BLENDING SURFACES BASED ON A THREE-PARAMETER FOURTH ORDER PARTIAL DIFFERENTIAL EQUATION

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ABSTRCT

The blending method based on partial differential equations (PDE) is an effective tool to generate a smooth, transition surface between base surfaces. The parameters in the partial differential equations have profound effect on the geometry of the blending surfaces. But in the existing research, only one vector-valued parameter is considered in the partial differential equations. To improve on the flexibility and versatility of the PDEs, we are proposing a fourth order partial differential equation with three vector-valued parameters to generate blending surfaces. This method actually includes the one-parameter fourth order partial differential equation as a special case, and is able to generate blends suitable for more geometric diversity. To demonstrate the proposed method, blends between two co-axial circular cylinders, between two intersecting planes, and between an elliptic cylinder and a horizontal plane are generated. The closed form solutions of the three-parameter fourth order partial differential equation for these three examples are obtained and are applied to generate the blending surfaces.

Key words: Three-parameter partial differential equation, closed form solutions, parametric blending

INTRODUCTION

Blending, a recurring operation in surface and solid modelling is to determine a smooth, transition surface between two and more base surfaces. Blending is widely applied in computer graphics, computer aided design and computer aided manufacturing.

Due to its extensive applications in geometric modelling, blending is a heavily researched area and many methods have been developed. Depending on the representation of the surfaces, blending methods can be either implicit or parametric. The implicit blends usually take the form of $F(x, y, z) = 0$. The blending surfaces are generated as the solution of such implicit equation. Solutions in rational polynomial representations are generally not obtainable. Details on implicit blending were given in the surveys conducted by Hoffmann (1987) and Warren (1989). The parametric blends can be written in the form of $F = \left(x(u,v),\ y(u,v),\ z(u,v)\right)$. For some simple cases, analytical solutions can be achieved. However, for most blending problems, approximation methods including rational polynomial representations are applied to generate the blending surfaces as explained by Rockwood (1987), Woodwark (1990) and Vida (1994).

The rolling-ball method (Sanglikar 1990) is one of the parametric methods. It generates blending surfaces by moving a ball along a spine curve, which is the intersection curve of the surfaces formed by offsetting each base surface by the radius of the ball. The rolling-ball blends can be further divided into fixed-radius rolling-ball blends and variable-radius rolling-ball blends. A most obvious advantage of this method is that the spine, the trimlines (also called linkage curves) and the profiles are automatically generated. However, this blending method can not be used in some blending situations, such as the base surfaces which or whose extension surfaces do not intersect. Also, it can not generate non-symmetrical blending surfaces and can not ensure curvature continuity on the trimlines. The spine-based blending method (Sanglikar 1993) is also a parametric one. To blend the base surfaces, three steps are usually taken. Firstly, some spine curves must be defined. Then, trimlines, or at least points lying on notional trimlines, on the base surface will be generated. At last, the profile curves are defined. The main problem of this method is it is very difficult for users to define the spine curve. The trimline-based blending method (Filkins 1993) defines trimlines first. Next, it either defines a spine curve or establishes the correspondence between the points of the trimlines. Following this, the profile information is generated to define the profile curves.

The blending method based on partial differential equations (Bloor 1994, 1995, 1996) is another case of the parametric blending methods. It generates blending surfaces by solving a partial differential equation. To do this, the boundary conditions on the trimlines must be first determined. Compared to the spine-based blending and trimline-based blending, it need not define a spine curve and profile curves. The profile of the blending surfaces is determined by the partial differential equation. Compared to the rolling-ball blending, it can deal with the blending problems that the rolling-ball blending cannot. In addition, it is able to generate non-symmetrical blending surfaces and has the potential to ensure the curvature continuity between the blending surface and base surfaces.

BASIC EQUATIONS

In the existing references, a one-parameter fourth order partial differential equation has been applied to deal with many blending problems between base surfaces. From (Bloor 1989, 1990) and Brown (1998), this equation can be written as

$$\left(\frac{\partial^2}{\partial u^2} + \mathbf{a}\frac{\partial^2}{\partial v^2}\right)^2 \mathbf{x} = \mathbf{0} \tag{1}$$

where u and v are the parameters, $\mathbf{a} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^T$ is the vector-valued parameter and $\mathbf{x}$ stands for the co-ordinates of the points on the blending surface that is the function of u and v.

The blending between two or more base surfaces is to solve Eqn (1) under the given boundary conditions, i.e., some or all of the positional, tangent and curvature continuity conditions at the interface curves, i.e. trimlines, between the base surfaces and the blending surface.

It has been discussed by Bloor et al. (1989) that the vector-valued parameter $\mathbf{a}$ in Eqn (1) affects the shape of the blending surface. From the viewpoint of design, manufacture and aesthetics, the designers would hope to obtain their desirable shape of the blending surfaces by adjusting the control parameters. For this reason, a three-parameter fourth order partial differential equation will be more desirable. Similar to that used by You (1997, 1999), we propose a PDE in the form of

$$\left(\mathbf{b}\frac{\partial^4}{\partial u^4}+\mathbf{c}\frac{\partial^4}{\partial \mathbf{u}^2\partial v^2}+\mathbf{d}\frac{\partial^4}{\partial \mathbf{v}^4}\right)\mathbf{x}=\mathbf{0} \tag{2}$$

where $\mathbf{b}=\begin{bmatrix}b_x & b_y & b_z\end{bmatrix}^T$, $\mathbf{c}=\begin{bmatrix}c_x & c_y & c_z\end{bmatrix}^T$ and $\mathbf{d}=\begin{bmatrix}d_x & d_y & d_z\end{bmatrix}^T$ are the vector-valued parameters.

It is easy to see that Eqn (2) is a general form of Eqn (1). In fact, when $b_i=1$, $c_i=2a_i$ and $d_i=a_i^2$ $(i=x,\ y,\ z)$, Eqn (2) becomes Eqn (1).

In the following, three numerical examples will be given to demonstrate the application of Eqn (2) in blending different base surfaces. For all the three examples, the closed form solutions of Eqn (2) were first sought and were implemented in C++ and OpenGL on a Silican Graphics workstation.

BLENDING TWO CO-AXIAL CIRCULAR CYLINDERS

The blend between two co-axial cylinders has many applications in the design of mechanical components, such as the transition surfaces between two adjacent segments with different radii of a shaft. For this problem, the solving region of the blending surface is defined in Ω: $0\le u\le 1,\ 0\le v\le 2\pi$. The radii of the upper and lower cylinders are taken as r_1 and r_2 respectively. At $u=0,\ z=h_0$ and the upper cylinder, whose height is h_1, meets the blending surface. At $u=1,\ z=0$ and the bottom cylinder of height h_2 joins the blending surface. According to the definition of trimlines (linkage curves), these two interface curves between the base surfaces and the blending surface are trimlines. The boundary conditions on the linkage curve of $u=0$ between the upper cylinder and the blending surface are

$$
u = 0 \qquad \begin{aligned} & x = r_1 \cos v \\ & y = r_1 \sin v \\ & z = h_0 \\ & \frac{\partial x}{\partial u} = 0 \\ & \frac{\partial y}{\partial u} = 0 \\ & \frac{\partial z}{\partial u} = -h_1 \end{aligned} \tag{3}
$$

and the boundary conditions on the linkage curve between the bottom cylinder and the blending surface are

$$
u = 1 \qquad \begin{aligned} & x = r_2 \cos v \\ & y = r_2 \sin v \\ & z = 0 \\ & \frac{\partial x}{\partial u} = 0 \\ & \frac{\partial y}{\partial u} = 0 \\ & \frac{\partial z}{\partial u} = -h_2 \end{aligned} \tag{4}
$$

The solution of Eqn (2) under the boundary conditions (3) and (4) can be obtained by using the method of variable separation. To do this, we assume that the solution of Eqn (2) has the form of

$$
\begin{aligned} & x = f_1(u) \cos v \\ & y = f_2(u) \sin v \\ & z = f_3(u) \end{aligned} \tag{5}
$$

Substituting Eqn (5) into Eqn (2) and integrating, we obtain

for $4b_i d_i < c_i^2 \quad (i = x,\ y)$

$$
\begin{aligned} & x = (a_1 e^{t_1 u} + a_2 e^{t_2 u} + a_3 e^{t_3 u} + a_4 e^{t_4 u}) \cos v \\ & y = (b_1 e^{q_1 u} + b_2 e^{q_2 u} + b_3 e^{q_3 u} + b_4 e^{q_4 u}) \cos v \\ & z = c_1 + c_2 u + c_3 u^2 + c_4 u^3 \end{aligned} \tag{6}
$$

where

$$
\begin{aligned}
t_{1,2} &= \pm\sqrt{\frac{c_x}{2b_x}\left[1+\sqrt{1-\frac{4b_x d_x}{c_x^2}}\right]} \\
t_{3,4} &= \pm\sqrt{\frac{c_x}{2b_x}\left[1-\sqrt{1-\frac{4b_x d_x}{c_x^2}}\right]} \\
q_{1,2} &= \pm\sqrt{\frac{c_y}{2b_y}\left[1+\sqrt{1-\frac{4b_y d_y}{c_y^2}}\right]} \\
q_{3,4} &= \pm\sqrt{\frac{c_y}{2b_y}\left[1-\sqrt{1-\frac{4b_y d_y}{c_y^2}}\right]}
\end{aligned}
\tag{7}
$$

and for $4b_i d_i = c_i^2 \quad (i = x,\ y)$

$$
\begin{aligned}
x &= \left[(a_1 + a_2 u)e^{t_1 u} + (a_3 + a_4 u)e^{t_2 u}\right]\cos v \\
y &= \left[(b_1 + b_2 u)e^{q_1 u} + (b_3 + b_4 u)e^{q_2 u}\right]\cos v \\
z &= c_1 + c_2 u + c_3 u^2 + c_4 u^3
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
t_{1,2} &= \pm\sqrt{\frac{c_x}{2b_x}} \\
q_{1,2} &= \pm\sqrt{\frac{c_y}{2b_y}}
\end{aligned}
\tag{9}
$$

The integrating constants a_j, b_j and c_j $(j = 1,\ 2,\ 3,\ 4)$ in Eqns (6) and (8) can be determined by substituting these equations into the boundary conditions (3) and (4).

Two numerical examples for this blending are given here. In both cases, the geometric parameters of the two circular cylinders are taken as: $r_1 = 1$, $r_2 = 1.5$, and $h_0 = h_1 = h_2 = 1$. For the first example, we take the parameters in Eqn (2) to be: $b_x = d_x = b_y = d_y = 1$ and $c_x = c_y = 2$ which give the same solution as Eqn (1) proposed by Bloor et al. The blending surface obtained is given in Fig. 1. For the second example, we choose the parameters to be: $b_x = b_y = 1$, $c_x = c_y = 22$, and $d_x = d_y = 100$ which give the solution of Eqn (2). The blending surface generated by these parameters is shown in Fig. 2. It can be seen from these two figures that for the same geometric parameters of the base surfaces to be blended, different parameters in Eqn (2) generate different shapes of the blending surface.

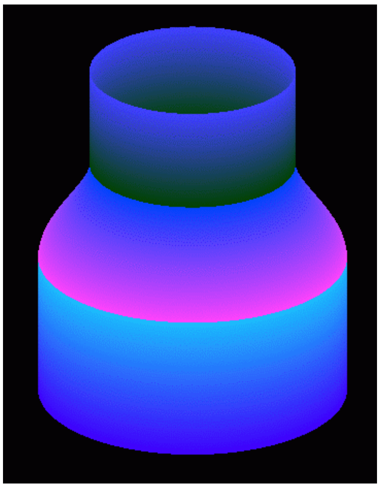

Fig. 1 Blending two cylinders ($b_x = d_x = b_y = d_y = 1$ and $c_x = c_y = 2$)

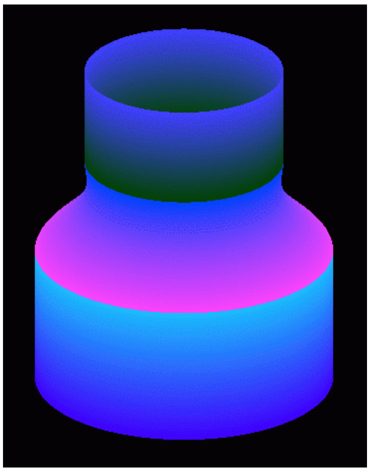

Fig. 2 Blending two cylinders ($b_x = b_y = 1$, $c_x = c_y = 22$, $d_x = d_y = 100$)

BLENDING TWO INTERSECTING PLANES

In this section we discuss the blending of two intersecting planes using the proposed PDE in eqn (2). For such a blending problem, the rolling-ball method can usually generate the blending surfaces which have a symmetrical structure, whereas the PDE method has greater flexibility. The PDE method can generate blending surfaces with various non-symmetrical features.

Let's consider two planes, one inclined and a horizontal. The intersecting angle between them is θ. The width of the inclined plane is h_2 and the width of the horizontal plane is h_4. The solving region is defined in Ω: $0 \le u \le 1$, $0 \le v \le 1$. The inclined plane intersects the blending surface at the trimline $u = 0$ where the value of y co-ordinate is h_0. The horizontal plane intersects the blending surface at the trimline $u = 1$ where the value of x co-ordinate is s_0. The boundary conditions given for this blending are

$$
\begin{array}{ll}
u = 0 & x = h_0 \tan\theta \\
 & y = h_0 \\
 & z = h_2 v \\
 & \dfrac{\partial x}{\partial u} = -h_1 \tan\vartheta \\
 & \dfrac{\partial y}{\partial u} = -h_1 \\
 & \dfrac{\partial z}{\partial u} = 0 \\
u = 1 & x = s_0 \\
 & y = 0 \\
 & z = h_4 v \\
 & \dfrac{\partial x}{\partial u} = h_3 \\
 & \dfrac{\partial y}{\partial u} = 0 \\
 & \dfrac{\partial z}{\partial u} = 0
\end{array}
\tag{10}
$$

where h_1 and h_3 are tangent values at the two trimlines, respectively.

The solution of Eqn (2) under the boundary conditions (10) is similarly derived as

$$
\begin{aligned}
& x = (h_0 - h_1 u)\tan\vartheta + \left[3s_0 - h_3 - (2h_1 - 3h_0)\tan\vartheta\right]u^2 \\
& +\left[h_3 - 2s_0 + (2h_0 - h_1)\tan\vartheta\right]u^3 \\
& y = h_0 - h_1 u + (2h_1 - 3h_0)u^2 + (2h_0 - h_1)u^3 \\
& z = \left[h_2 + 3(h_4 - h_2)u^2 + 2(h_2 - h_4)u^3\right]v
\end{aligned}
\tag{11}
$$

With Eqn (11), two numerical examples are studied. The first one gives a symmetrical blending surface whose geometric parameters are taken as: $h_0 = s_0 = 0.5$, $h_1 = h_3 = 1$, $h_2 = h_4 = 3$ and $\theta = 20^o$. Here θ is measured in clockwise direction. Substituting these parameters into Eqn (11), we obtain the blending surface between the two intersecting planes which is shown in Fig. 3. In the second example, we obtain a non-symmetrical blending surface by using unequal values of geometric parameters h_0 and s_0. The geometric parameters used in this case are: $h_0 = 0.4$, $s_0 = 0.8$, $h_1 = h_3 = 1$, $h_2 = h_4 = 5$ and $\theta = -30^o$. The blending surface is given in Fig. 4.

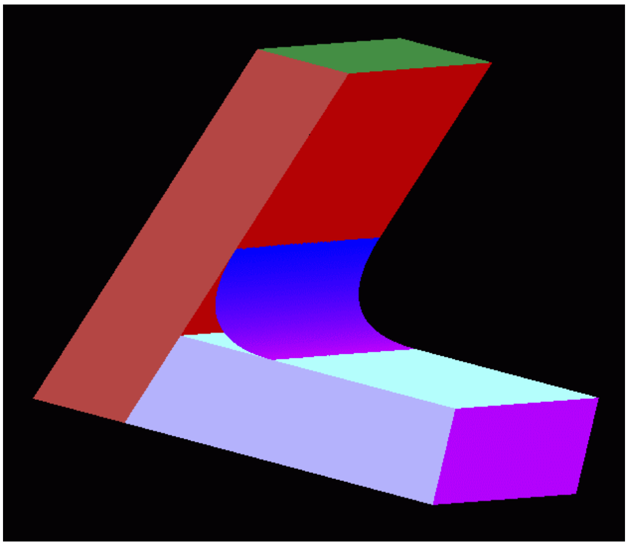

Fig. 3 Symmetrical blending

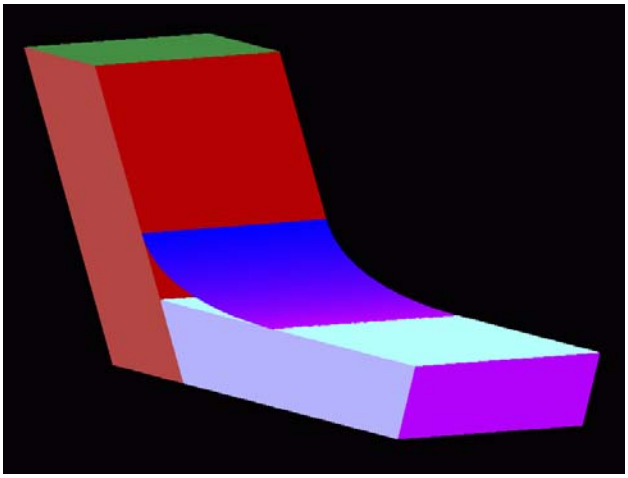

Fig. 4 Non-symmetrical blending

BLENDING AN ELLIPTIC CYLINDER AND A HORIZONTAL PLANE

Finally, we will study the blending between an elliptic cylinder and a horizontal plane with the linkage curve being an elliptic circle on it. The elliptic cylinder is placed in the vertical direction whose radii of the long and short axes are r_{sx} and r_{sy}, respectively, and whose height is h_1. The radii of the long and short axes of the elliptic circle on the horizontal plane are r_{bx} and r_{by}, respectively. The solving region is defined in Ω: $0 \le u \le 1$, $0 \le v \le 2\pi$. The blending surface meets the elliptic cylinder at $u = 0$ where the value of z co-ordinate is h_0 and meets the elliptic circle at $u = 1$. Both Eqns (6) and (8) are applied to generate the blending surface. For both equations, the geometric parameters for the blending are taken as: $r_{sy} = h_0 = h_1 = 0.5$, $r_{sx} = r_{by} = 1$ and $r_{bx} = 2$. The boundary conditions for this blending can be given as

$$
\begin{aligned}
u = 0 \qquad & x = r_{sx} \cos v \\
& y = r_{sy} \sin v \\
& z = h_0 \\
& \frac{\partial x}{\partial u} = 0 \\
& \frac{\partial x}{\partial u} = 0 \\
& \frac{\partial z}{\partial u} = -h_1 \\
u = 1 \qquad & x = r_{bx} \cos v \\
& y = r_{by} \sin v \\
& z = 0 \\
& \frac{\partial x}{\partial u} = r_{bx} \cos v \\
& \frac{\partial y}{\partial u} = r_{by} \sin v \\
& \frac{\partial z}{\partial u} = 0
\end{aligned}
\tag{12}
$$

Respectively substituting Eqn (6) and (8) into (12), all the integrating constants in these two equations will be determined. Then, these two equations can be applied to generate the blending surface.

For Eqn (8), the parameters in the equation are taken to be: $b_x = b_y = d_x = d_y = 1$ and $c_x = c_y = 2$ which generate the solution of the partial differential equation proposed by Bloor et al. The blending surface is given in Fig. 5. For Eqn (6), we take the parameters in the equation to be: $b_x = b_y = d_y = 1$, $c_y = 2$, $c_x = 20$ and $d_x = 90$. The blending surface is shown in Fig. 6. These examples show that the three-parameter partial differential equation proposed in this paper are able to generate more variety of blending surfaces than the one-parameter partial differential equation.

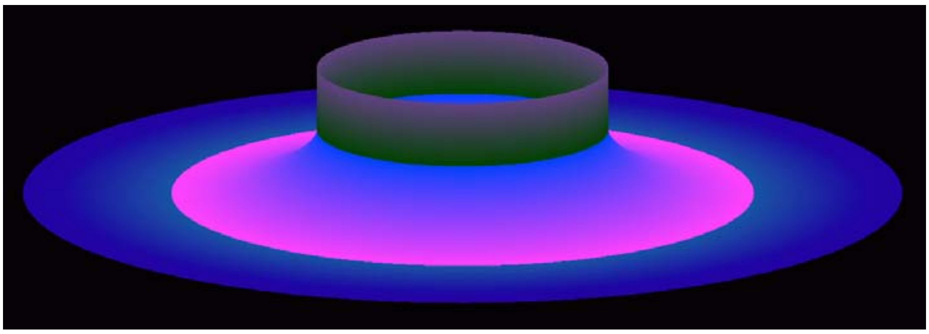

Fig. 5 Blending using equation (6)

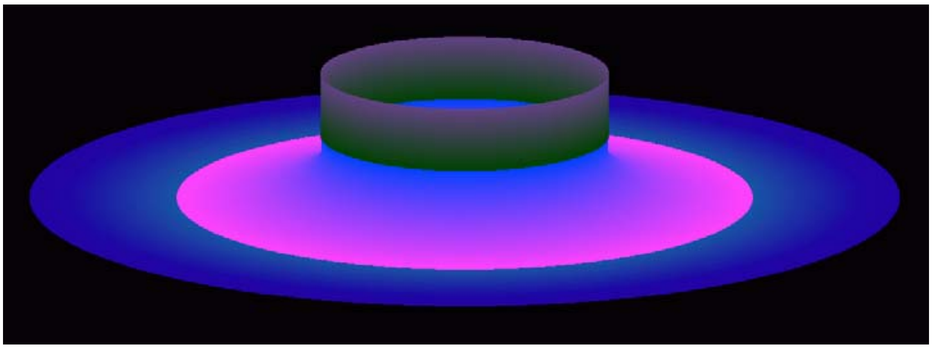

Fig. 5 Blending using equation (8)

CONCLUSION

In this paper, a fourth order partial differential equation with three vector-valued parameters was developed and applied to the generation of blending surfaces. This is a more general form of the PDEs for surface blending. The one-parameter fourth order partial differential equation proposed by Bloor et al. can be derived from the proposed PDE as a special case. Three examples were studied and the closed form solutions for these examples were implemented using OpenGL on a Silican Graphics workstation. The results indicate that the three-parameter fourth order partial differential equation presents more flexibility and geometric variety than the one-parameter fourth order partial differential equation.

ACKNOWLEDGEMENT

We would like to gratefully acknowledge Professor P Comninos at the National Centre for Computer Animation at Bournemouth University for his valuable discussions.

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