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QUANTALES AND (NONCOMMUTATIVE) LINEAR LOGIC

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It is the purpose of this paper to make explicit the connection between J.-Y. Girard's "linear logic" [4], and certain models for the logic of quantum mechanics, namely Mulvey's "quantales" [9]. This will be done not only in the case of commutative linear logic, but also in the case of a version of noncommutative linear logic suggested, but not fully formalized, by Girard in lectures given at McGill University in the fall of 1987 [5], and which for reasons which will become clear later we call "cyclic linear logic".

For many of our results on quantales, we rely on the work of Niefield and Rosenthal [10].

The reader should note that by "the logic of quantum mechanics" we do not mean the lattice theoretic "quantum logics" of Birkhoff and von Neumann [1], but rather a logic involving an associative (in general noncommutative) operation "and then". Logical validity is intended to embody empirical verification (whether a physical experiment, or running a program), and the validity of A & B (in Mulvey's notation) is to be regarded as "we have verified A, and then we have verified B". (See M. D. Srinivas [11] for another exposition of this idea.)

This of course is precisely the view of the "multiplicative conjunction", $\otimes$, in the phase semantics for Girard's linear logic [4], [5]. Indeed the quantale semantics for linear logic may be regarded as an element-free version of the phase semantics.

The main results of this paper may be summed up in the slogan

"quantales + a representable duality = linear logic".

However, the rigorously describable correspondences are not so close as we might wish to believe given the almost identical heuristic justifications for the two systems.

It is also disappointing that "cyclic linear logic" is still not "noncommutative enough" to properly express "time's arrow".

In the case of cyclic linear logic, we prove cut elimination and normalization theorems using a geometrization of proofs which appears to offer many of the advantages of proof-nets, while sticking sufficiently close to the usual style of sequent-calculus proofs so as to require little exposition.

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We conclude by observing that Girard's interpretation from intuitionistic logic to linear logic works equally well with cyclic linear logic, and by providing an interpretation from commutative linear logic into cyclic linear logic.

Throughout, we use lattice-theoretic notation for lattice operations, and Girard's notation for "multiplicative" operations, $\otimes$ (corresponding to Mulvey's &), and $⅋$ for its dual, and $\perp$ for the dualizing element ($=$ the absurd fact), but use the category-theoretic I for the $\otimes$-identity, instead of Girard's **1**.

§1. Definitions and elementary results on quantales. Following Niefield and Rosenthal [10], we make

DEFINITION 1.1. A *quantale*, $(\mathbf{Q}, \bigvee, \otimes)$, is a complete lattice $(\mathbf{Q}, \bigvee)$ equipped with an associative product $\otimes$ which distributes on both sides over arbitrary $\bigvee$'s.

A quantale $\mathbf{Q}$ is *unital* if it has an element I such that, for all $a \in \mathbf{Q}$, $a \otimes I = I \otimes A = a$. $\mathbf{Q}$ is *commutative* if $\otimes$ is commutative.

A *quantic nucleus* j on a quantale $\mathbf{Q}$ is an inflationary, idempotent, monotone (i.e. closure) operator, satisfying moreover $j(a) \otimes j(b) \leq j(a \otimes b)$.

Regarding $\mathbf{Q}$ as a category in the usual way, we have, by elementary abstract nonsense,

PROPOSITION 1.2. *The endofunctor* $(a \otimes -)$ *(resp.* $(- \otimes a)$*) for any* $a \in \mathbf{Q}$ *has a right adjoint, denoted* $- \rightarrow_r a$ *(resp.* $- \rightarrow_l a$*).*

We shall require a few new definitions related to quantales:

DEFINITION 1.3. An element D of a quantale $\mathbf{Q}$ is *dualizing* if for all $a \in \mathbf{Q}$ we have

$$(a \rightarrow_r D) \rightarrow_l D = a = (a \rightarrow_l D) \rightarrow_r D.$$

An element s is *cyclic* if for all $a \in \mathbf{Q}$ we have $a \rightarrow_r s = a \rightarrow_l s$.

This last terminology is justified by

PROPOSITION 1.4. *An element* s *of a quantale* $\mathbf{Q}$ *is cyclic if and only if* $a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n \leq s$ *implies* $a_n \otimes a_1 \otimes \cdots \otimes a_{n-1} \leq s$.

Note that the notion of cyclic element is strictly weaker than that of symmetric element as described in Niefield and Rosenthal [10].

The presence of a dualizing element makes the logic of a quantale "more classical" in the following sense:

PROPOSITION 1.5. *If* D *is a dualizing element in a quantale* $\mathbf{Q}$, *then* $\mathbf{Q}$ *is unital; in particular,* $I = (D \rightarrow_r D) = (D \rightarrow_l D)$. *Moreover, the bifunctors* $\rightarrow_r$ *and* $\rightarrow_l$ *are completely determined by their values with the second variable instantiated as* D, *in a way analogous to the expression of classical implication in terms of negation and conjunction in classical logic.*

PROOF. We begin with the second statement. Now, in general we have

$$a \rightarrow_r c = \bigvee\{b \mid b \otimes a \leq c\} \quad \text{and} \quad a \rightarrow_l c = \bigvee\{b \mid a \otimes b \leq c\}.$$

Now observe that is D is a dualizing element, then $b \otimes a \leq c$ is equivalent to $b \otimes a \otimes (c \rightarrow_l D) \leq D$. The forward implication is true for any element D, while the converse follows by observing that this latter implies that $b \otimes a \leq (c \rightarrow_l D) \rightarrow_r D$, the right-hand side of which equals c, since D is dualizing. But $\bigvee\{b \mid b \otimes a \otimes (c \rightarrow_l D) \leq D\}$ is just $(a \otimes (c \rightarrow_l D)) \rightarrow_r D$.

A similar argument shows that

$$a \rightarrow_r c = ((c \rightarrow_r D) \otimes a) \rightarrow_l D.$$

Now to see that $(D \to_r D) = (D \to_l D)$ is a unit for $\otimes$, note that

$$\begin{aligned} a \otimes (D \to_r D) &= ((a \otimes (D \to_r D)) \to_l D) \to_r D \\ &= \left[\bigvee\{b \mid b \otimes a \otimes (D \to_r D) \leq D\}\right] \to_r D \\ &= \left[\bigvee\{b \mid b \otimes a \leq (D \to_r D) \to_l D\}\right] \to_r D \\ &= \left[\bigvee\{b \mid b \otimes a \leq D\}\right] \to_r D = (a \to_l D) \to_r D = a. \end{aligned}$$

A similar argument shows that $(D \to_l D)$ is a left unit for $\otimes$. Of course, given any associative binary operation with right and left units, the units are equal and are a two-sided unit. $\square$

In particular, a cyclic dualizing element provides a quantale with a good involutory negation:

DEFINITION 1.6. A *Girard quantale* is a quantale, together with a cyclic dualizing element $\perp$. The operation $- \to_r \perp (= - \to_l \perp)$ will be called *linear negation*, and denoted $\perp_{(-)}$.

The categorically minded reader will note that in the commutative case a (necessarily cyclic) dualizing element is just what is needed to make the quantale into a $*$-autonomous category.

There is no shortage of examples of Girard quantales: the "facts" of any phase space in the sense of [4] form a Girard quantale; noncommutative examples can be similarly constructed by an analogous construction, starting with a noncommutative monoid M, and choosing $\perp \subset M$ to have the property $ab \in \perp$ iff $ba \in \perp$. Indeed, the following proposition shows that a Girard quantale can be constructed from any unital quantale and a cyclic element therein:

PROPOSITION 1.7. *Given any unital quantale* $\mathbf{Q}$ *and a cyclic element* $s \in \mathbf{Q}$, *let* $\to s$ *denote* $\to_r s = \to_l s$. *Then,* $j(-) = (- \to s) \to s$ *is a quantic nucleus,* s *is* j*-closed, and the lattice of* j*-closed elements of* $\mathbf{Q}$ *forms a Girard quantale with* $\perp = s$.

PROOF. That the j-closed elements form a quantale (under $j(\bigvee \cdots)$ and $j(- \otimes -)$) will follow immediately from a result of Niefield and Rosenthal [10] when it is shown that j is a quantic nucleus.

That j is a closure operator follows by an easy argument in Niefield and Rosenthal [10]. For the other condition, we adapt one of their arguments to the hypothesis of cyclicity.

Observe that

$$b \otimes [b \to_r (a \to s)] \otimes [(a \to s) \to s] \leq (a \to s) \otimes [(a \to s) \to s] \leq s$$

and thus

$$[b \to_r (a \to s)] \otimes [(a \to s) \to s] \leq (b \to s).$$

But the reader may easily check that $b \to_r (a \to s) = (a \otimes b) \to s$ (hint: treat $\to s$ as $\to_r s$). Thus we have

$$[(a \otimes b) \to s] \otimes [(a \to s) \to s] \leq (b \to s),$$

from which it follows that

$$[(a \otimes b) \to s] \otimes [(a \to s) \to s] \otimes [(b \to s) \to s] \leq s,$$

and thus finally

$$[(a \to s) \to s] \otimes [(b \to s) \to s] \leq [(a \otimes b) \to s] \to s.$$

Thus we verify that $j(-)$ is a quantic nucleus.

To see that s is j-closed, note that $s \to s \geq I$. Thus

$$j(s) = (s \to s) \to s = \bigvee\{a \mid a \otimes (s \to s) \leq s\} \leq \bigvee\{a \mid a \otimes I \leq s\} \leq s.$$

But $j(-)$ is a closure, so $s \leq j(s)$, and thus $j(s) = s$.

Given the preceding, it is easy to check that s remains cyclic in the quantale of j-closed elements, and that it becomes a dualizing element. □

Finally, to discuss Girard's "exponential" operations in the context of quantales, we need to define suitable modal operators on quantales:

DEFINITION 1.8. An *(open) modality* on a quantale $\mathbf{Q}$ is a map μ: $\mathbf{Q} \to \mathbf{Q}$ satisfying:

M1. $\mu(a) \leq a$ (deflationary),

M2. $a \leq b$ implies $\mu(a) \leq \mu(b)$ (monotone),

M3. $\mu(\mu(a)) = \mu(a)$ (idempotence), and

M4. $\mu(\mu(a) \otimes \mu(b)) = \mu(a) \otimes \mu(b)$.

A modality μ is *central* if it satisfies $\forall a, b \in \mathbf{Q}\, \mu(a) \otimes b = b \otimes \mu(a)$; μ is *idempotent* if it satisfies $\forall a \in \mathbf{Q}\, \mu(a) \otimes \mu(a) = \mu(a)$ (note this should not be confused with condition M3 above); and if $\mathbf{Q}$ is unital, μ is *weak* if $\forall a \in \mathbf{Q}\, \mu(a) \leq I$.

For any quantale $\mathbf{Q}$, the set $\mathbf{M}(\mathbf{Q})$ of modalities on $\mathbf{Q}$ is partially ordered by $\mu \leq \mu'$ whenever $\forall a \in \mathbf{Q}\; \mu(a) \leq \mu'(a)$.

We then have:

PROPOSITION 1.9. *Let $\mathbf{Q}$ be any quantale (resp. unital quantale); then there is a unique maximal central (resp. central weak idempotent) modality on $\mathbf{Q}$, which we denote c_{m} (resp. $!_{\mathrm{m}}$).*

PROOF. Let

$$c_{\mathrm{m}}(x) = \bigvee\{a \in \mathbf{Q} \mid a \leq x,\, a \in \mathbf{Z}(\mathbf{Q})\},$$
$$!_{\mathrm{m}}(x) = \bigvee\{a \in \mathbf{Q} \mid a \leq x,\, a \leq I,\, a \otimes a = a,\, a \in \mathbf{Z}(\mathbf{Q})\},$$

where $\mathbf{Z}(\mathbf{Q}) = \{a \in \mathbf{Q} \mid \forall b \in \mathbf{Q}\; a \otimes b = b \otimes a\}$. The actual verification that these are (open) modalities is easy, and so is left to the reader. Maximality is obvious. □

In the case of a Girard quantale, we can associate to each open modality μ a dual "closed modality": ${}^{\perp}\mu({}^{\perp}(-))$.

§2. Definitions from linear logic. We begin by making explicit a language for linear predicate calculi. In our formalism we adopt prefix notation in preference to the infix/postfix notation used by Girard [4], but follow his lead in giving a "negative-free" syntax.

DEFINITION 2.1. By a *linear predicate language*, $\mathbb{L}$, we mean a set $\mathscr{P} = \{A, B, \ldots\}$ of predicates, a map arity: $\mathscr{P} \to \mathbf{N}$, a set of variables $\mathscr{V} = \{x, y, \ldots\}$, a set of modal symbols $\mathscr{M}$ equipped with a fixed-point free involution dual: $\mathscr{M} \to \mathscr{M}$, and the symbols, $\mathbf{1}, \mathbf{0}, I, \perp, {}^{\perp}, \bigvee, \bigwedge, \vee, \wedge, \otimes$, and $⅋$.

We define the *formulae* of a linear predicate language recursively as follows:

F1. The symbols $\mathbf{1}, \mathbf{0}, I$, and $\perp$ are formulae.

F2. If P is a predicate symbol and $x_1, \ldots, x_{\mathrm{arity}(P)}$ are variables, then $Px_1 \cdots x_{\mathrm{arity}(P)}$ and ${}^{\perp}Px_1 \cdots x_{\mathrm{arity}(P)}$ are formulae.

F3. If F and G are formulae, then so are $\otimes FG$, $⅋ FG$, $\vee FG$, and $\wedge FG$.

F4. If F is a formula and m is a modal symbol, then mF is a formula.

F5. If F is a formula and x is a variable, then $\bigvee xF$ and $\bigwedge xF$ are formulae.

Of course, we call any occurrence of a variable x *free* unless it occurs in a (sub)formula of the form $\bigvee xF$ or $\bigwedge xF$, in which case it is called *bound*. Formulae of the form $Px_1 \cdots x_n$ will be called *elementary* formulae. We denote the set of formulae of $\mathbb{L}$ by form($\mathbb{L}$), and the set of elementary formulae by elf($\mathbb{L}$).

We can now define an operation ${}^{\perp}(-)$ on formulae by:

DEFINITION 2.2. The *syntactic linear negation* of a formula in a linear predicate language is defined recursively as follows:

N1. ${}^{\perp}(\mathbf{1}) = \mathbf{0}$, ${}^{\perp}(\mathbf{0}) = \mathbf{1}$, ${}^{\perp}(I) = \perp$, and ${}^{\perp}(\perp) = I$.

N2. ${}^{\perp}(Px\cdots y) = {}^{\perp}Px\cdots y$ and ${}^{\perp}({}^{\perp}Px\cdots y) = Px\cdots y$.

N3. ${}^{\perp}(\otimes FG) = \parr{}^{\perp}(F){}^{\perp}(G)$, ${}^{\perp}(\parr FG) = \otimes{}^{\perp}(F){}^{\perp}(G)$, ${}^{\perp}(\vee FG) = \wedge{}^{\perp}(F){}^{\perp}(G)$, and ${}^{\perp}(\wedge FG) = \vee{}^{\perp}(F){}^{\perp}(G)$.

N4. ${}^{\perp}(mF) = \mathrm{dual}(m){}^{\perp}(F)$.

N5. ${}^{\perp}(\bigvee xF) = \bigwedge x{}^{\perp}(F)$ and ${}^{\perp}(\bigwedge xF) = \bigvee x{}^{\perp}(F)$.

The other connectives of linear logic have their applications to formulae clearly given by the definition of formulae.

Of course for Girard's commutative linear logic we have $\mathcal{M} = \{!, ?\}$, with "dual" simply interchanging the two symbols. In the full "cyclic linear logic" we have $\mathcal{M} = \{!, ?, k, \kappa\}$, with "dual" interchanging ! and ?, and k and κ.

DEFINITION 2.3. A *sequent* of a linear predicate language is an expression of the form $\Rightarrow A_1, \ldots, A_n$, where $\Rightarrow$ is a metalinguistic symbol, and $A_1, \ldots, A_n$ are formulae of the language.

A *linear sequent calculus* $\mathbb{P}$ is a linear predicate language $\mathbb{L}$ together with a set of rules of the form

$$\frac{H}{C},$$

where H is a list (possibly empty) of sequents of $\mathbb{L}$ (the *hypotheses*) and C is a sequent of $\mathbb{L}$ (the *conclusion*). A rule with H empty is called an *axiom*. A *proof* in $\mathbb{P}$ is either an axiom, or a list of proofs, together with a rule so that the list of conclusions of the proofs in the list of hypotheses for the rule. A sequent is *provable* if there exists a proof with it as conclusion.

We are now in a position to recapitulate the definition of commutative linear logic in terms of the sequent calculus given in Girard [4], and to give a similar definition for cyclic linear logic (based on [5]). In the following, Latin letters denote formulae, Greek letters denote arbitrary lists of formulae, and a modal symbol preceding a Greek letter is shorthand for that modal symbol applied to each element of the list.

DEFINITION 2.4. The *commutative linear sequent calculus* is given by all rules of the following forms:

$$\text{Axiom 1} \quad \frac{}{\Rightarrow A, {}^{\perp}A}$$

$$\text{Axiom 2} \quad \frac{}{\Rightarrow \mathbf{1}, \Phi}$$

$$\text{Axiom 3} \quad \frac{}{\Rightarrow I}$$

$$\text{Cut} \quad \frac{\Rightarrow \Phi, A \qquad \Rightarrow {}^{\perp}A, \Psi}{\Rightarrow \Phi, \Psi}$$

$$\text{Exchange} \quad \frac{\Rightarrow \Phi}{\Rightarrow \Psi}$$

where Ψ is obtained by permuting the formulae of Φ.

$$\wedge \quad \frac{\Rightarrow A, \Phi \qquad \Rightarrow B, \Phi}{\Rightarrow \wedge AB, \Phi}$$

$$\vee 1 \quad \frac{\Rightarrow A, \Phi}{\Rightarrow \vee AB, \Phi}$$

$$\vee 2 \quad \frac{\Rightarrow A, \Phi}{\Rightarrow \vee BA, \Phi}$$

$$\perp \quad \frac{\Rightarrow \Phi}{\Rightarrow \perp, \Phi}$$

for Φ a nonempty list.

$$\otimes \quad \frac{\Rightarrow A, \Phi \qquad \Rightarrow B, \Psi}{\Rightarrow \otimes AB, \Psi, \Phi}$$

$$⅋ \quad \frac{\Rightarrow A, B, \Phi}{\Rightarrow ⅋AB, \Phi}$$

$$\text{Dereliction} \quad \frac{\Rightarrow A, \Phi}{\Rightarrow ?A, \Phi}$$

$$\text{Weakening} \quad \frac{\Rightarrow \Phi}{\Rightarrow ?A, \Phi}$$

$$\text{Contraction} \quad \frac{\Rightarrow ?A, ?A, \Phi}{\Rightarrow ?A, \Phi}$$

$$! \quad \frac{\Rightarrow A, ?\Phi}{\Rightarrow !A, ?\Phi}$$

$$\bigvee \quad \frac{\Rightarrow A[t \mid x], \Phi}{\Rightarrow \bigvee xA, \Phi}$$

$$\bigwedge \quad \frac{\Rightarrow A, \Phi}{\Rightarrow \bigwedge xA, \Phi}$$

whenever x is not free in Φ.

DEFINITION 2.5. The *cyclic linear sequent calculus* is given by all instances of the rules of 2.4, *except exchange and dereliction*, together with all instances of the following:

$$\text{Cycling} \quad \frac{\Rightarrow \Phi}{\Rightarrow \Psi}$$

where Ψ is obtained from Φ by a *cyclic* permutation.

$$?\text{-exchange} \quad \frac{\Rightarrow A, ?B, \Phi}{\Rightarrow ?B, A, \Phi}$$

$$\kappa\text{-exchange} \quad \frac{\Rightarrow A, \kappa B, \Phi}{\Rightarrow \kappa B, A, \Phi}$$

$$\kappa\text{-dereliction} \quad \frac{\Rightarrow A, \Phi}{\Rightarrow \kappa A, \Phi}$$

$$\kappa ?\text{-dereliction} \quad \frac{\Rightarrow \kappa A, \Phi}{\Rightarrow ?A, \Phi}$$

$$k \quad \frac{\Rightarrow A, \kappa\Phi}{\Rightarrow kA, \kappa\Phi}$$

In the cyclic calculus we refer to the old rules of Weakening and Contraction as ?-weakening and ?-contraction.

The reader should note that in terms of the semantics we will develop, the seemingly unnatural Cycling rule is forced by having a system with a single negation: ${}^{\perp}(-)$ (i.e. the cyclicity of $\perp$). In a totally noncommutative setting, where Cycling is dropped, it would be necessary (if any sensible lattice- or category-theoretic semantics is to be given) for there to be two negations, perhaps called "predictive" and "retrodictive" negation if the temporal (meta-)semantics suggested in the introduction is taken seriously.

DEFINITION 2.6. By the *subexponential fragment* of a linear predicate language (resp. linear sequent calculus), we mean the linear predicate language obtained by replacing the set of modal operators by the empty set (resp. the sequent calculus obtained by replacing the linear predicate language by its subexponential fragment, and deleting all rules in which modal operators occur in either the hypotheses or the conclusion).

We now turn from syntax to semantics:

DEFINITION 2.7. An *interpretation* of a linear predicate language $\mathbb{L}$ is an ordered triple $(\mathbf{A}, V, |-|_{\mathbf{A}})$, where $\mathbf{A}$ is an algebra for the theory with constants $\mathbf{1}$, $\mathbf{0}$, I, and $\perp$, unary operations${}^{\perp}$ and $m \in M$, binary operations $\vee$, $\wedge$, $\otimes$, and $⅋$; and infinitary (i.e. $\mathbf{P}(\mathbf{A}) \to \mathbf{A}$) operations $\bigvee$ and $\bigwedge$; V is a subset of $\mathbf{A}$ (called *valid* elements), and $|-|_{\mathbf{A}}$ is a map from elf($\mathbb{L}$) to $\mathbf{A}$.

Using the operations on $\mathbf{A}$, there is an obvious extension of $|-|_{\mathbf{A}}$ from elf($\mathbb{L}$) to form($\mathbb{L}$), which by abuse of notation we also denote $|-|_{\mathbf{A}}$.

A *semantics* for a linear predicate language is a class of interpretations. A semantics is *sound* with respect to a linear sequent calculus $\mathbb{P}$ if whenever $\Rightarrow A_1, \ldots, A_n$ is provable in $\mathbb{P}$, $|A_1 ⅋ \cdots ⅋ A_n|_S$ is valid for every interpretation S in the semantics.

A semantics is *complete* with respect to a linear sequent calculus if $|F|_S$ valid for all interpretations S in the semantics implies that $\Rightarrow F$ is provable in $\mathbb{P}$.

§3. Quantale semantics for linear logic. We are now in a position to describe a semantics for commutative and cyclic linear logic in terms of Girard quantales.

DEFINITION 3.1. A *quantale interpretation* of a linear predicate language $\mathbb{L}$ is a Girard quantale $\mathbf{Q}$ together with functions $i_M\colon M^- \to \mathbf{M}(\mathbf{Q})$ and $|-|_{\mathbf{Q}}$: elf($\mathbb{L}$) $\to \mathbf{Q}$, where the set M of modal operations in $\mathbb{L}$ decomposes as $M^- \cup M^+$, dual$(M^-) = M^+$.

An element $a \in \mathbf{Q}$ is valid if $a \geq I$. The constants and operations required by

Definition 2.7 are those already defined in the same notation, except for $\invamp$, which is defined by $a \invamp b = {}^{\perp}({}^{\perp}a \otimes {}^{\perp}b)$, and the modal operations which are given by i_M.

DEFINITION 3.2. The *quantale semantics* for cyclic (resp. commutative) linear logic consists of the class of all quantale interpretations of the relevant linear predicate language in which $M^- = \{!, k\}$, $i_M(k)$ is central, and $i_M(!)$ is weak, idempotent, central, and less than $i_M(k)$ in the ordering on $\mathbf{M(Q)}$ (resp. in which $\mathbf{Q}$ is commutative, $M^- = \{!\}$, and $i_M(!)$ is idempotent and weak).

We can now prove the main results of this section.

THEOREM 3.3. *The quantale semantics for cyclic* (*resp. commutative*) *linear logic is sound with respect to the cyclic* (*resp. commutative*) *linear sequent calculus.*

PROOF. The proof is essentially an element-free version of Girard's proof of the soundness of the phase semantics for (commutative) linear logic [4]. We actually give the proof only in the case of cyclic linear logic. The modification needed to give the commutative case is trivial.

For Axiom 1, consider

$${}^{\perp}({}^{\perp}a \otimes {}^{\perp\perp}a) = \bigvee\{x \mid x \otimes {}^{\perp}a \otimes {}^{\perp\perp}a \leq \perp\} \geq \bigvee\{x \mid x \otimes I \leq \perp\} \geq {}^{\perp}\perp = I.$$

The soundness of Axiom 3 is immediate, while Axiom 2 follows by

$${}^{\perp}({}^{\perp}\mathbf{1} \otimes {}^{\perp}a) = \bigvee\{x \mid {}^{\perp}\mathbf{1} \otimes {}^{\perp}a \otimes x \leq \perp\} = \bigvee\{x \mid {}^{\perp}a \otimes x \leq \mathbf{1}\} = \mathbf{1} \geq I.$$

For the soundness of Cut, first observe that the validity of the interpretation of $\Rightarrow \Phi, A$ (resp. $\Rightarrow {}^{\perp}A, \Psi$) is that statement

$$I \leq \bigvee\{y \mid y \otimes {}^{\perp}x \otimes {}^{\perp}a \leq \perp\} \leq \bigvee\{y \mid y \otimes {}^{\perp}x \leq a\}$$

$$(\text{resp. } I \leq \bigvee\{z \mid a \otimes {}^{\perp}w \otimes z \leq \perp\} \leq \bigvee\{z \mid {}^{\perp}w \otimes z \leq {}^{\perp}a\}),$$

where a is the interpretation of A, and x (resp. w) is the interpretation of (the $\invamp$ of the elements of) Φ (resp. Ψ). Thus we have

$$\begin{aligned} I = I \otimes I &\leq [\bigvee\{z \mid {}^{\perp}w \otimes z \leq {}^{\perp}a\}] \otimes [\bigvee\{y \mid y \otimes {}^{\perp}x \leq a\}] \\ &\leq \bigvee\{z \otimes y \mid {}^{\perp}w \otimes z \leq {}^{\perp}a \text{ and } y \otimes {}^{\perp}x \leq a\} \\ &\leq \bigvee\{z \otimes y \mid {}^{\perp}w \otimes z \otimes y \otimes {}^{\perp}x \leq {}^{\perp}a \otimes a\} \\ &\leq \bigvee\{z \otimes y \mid {}^{\perp}w \otimes z \otimes y \otimes {}^{\perp}x \leq \perp\} \\ &\leq \bigvee\{c \mid {}^{\perp}w \otimes c \otimes {}^{\perp}x \leq \perp\} \end{aligned}$$

and thus, by the cyclicity of $\perp$, $I \leq \bigvee\{c \mid {}^{\perp}x \otimes {}^{\perp}w \otimes c \leq \perp\}$, which is precisely the interpretation of the validity of $\Rightarrow \Phi, \Psi$.

For the validity of cycling, observe that the validity of $\Rightarrow A_1, \ldots, A_n$ is interpreted as

$$I \leq \bigvee\{x \mid x \otimes {}^{\perp}a_1 \otimes \cdots \otimes {}^{\perp}a_n \leq \perp\}$$

where a_i is $|A_i|$. Thus we have ${}^{\perp}a_1 \otimes \cdots \otimes {}^{\perp}a_n \leq \perp$ and hence, by the cyclicity of $\perp$, we have, for any cyclic permutation σ,

$$I \otimes {}^{\perp}a_{\sigma(1)} \otimes \cdots \otimes {}^{\perp}a_{\sigma(n)} \leq \perp$$

and hence

$$I \leq {}^{\perp}({}^{\perp}a_{\sigma(1)} \otimes \cdots \otimes {}^{\perp}a_{\sigma(n)});$$

that is, $\Rightarrow A_{\sigma(1)}, \ldots, A_{\sigma(n)}$ is valid.

The soundness of ?-exchange and κ-exchange follow immediately from the centrality of ! and k (and the fact that $^\perp(-)$ is an involution).

For $\wedge$, observe that by duality $⅋$ distributes over $\wedge$ (and indeed over $\bigwedge$). The soundness of $\vee$ 1 and $\vee$ 2 follows from the observation that $\vee$ is inflationary when either variable is fixed.

For $\perp$, note that $\perp$ is the dual of I, and thus is the identity for $⅋$. The soundness of $⅋$ is immediate from the definition of validity in the interpretation.

For the soundness of $\otimes$, note that the validity of $\Rightarrow \Phi, A$ (resp. $\Rightarrow B, \Psi$) is given by

$$I \leq \bigvee\{x \mid x \otimes {}^\perp\phi \otimes {}^\perp a \leq \perp\} \leq \bigvee\{x \mid x \otimes {}^\perp\phi \leq a\}$$

(resp. $I \leq \bigvee\{y \mid y \otimes {}^\perp\psi \leq b\}$). Thus we have ${}^\perp\phi \leq a$ and ${}^\perp\psi \leq b$, and so, by monotonicity of $\otimes$ in each variable, ${}^\perp\phi \otimes {}^\perp\psi \leq a \otimes b$. Thus we have

$$\begin{aligned} I &\leq \bigvee\{z \mid z \otimes {}^\perp\phi \otimes {}^\perp\psi \leq a \otimes b\} \leq \bigvee\{z \mid z \otimes {}^\perp\phi \otimes {}^\perp\psi \otimes {}^\perp(a \otimes b) \leq \perp\} \\ &= \phi ⅋ \psi ⅋ (a \otimes b). \end{aligned}$$

That is, $\Rightarrow \Phi, \Psi, \otimes AB$ is valid.

The soundness of κ-dereliction and κ?-dereliction follow from the fact that κ is inflationary, $k(a) \geq !(a)$, and $^\perp(-)$ is antitone.

Weakening is sound since $!(a) \leq I$. Contraction is sound because the $\otimes$-idempotence of $!({}^\perp a)$ implies the $⅋$-idempotence of $?(a)$.

$\bigvee$ is sound since the valid elements form a filter, while $\bigwedge$ is sound since the filter is principal (and thus closed under $\bigwedge$).

For !, suppose that $\Rightarrow A, ?\Phi$ is valid; that is,

$$I \leq \bigvee\{c \mid {}^\perp a \otimes {}^\perp ?\phi \otimes c \leq \perp\} \leq \bigvee\{c \mid {}^\perp ?\phi \otimes c \leq a\}.$$

Thus, $!{}^\perp\phi = {}^\perp ?\phi \leq a$. But by elementary properties of coclosure operations, this implies that ${}^\perp ?\phi \leq !a$, and therefore that $I \leq \bigvee\{c \mid {}^\perp ?\phi \otimes c \leq !a\}$, and thus that $\Rightarrow !A, ?\Phi$ is valid.

The proof of the soundness of k is similar. $\square$

Note in particular that Theorem 3.3 insures that a function from elf($\mathbb{L}$) to a Girard quantale provides a sound interpretation for cyclic linear logic, when k and ! are interpreted as c_m and $!_m$ respectively.

For completeness, however, it does not appear to be possible to restrict attention to the maximal modalities. We will comment further on this after proving:

THEOREM 3.4. *The quantale semantics for cyclic* (*resp. commutative*) *linear logic is complete with respect to the cyclic* (*resp. commutative*) *linear sequent calculus.*

PROOF. As in Girard [4], we construct the necessary interpretation from sets of formulae and the notion of provability in the sequent calculus:

Let $\mathbb{L}$ be a language for the cyclic linear predicate calculus. Let $\mathbf{M}$ be the monoid of lists of elements of form($\mathbb{L}$). In the following the phrase "is provable" means "is provable in the cyclic linear sequent calculus".

Consider the quantale $\mathbf{P}(\mathbf{M})$, whose elements are the subsets of $\mathbf{M}$, with $\bigvee$ as set-theoretic union, and $\otimes$ given by

$$A \otimes B = \{\Phi\Psi \mid \Phi \in A, \Psi \in B\}.$$

Let $\perp = \{\Phi \mid \Rightarrow \Phi \text{ is provable}\}$.

From the cyclicity rule of the sequent calculus, it follows that $\perp$ is a cyclic element of $\mathbf{P}(\mathbf{M})$, and thus by Proposition 1.7 that $(- \to \perp) \to \perp = j(-)$ is a quantic nucleus and that the j-closed elements form a Girard quantale, $\mathbf{Q}$ with dualizing element $\perp$.

We now construct a $\mathbf{Q}$-valued interpretation of $\mathbb{L}$ in which validity is equivalent to provability in the sequent calculus. Following Girard [4], we define a map $\mathrm{Pr}\colon \mathrm{form}(\mathbb{L}) \to \mathbf{P}(\mathbf{M})$ by setting $\mathrm{Pr}(A) = \{\Phi \mid \Rightarrow A\Phi \text{ is provable}\}$.

The following three lemmas then complete the proof.

LEMMA 3.4.1. $\mathrm{Pr}\colon \mathrm{form}(\mathbb{L}) \to \mathbf{P}(\mathbf{M})$ factors through $\mathbf{Q}$.

LEMMA 3.4.2. *For any formula A, $\mathrm{Pr}(A)$ is the interpretation of A in the $\mathbf{Q}$-valued interpretation of $\mathbb{L}$ defined by the restriction of* Pr *to* elf($\mathbb{L}$), *and the interpretation of modal operators by*

$$!(x) = \bigvee\{\mathrm{Pr}(!A) \mid \mathrm{Pr}(!A) \le x\} \quad \textit{and} \quad k(x) = \bigvee\{\mathrm{Pr}(kA) \mid \mathrm{Pr}(kA) \le x\}.$$

LEMMA 3.4.3. $\mathrm{Pr}(A) \ge I$ *in* $\mathbf{Q}$ *if and only if* $\Rightarrow A$ *is provable.*

PROOFS OF THE LEMMAS. The proof of Lemma 3.4.1 is essentially the proof in Girard [4] that the $\mathrm{Pr}(A)$'s are "facts"; the only modification necessary is that Cycling must be applied before Cut can be used. $\square$ (3.4.1)

For Lemma 3.4.3, note that

$$I = {}^{\perp}\perp = \{\Phi \mid \Rightarrow \Psi \text{ provable implies} \Rightarrow \Phi\Psi \text{ provable}\}$$

and thus $\phi \in I$. Thus if $\mathrm{Pr}(A) \ge I$, we have $\phi \in \mathrm{Pr}(A)$; that is, $\Rightarrow A$ is provable. Conversely, note that

$$I = \bigwedge\{\mathrm{Pr}(A) \mid \Rightarrow A \text{ is provable}\}$$

and thus $\Rightarrow A$ provable implies $\mathrm{Pr}(A) \ge 1$. $\square$ (3.4.3)

Finally, the proof of Lemma 3.4.2 for the subexponential fragment of $\mathbb{L}$ can be copied line by line from Girard [4]. For the modal operators, we must verify not only that $!(\mathrm{Pr}(A)) = \mathrm{Pr}(!A)$ and $k(\mathrm{Pr}(A)) = \mathrm{Pr}(kA)$, but also that the operators defined satisfy the definition of open modalities, and the requirements of the quantale semantics.

We begin by showing that ! as defined is an open modality on $\mathbf{Q}$. It is immediate that ! as defined satisfies M1, M2, and M3. For M4:

$$\begin{aligned} !(!a \otimes !b) &= \bigvee\{\mathrm{Pr}(!A) \mid \mathrm{Pr}(!A) \le \bigvee\{\mathrm{Pr}(!X) \mid \mathrm{Pr}(!X) \le a\} \\ &\qquad\qquad \otimes \bigvee\{\mathrm{Pr}(!Y) \mid \mathrm{Pr}(!Y) \le b\}\} \\ &= \bigvee\{\mathrm{Pr}(!A) \mid \mathrm{Pr}(!A) \le \bigvee\{\mathrm{Pr}(!X) \otimes \mathrm{Pr}(!Y) \mid \mathrm{Pr}(!X) \le a, \mathrm{Pr}(!Y) \le b\}\}. \end{aligned}$$

Thus it suffices to show that $\mathrm{Pr}(!X) \otimes \mathrm{Pr}(!Y) = \mathrm{Pr}(!Z)$ for some formula Z, since in this case the right-hand side becomes $!a \otimes !b$.

But $\mathrm{Pr}(!X) \otimes \mathrm{Pr}(!Y) = \mathrm{Pr}(\otimes !X!Y)$ by one of the induction steps in the proof for the subexponential fragment. We claim $\mathrm{Pr}(\otimes !X!Y) = \mathrm{Pr}(!(\otimes !X!Y))$. To see this, given a proof of $\Rightarrow \otimes !X!Y, \Phi$, there must be an application of the rule $\otimes$ to some pair of sequents $\Rightarrow ?\Phi_1, !X$ and $\Rightarrow !Y, ?\Phi_2$ to obtain $\Rightarrow \otimes !X!Y, ?\Phi_1, ?\Phi_2$; adding an application of ! at this point and then completing the proof as before gives a proof of $\Rightarrow !(\otimes !X!Y), \Phi$. Conversely, any proof of this latter must contain such an application of !; deleting it gives a proof of $\Rightarrow \otimes !X!Y, \Phi$.

Now we show that the dual closed modality is given by $?x = \bigwedge\{\mathrm{Pr}(?A) \mid \mathrm{Pr}(?A) \geq x\}$. That is, we wish to show that $?({}^{\perp}x) = {}^{\perp}!(x)$. Now:

$$\begin{aligned} ?({}^{\perp}x) &= \bigwedge\{\mathrm{Pr}(?A) \mid \mathrm{Pr}(?A) \geq {}^{\perp}x\} = \bigwedge\{\mathrm{Pr}(?A) \mid {}^{\perp}\mathrm{Pr}(?A) \leq x\} \\ &= \bigwedge\{\mathrm{Pr}(?A) \mid \mathrm{Pr}({}^{\perp}?A) \leq x\} = \bigwedge\{\mathrm{Pr}(?A) \mid \mathrm{Pr}(!{}^{\perp}A) \leq x\} \\ &= \bigwedge\{\mathrm{Pr}(?{}^{\perp}A) \mid \mathrm{Pr}(!A) \leq x\} = \bigwedge\{\mathrm{Pr}({}^{\perp}!A) \mid \mathrm{Pr}(!A) \leq x\} \\ &= \bigwedge\{{}^{\perp}\mathrm{Pr}(!A) \mid \mathrm{Pr}(!A) \leq x\} = {}^{\perp}\bigvee\{\mathrm{Pr}(!A) \mid \mathrm{Pr}(!A) \leq x\} = {}^{\perp}!x, \end{aligned}$$

each step being justified by duality in the quantale, duality of formulas in $\mathbb{L}$, or the observation that ${}^{\perp}\mathrm{Pr}(A) = \mathrm{Pr}({}^{\perp}A)$ from the proof of 3.4.1.

For the remaining verifications, it is most convenient to verify the dual statement concerning ?. We see that ! is weak by verifying $?x \geq \perp$: it suffices to show that for all A, $\mathrm{Pr}(?A) \geq \perp = \mathrm{Pr}(\perp)$. But any proof of $\Rightarrow \perp, \Phi$ must contain a step at which the $\perp$-rule is invoked; replacing this with an application of Weakening gives a proof of $\Rightarrow ?A, \Phi$.

For idempotence:

$$\begin{aligned} ?x \parr ?x &= \bigwedge\{\mathrm{Pr}(?A) \mid \mathrm{Pr}(?A) \geq x\} \parr \bigwedge\{\mathrm{Pr}(?A) \mid \mathrm{Pr}(?A) \geq x\} \\ &= \bigwedge\{\mathrm{Pr}(?A) \parr \mathrm{Pr}(?B) \mid \mathrm{Pr}(?A) \geq x, \mathrm{Pr}(?B) \geq x\} \quad \text{(by dual distributivity)} \\ &= \bigwedge\{\mathrm{Pr}(?A) \mid \mathrm{Pr}(?A) \geq x\} = ?x. \end{aligned}$$

This last holds since $\mathrm{Pr}(?B) \geq \perp$; $\perp$ is the identity for $\parr$; and $\mathrm{Pr}(?A)$ is $\parr$-idempotent by applications of contraction and weakening, and the fact that $\mathrm{Pr}(A \parr B) = \mathrm{Pr}(A) \parr \mathrm{Pr}(B)$ from the subexponential fragment.

Centrality follows from dual distributivity, the ?-exchange rule, the fact that $\mathrm{Pr}(\parr AB) = \mathrm{Pr}(A) \parr \mathrm{Pr}(B)$, and the observation that any $y \in \mathbf{Q}$ is of the form $y = \bigwedge\{\mathrm{Pr}(A) \mid A \in {}^{\perp}y\}$. Verifications for k are similar.

Finally we must show that $\mathrm{Pr}(!A) = !\mathrm{Pr}(A)$ and $\mathrm{Pr}(kA) = k\mathrm{Pr}(A)$. Again we work with the dual assertion. By the definition of $?x$ and the dereliction rules, it suffices to show that if $\mathrm{Pr}(?B) \geq \mathrm{Pr}(A)$, then $\mathrm{Pr}(?B) \geq \mathrm{Pr}(?A)$. Now, given a proof of $\Rightarrow ?A, \Phi$, if $?A$ arises by weakening, we can get a proof of $\Rightarrow ?B, \Phi$ by using a different instance of weakening. If, on the contrary, it arises by dereliction, it follows from $\mathrm{Pr}(?B) \geq \mathrm{Pr}(A)$ that we can find a proof of $\Rightarrow ?B, \Phi$. To handle contraction, work by induction on the number of contractions used to get the instance of $?A$ in $\Rightarrow ?A, \Phi$. We have done the cases with none. Now suppose we have replacements for all proofs with fewer than n contractions. Given a proof with n contractions, consider the step where the last contraction is applied: immediately before this we have $\Rightarrow ?A, ?A, \Psi$. By the induction hypothesis, we can replace this by a proof of $\Rightarrow ?B, ?A, \Psi$, then apply cycling to get a proof of $\Rightarrow ?A, \Psi, ?B$. Applying the induction hypothesis and cycling again gives a proof of $\Rightarrow ?B, ?B, \Psi$. Completing the proof as before (but this time contracting on $?B$) gives the desired proof of $\Rightarrow ?B, \Phi$.

Again the case of k is similar. □ (3.4.2) □ (3.4)

There is still a question of whether this result can be improved: is the semantics given by maps from elf($\mathbb{L}$) to Girard quantales with ! and k interpreted by $!_{\mathrm{m}}$ and c_{m} complete?

When it is observed that $\mathrm{Pr}(??A) = \mathrm{Pr}(\kappa?A) = \mathrm{Pr}(?)$, it is clear that this question can be reduced to the question of whether the quantale $\mathbf{Q}$ constructed in the proof of Theorem 3.4 can be embedded (by a map of Girard quantales) into a Girard quantale $\mathbf{Q}'$ in such a way that $k(\mathbf{Q})$ is the intersection of $\mathbf{Q}$ with the $\otimes$-center of $\mathbf{Q}'$. This problem appears to be quite difficult. (To get some idea of the difficulty, an analogous question in ring theory is to ask when a ring can be embedded into a larger ring so that a prescribed subring of the center is the intersection of the given ring with the center of the larger one.)

Before going on to consider the proof system associated to cyclic linear logic, we should note that since quantales have all sup's and inf's, not just those required for the definition of first-order quantifiers, it is clearly possible to formulate a quantale semantics for higher-order extensions of cyclic linear logic, once this has been correctly formulated.

§4. Proofs as drawings. Rather than adapting Girard's proof-nets to cyclic linear logic, we prefer to "geometrize" proofs in the sequent calculus for cyclic linear logic using a less radical approach, which, while retaining the flavor of the sequent calculus, is sufficient for "the removal of taxonomy".

The approach is suggested by the cyclicity rule of the sequent calculus: since validity of sequents is invariant under cyclic permutation of the formulas of the sequent, it is convenient to think of the (left-hand side of the) sequent as an arrangement of formulas on a circle, say oriented clockwise:

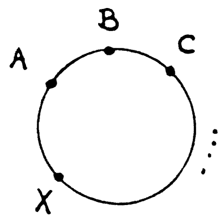

FIGURE 4.1

Or, better still, giving the formula a "duration" as an arrangement of radial lines (labeled by formulae) on an annulus.

Thinking in this way, it is possible to associate to each axiom a drawing on a disk, to each rule which replaces one sequent with another, a drawing on an annulus, to the cut rule and the $\otimes$-rule drawings on a disk with two holes, and to the $\wedge$-rule a drawing on the singular surface obtained by identifying the outer half of each of two annuli, schematically:

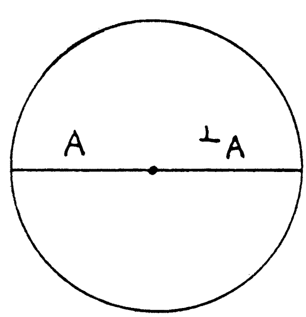

FIGURE 4.2. Axiom 1.

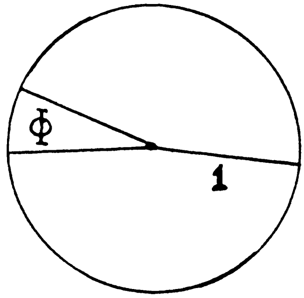

FIGURE 4.3. Axiom 2.

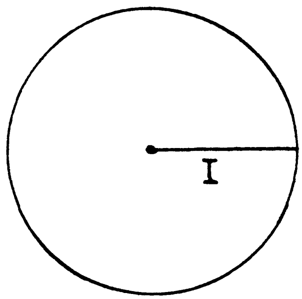

FIGURE 4.4. Axiom 3.

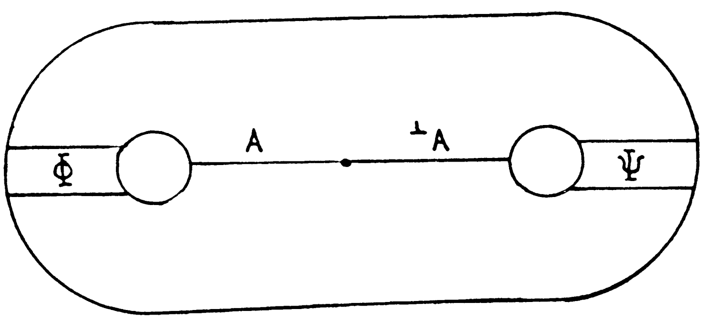

FIGURE 4.5. Cut.

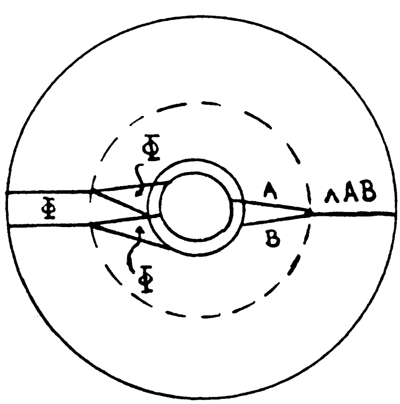

FIGURE 4.6. ∧.

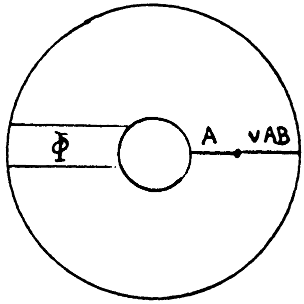

FIGURE 4.7. $\vee 1$.

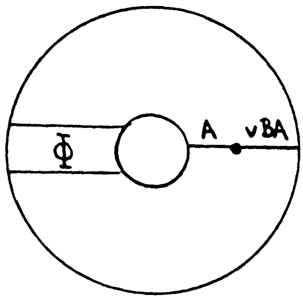

FIGURE 4.8. $\vee 2$.

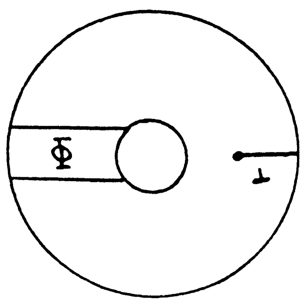

FIGURE 4.9. $\perp$.

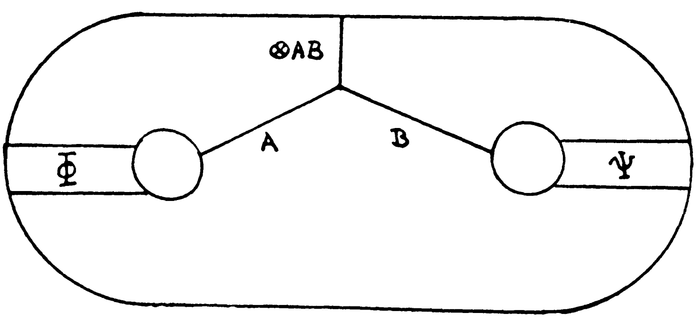

FIGURE 4.10. $\otimes$.

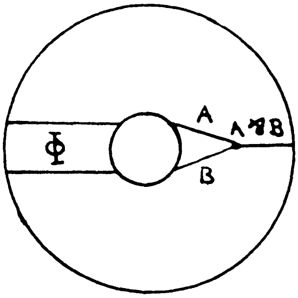

FIGURE 4.11. ⅋

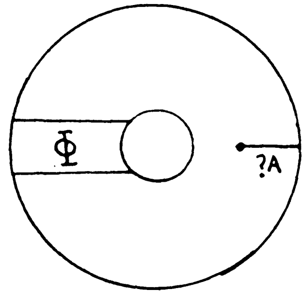

FIGURE 4.12. ?-weakening.

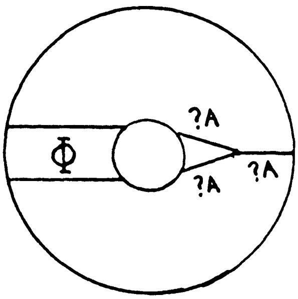

FIGURE 4.13. ?-contraction.

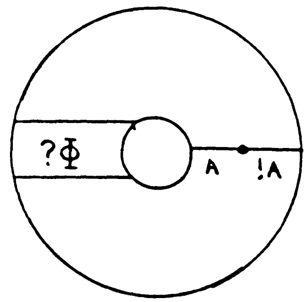

FIGURE 4.14. !.

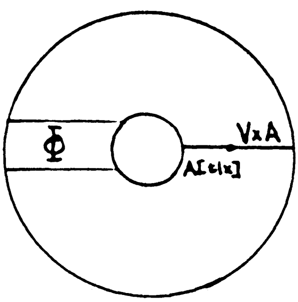

FIGURE 4.15. $\bigvee$.

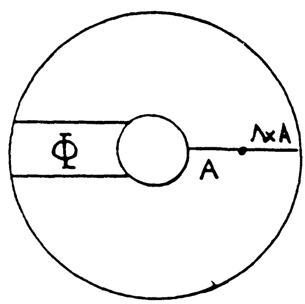

FIGURE 4.16. $\bigwedge$.

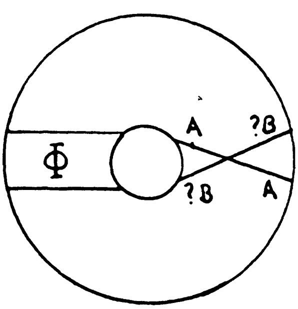

FIGURE 4.17. ?-exchange.

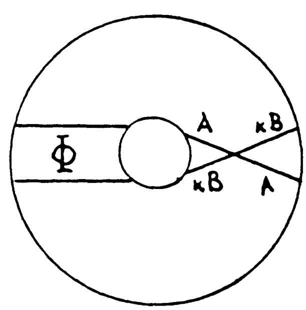

FIGURE 4.18. κ-exchange.

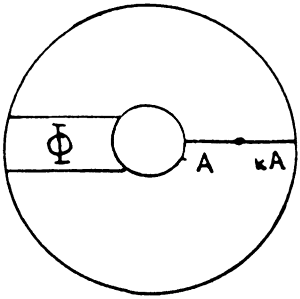

FIGURE 4.19. κ-dereliction.

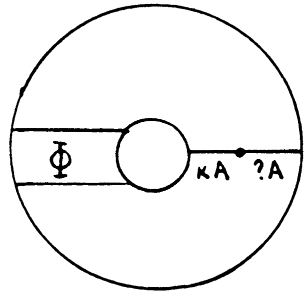

FIGURE 4.20. κ?-dereliction.

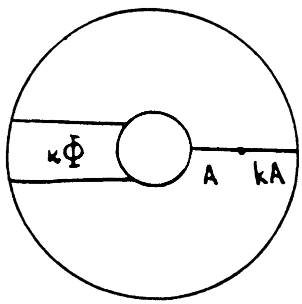

FIGURE 4.21. k.

As in our shorthand convention in the presentation of the sequent calculus, Latin letters stand for formulas, Greek letters for lists of formulas. A modal operator applied to a Greek letter indicates the list obtained by applying the operator to every element of the list. Bands labeled by lists of formulas are schematic for a sequence of radial lines labeled (in clockwise order) by the formulas of the list. Solid circles indicate edges of the underlying surface, while the dotted circle in $\wedge$ indicates the singular locus of the underlying surface. Moreover, the same restrictions apply to $\perp$ and $\bigwedge$ as in the sequent calculus.

The reader will note that no drawing is given for cycling. This is because the drawings are to be regarded as labeled *topological* structures, defined only up to orientation-preserving diffeomorphisms of the underlying surface. Cycling is thus no rule at all when considered geometrically!

A proof in the sequent calculus can then be translated as a drawing on a singular surface obtained by identifying disks along outer annuli, and such that the surface with drawing is covered by surfaces with drawings of the forms in Figures 4.2–4.23.

Observe that the entire system is easily handled in one pass without the unpleasantness of "boxes" in Girard's system of "proof-nets". The singular loci of the underlying surface serve as "synchronization markers" as do Girard's boxes.

We can now accomplish cut-elimination on the resulting system by using the system of rewrite rules given by Figures 4.22.1–8.

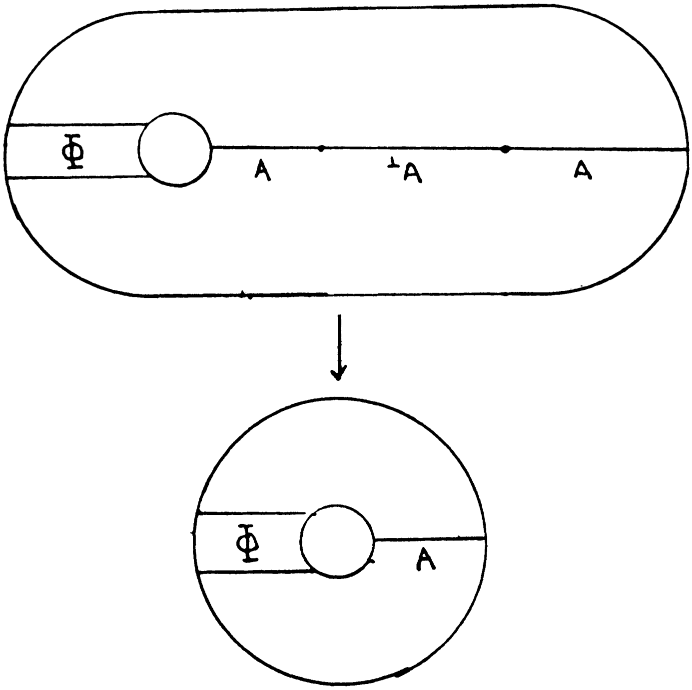

FIGURE 4.22.1. R.1.

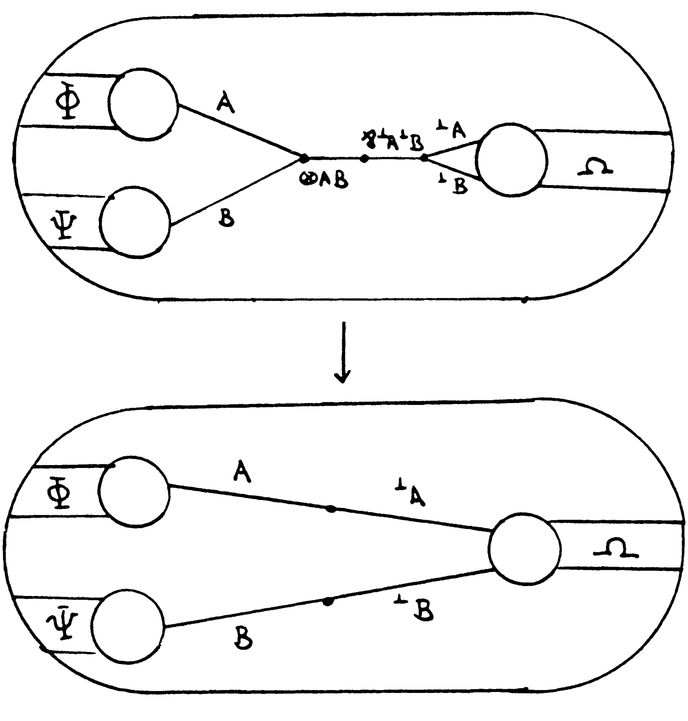

FIGURE 4.22.2. R.2.

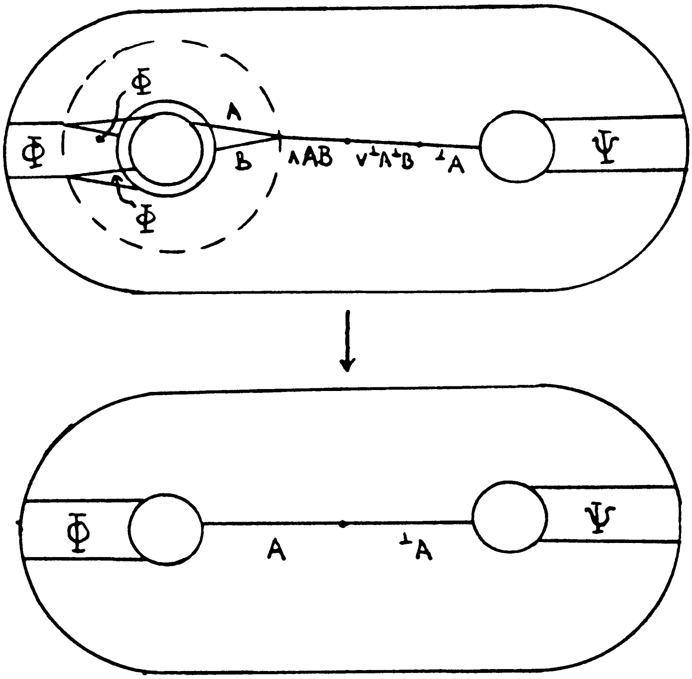

FIGURE 4.22.3. R.3.

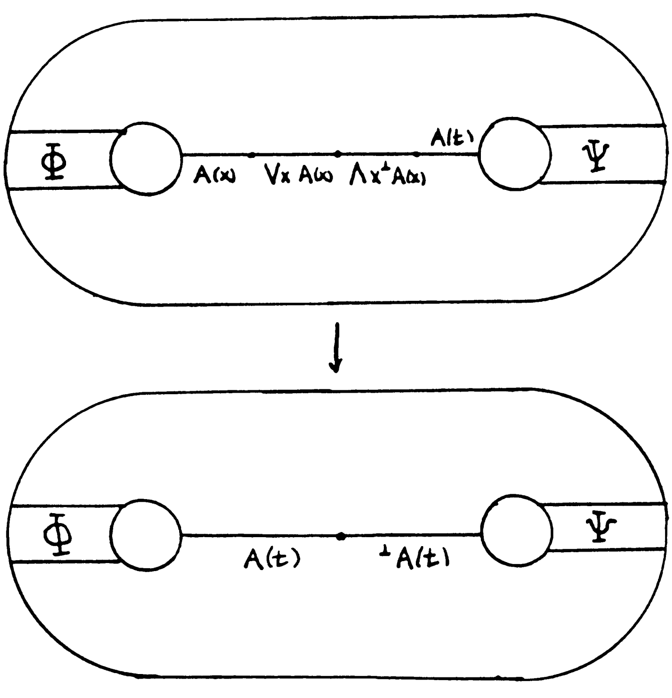

FIGURE 4.22.4. R.4.

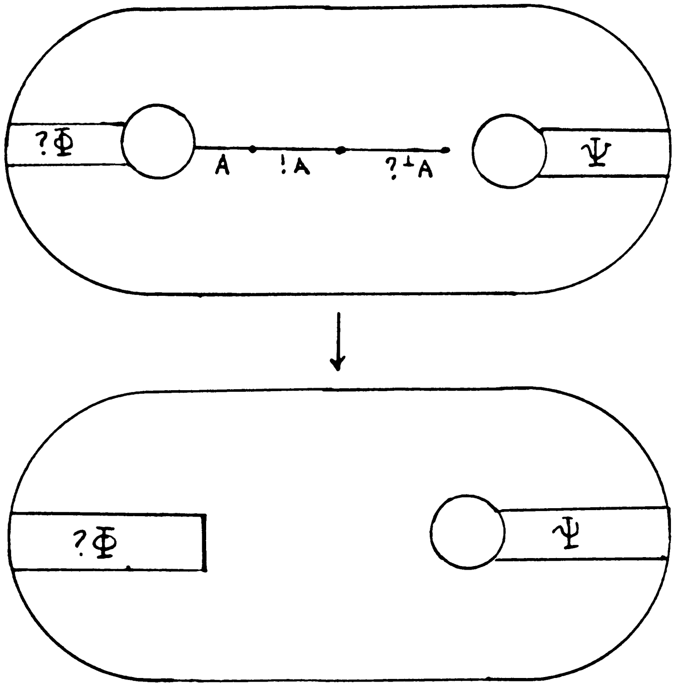

FIGURE 4.22.5. R.5.

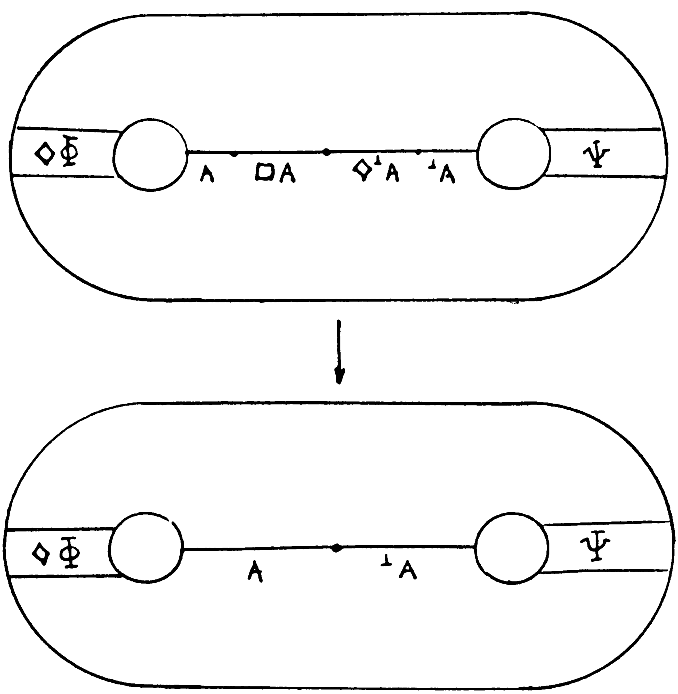

FIGURE 4.22.6. R.6.

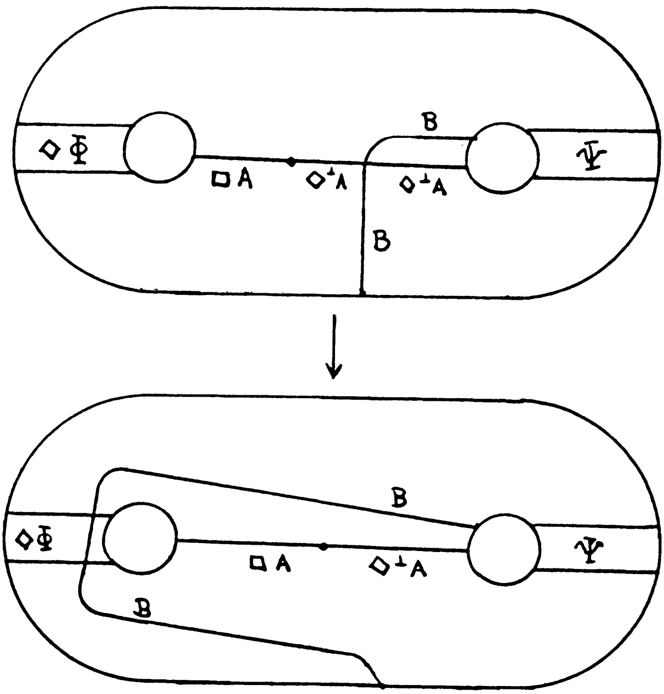

FIGURE 4.22.7. R.7.

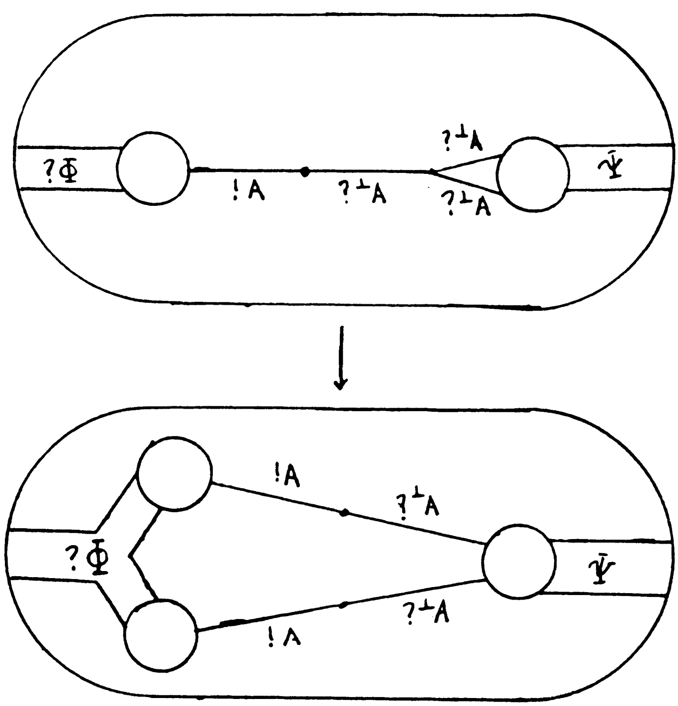

FIGURE 4.22.8. R.8.

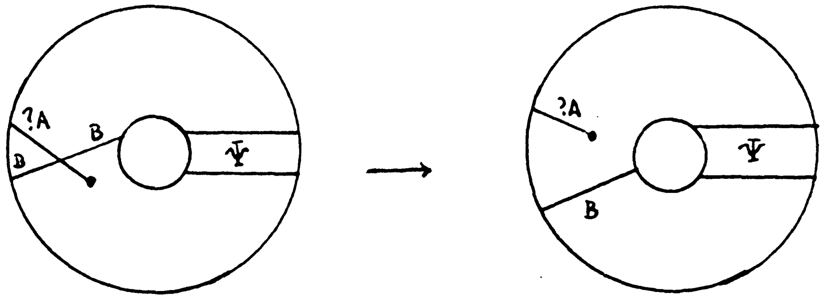

FIGURE 4.22.9. R.9.

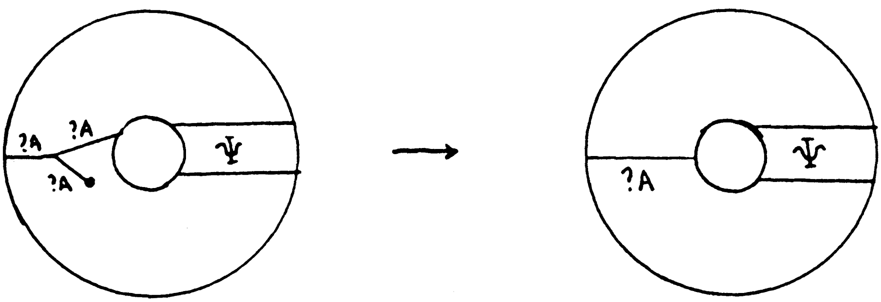

FIGURE 4.22.10. R.10.

In R.5 the abruptly beginning band labeled $?\Phi$ indicates an application of ?-weakening to introduce each of the elements of the list of formulas. In R.6 and R.7, the symbols $\Box$ and $\Diamond$ represent either ! and ? or k and κ. In R.8, the confluence of the two bands labeled $?\Phi$ indicates uses of ?-exchange to arrange corresponding copies of each formula next to each other, followed by use of contraction on each pair; and the part of the drawing inside the left-hand hole in the schematic is duplicated in the two left-hand holes of the rewritten drawing.

Using this rewrite system, one can then prove

THEOREM 4.1. *Any sequent provable in the cyclic linear sequent calculus has a cut-free proof.*

PROOF. Since any drawing involving the cut rule has one of R.1–R.10 applicable to it, this is implied by the well-foundedness of the rewrite system.

We leave it to the reader to devise his or her own ordinal-valued complexity measure on drawings which is decreased by all of the rules. (Hint: define complexities for the occurrences of each formula at a cut in terms of the rules lying above them, and use multiplication by an infinite ordinal when a contraction occurs above the cut.) $\Box$

Indeed it is possible to do better than this—the rewrite system used for cut-elimination is almost Church-Rosser. To be more precise, we make

DEFINITION 4.2. A *graphical proof* is an equivalence class of proofs as surfaces with drawings under the equivalence relation generated by orientation-preserving diffeomorphisms of the underlying surfaces, together with all moves of the form

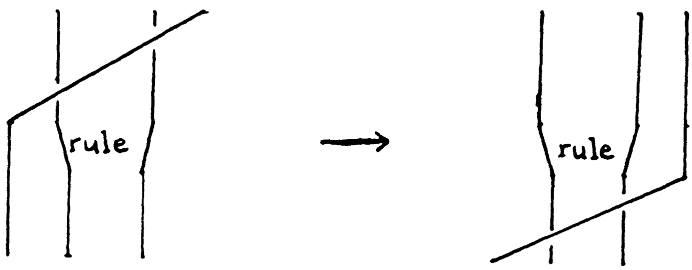

FIGURE 4.23. GP

where the band indicates any portion of a drawing with a rule of the sequent calculus occurring at the label, and the commutations indicated on both sides of the drawings are permitted.

The reader familiar with recent work relating coherence problems in category theory with low-dimensional topology (cf. [2], [3], and [7]) will recognize GP as a naturality condition. Indeed the deductive system associated to a pivotal category [3], [7] provides a semantics for cyclic linear logic in which $\perp = I$; the addition of a braiding or symmetry gives a semantics for commutative linear logic.

It is now a relatively easy (but tedious) task to show

THEOREM 4.3. *The rewrite system on graphical proofs defined by rules* R.1–R.10 *is Church-Rosser.*

Sketch of Proof. As in the word-theoretic case (cf. Knuth and Bendix [8]), it suffices to verify local confluence for "superpositions". The only superpositions occur when one of the "holes" in the subsurface for one possible rule application coincides with one of the "holes" for another—schematically,

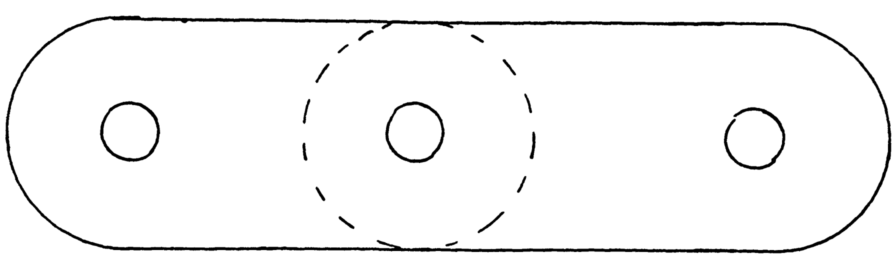

FIGURE 4.24.

For local confluence to fail, one of the rules would have to change the proof above the hole for one of the formulas involved in the other cut. Thus if one of the rules is R.1–R.4, or R.6, or the common hole is the right-hand hole of R.5, local confluence must hold. The remaining cases are left to the reader. It is only in some cases involving R.7 that the move GP is needed. □

We thus have strong normalization for graphical proofs, which can be translated into a kind of normalization result for the sequent calculus. The failure of uniqueness for cut-free normal forms is due entirely to the "naturality of commutativity" (captured geometrically as GP), and the lack of canonical sequentializations for the parallel portions of proofs (captured geometrically by the diffeomorphisms of the underlying surface).

§5. Interpretations of other logics in cyclic linear logic. We conclude by briefly noting interpretations from intuitionistic logic and commutative linear logic to cyclic linear logic.

For intutionistic logic, Girard's interpretation works equally well with cyclic linear logic. The crucial thing to note is that $!A$ is $\otimes$-central, so that, for any B,

$$!A \to_r B = !A \to_l B.$$

The proof of the following can then be copied from [4].

THEOREM 5.1. *The function* $\#(\ )$ *from formulas of intuitionistic first order logic to formulas of cyclic linear logic provides an interpretation which both preserves and reflects provability and extends to an inclusion of the set of proofs in first order intuitionistic logic into the set of proofs in the cyclic linear logic, where* $\#(\ )$ *is defined recursively by*:

$$\begin{aligned}
\#(A) &= A, \quad \textit{whenever } A \textit{ is atomic},\\
\#(A \wedge B) &= \#A \wedge \#B,\\
\#(A \vee B) &= !(\#A) \vee !(\#B),\\
\#(A \Rightarrow B) &= {}^{\perp}(!(\#A) \otimes (\#B)),\\
\#(\neg A) &= {}^{\perp}(!(\#A) \otimes \mathbf{0}),\\
\#(\perp) &= \mathbf{0},\\
\#(\top) &= \mathbf{1},\\
\#(\forall x.A) &= \textstyle\bigwedge x.\#(A),\\
\#(\exists x.A) &= \textstyle\bigvee x.\#(A).
\end{aligned}$$

The interpretation of commutative linear logic in cyclic linear logic is a rather dull modal interpretation: simply apply κ to everything in sight.

THEOREM 5.2. *The function* $C(\ )$ *from formulas of commutative linear logic to formulas of cyclic linear logic provides an interpretation which both preserves and reflects provability and extends to an inclusion of the set of proofs in commutative linear logic into the set of proofs in cyclic linear logic, where* $C(\ )$ *is defined recursively by*:

$$\begin{aligned}
C(A) &= \kappa A, \quad \textit{if } A \textit{ is atomic},\\
C(A * B) &= \kappa(C(A) * C(B)), \quad \textit{if } * \textit{ is any binary connective of commutative linear logic},\\
C(*A) &= \kappa(*C(A)), \quad \textit{if } * \textit{ is either } {}^{\perp}, \textit{ or } !, \textit{ or a quantifier followed by a variable},\\
C(?A) &= ?(C(A)).
\end{aligned}$$

PROOF. Given a proof of $\Rightarrow A_1, \dots, A_n$ in commutative linear logic, we can replace it by a proof of $\Rightarrow C(A_1), \dots, C(A_n)$ in cyclic linear logic by replacing all instances of introduction rules with the same rule followed by κ-dereliction, replacing Dereliction by κ-dereliction followed by κ?-dereliction, leaving instances of Weakening and Contraction unchanged, and replacing instances of Exchange by instances of κ- or ?-exchange as needed.

Conversely, observe that a proof of a formula of the form $C(A)$ must have an instance of κ-dereliction following all introduction rules, and thus the translation of the previous paragraph can be reversed.

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