(B)$ because we have

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow A \,\&\, B} \quad \dfrac{\dfrac{}{A \Rightarrow A}\,(\mathrm{id})}{A \,\&\, B \Rightarrow A}\,(\&,\mathrm{L1})}{\Gamma \Rightarrow A}\,(\mathrm{cut}), \qquad \dfrac{\dfrac{\vdots}{\Gamma \Rightarrow A \otimes B} \quad \dfrac{\dfrac{}{B \Rightarrow B}\,(\mathrm{id})}{A \,\&\, B \Rightarrow B}\,(\&,\mathrm{L2})}{\Gamma \Rightarrow B}\,(\mathrm{cut}),$$

i.e. $\Gamma \in \mathrm{PR}(A) \cap \mathrm{PR}(B)$.

Conversely, let $\Gamma \in \mathrm{PR}(A) \cap \mathrm{PR}(B)$; then $\Gamma \in \mathrm{PR}(A \,\&\, B)$ because

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow A} \quad \dfrac{\vdots}{\Gamma \Rightarrow B}}{\Gamma \Rightarrow A \,\&\, B}\,(\&,\mathrm{R}). \qquad \square$$

$\square$(vii) Let $\Gamma \in \mathrm{PR}(A \oplus B)$. Suppose $\Delta \in \mathrm{PR}(A) \cup \mathrm{PR}(B) \multimap \perp$, i.e. for every $\Theta \in \mathrm{PR}(A) \cup \mathrm{PR}(B)$ we have $\Theta * \Delta \in \perp$. But, by Lemma 2.8(i), $A \in \mathrm{PR}(A) \cup \mathrm{PR}(B)$

and $B \in \mathrm{PR}(A) \cup \mathrm{PR}(B)$, so that $A * \Delta \in \bot$ and $B * \Delta \in \bot$. Then $\Gamma * \Delta \in \bot$, because

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow A \oplus B} \qquad \dfrac{\dfrac{\vdots}{A * \Delta \Rightarrow} \quad \dfrac{\vdots}{B * \Delta \Rightarrow}}{(A \oplus B) * \Delta \Rightarrow}\,(\oplus,\mathrm{L})}{\Gamma * \Delta \Rightarrow}\,(\mathrm{cut}),$$

i.e. $\Gamma \in \bot \multimapinv (\mathrm{PR}(A) \cup \mathrm{PR}(B) \multimap \bot)$.

Let $\Gamma \in \bot \multimapinv (\mathrm{PR}(A) \cup \mathrm{PR}(B) \multimap \bot)$. Now, $(A \oplus B)^\bot \in (\mathrm{PR}(A) \cup \mathrm{PR}(B) \multimap \bot)$, because if $\Delta \in \mathrm{PR}(A) \cup \mathrm{PR}(B)$ then $\Delta * (A \oplus B)^\bot \in \bot$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Delta \Rightarrow A}}{\Delta \Rightarrow A \oplus B}\,(\oplus,\mathrm{R1}) \qquad \dfrac{\vdots}{A \oplus B, (A \oplus B)^\bot \Rightarrow}}{\Delta, (A \oplus B)^\bot \Rightarrow}\,(\mathrm{cut}),$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Delta \Rightarrow B}}{\Delta \Rightarrow A \oplus B}\,(\oplus,\mathrm{R2}) \qquad \dfrac{\vdots}{A \oplus B, (A \oplus B)^\bot \Rightarrow}}{\Delta, (A \oplus B)^\bot \Rightarrow}\,(\mathrm{cut}).$$

Thus $\Gamma * (A \oplus B)^\bot \in \bot$, and therefore $\Gamma \in \mathrm{PR}(A \oplus B)$ because

$$\dfrac{\dfrac{\Gamma, (A \oplus B)^\bot \Rightarrow}{\Gamma \Rightarrow {}^\bot((A \oplus B)^\bot)}\,({}^\bot(-),\mathrm{R}) \qquad \dfrac{\vdots}{{}^\bot((A \oplus B)^\bot) \Rightarrow A \oplus B}}{\Gamma \Rightarrow A \oplus B}\,(\mathrm{cut}). \qquad \square$$

2.10. THEOREM. *Let A be a formula of $\mathscr{L}$(PNCL). If $\Rightarrow A$ is PNCL-valid in every noncommutative classical phase space, then $\Rightarrow A$ is provable in PNCL.*

$\square$ Suppose $\Rightarrow A$ is PNCL-valid in every noncommutative classical phase space; then, in particular, $\Rightarrow A$ is PNCL-valid in the classical noncommutative classical phase space

$$\mathscr{S} = \langle \mathscr{A}, \subseteq, 1_\bot, \bot, \mathbb{T}, 0_\bot, (-)^\bot, {}^\bot(-), \otimes_\bot, \wp_\bot, \cap, \oplus_\bot \rangle$$

considered in Proposition 2.9. By Proposition 2.9(i), for every formula A of $\mathscr{L}$(PNCL) we have $\mathrm{PR}(A) \in \mathscr{A}$.

We define the following interpretation σ in $\mathscr{S}$: for every variable $\boldsymbol{a}$, $\sigma(\boldsymbol{a}) = \mathrm{PR}(\boldsymbol{a})$. By induction, we show that, for every formula B of $\mathscr{L}$(PNCL), $B^{\mathrm{PNCL},\mathscr{S},\sigma} = \mathrm{PR}(B)$. Indeed:

(i) $\boldsymbol{a}^{\mathrm{PNCL},\mathscr{S},\sigma} = \sigma(\boldsymbol{a})$, by definition, for every variable $\boldsymbol{a}$.

(ii) $\mathbf{1}^{\mathrm{PNCL},\mathscr{S},\sigma} = 1_\bot = \mathrm{PR}(\mathbf{1})$, by Definition 2.2 and Proposition 2.9(ii).

(iii) $\boldsymbol{T}^{\mathrm{PNCL},\mathscr{S},\sigma} = \mathbb{T} = \mathrm{PR}(\boldsymbol{T})$, by Definition 2.2 and Proposition 2.9(iii).

(iv) $(B^\bot)^{\mathrm{PNCL},\mathscr{S},\sigma} = (B^{\mathrm{PNCL},\mathscr{S},\sigma})^\bot$, by Definition 2.2,
$= \mathrm{PR}(B))^\bot$, by the induction hypothesis,
$= \mathrm{PR}(B^\bot)$, by Proposition 2.8(iii).

(v) $({}^{\perp}B)^{\mathrm{PNCL},\mathscr{S},\sigma} = {}^{\perp}(B^{\mathrm{PNCL},\mathscr{S},\sigma})$, by Definition 2.2,
$= {}^{\perp}(\mathrm{PR}(B))$, by the induction hypothesis,
$= \mathrm{PR}({}^{\perp}B)$, by Proposition 2.8(iv).

(vi) $(B \otimes C)^{\mathrm{PNCL},\mathscr{S},\sigma} = B^{\mathrm{PNCL},\mathscr{S},\sigma} \otimes_{\perp} C^{\mathrm{PNCL},\mathscr{S},\sigma}$, by Definition 2.2,
$= \mathrm{PR}(B) \otimes_{\perp} \mathrm{PR}(C)$, by the induction hypothesis,
$= \mathrm{PR}(B \otimes C)$, by Proposition 2.9(iv).

(vii) $(B \parr C)^{\mathrm{PNCL},\mathscr{S},\sigma} = B^{\mathrm{PNCL},\mathscr{S},\sigma} \parr_{\perp} C^{\mathrm{PNCL},\mathscr{S},\sigma}$, by Definition 2.2,
$= \mathrm{PR}(B) \parr_{\perp} \mathrm{PR}(C)$, by the induction hypothesis,
$= \mathrm{PR}(B \parr C)$, by Proposition 2.9(v).

(viii) $(B \,\&\, C)^{\mathrm{PNCL},\mathscr{S},\sigma} = B^{\mathrm{PNCL},\mathscr{S},\sigma} \cap C^{\mathrm{PNCL},\mathscr{S},\sigma}$, by Definition 2.2,
$= \mathrm{PR}(B) \cap \mathrm{PR}(C)$, by the induction hypothesis,
$= \mathrm{PR}(B \,\&\, C)$, by Proposition 2.9(vi).

(ix) $(B \otimes C)^{\mathrm{PNCL},\mathscr{S},\sigma} = B^{\mathrm{PNCL},\mathscr{S},\sigma} \oplus_{\perp} C^{\mathrm{PNCL},\mathscr{S},\sigma}$, by Definition 2.2,
$= \mathrm{PR}(B) \oplus_{\perp} \mathrm{PR}(C)$, by the induction hypothesis,
$= \mathrm{PR}(B \oplus C)$, by Proposition 2.9(vii).

Now, since $\Rightarrow A$ is PNCL-valid in $\mathscr{S}$, we have from Definitions 2.2 and 2.3 that $\mathbf{1}_{\perp} \subseteq A^{\mathrm{PNCL},\mathscr{S},\sigma} = \mathrm{PR}(A)$, so that $\bullet \in \mathrm{PR}(A)$, i.e. $\Rightarrow A$ is provable in PNCL. □

2.11. THEOREM. *Let A be a formula of $\mathscr{L}$(PNCL). Then $\Rightarrow A$ is provable in PNCL iff $\Rightarrow A$ is valid in every noncommutative classical phase space.*

□ From Theorems 2.10 and 2.6. □

2.12. REMARK. The following definitions and propositions are introductory steps in order to find a one-sided sequent calculus for the pure noncommutative classical linear propositional logic (§4).

2.13. DEFINITION. (i) If A is a formula of $\mathscr{L}$(PNCL), and $n \geq 0$, then $A^{\perp n}$ denotes the formula $A^{\perp \cdots \perp}$ (n times the connective $(-)^{\perp}$), and ${}^{\perp n}A$ denotes the formula ${}^{\perp \cdots \perp}A$ (n times the connective ${}^{\perp}(-)$).

(ii) The set of *normal formulas* of $\mathscr{L}$(PNCL) is defined as follows:

(a) For each propositional variable $\boldsymbol{a}$, the following formulas are normal formulas: $\boldsymbol{a}$, $\boldsymbol{a}^{\perp n}$ and ${}^{\perp n}\boldsymbol{a}$ for every $n > 0$.

(b) The following formulas are normal formulas: $\mathbf{1}$, $\mathbf{1}^{\perp}$, $\boldsymbol{T}$ and $\boldsymbol{T}^{\perp}$.

(c) If A and B are normal formulas, the the following formulas are normal formulas: $A \otimes B$, $A \parr B$, $A \,\&\, B$ and $A \oplus B$.

(d) Nothing else is a normal formula.

(iii) For every formula A of the language $\mathscr{L}$(PNCL), we define a normal formula $A^{\wedge}$ as follows, by induction:

(a) For every variable (propositional letter) $\boldsymbol{a}$,

$$(\boldsymbol{a}^{\perp n})^{\wedge} = \boldsymbol{a}^{\perp n}, \qquad ({}^{\perp n}\boldsymbol{a})^{\wedge} = {}^{\perp n}\boldsymbol{a},$$

$$({}^{\perp}(\boldsymbol{a}^{\perp n+1}))^{\wedge} = \boldsymbol{a}^{\perp n}, \qquad (({}^{\perp n+1}\boldsymbol{a})^{\perp})^{\wedge} = {}^{\perp n}\boldsymbol{a},$$

$$\mathbf{1}^{\wedge} = (\mathbf{1}^{\perp 2n+2})^{\wedge} = ({}^{\perp 2n+2}\mathbf{1})^{\wedge} = \mathbf{1},$$

$$(\mathbf{1}^{\perp 2n+1})^{\wedge} = ({}^{\perp 2n+1}\mathbf{1})^{\wedge} = \mathbf{1}^{\perp},$$

$$({}^{\perp}(\mathbf{1}^{\perp n}))^{\wedge} = (({}^{\perp n}\mathbf{1})^{\perp})^{\wedge} = (\mathbf{1}^{\perp n+1})^{\wedge},$$

$$\boldsymbol{T}^{\wedge} = (\boldsymbol{T}^{\perp 2n+2})^{\wedge} = ({}^{\perp 2n+2}\boldsymbol{T})^{\wedge} = \boldsymbol{T},$$

$$(\boldsymbol{T}^{\perp 2n+1})^{\wedge} = ({}^{\perp 2n+1}\boldsymbol{T})^{\wedge} = \boldsymbol{T}^{\perp},$$

$$({}^{\perp}(\boldsymbol{T}^{\perp n}))^{\wedge} = (({}^{\perp n}\boldsymbol{T})^{\perp})^{\wedge} = (\boldsymbol{T}^{\perp n+1})^{\wedge}.$$

(b) For all formulas A and B,

$$
\begin{array}{ll}
((A \otimes B)^{\perp})^{\wedge} = (B^{\perp})^{\wedge} \parr (A^{\perp})^{\wedge}, & ({}^{\perp}(A \otimes B))^{\wedge} = ({}^{\perp}B)^{\wedge} \parr ({}^{\perp}A)^{\wedge}, \\
((A \parr B)^{\perp})^{\wedge} = (B^{\perp})^{\wedge} \otimes (A^{\perp})^{\wedge}, & ({}^{\perp}(A \parr B))^{\wedge} = ({}^{\perp}B)^{\wedge} \otimes ({}^{\perp}A)^{\wedge}, \\
((A \mathbin{\&} B)^{\perp})^{\wedge} = (A^{\perp})^{\wedge} \oplus (B^{\perp})^{\wedge}, & ({}^{\perp}(A \mathbin{\&} B))^{\wedge} = ({}^{\perp}A)^{\wedge} \oplus ({}^{\perp}B)^{\wedge}, \\
((A \oplus B)^{\perp})^{\wedge} = (A^{\perp})^{\wedge} \mathbin{\&} (B^{\perp})^{\wedge}, & ({}^{\perp}(A \oplus B))^{\wedge} = ({}^{\perp}A)^{\wedge} \mathbin{\&} ({}^{\perp}B)^{\wedge}.
\end{array}
$$

(iv) For every finite sequence Γ of formulas of the language $\mathscr{L}$(PNCL), we define a finite sequence $\Gamma^{\wedge}$ as follows: if Γ is the empty sequence, then $\Gamma^{\wedge}$ is Γ, and if Γ is $A_1, \ldots, A_n$, then $\Gamma^{\wedge}$ is $(A_1)^{\wedge}, \ldots, (A_n)^{\wedge}$.

(v) A *normal sequent* of $\mathscr{L}$(PNCL) is a sequent $\Rightarrow \Gamma$, where Γ is a finite sequence of normal formulas.

(v) For every sequent $\Gamma \Rightarrow \Delta$ of $\mathscr{L}$(PNCL) we define a normal sequent $(\Gamma \Rightarrow \Delta)^{\wedge}$ as follows: if Γ is empty, then $(\Rightarrow \Delta)^{\wedge}$ is $\Rightarrow \Delta^{\wedge}$, and otherwise, $(\Gamma \Rightarrow \Delta)^{\wedge}$ is $\Rightarrow \mathrm{op}((\Gamma^{\perp})^{\wedge}), \Delta^{\wedge}$.

2.14. LEMMA. *If A is a formula of the language $\mathscr{L}$(PNCL), then the following sequents are provable in* PNCL:

$$
\begin{array}{ll}
A \Rightarrow A^{\wedge}, & A^{\wedge} \Rightarrow A, \\
(A^{\perp})^{\wedge} \Rightarrow (A^{\wedge})^{\perp}, & (A^{\wedge})^{\perp} \Rightarrow ({}^{\perp}A)^{\wedge}, \\
({}^{\perp}A)^{\wedge} \Rightarrow {}^{\perp}(A^{\wedge}), & {}^{\perp}(A^{\wedge}) \Rightarrow ({}^{\perp}A)^{\wedge}.
\end{array}
$$

□ Exercise. □

2.15. PROPOSITION. *Let $\Gamma \Rightarrow \Delta$ be a sequent of $\mathscr{L}$*(PNCL).

(i) *$\Gamma \Rightarrow \Delta$ is provable in* PNCL *iff $\Gamma^{\wedge} \Rightarrow \Delta^{\wedge}$ is provable in* PNCL.

(ii) *$\Gamma \Rightarrow \Delta$ is provable in* PNCL *iff $(\Gamma \Rightarrow \Delta)^{\wedge}$ is provable in* PNCL.

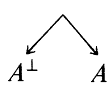
□ By Lemma 2.14 and the cut rule. □

§3. Noncommutative deduction nets.

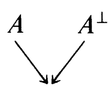
3.1. DEFINITION. By induction, to every proof π of a sequent $\Gamma \Rightarrow \Delta$ in the strict multiplicative fragment of PNCL, we associate an oriented planar graph π' of occurrences of formulas of PNCL, where the sequence of all the initial nodes (from left to right) is the finite sequence Γ and the sequence of all the terminal nodes (from left to right) is the finite sequence Δ.

The links of the graphs are the following ones:

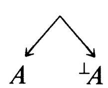
1) down $(-)^{\perp}$ (no premise, two conclusions):



2) up $(-)^{\perp}$ (two premises, no conclusion):



3) down ${}^{\perp}(-)$ (no premise, two conclusions):



4) up ${}^{\perp}(-)$ (two premises, no conclusion):

$$\begin{tikzcd} {}^{\perp}A \arrow[dr] & & A \arrow[dl] \\ & {} & \end{tikzcd}$$

5) down $\otimes$ (two premises, one conclusion):

$$\begin{tikzcd} A \arrow[dr] & & B \arrow[dl] \\ & A \otimes B & \end{tikzcd}$$

6) up $\otimes$ (one premise, two conclusions):

$$\begin{tikzcd} & A \otimes B \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

7) down $\parr$ (two premises, one conclusion):

$$\begin{tikzcd} A \arrow[dr] & & B \arrow[dl] \\ & A \parr B & \end{tikzcd}$$

8) up $\parr$ (one premise, two conclusions):

$$\begin{tikzcd} & A \parr B \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

(i) Let π be just (id), and let $\Gamma \Rightarrow \Delta$ be $A \Rightarrow A$. Then π' is the graph without links

$$A.$$

(ii) Let π be

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots \\ \Gamma \Rightarrow \Delta_1, A\end{array} \quad \begin{array}{c}\pi_2\\ \vdots \\ A, \Gamma_2 \Rightarrow \Delta\end{array}}{\Gamma, \Gamma_2 \Rightarrow \Delta_1, \Delta} \text{(cut)}.$$

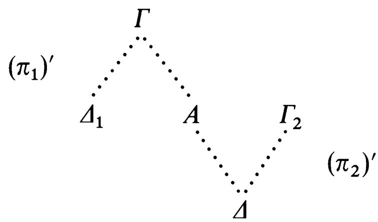
By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ and terminal nodes Δ_1, A:

$$\begin{tikzcd} & \Gamma \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & \\ \Delta_1 & & A \end{tikzcd}$$

and a graph $(\pi_2)'$ with initial nodes A, Γ_2 and terminal nodes Δ:

$$\begin{tikzcd} A \arrow[dr, dash, dotted] & & \Gamma_2 \arrow[dl, dash, dotted] \\ & \Delta & \end{tikzcd} .$$

Then π' is the graph

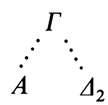


where the initial nodes are Γ,Γ_2 and the terminal nodes are Δ_1,Δ.

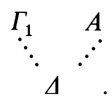
(iii) Let π be

$$\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \Gamma \Rightarrow A,\Delta_2\end{matrix} \qquad \begin{matrix}\pi_2 \\ \vdots \\ \Gamma_1,A \Rightarrow \Delta\end{matrix}}{\Gamma_1,\Gamma \Rightarrow \Delta,\Delta_2}\,(\text{cut}).$$

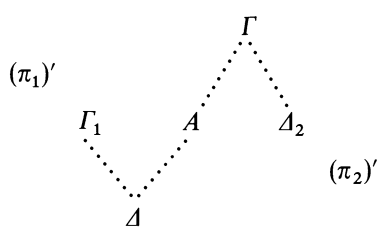
By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ and terminal nodes A,Δ_2:



and a graph $(\pi_2)'$ with initial nodes Γ_1,A and terminal nodes Δ:



Then π' is the graph



where the initial nodes are Γ_1,Γ and the terminal nodes are Δ,Δ_2.

(iv) Let π be

$$\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \Gamma \Rightarrow \Delta_1,A,\Delta_2\end{matrix} \qquad \begin{matrix}\pi_2 \\ \vdots \\ A \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta_1,\Delta,\Delta_2}\,(\text{cut}).$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ and terminal nodes Δ_1,A,Δ_2:

$$\begin{array}{ccc} & \Gamma & \\ \swarrow & \vdots & \searrow \\ \Delta_1 & A & \Delta_2 \end{array}$$

and a graph $(\pi_2)'$ with the initial node A only and terminal nodes Δ:

$$\begin{array}{c} A \\ \vdots \\ \Delta \end{array}$$

Then π' is the graph

$$\begin{array}{ccccc} & & \Gamma & & \\ (\pi_1)' & \swarrow & \vdots & \searrow & \\ \Delta_1 & & A & & \Delta_2 \\ & & \vdots & & (\pi_2)' \\ & & \Delta_2 & & \end{array}$$

where the initial nodes are Γ and the terminal nodes are Δ_1,Δ,Δ_2.

(v) Let π be

$$\frac{\begin{array}{c}\pi_1\\ \vdots \\ \Gamma \Rightarrow A\end{array} \quad \begin{array}{c}\pi_2\\ \vdots \\ \Gamma_1,A,\Gamma_2 \Rightarrow \Delta\end{array}}{\Gamma_1,\Gamma,\Gamma_2 \Rightarrow \Delta}\ (\text{cut}).$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ and the terminal node A only:

$$\begin{array}{c} \Gamma \\ \vdots \\ A \end{array}$$

and a graph $(\pi_2)'$ with initial nodes Γ_1,A,Γ_2 and terminal nodes Δ:

$$\begin{array}{ccc} \Gamma_1 & A & \Gamma_2 \\ \searrow & \vdots & \swarrow \\ & \Delta & \end{array}$$

Then π' is the graph

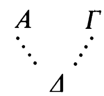
$$\begin{array}{ccccc} & & \Gamma & & \\ (\pi_1)' & & \vdots & & \\ \Gamma_1 & & A & & \Gamma_2 \\ & \searrow & \vdots & \swarrow & (\pi_2)' \\ & & \Delta & & \end{array}$$

where the initial nodes are Γ_1,Γ,Γ_2 and the terminal nodes are Δ.

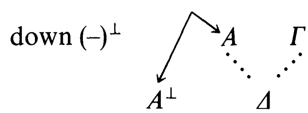
(vi) Let π be

$$\cfrac{\begin{matrix}\pi_1\\ \vdots \\ A,\Gamma \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow A^{\perp},\Delta}\ ((-)^{\perp},\mathrm{R}).$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes A,Γ and terminal nodes Δ:



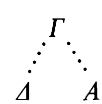
Then π' is the following graph where the initial nodes are Γ and terminal nodes are $A^{\perp},\Delta$:



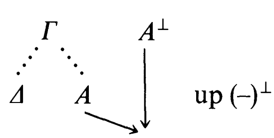
(vii) Let π be

$$\cfrac{\begin{matrix}\pi_1\\ \vdots \\ \Gamma \Rightarrow \Delta,A\end{matrix}}{\Gamma,A^{\perp} \Rightarrow \Delta}\ ((-)^{\perp},\mathrm{L}).$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ and terminal nodes Δ,A:



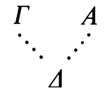
Then π' is the following graph with initial nodes $\Gamma,A^{\perp}$ and terminal nodes Δ:



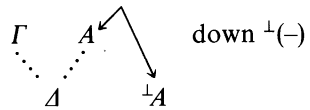
(viii) Let π be

$$\cfrac{\begin{matrix}\pi_1\\ \vdots \\ \Gamma,A \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta,{}^{\perp}A}\ ({}^{\perp}(-),\mathrm{R}).$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ,A and terminal nodes Δ:



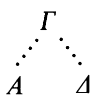
Then π' is the following graph with initial nodes Γ and terminal nodes $\Delta, {}^{\perp}A$:



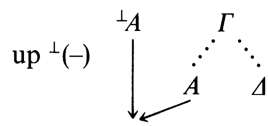
(ix) Let π be

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \Rightarrow A,\Delta\end{array}}{{}^{\perp}A,\Gamma \Rightarrow \Delta}\ ({}^{\perp}(-),\mathrm{L}).$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ and terminal nodes A,Δ:



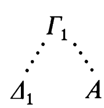
Then π' is the following graph with initial nodes ${}^{\perp}A,\Gamma$ and terminal nodes Δ:



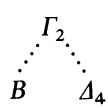
(x) Let π be

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma_1 \Rightarrow \Delta_1,A\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ \Gamma_2 \Rightarrow B,\Delta_4\end{array}}{\Gamma_1,\Gamma_2 \Rightarrow \Delta_1,A \otimes B,\Delta_4}\ (\otimes,\mathrm{R}).$$

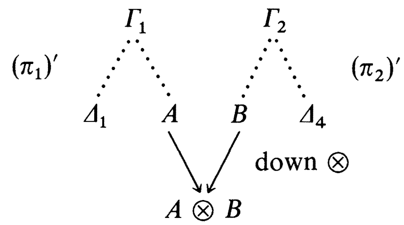
By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ_1 and terminal nodes Δ_1,A:



and a graph $(\pi_2)'$ with initial nodes Γ_2 and terminal nodes B,Δ_4:



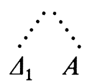
Then π is the following graph where the initial nodes are Γ_1, Γ_2 and the terminal nodes are $\Delta_1, A \otimes B, \Delta_4$:



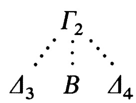
(xi) Let π be

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Rightarrow \Delta_1,A\end{matrix}\qquad \begin{matrix}\pi_2\\ \vdots\\ \Gamma_2 \Rightarrow \Delta_3,B,\Delta_4\end{matrix}}{\Gamma_2 \Rightarrow \Delta_3,\Delta_1,A \otimes B,\Delta_4}\,(\otimes,\mathrm{R}).$$

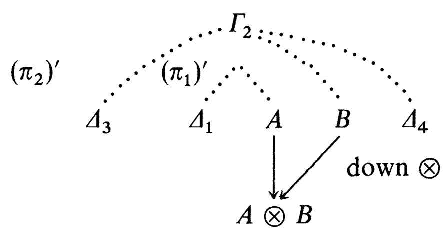
By the induction hypothesis, we have a graph $(\pi_1)'$ without initial nodes and with terminal nodes Δ_1, A:



and a graph $(\pi_2)'$ with initial nodes Γ_2 and terminal nodes Δ_3, B, Δ_4:



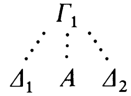
Then π' is the following graph with initial nodes Γ_2 and terminal nodes $\Delta_3, \Delta_1, A \otimes B, \Delta_4$:



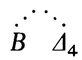
(xii) Let π be

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma_1 \Rightarrow \Delta_1,A,\Delta_2\end{matrix}\qquad \begin{matrix}\pi_2\\ \vdots\\ \Rightarrow B,\Delta_4\end{matrix}}{\Gamma_1 \Rightarrow \Delta_1,A \otimes B,\Delta_4,\Delta_2}\,(\otimes,\mathrm{R}).$$

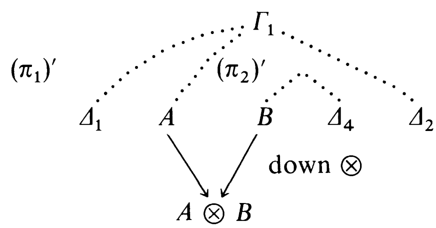
By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ_1 and terminal nodes Δ_1, A, Δ_2:



and a graph $(\pi_2)'$ without initial nodes and with terminal nodes B, Δ_4:



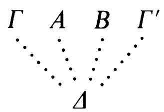
Then π' is the following graph with initial nodes Γ_1 and terminal nodes $\Delta_1, A \otimes B, \Delta_4, \Delta_2$:



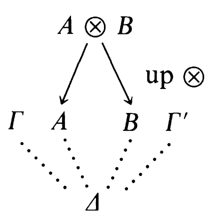
(xiii) Let π be

$$\begin{array}{c}\pi_1\\ \vdots\\ \dfrac{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, (A \otimes B), \Gamma_2 \Rightarrow \Delta}\,(\otimes, \mathrm{L}).\end{array}$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ, A, B, Γ' with terminal nodes Δ:



Then π is the following graph with initial nodes $\Gamma_1, (A \otimes B), \Gamma_2$ and terminal nodes Δ:



(xiv) Let π be

$$\begin{array}{c}\pi_1\\ \vdots\\ \dfrac{\Gamma \Rightarrow \Delta_1, A, B, \Delta_2}{\Gamma \Rightarrow \Delta_1, (A \parr B), \Delta_2}\,(\parr, \mathrm{L}).\end{array}$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ and terminal nodes Δ,A,B,Δ':

$$\begin{array}{cccc} & \Gamma & & \\ \swarrow & \vdots \quad \vdots & \searrow \\ \Delta & A \quad B & \Delta' \end{array}$$

Then π' is the following graph with initial nodes Γ and terminal nodes $\Delta_1,(A \parr B),\Delta_2$:

$$\begin{array}{ccccc} & & \Gamma & & \\ & \swarrow & \vdots \quad \vdots & \searrow & \\ & \Delta & A \quad B & \Delta' & \\ & & \searrow \ \swarrow & & \text{down } \parr \\ & & A \parr B & & \end{array}$$

(xv) Let π be

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots \\ \Gamma_1,A \Rightarrow \Delta_1\end{array} \quad \begin{array}{c}\pi_2\\ \vdots \\ B,\Gamma_4 \Rightarrow \Delta_2\end{array}}{\Gamma_1,(A \parr B),\Gamma_4 \Rightarrow \Delta_1,\Delta_2} (\parr,\mathrm{L}).$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ_1,A and terminal nodes Δ_1:

$$\begin{array}{ccc} \Gamma_1 & & A \\ & \searrow \ \swarrow & \\ & \Delta_1 & \end{array}$$

and a graph $(\pi_2)'$ with initial nodes B,Γ_4 and terminal nodes Δ_2:

$$\begin{array}{ccc} B & & \Gamma_4 \\ & \searrow \ \swarrow & \\ & \Delta_2 & \end{array}$$

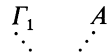
Then π' is the following graph with initial nodes $\Gamma_1,(A \parr B),\Gamma_4$ and terminal nodes Δ_1,Δ_2:

$$\begin{array}{ccccccc} & & & A \parr B & & & \\ & & \swarrow & & \searrow \ \text{up } \parr & & \\ (\pi_1)' \ \Gamma_1 & & A & & B & & \Gamma_4 \ (\pi_2)' \\ & \searrow \ \swarrow & & & & \searrow \ \swarrow & \\ & \Delta_1 & & & & \Delta_2 & \end{array}$$

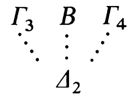
(xvi) Let π be

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots \\ \Gamma_1,A \Rightarrow\end{array} \quad \begin{array}{c}\pi_2\\ \vdots \\ \Gamma_3,B,\Gamma_4 \Rightarrow \Delta_2\end{array}}{\Gamma_3,\Gamma_1,(A \parr B),\Gamma_4 \Rightarrow \Delta_2} (\parr,\mathrm{L}).$$

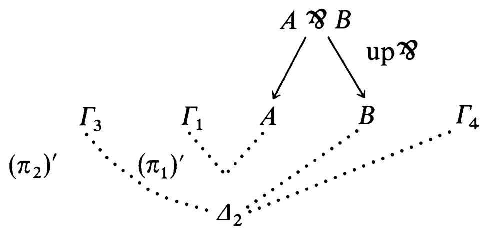
By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ_1,A and without terminal nodes:



and a graph $(\pi_2)'$ with initial nodes Γ_3,B,Γ_4 and terminal nodes Δ_2:



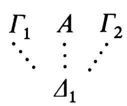
Then π' is the following graph with initial nodes $\Gamma_3,\Gamma_1,(A \parr B),\Gamma_4$ and terminal nodes Δ_2:



(xvii) Let π be

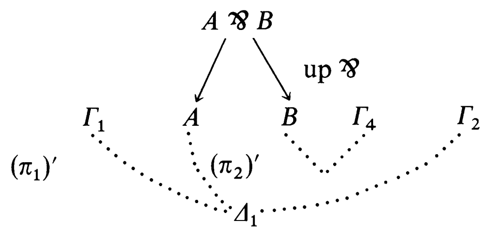
$$\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \vdots & \vdots \\ \Gamma_1,A,\Gamma_2 \Rightarrow \Delta_1 & B,\Gamma_4 \Rightarrow\end{array}}{\Gamma_1,(A \parr B),\Gamma_4,\Gamma_2 \Rightarrow \Delta_1}(\parr,\mathrm{L}).$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with initial nodes Γ_1,A,Γ_2 and terminal nodes Δ_1:



and a graph $(\pi_2)'$ with initial nodes B,Γ_4 and without terminal nodes:

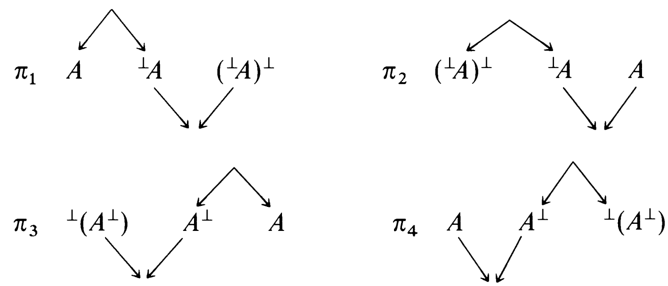
Then π' is the following graph with initial nodes $\Gamma_1,(A \parr B)$, Γ_4,Γ_2 and terminal nodes Δ_1:



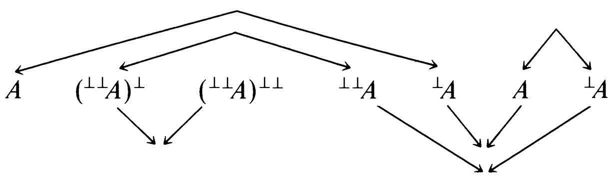
3.2. Definition. A *noncommutative deduction net* is a graph of formulas of $\mathscr{L}$(PNCL), associated to a proof in PNCL, according to Definition 3.1.

3.3. Remark. We exhibit some examples of noncommutative deduction nets.

(i) To the proofs given in 2.5(i) correspond the following noncommutative deduction nets:



(ii) To the proof given in 2.5(v) corresponds the following noncommutative deduction net:



3.4. Remark. The reader will observe that the conditions on the cut rule and binary rules for multiplicative connectives assure the planarity of the noncommutative deduction nets and the possibility of giving the initial nodes and terminal nodes as finite sequences.

§4. One-sided sequent calculus and noncommutative proof nets.

4.1. Definition. The language $\mathscr{L}$(SPNCL) of the one-sided sequent calculus for pure noncommutative classical linear propositional logic is defined as follows.

(i) The *alphabet* of $\mathscr{L}$(SPNCL) consists of the following symbols: denumerably many propositional variables, denoted by $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$; for every propositional variable $\boldsymbol{a}$, and for every $n \geq 0$, $\boldsymbol{a}^{\perp n}$ and $^{\perp n}\boldsymbol{a}$, with $\boldsymbol{a} = \boldsymbol{a}^{\perp 0} = {}^{\perp 0}\boldsymbol{a}$; the propositional constants $\perp$, $\mathbf{1}$, $\boldsymbol{T}$ and $\mathbf{0}$; the binary connectives $\otimes, \invamp, \&$ and $\oplus$; the sequent symbol $\vdash$; and the usual auxiliary symbols.

(ii) The *formulas* of $\mathscr{L}$(SPNCL) are defined inductively as follows:

(a) For each propositional variable $\boldsymbol{a}$, and for every $n \geq 0$, $\boldsymbol{a}^{\perp n}$ and $^{\perp n}\boldsymbol{a}$ are formulas.

(b) $\perp$, $\mathbf{1}$, $\boldsymbol{T}$ and $\mathbf{0}$ are formulas.

(c) If A and B are formulas, then $A \otimes B$, $A \invamp B$, $A \oplus B$ and $A \,\&\, B$ are formulas.

(d) Nothing else is a formula.

(iii) The *sequents* of $\mathscr{L}$(SPNCL) are defined as follows: if Γ is a finite sequence of formulas of the language $\mathscr{L}$(SPNCL), then $\vdash \Gamma$ is a sequent of $\mathscr{L}$(SPNCL).

(iv) For every formula A of $\mathscr{L}$(SPNCL), we define $A^{\perp}$ and $^{\perp}A$ as follows.

(a) For each propositional variable $\boldsymbol{a}$, and for every $n \geq 0$,

$$(\boldsymbol{a}^{\perp n})^{\perp} = \boldsymbol{a}^{\perp n+1}, \qquad ({}^{\perp n+1}\boldsymbol{a})^{\perp} = {}^{\perp n}\boldsymbol{a},$$
$${}^{\perp}({}^{\perp n}\boldsymbol{a}) = {}^{\perp n+1}\boldsymbol{a}, \qquad {}^{\perp}(\boldsymbol{a}^{\perp n+1}) = \boldsymbol{a}^{\perp n},$$
$$\mathbf{1}^{\perp} = {}^{\perp}\mathbf{1} = \perp, \qquad \perp^{\perp} = {}^{\perp}\perp = \mathbf{1},$$
$$\boldsymbol{T}^{\perp} = {}^{\perp}\boldsymbol{T} = \mathbf{0}, \qquad \mathbf{0}^{\perp} = {}^{\perp}\mathbf{0} = \boldsymbol{T},$$

(b) If A and B are formulas, then

$$(A \otimes B)^{\perp} = B^{\perp} \parr A^{\perp}, \qquad {}^{\perp}(A \otimes B) = {}^{\perp}B \parr {}^{\perp}A,$$
$$(A \parr B)^{\perp} = B^{\perp} \otimes A^{\perp}, \qquad {}^{\perp}(A \parr B) = {}^{\perp}B \otimes {}^{\perp}A,$$
$$(A \,\&\, B)^{\perp} = A^{\perp} \oplus B^{\perp}, \qquad {}^{\perp}(A \,\&\, B) = {}^{\perp}A \oplus {}^{\perp}B,$$
$$(A \oplus B)^{\perp} = A^{\perp} \,\&\, B^{\perp}, \qquad {}^{\perp}(A \oplus B) = {}^{\perp}A \,\&\, {}^{\perp}B.$$

4.2. LEMMA. *For every formula A of $\mathscr{L}$(SPNCL), $({}^{\perp}A)^{\perp} = {}^{\perp}(A^{\perp}) = A$.*

□ Immediate from the definition. □

4.3. DEFINITION. The one-sided sequent calculus SPNCL for classical linear propositional logic is given by the following rules concerning the sequents of $\mathscr{L}$(SPNCL).

(i) Identity rule:

$$\frac{}{\vdash A^{\perp}, A}\ \text{(id)}, \qquad \text{or, equivalently,} \qquad \frac{}{\vdash A, {}^{\perp}A}\ \text{(id)}'.$$

(ii) Cut rule:

$$\frac{\vdash \Gamma_1, A, \Gamma_2 \quad \vdash \Delta_1, A^{\perp}, \Delta_2}{\vdash \Delta_1, \Gamma_1, \Delta_2, \Gamma_2}\ \text{(cut)}$$

if $\Delta_1 = \varnothing$ or $\Gamma_2 = \varnothing$; or, equivalently,

$$\frac{\vdash \Gamma_1, {}^{\perp}A, \Gamma_2 \quad \vdash \Delta_1, A, \Delta_2}{\vdash \Delta_1, \Gamma_1, \Delta_2, \Gamma_2}\ \text{(cut)}'$$

if $\Delta_1 = \varnothing$ or $\Gamma_2 = \varnothing$.

(iii) **1**-rule:

$$\frac{}{\vdash \mathbf{1}}\ (\mathbf{1}).$$

(iv) $\perp$-rule:

$$\frac{\vdash \Gamma_1, \Gamma_2}{\vdash \Gamma_1, \perp, \Gamma_2}\ (\perp).$$

(v) $\boldsymbol{T}$-rule:

$$\frac{}{\vdash \Gamma_1, \boldsymbol{T}, \Gamma_2}\ (\boldsymbol{T}).$$

(vi) $\otimes$-rule:

$$\frac{\vdash \Gamma_1, A, \Gamma_2 \quad \vdash \Delta_1, B, \Delta_2}{\vdash \Delta_1, \Gamma_1, A \otimes B, \Delta_2, \Gamma_2}\ (\otimes)$$

if $\Delta_1 = \varnothing$ or $\Gamma_2 = \varnothing$.

(vii) ⅋-rule:

$$\frac{\vdash \Gamma_1, A, B, \Gamma_2}{\vdash \Gamma_1, A \parr B, \Gamma_2} \ (\parr).$$

(viii) &-rule:

$$\frac{\vdash \Gamma_1, A, \Gamma_2 \quad \vdash \Gamma_1, B, \Gamma_2}{\vdash \Gamma_1, A \,\&\, B, \Gamma_2} \ (\&).$$

(ix) $\oplus$-rules:

$$\frac{\vdash \Gamma_1, A, \Gamma_2}{\vdash \Gamma_1, A \oplus B, \Gamma_2} \ (\oplus 1), \qquad \frac{\vdash \Gamma_1, B, \Gamma_2}{\vdash \Gamma_1, A \oplus B, \Gamma_2} \ (\oplus 2).$$

A *proof* of a sequent $\vdash \Gamma$ in SPNCL is defined as usual. A sequent $\vdash \Gamma$ is *provable* in SPNCL iff there is a proof of $\vdash \Gamma$ in SPNCL.

The *strict multiplicative fragment* of SPNCL is obtained by restricting to the following rules: basic rules, $\otimes$-rules, ⅋-rules.

4.4. Definition. (i) For every formula A of $\mathscr{L}$(SPNCL), we define

$$A^{\#} = A[\mathbf{0}/\boldsymbol{T}^{\perp}, \perp/\mathbf{1}^{\perp}]$$

(i.e. replace $\mathbf{0}$ by $\boldsymbol{T}^{\perp}$ and $\perp$ by $\mathbf{1}^{\perp}$ everywhere in the formula A).

(ii) If $\Gamma = A_1, \ldots, A_m$ is a finite sequence of formulas of $\mathscr{L}$(SPNCL), then we put

$$\Gamma^{\#} = (A_1)^{\#}, \ldots, (A_m)^{\#}.$$

4.5. Proposition. (i) *For every formula A of* $\mathscr{L}$(SPNCL),

$$A^{\#} \text{ is a normal formula of } \mathscr{L}(\text{PNCL}),$$
$$(A^{\perp})^{\#} = ((A^{\#})^{\perp})^{\wedge}, \qquad ({}^{\perp}A)^{\#} = ({}^{\perp}(A^{\#}))^{\wedge}.$$

(ii) *If* $\vdash \Gamma$ *is a provable sequent in* SPNCL, then $\Rightarrow \Gamma^{\#}$ *is a provable normal sequent in* PNCL.

□(i) Exercise. □

□(ii) We argue by induction on the proof of $\vdash \Gamma$.

(ii1) The last rule is (id):

$$\frac{}{\vdash A^{\perp}, A} \ (\text{id}).$$

The sequent $\Rightarrow (A^{\#})^{\perp}, A^{\#}$ is provable in PNCL, and then by Lemma 2.14 and the cut rule the sequent $\Rightarrow ((A^{\#})^{\perp})^{\wedge}, A^{\#}$ is provable in PNCL, i.e. by (i) above $\Rightarrow (A^{\perp})^{\#}, A^{\#}$ is provable in PNCL.

(ii2) The last rule is (cut) in the form

$$\frac{\vdash \Gamma_1, A, \Gamma_2 \quad \vdash A^{\perp}, \Delta_2}{\vdash \Gamma_1, \Delta_2, \Gamma_2} \ (\text{cut}).$$

By the induction hypothesis, $\Rightarrow (A^{\perp})^{\#}, (\Delta_2)^{\#}$, i.e. $\Rightarrow ((A^{\#})^{\perp})^{\wedge}, (\Delta_2)^{\#}$, is provable in PNCL; so in PNCL we can prove the sequent $\Rightarrow (A^{\#})^{\perp}, (\Delta_2)^{\#}$, then the sequent ${}^{\perp}((A^{\#})^{\perp}) \Rightarrow (\Delta_2)^{\#}$ and finally the sequent $A^{\#} \Rightarrow (\Delta_2)^{\#}$. Moreover, by the induction hypothesis, $\Rightarrow (\Gamma_1)^{\#},A^{\#},(\Gamma_2)^{\#}$ is provable in PNCL. Thus we get in PNCL

$$\frac{\Rightarrow (\Gamma_1)^{\#},A^{\#},(\Gamma_2)^{\#} \quad A^{\#} \Rightarrow (\Delta_2)^{\#}}{\Rightarrow (\Gamma_1)^{\#},(\Delta_2)^{\#},(\Gamma_2)^{\#}} \text{ (cut).}$$

(ii2′) The last rule is (cut) in the form

$$\frac{\vdash \Gamma_1,A \quad \vdash \Delta_1,A^{\perp},\Delta_2}{\vdash \Delta_1,\Gamma_1,\Delta_2} \text{ (cut).}$$

By the induction hypothesis $\Rightarrow (\Delta_1)^{\#},(A^{\perp})^{\#},(\Delta_2)^{\#}$, i.e. $\Rightarrow (\Delta_1)^{\#},((A^{\#})^{\perp})^{\wedge},(\Delta_2)^{\#}$, is a provable sequent in PNCL; so in PNCL we can prove the sequent $\Rightarrow (\Delta_1)^{\#},(A^{\#})^{\perp},(\Delta_2)^{\#}$, then the sequent ${}^{\perp}((A^{\#})^{\perp}),{}^{\perp}((\Delta_1)^{\#}) \Rightarrow (\Delta_2)^{\#}$ and finally the sequent $A^{\#},{}^{\perp}((\Delta_1)^{\#}) \Rightarrow (\Delta_2)^{\#}$. Moreover, by the induction hypothesis, $\Rightarrow (\Gamma_1)^{\#},A^{\#}$ is provable in PNCL. Thus the sequent $\Rightarrow (\Delta_1)^{\#},(\Gamma_1)^{\#},(\Delta_2)^{\#}$ is provable in PNCL, because in PNCL we get

$$\cfrac{\cfrac{\Rightarrow (\Gamma_1)^{\#},A^{\#} \quad A^{\#},{}^{\perp}((\Delta_1)^{\#}) \Rightarrow (\Delta_2)^{\#}}{{}^{\perp}((\Delta_1)^{\#}) \Rightarrow (\Gamma_1)^{\#},(\Delta_2)^{\#}} \text{ (cut)}}{\Rightarrow ({}^{\perp}((\Delta_1)^{\#}))^{\perp},(\Gamma_1)^{\#},(\Delta_2)^{\#}} \text{ ((–)}^{\perp}\text{,R).}$$

The consideration of all the other cases is left to the reader. □

4.6. Definition. (i) For every formula A of $\mathscr{L}$(PNCL), we define

$$A^{+} = A^{\wedge}[\mathbf{1}^{\perp}/\perp, \boldsymbol{T}^{\perp}/\mathbf{0}].$$

(ii) If $\Gamma = A_1,\dots,A_m$ is a finite sequence of formulas of $\mathscr{L}$(PNCL), then we put

$$\Gamma^{+} = (A_1)^{+},\dots,(A_m)^{+}.$$

4.7. Proposition. (i) *For every formula A of $\mathscr{L}$*(PNCL),

$$A^{+} \textit{ is a formula of } \mathscr{L}\text{(SPNCL)},$$
$$(A^{\perp})^{+} = (A^{+})^{\perp}, \qquad ({}^{\perp}A)^{+} = {}^{\perp}(A^{+}).$$

(ii) *If $\Gamma \Rightarrow \Delta$ is provable in* PNCL, *then* $\vdash\mathrm{op}((\Gamma^{+})^{\perp}),\Delta^{+}$ *is provable in* SPNCL.

□ Exercise. □

4.8. Lemma. (i) *For every formula A of $\mathscr{L}$*(PNCL), $A^{\wedge+\#} = A^{\wedge}$.

(ii) *For every formula A of $\mathscr{L}$*(SPNCL), $A^{\#+} = A$.

□ Exercise. □

4.9. Proposition. (i) *Let Γ and Δ be finite sequences of formulas of $\mathscr{L}$*(PNCL). *Then $\Gamma^{\wedge} \Rightarrow \Delta^{\wedge}$ is provable in* PNCL *iff* $\vdash\mathrm{op}((\Gamma^{\wedge+})^{\perp}),\Delta^{\wedge+}$ *is provable in* SPNCL.

(ii) *Let Γ be a finite sequence of formulas of $\mathscr{L}$*(SPNCL). *Then $\vdash\Gamma$ is provable in* SPNCL *iff $\Rightarrow \Gamma^{\#}$ is provable in* PNCL.

□(i) If $\Gamma^{\wedge} \Rightarrow \Delta^{\wedge}$ is provable in PNCL, then, by Proposition 4.7(ii), $\vdash\mathrm{op}((\Gamma^{\wedge+})^{\perp}),\Delta^{\wedge+}$ is provable in SPNCL. If $\vdash\mathrm{op}((\Gamma^{\wedge+})^{\perp}),\Delta^{\wedge+}$ is provable in SPNCL, then the following are provable in PNCL:

the sequent $\Rightarrow (\mathrm{op}((\Gamma^{\wedge+})^{\perp}))^{\#},\Delta^{\wedge+\#}$, by Proposition 4.5(ii);

the sequent $\Rightarrow (\mathrm{op}((\Gamma^{\wedge})^{\perp}))^{+\#},\Delta^{\wedge+\#}$, by Proposition 4.7(i);

the sequent $\Rightarrow (\mathrm{op}((\Gamma)^{\perp}))^{\wedge+\#},\Delta^{\wedge+\#}$, trivially;

the sequent $\Rightarrow (\mathrm{op}((\Gamma)^{\perp}))^{\wedge},\Delta^{\wedge}$, by Lemma 4.8;
the sequent $\Rightarrow \mathrm{op}((\Gamma^{\wedge})^{\perp}),\Delta^{\wedge}$, by Lemma 2.14; and
the sequent $\Gamma^{\wedge} \Rightarrow \Delta^{\wedge}$, by Proposition 2.15(i). □

□(ii) If $\vdash\Gamma$ is provable in SPNCL, then $\Rightarrow \Gamma^{\#}$ is provable in PNCL by Proposition 4.5(ii). If $\Rightarrow \Gamma^{\#}$ is provable in PNCL, then, by Proposition 4.7(ii), $\vdash\Gamma^{\#+}$ is provable in SPNCL; but, by Lemma 4.8, $\Gamma^{\#+} = \Gamma$. □

4.10. DEFINITION. The one-sided sequent calculus SPNCL′ for pure noncommutative classical linear propositional logic is given by the following rules concerning the sequents of $\mathscr{L}$(SPNCL).

(i) Identity rule:

$$\frac{}{\vdash A^{\perp},A}\ (\mathrm{id}).$$

(ii) Cut rule:

$$\frac{\vdash\Gamma,A \quad \vdash\Delta,A^{\perp}}{\vdash\Delta,\Gamma}\ (\mathrm{cut}).$$

(iii) $(-)^{\perp\perp}$-rule:

$$\frac{\vdash\Gamma,A}{\vdash A^{\perp\perp},\Gamma}\ ((-)^{\perp\perp}).$$

(iv) ${}^{\perp\perp}(-)$-rule:

$$\frac{\vdash A,\Gamma}{\vdash\Gamma,{}^{\perp\perp}A}\ ({}^{\perp\perp}(-)).$$

(v) **1**-rule:

$$\frac{}{\vdash\mathbf{1}}\ (\mathbf{1}).$$

(vi) $\perp$-rule:

$$\frac{\vdash\Gamma}{\vdash\Gamma,\perp}\ (\perp).$$

(vii) $\boldsymbol{T}$-rule:

$$\frac{}{\vdash\Gamma,\boldsymbol{T}}\ (\boldsymbol{T}).$$

(viii) $\otimes$-rule:

$$\frac{\vdash\Gamma,A \quad \vdash\Delta,B}{\vdash\Delta,\Gamma,A\otimes B}\ (\otimes).$$

(ix) ⅋-rule:

$$\frac{\vdash\Gamma,A,B}{\vdash\Gamma,A ⅋ B}\ (⅋).$$

(x) &-rule:

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ (\&).$$

(xi) $\oplus$-rules:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ (\oplus 1), \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ (\oplus 2).$$

A *proof* of a sequent $\vdash \Gamma$ in SPNCL′ is defined as usual. A sequent $\vdash \Gamma$ is *provable* in SPNCL′ iff there is a proof of $\vdash \Gamma$ in SPNCL′.

4.11. PROPOSITION. (i) *If* $\vdash \Gamma$ *is (cut-free) provable in* SPNCL, *then* $\vdash \Gamma$ *is (cut-free) provable in* SPNCL′.

(ii) *If* $\vdash \Gamma$ *is provable in* SPNCL′, *then* $\vdash \Gamma$ *is provable in* SPNCL.

$\square$(i) We show how all the rules of SPNCL are derived rules of SPNCL′.

(id) is a rule of SPNCL′.

Cut-rule, with $\Delta_1 = \varnothing$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \Gamma_1, A, \Gamma_2}}{\vdash (\Gamma_2)^{\perp\perp}, \Gamma_1, A}((-)^{\perp\perp}) \quad \dfrac{\dfrac{\vdots}{\vdash A^\perp, \Delta_2}}{\vdash (\Delta_2)^{\perp\perp}, A^\perp}((-)^{\perp\perp})}{\dfrac{\vdash (\Delta_2)^{\perp\perp}, (\Gamma_2)^{\perp\perp}, \Gamma_1}{\vdash \Gamma_1, \Delta_2, \Gamma_2}({}^{\perp\perp}(-))}(\text{cut})$$

Cut rule, with $\Gamma_2 = \varnothing$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \Gamma_1, A} \quad \dfrac{\dfrac{\vdots}{\vdash \Delta_1, A^\perp, \Delta_2}}{\vdash (\Delta_2)^{\perp\perp}, \Delta_1, A^\perp}((-)^{\perp\perp})}{\vdash (\Delta_2)^{\perp\perp}, \Delta_1, \Gamma_1}(\text{cut})}{\vdash \Delta_1, \Gamma_1, \Delta_2}({}^{\perp\perp}(-))$$

The **1**-rule is a rule of SPNCL′.

$\perp$-rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash \Gamma_1, \Gamma_2}}{\vdash (\Gamma_2)^{\perp\perp}, \Gamma_1}((-)^{\perp\perp})}{\vdash (\Gamma_2)^{\perp\perp}, \Gamma_1, \perp}(\perp)}{\vdash \Gamma_1, \perp, \Gamma_2}({}^{\perp\perp}(-))$$

The $\boldsymbol{T}$-rule is left to the reader as an exercise.

$\otimes$-rule with $\Delta_1 = \varnothing$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash \Gamma_1, A, \Gamma_2}}{\vdash (\Gamma_2)^{\perp\perp}, \Gamma_1, A}((-)^{\perp\perp}) \quad \dfrac{\dfrac{\vdots}{\vdash B, \Delta_2}}{\vdash (\Delta_2)^{\perp\perp}, B}((-)^{\perp\perp})}{\vdash (\Delta_2)^{\perp\perp}, (\Gamma_2)^{\perp\perp}, \Gamma_1, A \otimes B}(\otimes)}{\vdash \Gamma_1, A \otimes B, \Delta_2, \Gamma_2}({}^{\perp\perp}(-))$$

$\otimes$-rule with $\Gamma_2 = \varnothing$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash \Gamma_1, A\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \vdash \Delta_1, B, \Delta_2\end{array}}{\vdash (\Delta_2)^{\perp\perp}, \Delta_1, B}\,((-)^{\perp\perp})}{\vdash (\Delta_2)^{\perp\perp}, \Delta_1, \Gamma_1, A \otimes B}\,(\otimes)}{\vdash \Delta_1, \Gamma_1, A \otimes B, \Delta_2}\,({}^{\perp\perp}(-))$$

The ⅋-rule, &-rule and $\oplus$-rules are exercises. □

□(ii) The $(-)^{\perp\perp}$-rule is a derived rule in SPNCL, since

$$\dfrac{\begin{array}{c}\vdots\\ \vdash \Gamma, A\end{array} \quad \dfrac{}{\vdash A^{\perp\perp}, A^{\perp}}\,(\mathrm{id})}{\vdash A^{\perp\perp}, \Gamma}\,(\mathrm{cut})$$

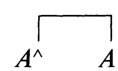
The ${}^{\perp\perp}(-)$-rule is a derived rule in SPNCL, since

$$\dfrac{\dfrac{}{\vdash {}^{\perp}A, {}^{\perp\perp}A}\,(\mathrm{id}) \quad \begin{array}{c}\vdots\\ \vdash A, \Gamma\end{array}}{\vdash \Gamma, {}^{\perp\perp}A}\,(\mathrm{cut}).$$ □

4.12. Definition. To every proof π of a sequent $\vdash \Gamma$ in the strict multiplicative fragment of SPNCL, we associate an oriented planar graph π' of occurrencies of formulas of SPNCL, without initial nodes and where the sequence of all the terminal nodes (from left to right) is the finite sequence Γ.

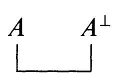
The graph is constructed by means of the following links:

Ax-link:



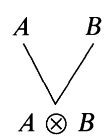
(two conclusions, no premise).

Cut-link:



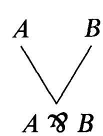
(two premises, no conclusion).

$\otimes$-link:



(two premises, one conclusion).

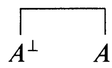
⅋-link:



(two premises, one conclusion).

We now proceed by induction.

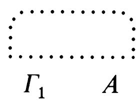
(i) Let π be just (id) and let Γ be ${}^{\perp}A, A$. Then π' is the graph



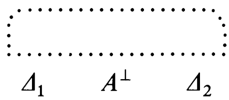
(ii) Let π be

$$\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \vdash \Gamma_1, A\end{matrix} \qquad \begin{matrix}\pi_2 \\ \vdots \\ \vdash \Delta_1, A^{\perp}, \Delta_2\end{matrix}}{\vdash \Delta_1, \Gamma_1, \Delta_2}\ \text{(cut)}$$

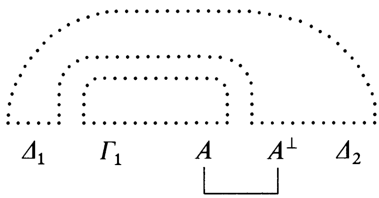
By the induction hypothesis, we have a graph $(\pi_1)'$ with terminal nodes Γ_1, A



and a graph $(\pi_2)'$ with terminal nodes $\Delta_1, A^{\perp}, \Delta_2$



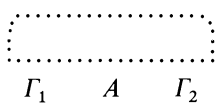
Then π' is the graph



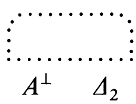
(iii) Let π be

$$\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \vdash \Gamma_1, A, \Gamma_2\end{matrix} \qquad \begin{matrix}\pi_2 \\ \vdots \\ \vdash A^{\perp}, \Delta_2\end{matrix}}{\vdash \Delta_1, \Gamma_1, \Delta_2}\ \text{(cut)}.$$

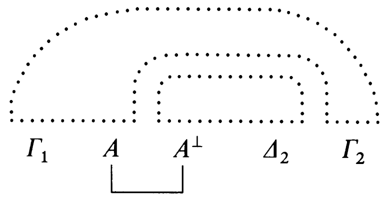
By the induction hypothesis, we have a graph $(\pi_1)'$ with terminal nodes Γ_1, A, Γ_2



and a graph $(\pi_2)'$ with terminal nodes $A^{\perp}, \Delta_2$



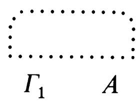
Then π' is the graph



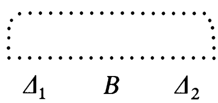
(iv) Let π be

$$\frac{\begin{matrix}\pi_1 & \quad & \pi_2\\ \vdots & & \vdots\\ \vdash \Gamma_1, A & & \vdash \Delta_1, B, \Delta_2\end{matrix}}{\vdash \Delta_1, \Gamma_1, A \otimes B, \Delta_2}\,(\otimes).$$

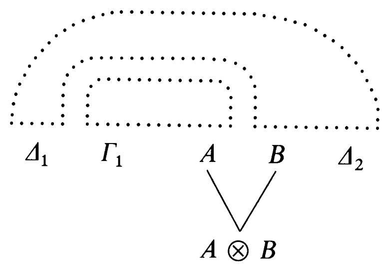
By the induction hypothesis, we have a graph $(\pi_1)'$ with terminal nodes Γ_1, A



and a graph $(\pi_2)'$ with terminal nodes Δ_1, B, Δ_2



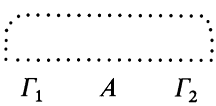
Then π' is the graph



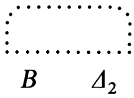
(v) Let π be

$$\frac{\begin{matrix}\pi_1 & \quad & \pi_2\\ \vdots & & \vdots\\ \vdash \Gamma_1, A, \Gamma_2 & & \vdash B, \Delta_2\end{matrix}}{\vdash \Gamma_1, \Delta_2, \Gamma_2}\,(\otimes).$$

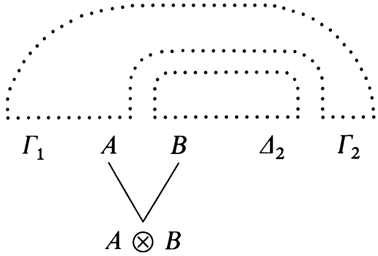
By the induction hypothesis, we have a graph $(\pi_1)'$ with terminal nodes Γ_1, A, Γ_2



and a graph $(\pi_2)'$ with terminal nodes B, Δ_2



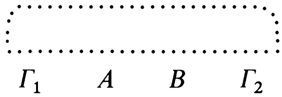
Then π' is the graph



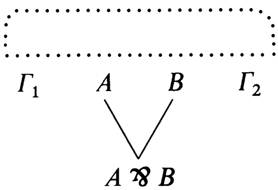
(vi) Let π be

$$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{\vdash \Gamma_1, A, B, \Gamma_2}{\vdash \Gamma_1, A \parr B, \Gamma_2}\ (\parr). \end{array}$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with terminal nodes Γ_1, A, B, Γ_2



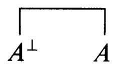
Then π' is the graph



4.13. REMARKS. To every proof π of a sequent $\vdash \Gamma$ in the strict multiplicative fragment of SPNCL′, we associate an oriented planar graph π' of occurrences of formulas of SPNCL′, without initial nodes and where the sequence of all the terminal nodes (from left to right) is the finite sequence Γ.

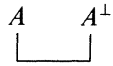
The graph is constructed by means of the following links:

Ax-link:



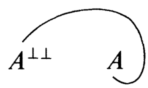
(two conclusions, no premise).

Cut-link:



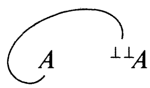
(two premises, no conclusion).

$(-)^{\perp\perp}$-link:



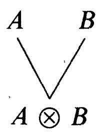
(one premise, one conclusion).

${}^{\perp\perp}(-)$-link:



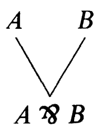
(one premise, one conclusion).

$\otimes$-link:



(two premises, one conclusion).

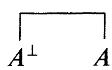
⅋-link:



(two premises, one conclusion).

We now proceed by induction.

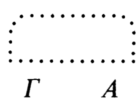
(i) Let π be just (id) and Γ be ${}^{\perp}A, A$. Then π' is the graph



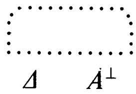
(ii) Let π be

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \vdots & \vdots \\ \vdash \Gamma, A & \vdash \Delta, A^{\perp}\end{array}}{\vdash \Delta, \Gamma}\ (\text{cut}).$$

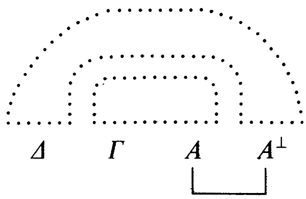
By the induction hypothesis, we have a graph $(\pi_1)'$ with terminal nodes Γ, A



and a graph $(\pi_2)'$ with terminal nodes $\Delta, A^\perp$



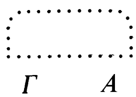
Then π' is the graph



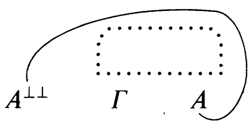
(iii) Let π be

$$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{\vdash \Gamma, A}{\vdash A^{\perp\perp}, \Gamma}\,((-)^{\perp\perp}). \end{array}$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with terminal nodes Γ, A



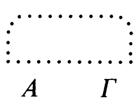
Then π' is the graph



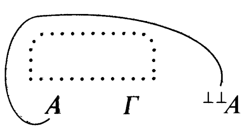
(iv) Let π be

$$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{\vdash A, \Gamma}{\vdash \Gamma, {}^{\perp\perp}A}\,({}^{\perp\perp}(-)). \end{array}$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with terminal nodes A, Γ



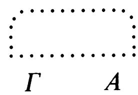
Then π' is the graph



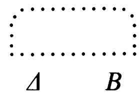
(v) Let π be

$$\dfrac{\begin{matrix}\pi_1 & & \pi_2 \\ \vdots & & \vdots \\ \vdash \Gamma, A & & \vdash \Delta, B\end{matrix}}{\vdash \Delta, \Gamma, A \otimes B}\,(\otimes).$$

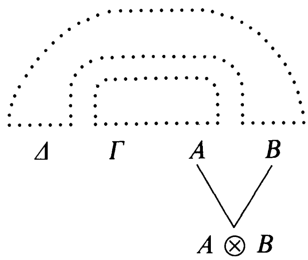
By the induction hypothesis, we have a graph $(\pi_1)'$ with terminal nodes Γ, A



and a graph $(\pi_2)'$ with terminal nodes Δ, B



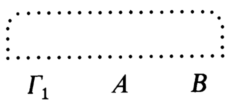
Then π' is the graph



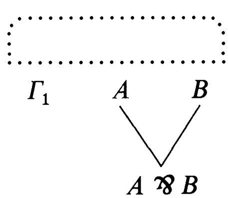
(vi) Let π be

$$\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ \vdash \Gamma, A, B\end{matrix}}{\vdash \Gamma, A \parr B}\,(\parr).$$

By the induction hypothesis, we have a graph $(\pi_1)'$ with terminal nodes Γ_1, A, B



Then π' is the graph



4.14. Definition. (i) A *noncommutative proof net* is a graph of occurrences of formulas of $\mathscr{L}$(SPNCL), associated to a proof in SPNCL, according to Definition 4.12.

(ii) A *noncommutative enriched proof net* is a graph of occurrences of formulas of $\mathscr{L}$(SPNCL), associated to a proof in SPNCL′, according to Definition 4.13.

§5. Appendix. Cut-elimination theorem. Let us consider the formulations of the sequent calculus for the pure noncommutative classical linear propositional logic: a two-sided sequent calculus, PNCL, and two one-sided sequent calculi, SPNCL and SPNCL′.

The dropping of the exchange rule involves some features of these sequent calculi. One of these features deals with the cut rule and with binary rules for multiplicative connectives.

In the sequent calculi PNCL and SPNCL the cut rule is subject to a condition on the contexts:

cut rule in PNCL $$\frac{\Gamma \Rightarrow \Delta_1,A,\Delta_2 \quad \Gamma_1,A,\Gamma_2 \Rightarrow \Delta}{\Gamma_1,\Gamma,\Gamma_2 \Rightarrow \Delta_1,\Delta_2}$$

if $\Delta_1 = \Gamma_2 = \varnothing$ or $\Delta_2 = \Gamma_1 = \varnothing$ or $\Gamma_1 = \Gamma_2 = \varnothing$ or $\Delta_1 = \Delta_2 = \varnothing$; and

cut rule in SPNCL $$\frac{\vdash \Gamma_1,A,\Gamma_2 \quad \vdash \Delta_1,A^{\perp},\Delta_2}{\vdash \Delta_1,\Gamma_1,\Delta_2,\Gamma_2}$$

if $\Delta_1 = \varnothing$ or $\Gamma_2 = \varnothing$. Roughly, these conditions express that a communication between an input A and an output A is possible if there is no obstacle (there is an obstacle when the condition on the contexts is not satisfied) between the input A and the output A.

Similar conditions on the contexts are imposed in PNCL and SPNCL to all the binary rules for the multiplicative connectives.

The rules for the connectives $(-)^{\perp}$ and ${}^{\perp}(-)$ (the two negations of the pure noncommutative linear propositional logic) in the sequent calculus PNCL allow us to remove the obstacles, or, better, to do as if the obstacles are removed; these rules are not present in the sequent calculus SPNCL. So, the difference between PNCL and SPNCL is that in PNCL we express the process of removing the obstacles whereas in SPNCL there is no way to express this process.

In the sequent calculus SPNCL′ there is no condition on the contexts, e.g. the cut rule is expressed in the following apparently weaker way:

cut rule in SPNCL′ $$\frac{\vdash \Gamma,A \quad \vdash \Delta,A^{\perp}}{\vdash \Delta,\Gamma},$$

but there are two special rules concerning double negations and allowing us to get in SPNCL′ the same power of SPNCL:

$$\frac{\vdash \Gamma,A}{\vdash A^{\perp\perp},\Gamma}, \qquad \frac{\vdash A,\Gamma}{\vdash \Gamma,{}^{\perp\perp}A}.$$

Due to these features, the cut-elimination theorem for the pure noncommutative classical linear propositional logic is not straightforward—not a trivial repetition of the usual cut-elimination theorems.

5.1. THEOREM. SPNCL *enjoys cut elimination. In other words, if π is a proof of a sequent $\vdash\Gamma$ in* SPNCL, *then π can be transformed into a cut-free proof π' of $\vdash\Gamma$ in* SPNCL.

□ The proof is, as usual, by induction on the number of cut rules in the proof of $\Gamma \Rightarrow \Delta$, from the following main lemma:

LEMMA. *Let π be a proof in* SPNCL *having the form*

$$\frac{\pi_1\{\vdash\Gamma_1,C,\Gamma_2 \quad \pi_2\{\vdash\Delta_1,C^{\perp},\Delta_2}{\vdash\Delta_1,\Gamma_1,\Delta_2,\Gamma_2}\,(cut)$$

($\Gamma_2 = \varnothing$ or $\Delta_1 = \varnothing$), where π_1 and π_2 are cut-free proofs. Then we can transform π into a cut-free proof of $\vdash\Delta_1,\Gamma_1,\Delta_2,\Gamma_2$.

The proof of the lemma is, as usual, by induction on the degree of π and subinduction on the rank of π.

The definitions of degree and rank of π, and all the details of the proof, are given in [2]. From the failure of the exchange rule and from the conditions on the cut rule and ($\otimes$)-rule, nontrivial and unusual considerations arise: in particular, we have to check that in every reduction of a cut to other cuts (with lower degree or lower rank) the conditions on the contexts are satisfied.

We show two typical cases.

Case 1). $\Gamma_2 = \varnothing$ and π is

$$\frac{\pi_1\left\{\dfrac{\psi_1\{\vdash\Theta_1,A \quad \psi_2\{\vdash\Theta_2,B,\Theta_3,C}{\vdash\Theta_2,\Theta_1,A\otimes B,\Theta_3,C}\,(\otimes)\right. \quad \pi_2\{\vdash\Delta_1,C^{\perp},\Delta_2}{\vdash\Delta_1,\Theta_2,\Theta_1,A\otimes B,\Theta_3,\Delta_2,}\,(\mathrm{cut})$$

Then we construct the following proof:

$$\frac{\psi_1\{\vdash\Theta_1,A \quad \psi\left\{\dfrac{\psi_2\{\vdash\Theta_2,B,\Theta_3,C \quad \pi_2\{\vdash\Delta_1,C^{\perp},\Delta_2}{\vdash\Delta_1,\Theta_2,B,\Theta_3,\Delta_2}\,(\mathrm{cut})\right.}{\vdash\Delta_1,\Theta_2,\Theta_1,A\otimes B,\Theta_3,\Delta_2}\,(\otimes)$$

where ψ_2 and π_2 are cut-free proofs and $d(\psi) = d(\pi)$ and $\mathrm{rank}(\psi) < \mathrm{rank}(\pi)$, so that by the induction hypothesis we get a cut-free proof ψ' of $\vdash\Delta_1,\Theta_2,B,\Theta_3,\Delta_2$, and the final cut-free proof is

$$\frac{\psi_1\{\vdash\Theta_1,A \quad \psi'\{\vdash\Delta_1,\Theta_2,B,\Theta_3,\Delta_2}{\vdash\Delta_1,\Theta_2,\Theta_1,A\otimes B,\Theta_3,\Delta_2}\,(\otimes).$$

Case 2). $C = A \otimes B$, $\Gamma_2 = \varnothing$ and π is

$$\dfrac{\pi_1\left\{\dfrac{\psi_1\{\vdash \Theta_1, A \quad \psi_2\{\vdash \Theta_2, B}{\vdash \Theta_2, \Theta_1, A \otimes B}(\otimes)\right. \quad \pi_2\left\{\dfrac{\psi_3 \vdash \Delta_1, B^{\perp}, A^{\perp}, \Delta_2}{\vdash \Delta_1, B^{\perp} \parr A^{\perp}, \Delta_2}(\parr).\right.}{\vdash \Delta_1, \Theta_2, \Theta_1, \Delta_2}\text{(cut)}.$$

Then we construct the following proof:

$$\dfrac{\psi_1\{\vdash \Theta_1, A \quad \psi\left\{\dfrac{\psi_2\{\vdash \Theta_2, B \quad \psi_3 \vdash \Delta_1, B^{\perp}, A^{\perp}, \Delta_2}{\vdash \Delta_1, \Theta_2, A^{\perp}, \Delta_2}\text{(cut)}\right.}{\vdash \Delta_1, \Theta_2, \Theta_1, \Delta_2}\text{(cut)}$$

where ψ_2 and ψ_3 are cut-free proofs and $d(\psi) < d(\pi)$, so that by the induction hypothesis we can construct a cut-free proof ψ' of $\vdash \Delta_1, \Theta_2, A^{\perp}, \Delta_2$; now we construct the following proof:

$$\chi\left\{\dfrac{\psi_1\{\vdash \Theta_1, A \quad \psi'\{\vdash \Delta_1, \Theta_2, A^{\perp}, \Delta_2}{\vdash \Delta_1, \Theta_2, \Theta_1, \Delta_2}\text{(cut)}\right.$$

where ψ_1 and ψ' are cut-free proofs and $d(\chi) < d(\pi)$, so that by the induction hypothesis we can construct a cut-free proof π' of $\vdash \Delta_1, \Theta_2, \Theta_1, \Delta_2$. □

5.2. THEOREM. SPNCL′ *enjoys cut elimination. In other words, if* π *is a proof of a sequent* $\vdash \Gamma$ *in* SPNCL′, *then* π *can be transformed into a cut-free proof* π' *of* $\vdash \Gamma$ *in* SPNCL′.

□ We give the following cut-elimination procedure.

First, we transform π into a proof ψ of $\vdash \Gamma$ in SPNCL, according to the procedure given in 4.11(ii).

Second, we transform ψ into a cut-free proof ψ' of $\vdash \Gamma$ in SPNCL, according to the procedure given in Theorem 5.1.

Third, we transform ψ' into a proof π' of $\vdash \Gamma$ in SPNCL′, according to the procedure given in 4.11(i): since ψ' is cut-free, by 4.11(i), π' is cut-free. □

5.3. PROBLEM. Find a *direct* cut-elimination procedure for SPNCL′; i.e., find a cut-elimination procedure such that every step consists in a transformation of the proof in SPNCL′ into a proof in SPNCL′, without the intermediary step of transforming a proof in SPNCL′ into a proof in SPNCL.

The principal point is to say how the cut-elimination procedure acts when the last rule before a cut-rule is a $(-)^{\perp\perp}$-rule or a ${}^{\perp\perp}(-)$-rule.

5.4. REMARK. PNCL does not enjoy cut elimination, at least not by the usual cut-elimination procedures.

This fact depends on the main property of the connectives $(-)^{\perp}$ and ${}^{\perp}(-)$: these connectives allow us to arrange the sequents in order to use the cut rule (or other rules requiring the satisfaction of conditions on the context), so that we cannot reverse the order of the rules, as required by the usual cut-elimination procedures. In other words, if there are obstacles to communicating between an input C and an output C, we first have to remove the obstacles by means of the rules for $(-)^\perp$ and ${}^\perp(-)$, and then we can communicate; but the usual procedures for the cut elimination require that we first communicate and then remove the obstacles!

For example, let us consider a proof π in PNCL

$$\dfrac{\pi_1\left\{\dfrac{\psi_1\{\overset{\vdots}{\Theta \Rightarrow \Delta_2, C, A}}{\Theta, A^\perp \Rightarrow \Delta_2, C}\,((-)^\perp,\mathrm{L})\right. \qquad \pi_2\{\overset{\vdots}{C, \Gamma_2 \Rightarrow \Delta}}{\Theta, A^\perp, \Gamma_2 \Rightarrow \Delta_2, \Delta}\,(\mathrm{cut})$$

with $\Theta, \Delta_2, \Delta, \Gamma_2$ not empty, and π_1 and π_2 cut-free. The usual procedure for cut-elimination is to construct a proof

$$\dfrac{\psi\left\{\dfrac{\psi_1\{\overset{\vdots}{\Theta \Rightarrow \Delta_2, C, A} \qquad \pi_2\{\overset{\vdots}{C, \Gamma_2 \Rightarrow \Delta}}{\vdash \Theta, \Gamma_2 \Rightarrow \Delta_2, \Delta, A}\,(\mathrm{cut})\right.}{\vdash \Theta, A^\perp, \Gamma_2 \Rightarrow \Delta_2, \Delta}\,((-)^\perp,\mathrm{L})$$

with $d(\psi) = d(\pi)$ and $\mathrm{rank}(\psi) < \mathrm{rank}(\pi)$. But the proof is not a proof in PNCL, because:

(a) the cut rule is not allowed, because a formula A occurs after C and Γ_2 is not empty; and

(b) the $((-)^\perp,\mathrm{L})$-rule is incorrect, because the correct conclusion is $\Theta, \Gamma_2, A^\perp \Rightarrow \Delta_2, \Delta$.

A typical proof where the cut rule cannot be eliminated is the proof of the sequent $({}^{\perp\perp}A)^{\perp\perp} \Rightarrow A$, given in 2.5(v). If we apply the usual procedure, we get

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,(\mathrm{id})}{\Rightarrow A, {}^\perp A}\,({}^\perp(-),\mathrm{R}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,(\mathrm{id})}{{}^\perp A, A \Rightarrow}\,({}^\perp(-),\mathrm{L})}{{}^\perp A \Rightarrow {}^\perp A}\,({}^\perp(-),\mathrm{R})}{{}^{\perp\perp}A, {}^\perp A \Rightarrow}\,({}^\perp(-),\mathrm{L})}{{}^{\perp\perp}A \Rightarrow A}\,(\mathrm{cut})}{\Rightarrow A, ({}^{\perp\perp}A)^\perp}\,((-)^\perp,\mathrm{R})}{({}^{\perp\perp}A)^{\perp\perp} \Rightarrow A}\,((-)^\perp,\mathrm{L})$$

but the rules (cut) and $((-)^\perp,\mathrm{R})$ are incorrect!

Suppose we have a proof of the failure of cut-elimination for PNCL. Then we can distinguish cut-free provable sequents and non-cut-free provable sequents. Now, in order to build the one-sided sequent calculus SPNCL, we make an identification of formulas by using both cut-free and non-cut-free provable sequents. For example, a formula A of $\mathscr{L}$(SPNCL) is literally the same as

$$({}^\perp A)^\perp,\ ({}^{\perp\perp}A)^{\perp\perp},\ ({}^{\perp\perp\perp}A)^{\perp\perp\perp}, \ldots,\ {}^\perp(A^\perp),\ {}^{\perp\perp}(A^{\perp\perp}),\ {}^{\perp\perp\perp}(A^{\perp\perp\perp}), \ldots,$$

but in PNCL the sequents

$$({}^{\perp}A)^{\perp} \Rightarrow A, \quad A \Rightarrow ({}^{\perp}A)^{\perp}, \quad {}^{\perp}(A^{\perp}) \Rightarrow A, \quad A \Rightarrow {}^{\perp}(A^{\perp})$$

are cut-free provable, whereas the sequents

$$({}^{\perp\perp}A)^{\perp\perp} \Rightarrow A, \quad A \Rightarrow ({}^{\perp\perp}A)^{\perp\perp}, \quad {}^{\perp\perp}(A^{\perp\perp}) \Rightarrow A, \quad A \Rightarrow {}^{\perp\perp}(A^{\perp\perp})$$

are not.

However, we have to better understand the meaning and the consequences of these results on the cut-elimination theorem.

REFERENCES

[1] V. M. Abrusci, *Noncommutative intuitionistic linear propositional logic*, ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik***, vol. 36 (1990), pp. 297–318.

[2] ———, *Cut-elimination theorem for pure noncommutative classical linear propositional logic*, preprint, Università di Bari, Bari, 1990.

[3] J.-Y. Girard, *Linear logic*, ***Theoretical Computer Science***, vol. 50 (1987), pp. 1–102.

[4] ———, *Towards a geometry of interaction*, ***Categories in computer science and logic***, Contemporary Mathematics, vol. 92, American Mathematical Society, Providence, Rhode Island, 1989, pp. 69–108.

[5] S. B. Niefeld and K. I. Rosenthal, *Constructing locales from quantales*, ***Mathematical Proceedings of the Cambridge Philosophical Society***, vol. 104 (1988), pp. 215–234.

[6] D. N. Yetter, *Quantales and (noncommutative) linear logic*, this Journal, vol. 55 (1990), pp. 41–64.

FILOSOFIA DELLA SCIENZA—LOGICA MATEMATICA
DIPARTIMENTO DI SCIENZE FILOSOFICHE
UNIVERSITÀ DI BARI
70121 BARI, ITALY