"Basic DE's: Separation of Variables"

$$dy/dx = 2y + 1 \tag{1}$$
$$\frac{dy}{2y+1} = dx \tag{2}$$
$$\int_a^b \frac{1}{2y+1} dy = \int_a^b dx \tag{3}$$
$$\tag{4}$$

$u := 2y + 1, f(u) := \frac{1}{2y+1}$

$$\therefore \frac{dy}{du} \int_a^b \frac{1}{u} \frac{du}{dy} dy = \int \frac{1}{u} \frac{du}{dy} \frac{dy}{du} dy = \int \frac{1}{u} dy = \int_a^b \frac{1}{2y+1} dy = \int_a^b dx \tag{5}$$
$$\frac{dy}{du} \int_a^b \frac{1}{u} \frac{du}{dy} dy = \int_a^b dx \tag{6}$$
$$\tag{7}$$

$\because \int (f \circ g) g' = \int f$
$\therefore \int f(g(y)) g'(y) dy = \int f(u) u' dy = \int f(u) \frac{du}{dy} dy = \int f(u) du$

$$\frac{dy}{du} \int_a^b \frac{1}{u} \frac{du}{dy} dy = \frac{dy}{du} \int_a^b f(u) du \tag{8}$$
$$= \frac{dy}{du} \int_a^b \frac{1}{u} du \tag{9}$$
$$\tag{10}$$

$$\therefore \frac{dy}{du} \int_a^b \frac{1}{u} \frac{du}{dy} dy = \frac{dy}{du} \int_a^b \frac{1}{u} du = \int_a^b dx \tag{11}$$
$$\frac{dy}{du} \int_a^b \frac{1}{u} du = \int_a^b dx \tag{12}$$

$\because u = 2y + 1 \therefore \frac{du}{dy} = 2$
$\because \frac{dy}{du} = (\frac{du}{dy})^{-1} = 2^{-1} = \frac{1}{2}$

$$\frac{1}{2}\int_a^b \frac{1}{u}du = \frac{dy}{du}\int_a^b \frac{1}{u}du = \int_a^b dx \tag{13}$$

$$\frac{1}{2}\int_a^b \frac{1}{u}du = \int_a^b dx \tag{14}$$

$$\frac{1}{2}\ln|u| + c_1 = x + c_2 \tag{15}$$

$$\frac{1}{2}\ln|u| = x + c_2 - c_1 \tag{16}$$

$$\ln|u| = 2x + 2(c_2 - c_1) \tag{17}$$

$$\ln|u| = 2x + c_3 \tag{18}$$

$$\ln|2y+1| = 2x + c_3 \because u = 2y+1 \tag{19}$$

$$\because e^{\ln|2y+1|} = e^{2x+c_3} \tag{20}$$

$$\therefore 2y+1 = e^{2x}e^{c_3} \tag{21}$$

$$2y = e^{2x}e^{c_3} - 1 \tag{22}$$

$$2y = e^{2x}e^{c_3} - 1 \tag{23}$$

$$2y = Ce^{2x} - 1 \tag{24}$$

$$y = \frac{Ce^{2x} - 1}{2} \quad \blacksquare \tag{25}$$