The Visual Computer

Parametric G^n blending of curves and surfaces

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We introduce a simple blending method for parametric curves and surfaces that produces families of parametrically defined, G^n–continuous blending curves and surfaces. The method depends essentially on the parameterizations of the curves/surfaces to be blended. Hence, the flexibility of the method relies on the existence of suitable parameter transformations of the given curves/surfaces. The feasibility of the blending method is shown by several examples. The shape of the blend curve/surface can be changed in a predictable way with the aid of two design parameters (thumb weight and balance).

Key words: G^n Continuity – G^n blending – Parametric blending curves – Parametric blending surfaces – Thumb weight

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1 Introduction

1.1 Parametric blending

The blending of curves and surfaces is an essential task in geometric modeling. It is mostly used for smoothing corners and edges. However, it can also be considered as a technique for the controlled design of new curves and surfaces. The classical method for blending two surfaces generates a suitable pipe surface by a rolling ball, which has G^1–continuous contact (the tangent planes are continuous) to the given surfaces. The literature describes many attempts to simulate this method with free-form surfaces, for example Choi and Ju (1989). One gets more flexibility by varying the radius of the rolling ball (VRRB). Vida et al. (1994) give an extended survey of blending methods that use parametric surfaces. More recent papers that deal with parametric blending are those of Braid (1997), Chuang and Hwang (1997), Farouki and Sverrisson (1996), Hermann et al. (1995), Lukács (1998), Várady and Rockwood (1997), and Wallner and Pottmann (1997).

1.2 Higher continuity

The raising of the continuity to G^n, $n > 1$, within existing methods is difficult and rarely tackled in the literature. For the continuity of the normal curvatures (G^2 continuity) Bloor and Wilson (1989) have to solve partial differential equations of the sixth order, and Filip (1989) needs higher-order derivatives of the surfaces to be blended. However, because of the complexity, no examples are given. By using implicit methods, it is quite easy to achieve greater smoothness. Such methods can be applied to parametric surfaces via the normal form of a surface Hartmann (1998). The result is, in any case, an implicit surface, possibly one that is implicitized numerically. Implicit surfaces are, in general, not popular with users because of their disadvantages. For example, determining surface points or curves is not trivial.

1.3 Aim of the paper

The aim of this paper is to introduce a rather simple blending method that produces G^n–continuous parametric blend curves and surfaces, given two design parameters with which users can manipulate the solution. The blending *curves* are just linear combinations of the given curves (base curves) with suitably chosen coefficients (blending functions). The blending *surfaces* are linear combinations of the base surfaces with blending functions depending on one of the common parameters (not to be confused with Coons patches – see Sect. 3.1). The examples used as blending functions are rational, and their degree determines the smoothness. The shape of the blending curve or surface can be modified by two parameters: the *balance* contained in the blending functions and the *thumb weight* contained in rational parameter transformations. Besides the simplicity of the method, there is a further advantage: the blending curves or surfaces are *rational* if the base surfaces are rational.

The paper is organized as follows. Section 2 repeats the definitions of C^n and G^n continuity, introduces the new method to planar and spatial curves, and gives some illustrative examples showing the effect of the design parameters. Section 3 deals with the extension to surfaces and gives numerous examples. An essential assumption for applying the method is the existence of suitable parameterizations of the patches to be blended. The examples show that simple parameter transformations are sufficient in many cases. For more general cases, a numerical parameterization is needed and is applied in examples 8 and 9; this is based on Hartmann's (2000) results. The new parametric blending method is applicable to implicit or more general surfaces as well.

2 G^n blending of parametric curves

2.1 G^n–continuous curves

A C^n–continuous curve Γ considered as a set of points can be described by several parametric representations. The essentials of a representation are its derivatives, with which geometrical attributes of the curve such as tangent line, curvature, etc., can be determined. Such geometrical attributes are independent of the representation, that it is necessary to distinguish between the continuity of the representations (C^n continuity) and the continuity of the geometric attributes (G^n continuity):

Definition. Two regular C^n–continuous curves $\Gamma_1 : \boldsymbol{x} = \gamma_1(s)$, $\Gamma_2 : \boldsymbol{x} = \gamma_2(t)$, have C^n *contact* at a common point $\boldsymbol{p}$ if the derivatives $\gamma_1^{(k)}, \gamma_2^{(k)}, k = 0, 1, \ldots n$ at point $\boldsymbol{p}$ are the same.

Definition. Two regular C^n–continuous curves Γ_1, Γ_2 have G^n *contact* at a common point $\boldsymbol{p}$ if they have C^n contact at point $\boldsymbol{p}$ with respect to some local reparameterizations of the two curves.

For further discussions and criteria for G^n continuity, see Mazure (1994).

Definition. A regular C^n–continuous curve Γ that has G^n contact to two regular C^n–continuous curves Γ_1 and Γ_2 at points $\boldsymbol{p}_1 \in \Gamma_1$ and $\boldsymbol{p}_2 \in \Gamma_2$, respectively, is called a G^n *blending curve* of Γ_1 and Γ_2.

2.2 The blending method

A simple G^0 blending method for curves that suggests itself works as follows:

Let $\Gamma_1 : \boldsymbol{x} = \boldsymbol{\gamma}_1(t),\ t \in [0, 1]$ and $\Gamma_2 : \boldsymbol{x} = \boldsymbol{\gamma}_2(t),\ t \in [0, 1]$ be two C^n–continuous planar or spatial curves, then $\Gamma : \boldsymbol{x} = (1-t)\boldsymbol{\gamma}_1(t) + t\boldsymbol{\gamma}_2(t),\ t \in [0, 1]$ is a G^0 blending curve of Γ_1 and Γ_2 with $\gamma_1(0)$ and $\gamma_2(1)$ as points of contact. In order to smooth the contact of Γ to the curves Γ_1 and Γ_2, we generalize this simple blending method by replacing the linear function $1-t$ with a suitable function $f(t)$.

Theorem 1. *Let $\Gamma_1 : \boldsymbol{x} = \boldsymbol{\gamma}_1(t),\ t \in [0, 1]$ and $\Gamma_2 : \boldsymbol{x} = \boldsymbol{\gamma}_2(t),\ t \in [0, 1]$ be two regular C^n continuous planar or spatial curves (base curves), and let $f(t), t \in [0, 1]$ be a C^n–continuous real function (blending function) with the following properties:*

$$f(0) = 1,\ \ f(1) = 0,$$
$$f^{(k)}(0) = f^{(k)}(1) = 0 \ \text{for}\ k = 1, \ldots, n.$$

Then the curve (blending curve)

$$\Gamma : \boldsymbol{x} = \gamma(t) := f(t)\boldsymbol{\gamma}_1(t) + (1 - f(t))\boldsymbol{\gamma}_2(t),\ \ t \in [0, 1]$$

has C^n contact to the curves Γ_1 and Γ_2 at the points $\gamma_1(0)$ and $\gamma_2(1)$, i.e. $\gamma^{(k)}(0) = \gamma_1^{(k)}(0)$, $\gamma^{(k)}(1) = \gamma_2^{(k)}(1)$ for $k = 0, 1, \ldots, n$.

Proof. The derivatives of γ are:

$$\begin{aligned}
\gamma' &= f'\gamma_1 + f\gamma_1' - f'\gamma_2 + (1-f)\gamma_2', \\
\gamma'' &= f''\gamma_1 + 2f'\gamma_1' + f\gamma_1'' \\
&\quad - f''\gamma_2 - 2f'\gamma_2' + (1-f)\gamma_2'', \\
\gamma''' &= f'''\gamma_1 + 3f''\gamma_1' + 3f'\gamma_1'' + f\gamma_1''' \\
&\quad - f'''\gamma_2 - 3f''\gamma_2' - 3f'\gamma_2'' + (1-f)\gamma_2''', \\
&\quad \ldots
\end{aligned}$$

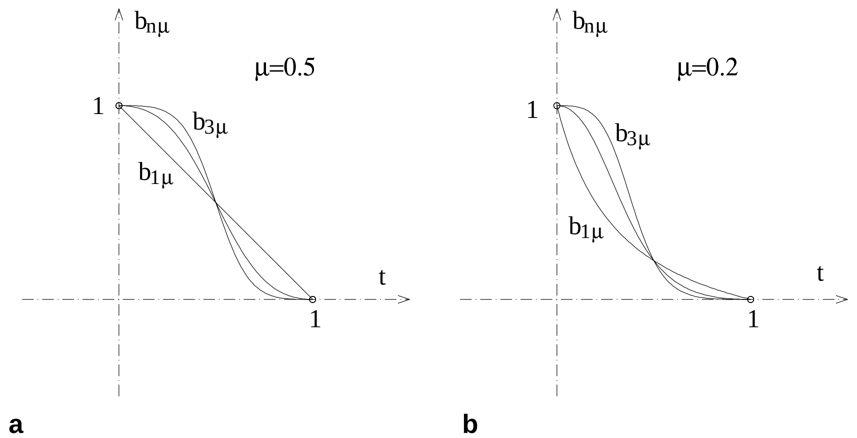


Fig. 1a,b. Graphs of blending functions $b_{n,\mu}$ for $n = 1, 2, 3$ and balance: **a** $\mu = 0.5$; **b** $\mu = 0.2$

From the properties of function f, we get $\gamma^{(k)}(0) = \gamma_1^{(k)}(0), \gamma^{(k)}(1) = \gamma_2^{(k)}(1)$ for $k = 0, 1, \dots, n$. □

Remark. For the application of Theorem 1 to the examples given later, we need parameter transformations to meet the precondition "$t \in [0, 1]$". Thus, the blend curve Γ will have only a G^n– (instead of C^n–) continuous contact with the base curves.

Remark. In CAD, it is a common technique to consider linear combinations of curves (as in Theorem 1) for modification or generation of curves. Further examples are as follows.

- $\Gamma : \boldsymbol{x} = \gamma(t) := (1-\alpha)\gamma_0(t) + \alpha\lambda(t), t \in [0, 1]$, $0 \le \alpha \le 1$.
  Coons (1977) uses this for the smooth modification of the shape of a given curve segment $\Gamma_0 : \boldsymbol{x} = \gamma_0(t)$. $\lambda(t) = \gamma_0(0) + (6t^5 - 15t^4 + 10t^3)(\gamma_0(1) - \gamma_0(0))$. This is the nonlinear parametrized *chord* between $\gamma_0(0)$ and $\gamma_0(1)$. Γ and Γ_0 have G^2 contact at the common points $\gamma_0(0)$, $\gamma_0(1)$. The G^2 continuity between Γ and Γ_0 can be maintained, while parameter α is allowed to be a C^2 function of t. Coons applies this technique to the modification of a C^2–continuous curve consisting of a sequence of segments, such that the modified curve is C^2 continuous as well, and it has a G^2 contact with the original curve at the boundaries of the segments (Coons 1977; Faux and Pratt 1979).
- $\Gamma : \boldsymbol{x} = (1-t)\boldsymbol{\pi}_1(t) + t\boldsymbol{\pi}_2(t),\ t \in [0, 1]$.
  Overhauser uses this for a cubic C^1 blending curve between two intersecting parabolas $\pi_1(t)$, $\pi_2(t)$ (Creasy and Craggs 1990).

2.3 Blending functions

Though polynomial blending functions exist (see the remarks later in this section) we choose the following *rational* G^n blending functions because they provide a design parameter (μ) for any n. Another important design parameter, the thumb weight (Sect. 2.4), will be available via suitable rational parameter transformations.

Definition. $b_{n,\mu}(t) = \frac{\mu(1-t)^{n+1}}{\mu(1-t)^{n+1} + (1-\mu)t^{n+1}},\ t \in [0, 1]$, $0 < \mu < 1, n \ge 0$.

Figure 1 shows graphs of blending functions $b_{n,\mu}$ for various n and μ. The graphs can be considered as G^n–continuous transition curves between the lines $y = 1$ and $y = 0$. For $\mu = 0.5$, the curves are point symmetric about the point $(0.5, 0.5)$. If we choose $\mu = 0.2$, for example, the curves are asymmetric and closer to the line $y = 0$. We call μ the *balance* of the blending function $b_{n,\mu}$.

Functions $b_{n,\mu}$ fulfill the following necessary conditions.

Lemma. $b_{n,\mu}(0) = 1,\ b_{n,\mu}(1) = 0,\ b^{(k)}_{n,\mu}(0) = b^{(k)}_{n,\mu}(1) = 0$ for $k = 1, \dots, n$.

Proof. Let be $f := b_{n,\mu}$ and $u(t) := \mu(1-t)^{n+1}$, $v(t) := (1-\mu)t^{n+1}$. Functions u, v have the properties:

$$u^{(k)}(0) \neq 0,\ v^{(k)}(0) = 0,$$
$$u^{(k)}(1) = 0,\ v^{(k)}(1) \neq 0, \ \text{for } k = 0, \ldots, n.$$

Differentiating the identity $f(t)(u(t)+v(t)) = u(t)$ yields

$$f'(u+v) + f(u'+v') = u',$$
$$f''(u+v) + 2f'(u'+v') + f(u''+v'') = u'', \ldots$$

From $f(0) = 1$ and $u^{(k)}(0) \neq 0$, $v^{(k)}(0) = 0$, we get recursively $f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0$.
From $f(1) = 0$ and $u^{(k)}(1) = 0$, $v^{(k)}(1) \neq 0$, we get recursively $f'(1) = f''(1) = \cdots = f^{(n)}(1) = 0$. □

Remark. The derivatives of the function $b_{n,\mu}$ can easily be determined recursively (cf. the proof just given).

From Theorem 1 and the Lemma just stated, we get the

Corollary. *Let $\Gamma_1 : \boldsymbol{x} = \boldsymbol{\gamma}_1(t)$, $t \in [0,1]$ and $\Gamma_2 : \boldsymbol{x} = \boldsymbol{\gamma}_2(t)$, $t \in [0,1]$ be two regular C^n–continuous planar or spatial curves.*
Then for any μ with $0 < \mu < 1$ curve

$$\Gamma : \ \boldsymbol{x} = \gamma(t) := \frac{\mu(1-t)^{n+1}}{\mu(1-t)^{n+1} + (1-\mu)t^{n+1}}\gamma_1(t) + \frac{(1-\mu)t^{n+1}}{\mu(1-t)^{n+1} + (1-\mu)t^{n+1}}\gamma_2(t), \ t \in [0,1],$$

has C^n contact with curve Γ_1 at point $\gamma_1(0)$, and C^n contact with curve Γ_2 at point $\gamma_2(1)$. If γ_1 and γ_2 are rational, then γ is too.

Remarks.

1. *Polynomial* blending functions do exist as well. For example, $f(t) := 1 - 3t^2 + 2t^3$ is a C^1 blending function and $f(t) := 1 - (6t^5 - 15t^4 + 10t^3)$ is a C^2 blending function.
   The advantage of polynomial blending functions is that, if the blended curves γ_1 and γ_2 are polynomial, then the blending curve is polynomial as well.
   The disadvantage of these blending functions is the lack of a design parameter.
2. Due to the definition (see Theorem 1), a blending curve is contained within the *convex hull* of the arcs to be blended.

Example 1: planar curves. Let Γ_1 and Γ_2 be the arcs of two Bézier curves $\beta_1(u)$ and $\beta_2(v)$ of degree 4 with control points $(-1, 1)$, $(-0.5, 0)$, $(0, 1)$, $(1, 1.5)$, $(2, 1)$ and $(-1, 0)$, $(-0.5, 0)$, $(0, -1)$, $(1, -0.5)$, $(2, -1)$, respectively.
Γ_1 is the arc $\gamma_1(t) := \beta_1(u_1 + t(u_2 - u_1))$ with $u_1 = 0.2$, $u_2 = 0.8$, $t \in [0,1]$.
Γ_2 is the arc $\gamma_2(t) := \beta_2(v_1 + t(v_2 - v_1))$ with $v_1 = 0.2$, $v_2 = 0.8$, $t \in [0,1]$.
(The applied parameter transformations are linear.)

Figure 2 shows G^2 blending curves for various parameters μ, the bounding points of the arcs and the effect for the case in which the orientation of the lower arc is reversed, i.e., $\gamma_2(t) := \beta_2(v_2 + t(v_1 - v_2))$.
Figure 3 shows the effect of shortening the upper arc: $u_2 = 0.8,\ 0.6,\ 0.4$. The interval in which the blending curves are close to the upper curve is shortened as well. Parameter μ is 0.5.

2.4 Thumb weight

Further design parameters are provided by suitable nonlinear parameter transformations for the given curves. Simple and effective transformations are the following ones:

$$p_1(t) := \frac{t}{1-\lambda+\lambda t} \quad \text{and}$$
$$p_2(t) := \frac{t(1-\lambda)}{1-\lambda t} \quad 0 \le \lambda < 1.$$

The relation between p_1 and p_2 is $p_2(t) = 1 - p_1(1-t)$. Figure 4a shows graphs of p_1 and p_2. For $\lambda = 0$, we get $p_i(t) = t$, $i = 1, 2$. Applying p_1 to the Bézier curve (see example 1) γ_1 and p_2 to γ_2, we find that

$$\Gamma : \boldsymbol{x} = \gamma(t) := f(t)\boldsymbol{\gamma}_1(p_1(t)) + (1 - f(t))\boldsymbol{\gamma}_2(p_2(t)), \ \ t \in [0,1]$$

yields the blending curves in Fig. 4b. The effect of parameter λ is the following. Raising λ yields blending curves closer to the line between the uninterpolated bounding points (Fig. 4b). An analogous influence of λ occurs for blending surfaces. Therefore, λ is a design parameter with predictible effects. It plays the role of a *thumb weight* (Vida et al. 1994).

Remark. It is possible to choose parameters λ_1, λ_2 separately for functions p_1, p_2, which yields another design parameter.

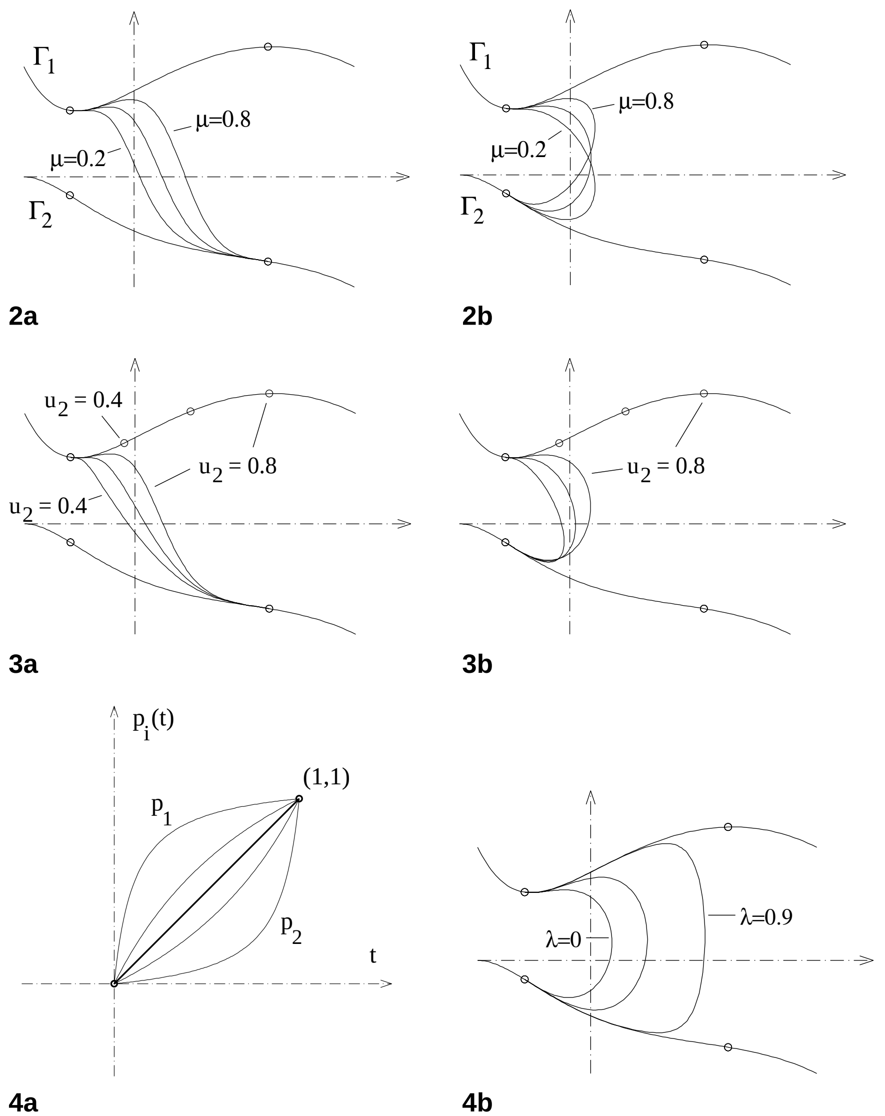


Fig. 2a,b. G^2 blending curves of two Bézier curves for balance $\mu = 0.2, 0.5, 0.8$ (**a**) and after reversing the orientation of the lower arc (**b**)
Fig. 3a,b. G^2 blending curves of two Bézier curves for balance $\mu = 0.5$ and various arcs on the *upper curve* (**a**) and after reversing the orientation of the lower arc (**b**)
Fig. 4a. Graphs of the parameter transformations p_1 (*upper curves*) and p_2 (*lower curves*) for $\lambda = 0,\ 0.5,\ 0.9$; **b** G^2 blending curves of two Bézier curves for balance $\mu = 0.5$ and thumb weight $\lambda = 0, 0.5, 0.9$

Example 2: space curves. Figure 5 shows a G^3 blending curve between an arc on a helix and an arc on a horizontal line, together with the corresponding pipe surfaces. Because the blending curve has G^3 contact with the helix and the line, the whole pipe surface is G^2 continuous.

Helix: $\boldsymbol{x}(t) = (r_0 \cos(t), r_0 \sin(t), 5t),\ r_0 = 50,\ t_1 = \pi + 0.2, t_2 = \pi + 0.6.$

Line: $\boldsymbol{x}(t) = (1-t)\boldsymbol{p}_1 + t\boldsymbol{p}_2,\ \boldsymbol{p}_1 = (r_0, 0, 40),\ \boldsymbol{p}_2 = (r_0, 100, 40), t_1 = 0.2, t_2 = 0.5.$

Radius of the pipe: $r=10$. Design parameter $\mu=0.5$.

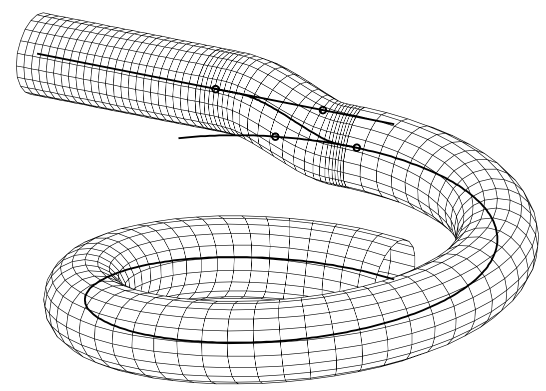


Fig. 5. G^3 blending curve between a helix and a line and a corresponding G^2–continuous pipe surface

The pipe surface is defined as a *ringed surface* $\boldsymbol{x} = \boldsymbol{\gamma}(t) + r\cos(s)\boldsymbol{a}(t) + r\sin(s)\boldsymbol{b}(t)$, where $\boldsymbol{x} = \boldsymbol{\gamma}(t)$ is the *directrix curve* and $\{\dot{\gamma}(t), \boldsymbol{a}(t), \boldsymbol{b}(t)\}$ is a right-handed orthonormal system. One can even prove that the pipe surface of this example is G^3, too (Hartmann 2000c).

3 G^n blending of parametric surfaces

3.1 The blending method

Analogous to the curve case, we define C^n and G^n contact of surfaces.

Definition. Two regular C^n–continuous surfaces $\Phi_1 : \boldsymbol{x} = \boldsymbol{S}_1(s,t)$, $\Phi_2 : \boldsymbol{x} = \boldsymbol{S}_2(s,t)$ have C^n *contact* at a common point $\boldsymbol{p}$ if all partial derivatives up to the nth order of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ at point $\boldsymbol{p}$ are the same.

Definition. Two regular C^n–continuous surfaces Φ_1, Φ_2 have G^n *contact* at a common point $\boldsymbol{p}$ if they have C^n contact at point $\boldsymbol{p}$ with respect to some local reparameterizations of the two surfaces. (cf. Mazure 1994; Wolter and Tuohy 1992).

Definition. A regular C^n–continuous surface Φ that has G^n contact with two regular C^n–continuous surfaces Φ_1, Φ_2 at curves $\Gamma_1 \subset \Phi_1$, $\Gamma_2 \subset \Phi_2$, respectively, is called a G^n *blending surface* of Φ_1 and Φ_2.

The G^n blending method introduced for curves can be extended to surfaces. As for curves, the flexibility of the method depends essentially on suitable parameter transformations.

Theorem 2. *Let $\Phi_1 : \boldsymbol{x} = \boldsymbol{S}_1(s,t)$, $s \in [s_1, s_2]$, $t \in [0,1]$ and $\Phi_2 : \boldsymbol{x} = \boldsymbol{S}_2(s,t)$, $s \in [s_1, s_2]$, $t \in [0,1]$ be two regular C^n–continuous surface patches (base surfaces), and let $f(t)$, $t \in [0,1]$ be a C^n–continuous real function (blending function) with the following properties:*

$$f(0) = 1, \;\; f(1) = 0,$$
$$f^{(k)}(0) = f^{(k)}(1) = 0 \text{ for } k = 1, \ldots, n.$$

Then the surface patch (blending surface)

$$\Phi : \boldsymbol{x} = \boldsymbol{S}(s,t) := f(t)\boldsymbol{S}_1(s,t) + (1 - f(t))\boldsymbol{S}_2(s,t), \quad s \in [s_1, s_2], \; t \in [0,1]$$

has C^n contact with the surface patch Φ_1 along the curve $\boldsymbol{S}_1(s,0)$, $s \in [s_1, s_2]$ and with Φ_2 along the curve $\boldsymbol{S}_2(s,1)$, $s \in [s_1, s_2]$.

The proof is straightforward, differentiating analogously to the curve case (Theorem 1).

As blending functions for the following examples, we use the functions $b_{n,\mu}$, which have already been introduced.

Remarks.

1. The curves $\boldsymbol{S}_1(s,0)$ and $\boldsymbol{S}_2(s,1)$ $s \in [s_1, s_2]$ are the *contact curves* between the base surfaces and the blend surface. The curves $\boldsymbol{S}_1(s,1)$ and $\boldsymbol{S}_2(s,0)$ $s \in [s_1, s_2]$ are called *auxiliary curves*. Contact curves and auxiliary curves are boundaries of the surface patches to be blended.
2. Curves $\boldsymbol{x} = \boldsymbol{S}(s = const, t)$ are *profile curves*, which locally define the shape of the blend surface.

Remark. For the application of Theorem 2 to the examples in Sect. 3.2, we need parameter transformations to meet the precondition "$s \in [s_1, s_2]$, $t \in [0,1]$". Thus, the blend surface $\boldsymbol{S}$ will have only G^n (instead of C^n) contact with the base surfaces.

Remark. The given blending method should not be confused with *Coons patches* (Farin 1993). Coons considers linear combinations of space *curves* $\gamma_1(s)$, $\gamma_2(s), \ldots$ with a common parameter s and coefficients (blending functions) depending on a second parameter t : $\boldsymbol{S}(u,v) := f_1(t)\gamma_1(s) + f_2(t)\gamma_2(s) + \cdots$. Theorem 2 considers linear combinations of *surfaces* with common parameters, and the coefficients depend on one of the parameters.

3.2 Examples

Example 3: Bézier patches. Let Φ_1 and Φ_2 be two tensor-product Bézier patches with parametric representations:

$$\Phi_1 : \boldsymbol{x} = \boldsymbol{B}_1(u, v) \\ = (10v - 5, 10u - 5, 6(u - u^2 + v - v^2)),$$
$$\Phi_2 : \boldsymbol{x} = \boldsymbol{B}_2(u, v) \\ = (6(u - u^2 + v - v^2), 10u - 5, 10v - 5).$$

The boundary curves on Φ_1 are:
Curve of contact: $v = v_{11} = 0.5$, auxiliary curve: $v = v_{12} = 0.3$, boundary curves: $u = 0$ and $u = 0.75$.
The boundary curves on Φ_2 are:
Curve of contact: $v = v_{21} = 0.9$, auxiliary curve: $v = v_{22}(u) = 0.8(1 - (0.5 - u)^2)$, boundary curves: $u = 0$ and $u = 0.75$.
In order to meet the preconditions of Theorem 2, we use the following parameter transformations:

$$\boldsymbol{S}_1(s, t) := \boldsymbol{B}_1(s, v_{11} + (v_{12} - v_{11})t), \quad t \in [0, 1]$$
$$\boldsymbol{S}_2(s, t) := \boldsymbol{B}_2(s, v_{21} + (v_{22}(s) - v_{21})t), \quad t \in [0, 1]$$
$$s \in [0, 0.75].$$

Hence, $\Phi : \boldsymbol{x} = \boldsymbol{S}(s, t) := b_{n,\mu}(t)\boldsymbol{S}_1(s, t) + (1 - b_{n,\mu}(t))\boldsymbol{S}_2(s, t)$, $s \in [0, 0.75]$, $t \in [0, 1]$ is, for any μ with $0 < \mu < 1$, a G^n blending surface ($b_{n,\mu}$, a G^n blending function, has already been defined).
Figure 6a shows the base surfaces, the curves of contact (thick) and the auxiliary curves (dashed). Figure 6b shows a G^2 blending surface for the balance $\mu = 0.5$ (for the meaning of μ, see Sect. 2.3).
In order to get a *thumb weight* parameter, we apply parameter transformations p_1, p_2 (see Sect. 2.4) on parameter t before blending:

$$\Phi : \boldsymbol{x} = \boldsymbol{S}(s, t) := b_{n,\mu}(t)\boldsymbol{S}_1(s,\ p_1(t)) \\ + (1 - b_{n,\mu}(t))\boldsymbol{S}_2(s,\ p_2(t)),$$
$$s \in [0, 0.75], \ t \in [0, 1].$$

p_1 and p_2 depend on the thumb weight λ.
Figure 7a shows blending surfaces of the given Bézier patches for balance $\mu = 0.5$ and thumb weight $\lambda = 0.65$, and for Fig. 7b, $\mu = 0.05$, and $\lambda = 0.5$. Increasing $\lambda \in [0, 1)$ brings the blend surface closer to the auxiliary curves (compare Figs. 6 and 7). For a balance of $\mu \neq 0.5$, the blend surface is closer to one of the base surfaces (compare Fig. 7a and b).

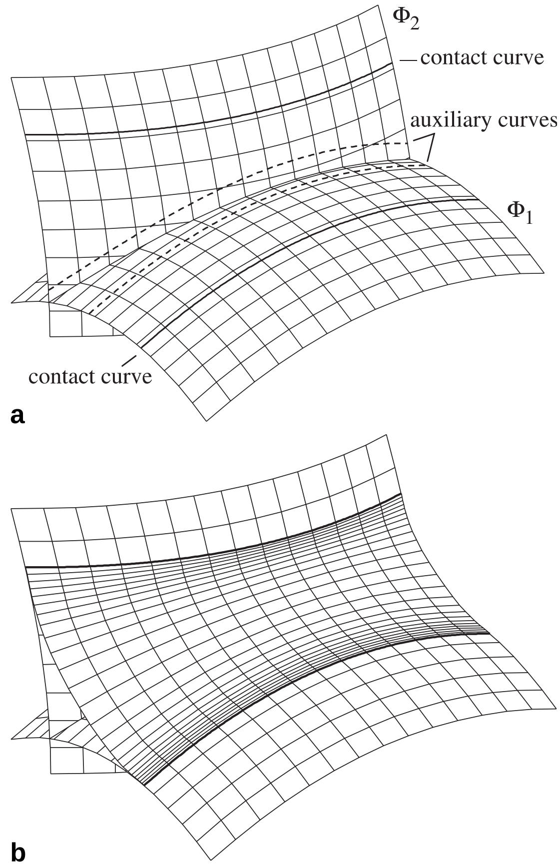
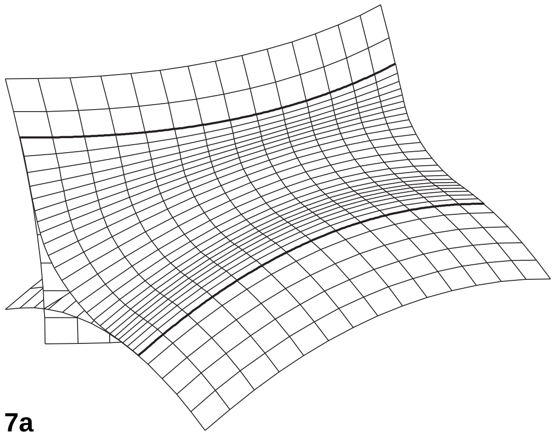


Fig. 6a. The boundaries of the patches used for blending; **b** G^2 blending surface between two tensor product Bézier surfaces with balance $\mu = 0.5$ and thumb weight $\lambda = 0$

Remarks.

1. The surfaces to be blended need not have an intersection curve.
2. The curves of contact are always the parameter lines $\boldsymbol{S}_1(s, 0)$ and $\boldsymbol{S}_2(s, 1)$. This looks rather rigid. However, suitable (more or less simple) parameter transformations give the user the opportunity to determine the curves of contact, within a wide range, in advance.
3. Curves of contact can be retained while moving the surface patches (see Fig. 8).



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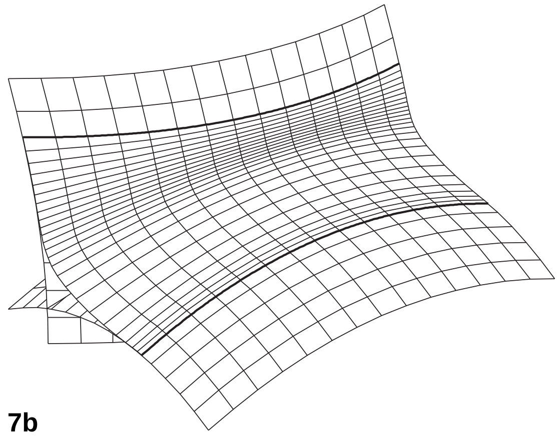

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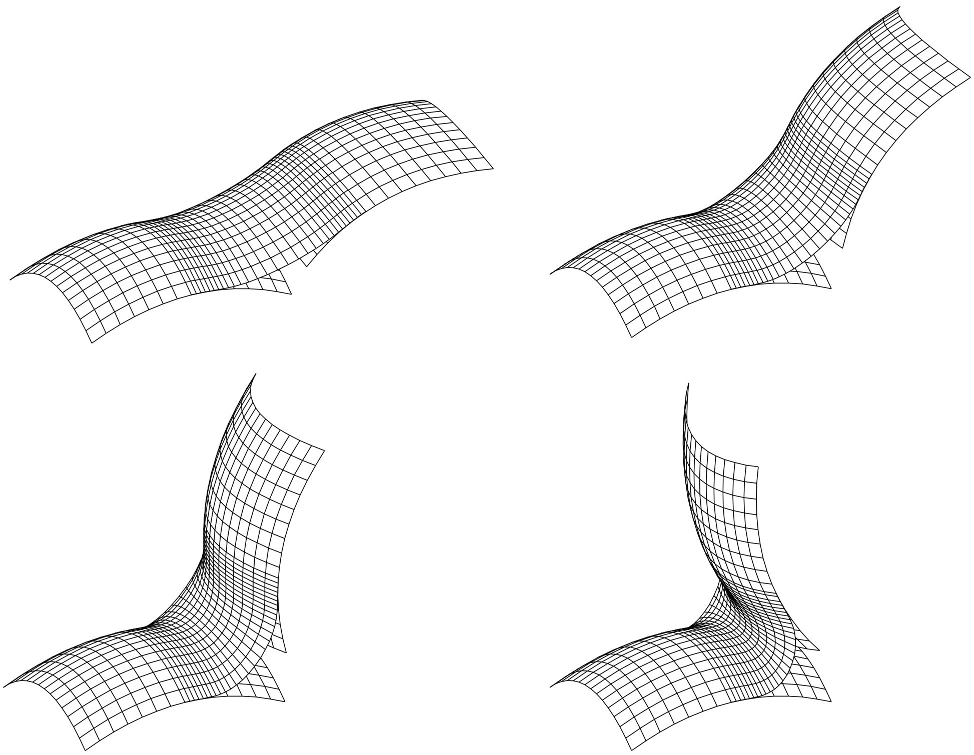

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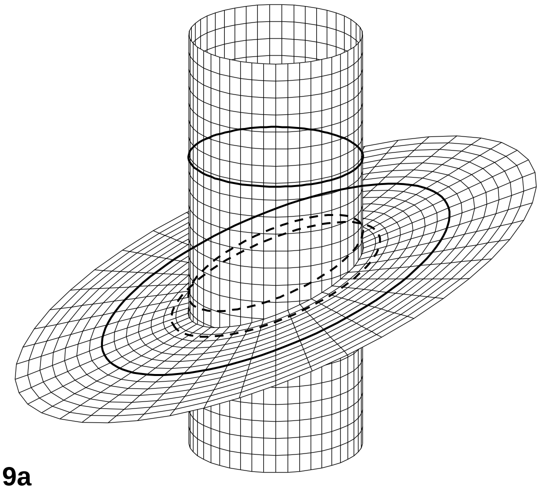
Fig. 7a,b. G^2 blending surface between two tensor-product Bézier surfaces with balance $\mu = 0.5$, thumb weight $\lambda = 0.65$ (**a**) and $\mu = 0.05$, $\lambda = 0.5$ (**b**)
Fig. 8. Blending moving surfaces

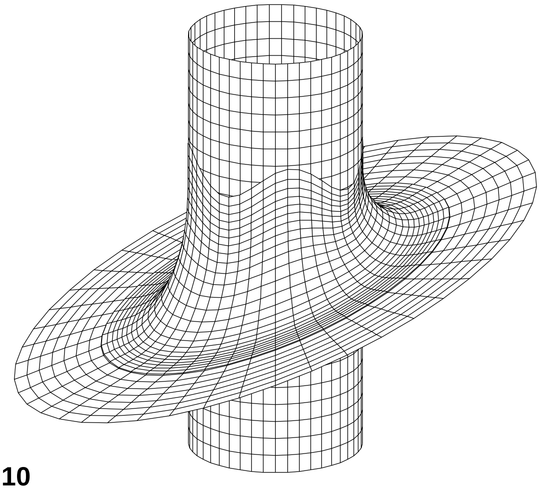
Example 4: cylinder and plane. Let Φ_1 be a cylinder and Φ_2 an inclined plane.

$\Phi_1 : \boldsymbol{x} = (\cos(u), \sin(u), v)$,
$\Phi_2 : \boldsymbol{x} = (v\cos(u), v\sin(u), cv\cos(u)),\ \ c = 0.4.$

The boundary curves on Φ_1 are the curve of contact $v = v_{11} = 1.5$ and the auxiliary curve $v = v_{12}(u) = 0.2 + c\cos u$.
The boundary curves on Φ_2 are the curve of contact $v = v_{21} = 1.2$, and the auxiliary curve $v = v_{22} = 2$.



9a



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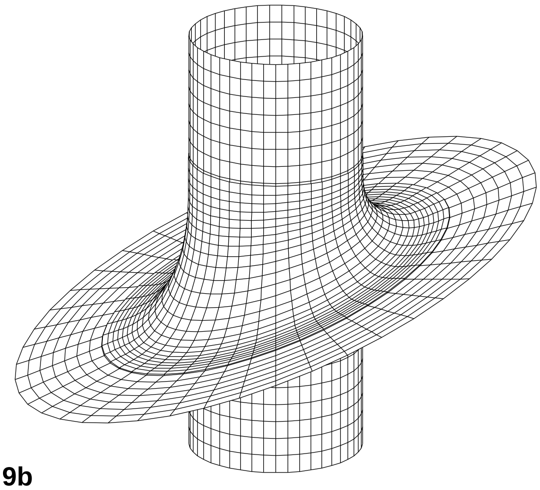


9b

Fig. 9a. The boundaries of the patches used for blending; **b** G^2 blending surface between a cylinder and a plane

Fig. 10. Further G^2 blending surface between a cylinder and a plane

Figure 9a shows the curves of contact and auxiliary curves for a G^2 blending surface (Fig. 9b). The design parameters are balance $\mu = 0.5$ and thumb weight $\lambda := 0.5$.
Figure 10 shows a G^2 blending surface with a changed curve of contact for $\Phi_1 : v = v_{11}(u) = 1.5 + 0.2\cos(4u)$.

Example 5: two cylinders. Let Φ_1 and Φ_2 be two cylinders.

$\Phi_1 : \boldsymbol{x} = (\cos(u), \sin(u), v)$,

$\Phi_2 : \boldsymbol{x} = (2\cos(u), v, 2\sin(u))$.

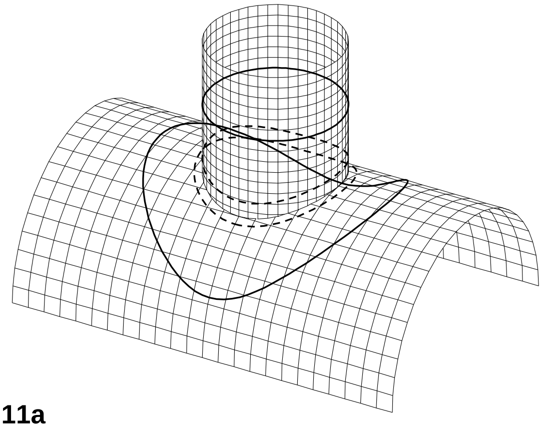
The curve of contact on Φ_1 is the circle with $z = 3$ and the auxiliary curve is the intersection curve with the cylinder $x^2 + z^2 = 2.2^2$.

The curve of contact on Φ_2 is the intersection curve with the cylinder $x^2 + y^2 = 1.8^2$; the auxiliary curve is the intersection curve with the cylinder $x^2 + y^2 = 1.2^2$. The blended patch on Φ_2 is described by polar coordinates $x = v\cos(u)$ and $y = v\sin(u)$.
Figure 11a shows the curves, and Fig. 11b shows a G^2 blending surface.

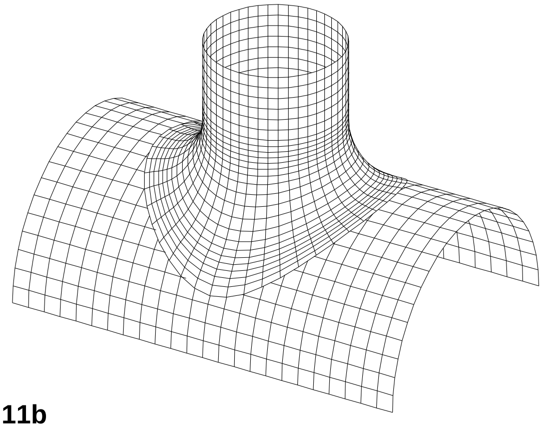
Example 6: two cylinders of equal radii. Another classical example for blending surfaces is the blending of two cylinders with orthogonal intersecting axes and equal radii. Let Φ_1 and Φ_2 be the two cylinders

$\Phi_1 : \boldsymbol{x} = (\cos(u), \sin(u), v)$ and

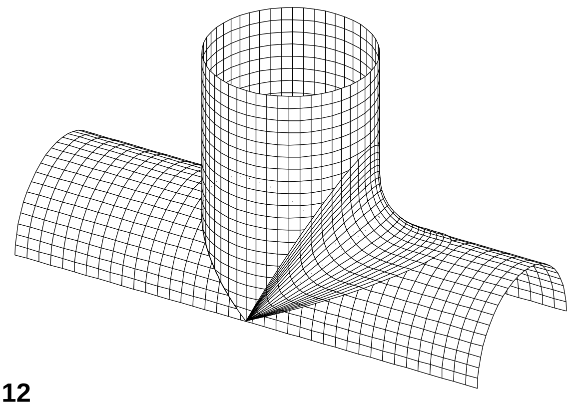
$\Phi_2 : \boldsymbol{x} = (\cos(u), v, \sin(u))$.



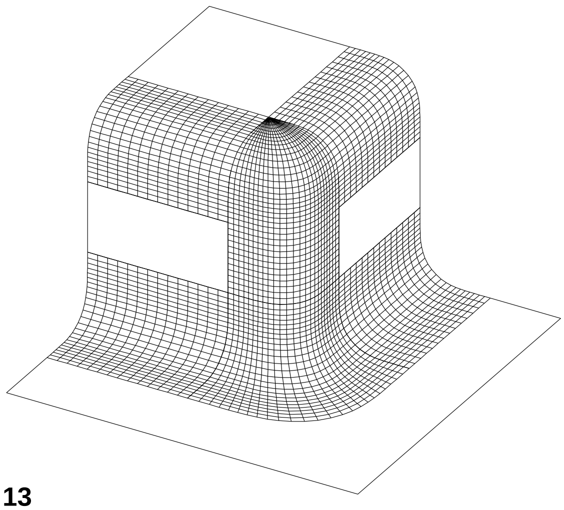
11a



11b



12



13

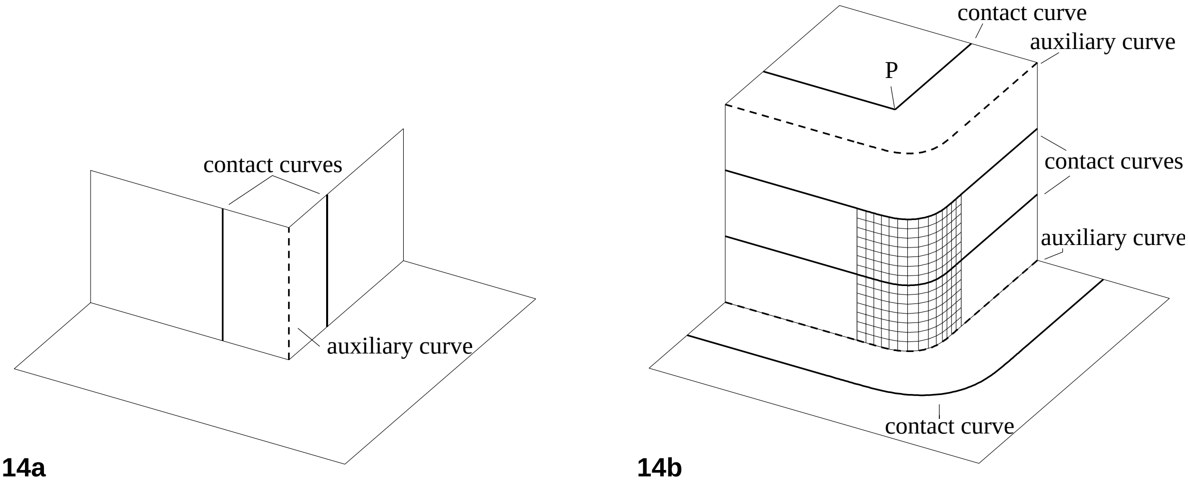
Fig. 11a. The boundaries of the patches used for blending; **b** G^2 blending surface between two cylinders
Fig. 12. G^2 blending surface between two cylinders of equal radii
Fig. 13. G^2 blending of a “house corner” and a “suitcase corner”

For both Φ_1 and Φ_2, the curve of contact has the equation $v = 2\sin(u)$ in the parameter plane and the auxiliary curve $v = \sin(u)$ (intersection curve of the two cylinders). The four bounding curves are ellipses!
Figure 12 shows a G^2 blending surface. The blending surface is singular at the double point of the intersection curve of the cylinders.

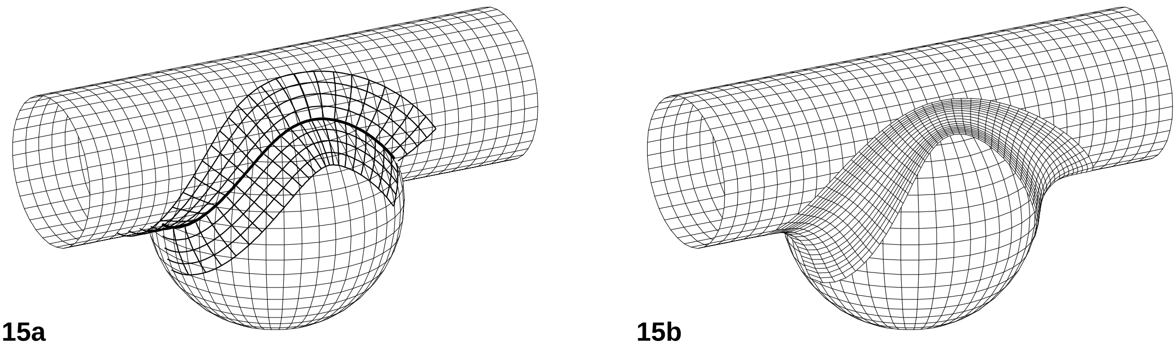
Example 7: suitcase corner, house corner. Figure 13 shows the rounding of a cube standing on a plane. The blending procedure consists of three steps.

Step 1. The vertical edge is blended with the curves of contact (thick) and the auxiliary curves (dashed) drawn in Fig. 14a.

Steps 2 and 3. For the remaining two patches (bottom and top) the curves in Fig. 14b, are used. The triangular patch (Fig. 13) on the top is created with a point of contact P instead of a curve of contact. The parametric representation at P is singular. However, the surface has G^n continuous contact with the plane. This can be seen if one considers the pencil of parameter curves passing point P. Any curve of this



14a **14b**



15a **15b**

Fig. 14a,b. The boundaries of the patches used for G^2 blending: **a** a “house corner”; **b** a “suitcase corner”
Fig. 15a,b. Numerical parametric blending of a cylinder and a sphere: **a** parameter lines of the (numerical) reparameterization; **b** G^2 blending surface

pencil has G^n contact with a line of the top plane at P. From the $(n+1)$ Tangents theorem, Corollary 1 (Wolter and Tuohy 1992), we get the G^n contact of the corner blend with the plane at the top. Figure 13 shows the total G^2 blending.

3.3 Numerical method

In general, it may be difficult to find a suitable parameterization of both surfaces to be blended. A numerical way that works in most cases is the following one: let Φ_1, Φ_2 be two surfaces, parametric or implicit, differentiable enough, with a regular intersection curve Γ (the normals at common points are not linearly dependent). Let curve Γ be parameterized numerically by a suitable Bézier curve $\beta(t)$ as introduced by (Hartmann 2000a) and let $\gamma(t),\ t \in [0, 1]$ be this (numerical) parameterization (see Step 2). The family of pipe surfaces with a spine curve Γ and the radius as family parameter induces a regular parameterization in the vicinity of Γ on both surfaces Φ_1 and Φ_2 (Fig. 15). Its differentiability is one less than that of the base surfaces Φ_1, Φ_2. Hence, we get (numerical) reparameterizations $\boldsymbol{S}_1(t, u),\ t \in [0, 1], u \in [0, u_0]$ and $\boldsymbol{S}_2(t, v),\ t \in [0, 1], v \in [0, v_0]$ of Φ_1 and Φ_2, respectively. The parameters u, v are the distances of the the surface points $\boldsymbol{S}_1(t, u)$ and $\boldsymbol{S}_2(t, v)$ to the inter-

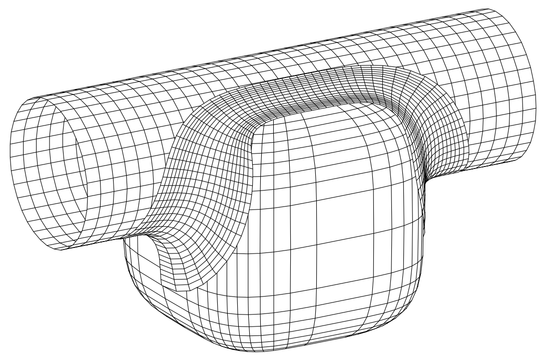


Fig. 16. Parametric G^2 blending of two implicit surfaces

section curve Γ. (The intersection curve divides the surfaces, so one has to take care that the "right" patches are chosen.) Hence, point $\boldsymbol{S}_1(t,u)$ is determined by the following (numerical) steps:

1. Calculate $\beta(t)$.
2. Determine the *foot point* of $\beta(t)$ on Γ, which is $\gamma(t)$.
   Point $\boldsymbol{q}$ is foot point of point $\boldsymbol{p}$ on curve Γ if $\|\boldsymbol{p}-\boldsymbol{q}\|$ is minimal and $\boldsymbol{p}-\boldsymbol{q}$ orthogonal to Γ (Hartmann 2000a).
3. Intersect the circle with midpoint $\gamma(t)$, radius u, and normal to curve Γ with surface Φ_1 (Make the "right" choice of the two possible points!). The intersection point is $\boldsymbol{S}_1(t,u)$.

Point $\boldsymbol{S}_2(t,v)$ is determined analogously.

Example 8: sphere and cylinder. Let be Φ_1 a sphere and Φ_2 a cylinder.

$$\Phi_1 : \boldsymbol{x} = (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)),$$
$$\Phi_2 : \boldsymbol{x} = (v, z_0\cos(u), z_0 + z_0\sin(u)),\ z_0 = 0.6.$$

Figure 15a shows parameter lines of the numerical reparameterization of both the surfaces and a parametric blending surface (Fig. 15b) with curves of contact that depend on the angle of intersection.

Example 9: implicit surfaces. Figure 16 shows an application of the numerical blending to the implicit surfaces

$$\Phi_1 : x^4 + y^4 + z^4 = 1 \text{ and}$$
$$\Phi_2 : y^2 + (z-0.6)^2 = 0.6^2 \text{ (cylinder).}$$

Remark. Hartmann (2000b) gives a numerical reparameterization of two intersecting surfaces such that parameter lines are curves of contact of a rolling ball. Applying the parametric G^n blending method of Theorem 2, one gets G^n blending surfaces with *rolling ball contact curves*.

4 Conclusion

A simple blending method is introduced, producing parametric blending curves and surfaces that have G^n contact with the base curves and surfaces. Two design parameters (balance and thumb weight), together with suitable reparameterizations of the base curves and surfaces, allow flexibility in a wide range. Using numerical methods, the blending is applicable to nearly arbitrary regular curves and surfaces too. The practical use of the method and the meaning of the design parameters are shown by several examples.

References

1. Bloor, MIG, Wilson, MJ (1989) Generating blend surfaces using differential equations. Comput Aided Des 21:165–171
2. Braid, IC (1997) Non-local blending of boundary models. Comput Aided Des 29:89–100
3. Choi, BK, Ju, SY (1989) Constant-radius blending in surface modelling. Comput Aided Des 21:213–220
4. Chuang, JH, Hwang, WC (1997) Variable-radius blending by constrained spine generation. Visual Comput 13:316–329
5. Coons, SA (1977) Modification of the Shape of Piecewise Curves. Comput Aided Des 9:178–180
6. Creasy, CFM, Craggs, C (1990) Applied Surface Modelling. Ellis Horwood, Chichester, Chap 13, pp 139–149
7. Farin, G (1993) Curves and Surfaces for CAGD. Orlando, Academic Press
8. Farouki, RAMT, Sverrisson, R (1996) Approximation of rolling-ball blends for free-form parametric surfaces. Comput Aided Des 21:871–878
9. Faux, ID, Pratt, MJ (1979) Computational Geometry for Design and Manufacture. Ellis Horwood, Chichester, Sect 6.5, pp 192–197
10. Filip, DJ (1989) Blending parametric surfaces. ACM Trans Graph 8:164–173
11. Hartmann, E (1998) Numerical implicitization for intersection and G^n-continuous blending of surfaces. Comput Aided Geom Des 15:377–397
12. Hartmann, E (2000a) Numerical parameterization of curves and surfaces. Comput Aided Geom Des 17:251–266

13. Hartmann, E (2000b) G^n-blending with rolling ball contact curves. In: Martin R, Wang W (eds) Proceedings of Geometric Modeling and Processing 2000, Hongkong, IEEE Comput Soc, Los Alamitos, pp 385–389
14. Hartmann, E (2000c) The normal form of a space curve and its application to surface design. submitted
15. Hermann, T, Lukács, G, Várady, T (1995) Techniques for Variable Radius Rolling Ball Blends. In: Daehlen M, Lyche T, Schumaker L (eds) Mathematical methods for curves and surfaces. Vanderbilt University Press, Nashville, pp 225–236
16. Lukács, G (1998) Differential geometry of G^1 variable radius rolling ball blend surfaces. Comput Aided Geom Des 15:585–613
17. Mazure, M-L (1994) Geometric contact curves and surfaces. Comput Aided Geom Des 11:177–195
18. Várady, T, Vida, J, Martin, RR (1989) Parametric blending in a boundary representation solid modeller. In: Handscomb, DC (ed) The mathematics of surfaces III. Oxford University Press (1988). Inst Math Appl Conf Ser, New Ser 23:171–197
19. Várady, T, Rockwood, A (1997) Geometric construction for setback vertex blending. Comput Aided Des 26:341–365
20. Vida, J, Martin, RR, Várady, T (1994) A survey of blending methods that use parametric surfaces. Comput Aided Des 26:341–365
21. Wallner, J, Pottmann, H (1997) Rational blending surfaces between quadrics. Comput Aided Geom Des 14:407–419
22. Wolter, F-E, Tuohy, ST (1992) Curvature computations for degenerate surface patches. Comput Aided Geom Des 9:241–270

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